The smash product of monoidal theories

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Abstract. The tensor product of props was defined by Hackney and Robertson as an extension of the Boardman–Vogt product of operads to more general monoidal theories. Theories that factor as tensor products include the theory of commutative monoids and the theory of bialgebras. We give a topological interpretation (and vast generalisation) of this construction as a low-dimensional projection of a "smash product of pointed directed spaces". Here directed spaces are embodied by combinatorial structures called diagrammatic sets, while Gray products replace cartesian products. The correspondence is mediated by a web of adjunctions relating diagrammatic sets, pros, probs, props, and Gray-categories. The untruncated smash product applies to higher-dimensional theories and systematically produces higher-dimensional coherence cells.

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Introduction

In a categorical tradition of universal algebra dating back to F. William Lawvere's thesis [Law63], algebraic theories are embodied by cartesian monoidal categories whose objects are freely generated from a set of sorts. Models of the

theory, embodied by strong monoidal functors, may live in an arbitrary cartesian monoidal category: we specialise to the category of sets and functions to recover the classical notion.

Following this fundamental shift in perspective, and considering that cartesianness of a monoidal structure can be defined equationally [Fox76], it is a relatively small step to consider more general *monoidal theories* whose models live in arbitrary monoidal categories.

Monoidal theories are embodied by structures called *pros* [ML65].[1] Intermediate between monoidal theories and algebraic theories, there are *braided* and *symmetric* monoidal theories, embodied respectively by *probs* and *props*. The familiar term-algebraic calculus is inadequate for these generalised theories, and is commonly replaced with a calculus of string diagrams [Sel10].

Given theories T and S, a common occurrence is a mathematical object having both a structure of T-model and of S-model, satisfying some compatibility condition. For example, a bimodule is both a left and a right module, in such a way that the left and right actions commute. A natural question is: can we systematically *compose* theories, so that a model of the composite of T and S is an object with compatible T and S-model structures?

In a line of work that has been attracting attention in theoretical computer science [BSZ14, BSZ18], composition of monoidal theories is mediated by distributive laws which specify a factorisation system between operations of T and S, as described by Steve Lack [Lac04].

A less flexible, yet more uniform composition is the *tensor product*, which applies to all props and does not require additional data. The tensor product of props was defined and studied by Philip Hackney and Marcy Robertson [HR15],[2] who also proved that it extends, in a precise sense, the product of symmetric operads introduced by J. M. Boardman and R. M. Vogt [BV06].

Some intuition about the tensor product may be gained as follows. If M is a symmetric monoidal category, the category of T-models in M inherits a symmetric monoidal structure: to compose two models, we "run their operations in parallel", using the symmetric structure of M to rearrange inputs and outputs as needed. The data of a model in M of the tensor product $T \otimes_{\mathbb{S}} S$ is equivalent to the data of an S-model in the category of T-models in M.

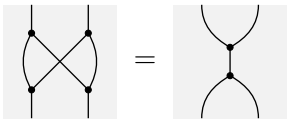
As remarked in [Lac04, §4.2], there is something mysterious about the rôle of symmetric braidings in the composition of monoidal theories. From a certain

[1] In some sources, the term *pro* or *PRO* is reserved for a one-sorted theory, and the multi-sorted variant is called a *coloured* pro.

[2] Although its possibility was noticed earlier by John C. Baez [Bae06]. This is also a nice survey of the relations between pros, probs, props, and algebraic theories.

perspective, a symmetric braiding is just another operation in a pro, yet it plays an inescapable structural rôle in the tensor product.

Consider the theories of *monoids* and *comonoids*. These are *planar* monoidal theories, naturally embodied by a pro: in the corresponding prop, symmetric braidings are added freely, so that models in the sense of props are equivalent to models in the sense of pros. Nevertheless, their tensor product — the theory of *bialgebras* — features the non-planar equation



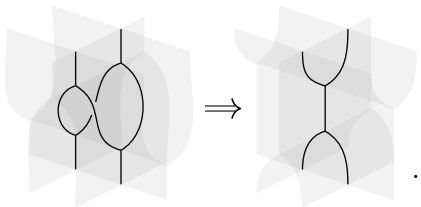
(1)

where the symmetric braiding in the left-hand side cannot clearly be attributed to either factor.

In particular, the tensor product of props does not restrict to a monoidal structure on pros. At most, as shown in Section 5.1, we can define an "external" tensor product which takes two pros and returns a prob, from which we can then universally reconstruct the tensor product of props.

A few years ago, we noticed that equation (1) admits the following topological interpretation.[3] Take the string diagrams corresponding to monoid multiplication and comonoid comultiplication, and extend them along perpendicular directions in the plane so that they form *branching surfaces*:

Intersect the two branching surfaces and "slide" one past another along the vertical axis. As one branching slides past the other branching, the intersection — a "string diagram in 3-dimensional space" — evolves as in the following figure:



(2)

The two sides of (1) arise as planar projections of the two sides of (2).

[3] A similar observation was made, around the same time, by J. Scott Carter [Car18].

This interpretation extends to all "compatibility" equations in the tensor product of props, and recasts the tensor product as a *dimension-raising* construction: given planar diagrams, it produces equations of 3-dimensional diagrams. This solves our conundrum about symmetric braidings: they are absent in the 3-dimensional picture, and only appear in the 2-dimensional picture as an artefact of planar projection.

What is going on? As we first detailed in [Had17, Section 2.3], the correct interpretation of (2) is that it arises from a *smash product of pointed directed spaces*, in a sense that we will soon explain. To make this precise, we needed a framework for diagrammatic reasoning in weak higher categories of arbitrary dimension, which was not available at the time. We have since developed a suitable foundation: the theory of *diagrammatic sets* [Had20b].

The aim of this article is the statement and proof of Theorem 5.29: the "external" tensor product of pros factors functorially through the smash product of pointed diagrammatic sets. Through this result, we can attribute a precise meaning to our earlier statements, such as the assertion that equation (1) arises from (2). On our way, we develop a great deal of combinatorics in order to relate diagrammatic sets, pros, and probs via a web of adjunctions involving a few "ancillary" higher structures.

We see this result not as an end point, but as opening. Far from being just a reinterpretation, our smash product is a vast generalisation of the tensor product of props, and transitively of the Boardman–Vogt product of operads.

Indeed, pointed diagrammatic sets can embody higher-dimensional theories with non-invertible generators in arbitrarily high dimension.[4] From these, the smash product generates non-invertible higher-dimensional cells rather than equations. Already when applied to 3-dimensional presentations of monoidal theories [Mim14], this construction produces interesting *coherence* cells up to dimension 6, of the kind that is typically constructed with methods of rewriting theory such as the computation of critical pairs.

Monoidal theories, directed spaces, and diagrammatic sets

The connection between monoidal theories and directed spaces is founded on four conceptual leaps. The first leap, as mentioned, is Lawvere's realisation that monoidal categories can embody algebraic theories.

The second is John C. Baez and James Dolan's *periodic table of n-categories*, by which a monoidal category is equivalent to a bicategory with a single 0-cell, but a braided monoidal category is equivalent to a *tricategory* with a single

[4] As opposed to structures used in homotopical algebra, such as ∞-operads, that embody theories with *invertible* higher data.

0-cell and 1-cell [BD95]. A variant of this result implies that a pro is a special kind of 2-category, while a prob is a special kind of Gray-category, a semistrict notion of tricategory [GPS95]. This matches the intuition that tensoring pros to obtain a prob is dimension-raising.

The third is Alexander Grothendieck's *homotopy hypothesis*, that "spaces", more precisely homotopy types, are modelled in a precise combinatorial sense by higher groupoids. In models where higher groupoids are higher categories whose cells are all invertible in a weak sense, this leaves open the possibility of interpreting higher categories as "spaces of directed cells".

The fourth and final leap is Albert Burroni's observation that various notions of presentations by generators and relations, or rewrite systems, can be unified as presentations of "cell complexes in a category of higher categories", a notion of directed space with combinatorial structure [Bur93].

Following the sequence, we can reinterpret a monoidal theory with its set of sorts as a kind of directed 2-dimensional space containing a 1-dimensional cell complex. A braided monoidal theory is the same thing one dimension up.

These spaces are canonically pointed with the unique 0-cell in the cell complex structure. It is natural, at this point, to wonder about a directed counterpart of the classical smash product of pointed spaces. The correct generalisation replaces the cartesian product of spaces with a version of the *Gray product* [Ste04].

In [Had17, Section 2.3], we considered smash products in the context of Burroni and Ross Street's theory of polygraphs, based on strict ω-categories. This had the advantage that a theory of Gray products had already been developed [AM20], and that we could identify a pro directly with a pointed 2-category. However, in this context the smash product of pros produces a strict 3-category equivalent not to a braided monoidal category, but to a highly degenerate *commutative* monoidal category.[5]

In [Had20b], based on an abandoned idea of Mikhail M. Kapranov and Vladimir V. Voevodsky, we developed the theory of diagrammatic sets as an alternative to polygraphs that would avoid this pitfall and support rewriting and diagrammatic reasoning in weak higher categories.

While the model of a directed cell in a polygraph is algebraic, diagrammatic sets adopt a combinatorial model. Roughly, a model of a directed n-cell is the face poset of a regular CW-decomposition of the topological closed n-ball, together with an orientation subdividing the boundary of each cell into an *input* and *output* half, in such a way that the input and output half are also

[5] This is connected to the known failure of the homotopy hypothesis for strict ω-categories, see [Sim09, Chapter 4].

face posets of regular CW-balls, and their orientations determine a composable pasting diagram in a strict ω-category. Common higher-categorical shapes such as oriented simplices and cubes, and even less common ones such as positive opetopes [Zaw17] appear as special cases.

A pleasant outcome of the transition to the combinatorial setup is that Gray products and smash products are much easier to define and compute. On the other hand, the identification of pros or probs with certain pointed diagrammatic sets is non-trivial. The technical core of this article is the definition of a full and faithful diagrammatic nerve of pros (Section 4.1), and then of a non-trivial realisation functor of diagrammatic sets in Gray-categories (Section 4.2), which allows us to recover the tensor product of two pros as the realisation of the smash product of their nerves.[6]

Diagrammatic sets are related to (compactly generated Hausdorff) spaces by a nerve-realisation pair [Had20b, Section 8.3], where the nerve actualises a version of the homotopy hypothesis. As detailed in Section 5.2, the geometric realisation sends Gray products to cartesian products, so it sends smash products to smash products.

Altogether, our results amount to the surprising fact that *the tensor product of pros and the smash product of pointed spaces are two facets of the same construction.*

Related work

We have already paid tribute to our main influences on the conceptual side. On the other hand, this article is technically most indebted to three sources.

The first is Hackney and Robertson's article [HR15]: beyond the fact that they introduced the tensor product of props, our proofs in Section 2 that certain categories of pros and probs have small limits and colimits are essentially lifted from their work on the category of props, with minor tweaks.

The second is John Power's work on pasting diagrams [Pow91]. While our formalisation of diagrams is based on Richard Steiner's combinatorial framework [Ste93], an analogue of Power's *domain replacement condition* turns out to be key to the constructions of Section 4, and the technical Section 3 is devoted to showing that it holds for all our 3-dimensional diagrams.

In particular, our Theorem 3.25 is roughly equivalent in meaning to [Pow91, Theorem 4.14]. Interestingly, Power's topological setup seems to have completely different strengths and weaknesses compared to our combinatorial setup: when translating Power's proofs, we discovered that every single non-

[6]More precisely, of their nerves with an orientation reversal in the second factor, as explained in Section 5.3.

trivial step in his proofs followed easily from our definitions, whereas the trivial steps would require non-trivial proofs, as in Proposition 3.22. No formal comparison has been made, to our knowledge, between Steiner's and Power's theory, so we think it is justified to consider our results original.

The third is Simon Forest and Samuel Mimram's article on the rewriting theory of Gray-categories [FM18]. Not only we learned from them a convenient axiomatisation of Gray-categories, but the construction of Section 4.2 draws directly on their ideas and results and can be seen as a continuation of their work, showing that every diagrammatic set presents a Gray-category.

Structure of the article

Notation: say that I use diagrammatic order for morphism composition (but not for functors).

Outlook and open problems

Mention the extension of the "realisation in Gray-categories" to associative n-categories.

1. Some higher structures

1.1. Directed complexes and diagrammatic sets

We quickly go through the main definitions, and refer the reader to [Had20b] for an in-depth development.

1.1 (Graded poset)**.** Let P be a finite poset with order relation $\leq$. For all elements $x, y \in P$, we say that y *covers* x if $x < y$ and, for all $y' \in X$, if $x < y' \leq y$ then $y' = y$.

The *Hasse diagram* of P is the finite directed graph $\mathscr{H}P$ with $\mathscr{H}P_0 \coloneqq P$ as set of vertices and $\mathscr{H}P_1 \coloneqq \{y \to x \mid y \text{ covers } x\}$ as set of edges.

Let $P_\perp$ be P extended with a least element $\perp$. We say that P is *graded* if, for all $x \in P$, all directed paths from x to $\perp$ in $\mathscr{H}P_\perp$ have the same length. If this length is $n+1$, we let $\dim(x) \coloneqq n$ be the *dimension* of x.

1.2 (Closed and pure subsets)**.** Let P be a poset and $U \subseteq P$. The *closure* of U is the subset $\mathrm{cl}U \coloneqq \{x \in P \mid \exists y \in U\ x \leq y\}$ of P. We say that U is *closed* if $U = \mathrm{cl}U$.

Suppose P is graded and $U \subseteq P$ is closed. Then U is graded with the partial order inherited from P. The *dimension* $\dim(U)$ of U is $\max\{\dim(x) \mid x \in U\}$ if U is inhabited, -1 otherwise. In particular, $\dim(\mathrm{cl}\{x\}) = \dim(x)$.

We say that U is *pure* if its maximal elements all have dimension $\dim(U)$.

1.3 (Oriented graded poset)**.** An *orientation* on a finite poset P is an edge-labelling $o\colon \mathscr{H}P_1 \to \{+,-\}$ of its Hasse diagram.

An *oriented graded poset* is a finite graded poset with an orientation.

1.4. We will often let variables α, β range implicitly over $\{+,-\}$.

1.5 (Boundaries)**.** Let P be an oriented graded poset and $U \subseteq P$ a closed subset. Then U inherits an orientation from P by restriction.

For all $\alpha \in \{+,-\}$ and $n \in \mathbb{N}$, we define

$$\begin{aligned}
\Delta^\alpha_n U &\coloneqq \{x \in U \mid \dim(x) = n \text{ and if } y \in U \text{ covers } x, \text{ then } o(y \to x) = \alpha\},\\
\partial^\alpha_n U &\coloneqq \mathrm{cl}(\Delta^\alpha_n U) \cup \{x \in U \mid \text{for all } y \in U, \text{ if } x \leq y, \text{ then } \dim(y) \leq n\},\\
\Delta_n U &\coloneqq \Delta^+_n U \cup \Delta^-_n U, \qquad\qquad \partial_n U \coloneqq \partial^+_n U \cup \partial^-_n U.
\end{aligned}$$

We call $\partial^-_n U$ the *input n-boundary*, and $\partial^+_n U$ the *output n-boundary* of U.

If U is $(n+1)$-dimensional, we write $\Delta^\alpha U \coloneqq \Delta^\alpha_n U$ and $\partial^\alpha U \coloneqq \partial^\alpha_n U$. For each $x \in P$, we write $\Delta^\alpha_n x \coloneqq \Delta^\alpha_n \mathrm{cl}\{x\}$ and $\partial^\alpha_n x \coloneqq \partial^\alpha_n \mathrm{cl}\{x\}$.

1.6 (Atoms and molecules)**.** Let P be an oriented graded poset. We define a family of closed subsets of P, the *molecules* of P, by induction on proper subsets. If U is a closed subset of P, then U is a molecule if either

- U has a greatest element, in which case we call it an *atom*, or
- there exist molecules U_1 and U_2, both properly contained in U, and $n \in \mathbb{N}$ such that $U_1 \cap U_2 = \partial^+_n U_1 = \partial^-_n U_2$ and $U = U_1 \cup U_2$.

We define $\sqsubseteq$ to be the smallest partial order relation such that, if U_1 and U_2 are molecules and $U_1 \cap U_2 = \partial^+_n U_1 = \partial^-_n U_2$, then $U_1, U_2 \sqsubseteq U_1 \cup U_2$.

We say *n-molecule* for an n-dimensional molecule. We say that P itself is a molecule if $P \subseteq P$ is a molecule.

1.7 (Spherical boundary)**.** An n-molecule U in an oriented graded poset *has spherical boundary* if, for all $k < n$,

$$\partial^+_k U \cap \partial^-_k U = \partial_{k-1} U.$$

1.8 (Regular directed complex)**.** An oriented graded poset P is a *regular directed complex* if, for all $x \in P$ and $\alpha, \beta \in \{+,-\}$,

1. $\mathrm{cl}\{x\}$ has spherical boundary,
2. $\partial^\alpha x$ is a molecule, and
3. $\partial^\alpha(\partial^\beta x) = \partial^\alpha_{n-2} x$ if $n \coloneqq \dim(x) > 1$.

A *map* $f\colon P \to Q$ of regular directed complexes is a function of their underlying sets that satisfies

$$\partial^\alpha_n f(x) = f(\partial^\alpha_n x)$$

for all $x \in P$, $n \in \mathbb{N}$, and $\alpha \in \{+,-\}$. We call an injective map an *inclusion*. With their maps, regular directed complexes form a category $\mathbf{DCpx}^{\mathcal{R}}$.

1.9 (Regular molecule)**.** A *regular molecule* is a molecule in a regular directed complex.

By [Had20b, Proposition 1.38], if two regular molecules are isomorphic in $\mathbf{DCpx}^{\mathcal{R}}$, they are isomorphic in a unique way. As customary in these situations, we will treat isomorphic regular molecules as "equal" under appropriate circumstances.

1.10 (Globe)**.** For each $n \in \mathbb{N}$, let O^n be the poset with a pair of elements $\underline{k}^+, \underline{k}^-$ for each $k < n$ and a greatest element $\underline{n}$, with the partial order defined by $\underline{j}^\alpha \le \underline{k}^\beta$ if and only if $j \le k$. This is a graded poset, with $\dim(\underline{n}) = n$ and $\dim(\underline{k}^\alpha) = k$ for all $k < n$.

With the orientation $o(y \to \underline{k}^\alpha) \coloneqq \alpha$ if y covers $\underline{k}^\alpha$, O^n becomes a regular directed complex, in particular a regular atom. We call O^n the *n-globe*.

1.11 (Pasting of molecules)**.** Let U_1, U_2 be regular molecules and suppose that $\partial^+_k U_1$ and $\partial^-_k U_2$ are isomorphic in $\mathbf{DCpx}^{\mathcal{R}}$. Given an isomorphic copy V of the two, there is a unique span of inclusions $V \hookrightarrow U_1$ and $V \hookrightarrow U_2$ whose images are, respectively, $\partial^+_k U_1$ and $\partial^-_k U_2$. We let $U_1 \#_k U_2$ be the pushout

$$\begin{array}{ccc} V & \hookrightarrow & U_2 \\ \downarrow & & \downarrow \\ U_1 & \hookrightarrow & U_1 \#_k U_2 \end{array}$$

in $\mathbf{DCpx}^{\mathcal{R}}$. Then $U_1 \#_k U_2$ is a regular molecule, decomposing as $U_1 \cup U_2$ with $U_1 \cap U_2 = \partial^+_k U_1 = \partial^-_k U_2$.

1.12 ($- \Rightarrow -$ construction)**.** Let U, V be regular n-molecules with spherical boundary such that $\partial^\alpha U$ is isomorphic to $\partial^\alpha V$ for all $\alpha \in \{+,-\}$.

Form the pushout $U \cup V$ of the span of inclusions $\partial U \hookrightarrow U$, $\partial U \hookrightarrow V$ whose images are ∂U and ∂V, respectively. We define $U \Rightarrow V$ to be the oriented graded poset obtained from $U \cup V$ by adjoining a greatest element $\top$ with $\partial^- \top \coloneqq U$ and $\partial^+ \top \coloneqq V$. Then $U \Rightarrow V$ is an $(n+1)$-dimensional atom with spherical boundary.

1.13 ($\langle - \rangle$ construction)**.** Let U be a regular molecule with spherical boundary. Then $\partial^- U \Rightarrow \partial^+ U$ is defined, and we denote it by $\langle U \rangle$.

1.14. There is a unique 0-atom, namely, the 0-globe $1 \coloneqq O^0$, which is also the terminal object of $\mathbf{DCpx}^{\mathcal{R}}$.

We define a sequence $\{I_n\}_{n>0}$ of 1-molecules by

$$I_1 \coloneqq O^1, \qquad I_n \coloneqq I_{n-1} \#_0 O^1 \text{ for } n > 1.$$

Every regular 1-molecule is of the form I_n for some $n > 0$.

For each pair $n, m > 0$, let $U_{n,m} \coloneqq (I_n \Rightarrow I_m)$. Every regular 2-atom is of the form $U_{n,m}$ for some $n, m > 0$. Regular 2-molecules are then generated by I_1 and the $U_{n,m}$ under the pasting operations $\#_0$, $\#_1$.

1.15 (Diagrammatic set)**.** We write $\odot$ for a skeleton of the full subcategory of $\mathbf{DCpx}^{\mathcal{R}}$ on the atoms of every dimension.

A *diagrammatic set* is a presheaf on $\odot$. Diagrammatic sets and their morphisms of presheaves form a category $\odot\mathbf{Set}$.

Comment 1.16. The definition in [Had20b] is relative to a fixed "convenient" class of molecules; for simplicity, here we pick the class of all molecules with spherical boundary.

1.17. We identify $\odot$ with a full subcategory $\odot \hookrightarrow \odot\mathbf{Set}$ via the Yoneda embedding. With this identification, we use morphisms in $\odot\mathbf{Set}$ as our notation for both elements and structural operations of a diagrammatic set X:

- $x \in X(U)$ becomes $x \colon U \to X$, and
- for each map $f \colon V \to U$ in $\odot$, $X(f)(x) \in X(V)$ becomes $f; x \colon V \to X$.

As described in [Had20b, §4.4], the embedding $\odot \hookrightarrow \odot\mathbf{Set}$ extends to an embedding $\mathbf{DCpx}^{\mathcal{R}} \hookrightarrow \odot\mathbf{Set}$.

1.18 (Diagrams and cells)**.** Let X be a diagrammatic set and U a molecule in $\mathbf{DCpx}^{\mathcal{R}}$. A *diagram of shape U in X* is a morphism $x \colon U \to X$. It is *composable* if U has spherical boundary and a *cell* if U is an atom. For all $n \in \mathbb{N}$, we say that x is *n-diagram* or an *n-cell* when $\dim(U) = n$.

Given an inclusion $\imath \colon V \hookrightarrow U$ of molecules, we write $x|_V$ for the diagram $\imath; x \colon V \to X$. If U decomposes as $U_1 \#_k U_2$, we write $x = x_1 \#_k x_2$ for $x_i \coloneqq x|_{U_i}$, $i \in \{1, 2\}$.

If $x \colon U \to X$ is a diagram in X and $f \colon X \to Y$ a morphism of diagrammatic sets, we may write $f(x)$ for the diagram $x; f \colon U \to Y$.

1.19 (Boundaries of diagrams)**.** Let X be a diagrammatic set, $x \colon U \to X$ a diagram, and let $\imath^\alpha_k \colon \partial^\alpha_k U \hookrightarrow U$ be the inclusions of the k-boundaries of U. The *input k-boundary* of x is the diagram $\partial^-_k x \coloneqq \imath^-_k; x$ and the *output*

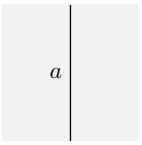
k-boundary of x is the diagram $\partial_k^+ x \coloneqq \imath_k^+ ; x$. We may omit the index k when $k = \dim(U) - 1$.

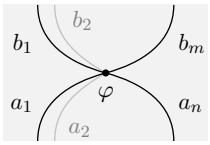
We write $x \colon y^- \Rightarrow y^+$ to express that $\partial_k^\alpha x = y^\alpha$ for each $\alpha \in \{+, -\}$, and say that x is of *type* $y^- \Rightarrow y^+$. We say that two diagrams x_1, x_2 are *parallel* if they have the same type.

1.20. A 1-cell a in a diagrammatic set has shape I_1. A 2-cell φ has shape $U_{n,m}$ for some $n, m > 0$, so it is of type

$$a_1 \#_0 \ldots \#_0 a_n \Rightarrow b_1 \#_0 \ldots \#_0 b_m$$

for some 1-cells $a_1, \ldots, a_n, b_1, \ldots, b_m$. We may depict such cells as string diagrams

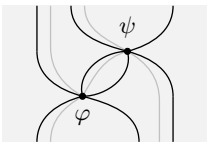


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where a lighter shade indicates a repeated pattern. Each region of space bounded by wires corresponds to a potentially different 0-cell; in practice, we will mostly work with diagrammatic sets that have a single 0-cell. Labels will be omitted when irrelevant, or implied by the shape of a cell.

A 2-diagram decomposes into 1-cells and 2-cells under the $\#_0$, $\#_1$ operations. In string diagrams, $\#_0$ is horizontal juxtaposition and $\#_1$ is vertical juxtaposition with the output wires of one diagram connecting to the input wires of another. For example,



depicts a generic 2-diagram of the form

$$(\varphi \#_0 a_1 \#_0 \ldots \#_0 a_n) \#_1 (b_1 \#_0 \ldots \#_0 b_m \#_0 \psi)$$

for some 2-cells φ, ψ and 1-cells $a_1, \ldots, a_n, b_1, \ldots, b_m$.

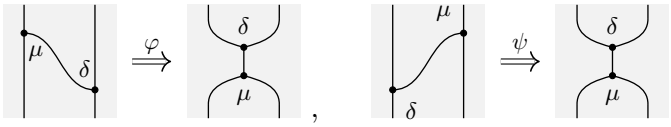
Observe that, in our setting, there is no need to attribute a topological nature to string diagrams, *à la* Joyal and Street [JS91]: they should instead be interpreted as compact encodings of regular molecules – a discrete, combinatorial structure – and their morphisms into diagrammatic sets.

A 2-diagram is composable if and only if it is *connected* as a string diagram. For example, of the diagrams

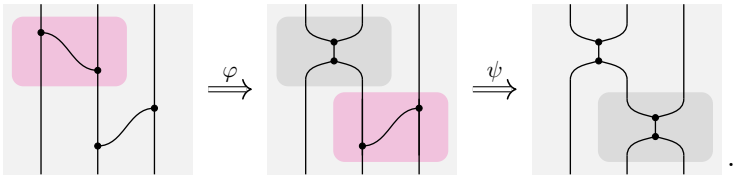


only the first one is composable.

We may depict a 3-diagram as a *sequence of rewrites* on composable subdiagrams of a diagram. For example, a diagrammatic set with a single 0-cell, a single 1-cell, and 3-cells φ, ψ of the form



will have a 3-diagram of the form



(3)

where the input boundary of each 3-cell is highlighted in pink.

We will also use string diagrams to describe certain regular molecules. This is justified by the interpretation of a molecule U as the "tautologous" diagram $\mathrm{id}_U : U \to U$ in $\dot{\odot}\mathbf{Set}$.

1.21 (Dual diagrammatic set)**.** Let U be a regular atom. The oriented graded poset U° with the same underlying poset as U and the opposite orientation $o^\circ(y \to x) \coloneqq -o(y \to x)$ is a regular atom. If $f : U \to V$ is a map in $\dot{\odot}$, its underlying function also defines a map $f^\circ : U^\circ \to V^\circ$. This determines an involution $-^\circ$ on $\dot{\odot}$.

Let X be a diagrammatic set. Its *dual* X° is the diagrammatic set defined by $X^\circ(-) \coloneqq X(-^\circ)$. This extends to morphisms in the obvious way, and extends the involution on $\dot{\odot}$ to an involution on $\dot{\odot}\mathbf{Set}$.

Remark 1.22. If $x : U \to X$ is a 2-diagram, the depiction of $x^\circ : U^\circ \to X^\circ$ in string diagrams is the horizontal and vertical reflection of the depiction of x.

1.2. Higher-categorical structures

1.23 (Reflexive ω-graph)**.** Let $\mathbf{O}$ be the full subcategory of $\odot$ whose objects are the globes O^n. For all n and $k < n$,

- the k-boundary inclusions ι_k^+, ι_k^- are the only inclusions of O^k into O^n;
- the map $\tau\colon O^n \twoheadrightarrow O^k$, defined by $\tau(\underline{n}), \tau(\underline{j}^\alpha) \coloneqq \underline{k}$ if $j \geq k$ and $\tau(\underline{j}^\alpha) \coloneqq \underline{j}^\alpha$ if $j < k$, is the only surjective map from O^n onto O^k.

A *reflexive ω-graph* is a presheaf X on $\mathbf{O}$. With their morphisms of presheaves, reflexive ω-graphs form a category $\omega\mathbf{Gph}_{ref}$.

1.24. The embedding $\mathbf{O} \hookrightarrow \odot$ induces a restriction functor $\odot\mathbf{Set} \to \omega\mathbf{Gph}_{ref}$ with a full and faithful left adjoint $\omega\mathbf{Gph}_{ref} \hookrightarrow \odot\mathbf{Set}$; we can thus identify reflexive ω-graphs with particular diagrammatic sets, and use for them the same terminology and notation.

Because all n-cells in a reflexive ω-graph X have the same shape O^n, we leave it implicit and write $X_n \coloneqq X(O^n)$.

1.25 (Units)**.** Let x be a k-cell in a reflexive ω-graph X. For $n > k$, we let $\varepsilon_n x \coloneqq \tau; x$ where τ is the unique surjective map $O^n \twoheadrightarrow O^k$. We call $\varepsilon_n x$ a *unit* on x. We may omit the index when $n = k + 1$.

1.26 (Rank of a cell)**.** Let x be an n-cell in a reflexive ω-graph. The *rank* $\mathrm{rk}(x)$ of x is defined inductively on n by

- if $n = 0$, then $\mathrm{rk}(x) \coloneqq 0$;
- if $n > 0$, if $x = \varepsilon y$ for an $(n-1)$-cell y, then $\mathrm{rk}(x) \coloneqq \mathrm{rk}(y)$, otherwise $\mathrm{rk}(x) \coloneqq n$.

1.27 (Partial ω-category)**.** A *partial ω-category* is a reflexive ω-graph X together with partial k-composition operations

$$\#_k : X_n \times X_n \rightharpoonup X_n$$

for all $n \in \mathbb{N}$ and $k < n$, satisfying the following axioms:

1. for all n-cells x, y and all $k < n$ such that $x \#_k y$ is defined,

$$\partial_k^+ x = \partial_k^- y \text{ and } \varepsilon(x \#_k y) = \varepsilon x \#_k \varepsilon y;$$

2. for all n-cells x and all $k < n$, the k-compositions

$$x \#_k \varepsilon_n(\partial_k^+ x) \text{ and } \varepsilon_n(\partial_k^- x) \#_k x$$

are defined and equal to x;

3. for all $(n+1)$-cells x, y and $k < n$, whenever the left-hand side is defined,

$$\begin{aligned}\partial^-(x \#_n y) &= \partial^- x,\\ \partial^+(x \#_n y) &= \partial^+ y,\\ \partial^\alpha(x \#_k y) &= \partial^\alpha x \#_k \partial^\alpha y;\end{aligned}$$

4. for all cells x, y, z and all k such that both sides are defined,

$$(x \#_k y) \#_k z = x \#_k (y \#_k z);$$

5. for all cells x, y, x', y', all n and all $k < n$ such that both sides are defined,

$$(x \#_n x') \#_k (y \#_n y') = (x \#_k y) \#_n (x' \#_k y'). \tag{4}$$

A *functor* $f\colon X \to Y$ of partial ω-categories is a morphism of the underlying reflexive ω-graphs such that, for all cells x, y in X, if $x \#_n y$ is defined in X then $f(x) \#_n f(y)$ is defined and equal to $f(x \#_n y)$ in Y. Partial ω-categories and their functors form a category $\mathbf{p}\omega\mathbf{Cat}$.

1.28. We will generally confuse the notation for a k-cell and the units on it when using it to form composites: that is, if x is an n-cell and y a k-cell, $k < n$, such that $x \#_m \varepsilon_n y$ is defined, we will write $x \#_m y \coloneqq x \#_m \varepsilon_n y$.

1.29 (ω-Precategory)**.** An *ω-precategory* is a partial ω-category X such that, for all n-cells x, y in X, the k-composition $x \#_k y$ is defined if and only if $\partial^+_k x = \partial^-_k y$ and $\min\{\mathrm{rk}(x), \mathrm{rk}(y)\} \leq k+1$. With their functors, ω-precategories form a category $\omega\mathbf{PreCat}$.

1.30 (ω-Category)**.** An *ω-category* is a partial ω-category such that, for all n-cells x, y in X, the k-composition $x \#_k y$ is defined if and only if $\partial^+_k x = \partial^-_k y$. With their functors, ω-categories form a category $\omega\mathbf{Cat}$.

1.31. The inclusion $\omega\mathbf{Cat} \hookrightarrow \mathbf{p}\omega\mathbf{Cat}$ has a left adjoint $-^*\colon \mathbf{p}\omega\mathbf{Cat} \to \omega\mathbf{Cat}$; if X is a partial ω-category, then X^* is the free ω-category on the underlying reflexive ω-graph of X, quotiented by all the equations involving compositions that are defined in X.

By [Had20b, Proposition 1.23], if P is a regular directed complex, there is a partial ω-category $\mathcal{M}o\ell P$ where

1. the set $\mathcal{M}o\ell P_n$ of n-cells is the set of molecules $U \subseteq P$ with $\dim(U) \leq n$,
2. $\partial^\alpha_k\colon \mathcal{M}o\ell P_n \to \mathcal{M}o\ell P_k$ is $U \mapsto \partial^\alpha_k U$,
3. $\varepsilon_n\colon \mathcal{M}o\ell P_k \to \mathcal{M}o\ell P_n$ is $U \mapsto U$,

4. $U \#_k V$ is defined if and only if $U \cap V = \partial_k^+ U = \partial_k^- V$, and in that case it is equal to $U \cup V$.

We will write $W = U \#_k V$ to signify that W is a molecule decomposing as $U \cup V$, where U and V are molecules with $U \cap V = \partial_k^+ U = \partial_k^- V$.

As detailed in [Section 7, *ibid.*], the assignment $P \mapsto \mathcal{M}o\ell P^*$ extends to a functor $\mathcal{M}o\ell -^* \colon \mathbf{DCpx}^{\mathcal{R}} \to \omega\mathbf{Cat}$ which is faithful and injective on objects.

1.32 (Principal composition)**.** Let x, y be n-cells in an ω-precategory or an ω-category, and let $k \coloneqq \min\{\mathrm{rk}(x), \mathrm{rk}(y)\} - 1$. If $\partial_k^+ x = \partial_k^- y$, the *principal composition* of x and y is

$$x \# y \coloneqq x \#_k y.$$

Comment 1.33. Given cells x, y in an ω-precategory, suppose that $x \#_k y$ is defined. Then either

- $\min\{\mathrm{rk}(x), \mathrm{rk}(y)\} = k + 1$, in which case $x \#_k y = x \# y$, or
- $\mathrm{rk}(x) \leq k$, in which case $x \#_k y = y$, or
- $\mathrm{rk}(y) \leq k$, in which case $x \#_k y = x$.

This implies that ω-precategories admit an axiomatisation involving only principal compositions, at the cost of explicitly handling some corner cases in the axioms.

Moreover, the two sides of (4) can both be defined only if

- both x and x' have rank lower or equal than $k + 1$, in which case (4) is equivalent to
$$x \#_k (y \#_n y') = (x \#_k y) \#_n (x \#_k y'),$$
or dually
- both y and y' have rank lower or equal than $k + 1$, in which case (4) is equivalent to
$$(x \#_n x') \#_k y = (x \#_k y) \#_n (x' \#_k y).$$

These two observations can be used to establish an equivalence between our definition of ω-precategory and the one given in [FM18, Section 4.1].

1.34. There is a forgetful functor $\mathsf{U} \colon \omega\mathbf{Cat} \to \omega\mathbf{PreCat}$ which makes $x \#_k y$ undefined whenever $\min\{\mathrm{rk}(x), \mathrm{rk}(y)\} > k + 1$.

Proposition 1.35 — *The functor $\mathsf{U} \colon \omega\mathbf{Cat} \to \omega\mathbf{PreCat}$ is full and faithful. Its image consists of the ω-precategories satisfying*

$$(x \#_{k-1} \partial_k^- y) \#_k (\partial_k^+ x \#_{k-1} y) = (\partial_k^- x \#_{k-1} y) \#_k (x \#_{k-1} \partial_k^+ y) \tag{5}$$

for all cells x, y with $\min\{\mathrm{rk}(x), \mathrm{rk}(y)\} = k + 1$ and $\partial_{k-1}^+ x = \partial_{k-1}^- y$.

Proof. First of all, observe that both sides of (5) are defined in an ω-precategory when x, y satisfy the conditions of the statement. If this precategory is of the form $\mathsf{U}X$ for some ω-category X, then both sides are equal to $x \#_{k-1} y$ in X, so (5) is satisfied.

Let X' be an ω-precategory such that (5) holds for all cells in X' in the conditions of the statement. We will define an ω-category X such that $\mathsf{U}X = X'$, which necessarily has the same underlying reflexive ω-graph as X'.

Let x, y be cells such that $\partial^+_{k-1}x = \partial^-_{k-1}y$ for some $k > 0$. We must define $x \#_{k-1} y$ in X. If $\min\{\mathrm{rk}(x), \mathrm{rk}(y)\} \leq k$, then $x \#_{k-1} y$ is defined in X', and we declare it to be the same in X.

Otherwise, $\min\{\mathrm{rk}(x), \mathrm{rk}(y)\} = k + 1 + m$ for some $m \geq 0$. If $m = 0$, we define $x \#_{k-1} y$ to be equal to either side of (5). For $m > 0$, observe that

$$\min\{\mathrm{rk}(x), \mathrm{rk}(\partial^\alpha_{k+m}y)\}, \min\{\mathrm{rk}(\partial^\alpha_{k+m}x), \mathrm{rk}(y)\} \leq k + m$$

for all $\alpha \in \{+, -\}$, but

$$\partial^+_{k-1}x = \partial^+_{k-1}(\partial^\alpha_{k+m}x) = \partial^-_{k-1}(\partial^\alpha_{k+m}y) = \partial^-_{k-1}y.$$

We may thus assume, inductively, that $x \#_{k-1} \partial^\alpha_{k+m}y$ and $\partial^\alpha_{k+m}x \#_{k-1} y$ have already been defined, and let

$$x \#_{k-1} y \coloneqq (x \#_{k-1} \partial^-_{k+m}y) \#_{k+m} (\partial^+_{k+m}x \#_{k-1} y).$$

It is an exercise to show, by induction, that this is equal to

$$(\partial^-_{k+m}x \#_{k-1} y) \#_{k+m} (x \#_{k-1} \partial^+_{k+m}y)$$

and derive that X is an ω-category. Because all the definitions are forced by the axioms of ω-categories, X is unique with the property that $\mathsf{U}X = X'$.

Since all compositions in X are defined in terms of compositions in $\mathsf{U}X$, every functor $f' \colon \mathsf{U}X \to \mathsf{U}Y$ of ω-precategories lifts to a functor $f \colon X \to Y$ of ω-categories. This proves fullness; faithfulness is immediate from the fact that f and $\mathsf{U}f$ have the same underlying morphism of reflexive ω-graphs. ∎

1.36 (Skeleta)**.** Let X be an ω-(pre)category, $n \in \mathbb{N}$. The *n-skeleton* $\sigma_{\leq n}X$ of X is the restriction of X to cells of rank $\leq n$. We let $\sigma_{\leq -1}X \coloneqq \emptyset$, the initial ω-(pre)category.

1.37 (n-Category)**.** An ω-(pre)category is an *n-(pre)category* if it is equal to its n-skeleton. An n-(pre)category is determined by its restriction to k-cells with $k \leq n$.

Let $n\mathbf{PreCat}$ denote the full subcategory of $\omega\mathbf{PreCat}$ and $n\mathbf{Cat}$ the full subcategory of $\omega\mathbf{Cat}$ on the n-(pre)categories. In both cases, $\sigma_{\leq n}$ is right adjoint to the inclusion of subcategories.

The inclusion also has a left adjoint $\tau_{\leq n}$: given an ω-(pre)category X, the n-(pre)category $\tau_{\leq n}X$ is obtained from $\sigma_{\leq n}X$ by identifying all pairs of n-cells x, y such that there exists an $(n+1)$-cell $e\colon x \Rightarrow y$ in X.

Remark 1.38. Both $\omega\mathbf{PreCat}$ and $\omega\mathbf{Cat}$ are categories of algebras for finitary monads on $\omega\mathbf{Gph}_{ref}$, a presheaf topos. By the Remark at the end of [AR94, §2.78], they are locally finitely presentable, and in particular have all small limits and colimits. The same applies to their reflective subcategories $n\mathbf{PreCat}$ and $n\mathbf{Cat}$ for all $n \in \mathbb{N}$.

1.39 (Polygraph)**.** Let $\partial O^n \coloneqq \sigma_{\leq n-1}O^n$.

A *(pre)polygraph* is an ω-(pre)category X together with a set $\mathscr{X} = \sum_{n\in\mathbb{N}} \mathscr{X}_n$ of *generating* cells such that, for all $n \in \mathbb{N}$,

$$\begin{array}{ccc} \coprod_{x\in\mathscr{X}_n} \partial O^n & \hookrightarrow & \coprod_{x\in\mathscr{X}_n} O^n \\ \downarrow & & \downarrow{\scriptstyle (x)_{x\in\mathscr{X}_n}} \\ \sigma_{\leq n-1}X & \hookrightarrow & \sigma_{\leq n}X \end{array}$$

is a pushout in $\omega\mathbf{PreCat}$ or $\omega\mathbf{Cat}$.

An *n-(pre)polygraph* is a (pre)polygraph whose underlying ω-(pre)category is an n-(pre)category. In an n-(pre)polygraph, $\mathscr{X}_m = \emptyset$ for $m > n$.

Remark 1.40. In every (pre)polygraph $(X, \mathscr{X})$, all cells in $\mathscr{X}_n$ have rank n. The set $\mathscr{X}_0$ is the entire set X_0 of 0-cells.

Example 1.41. Both 1-precategories and 1-categories coincide with small categories; a 1-(pre)polygraph is a category free on a graph.

2. Monoidal theories

2.1. Planar monoidal theories

For us, monoidal theories are embodied by *pros*. For technical reasons, we treat these as a special case of a more general structure of *bicoloured pro*, whose relation to pros is the same as the relation of bicategories to monoidal categories.

2.1 (Bicoloured pro)**.** A *bicoloured pro* is a 2-category T together with the structure of a 1-polygraph $(\sigma_{\leq 1}T, \mathscr{T})$ on its 1-skeleton.

A *morphism* $f\colon (T, \mathscr{T}) \to (S, \mathscr{S})$ of bicoloured pros is a functor $f\colon T \to S$ of 2-categories with the property that $f(a) \in \varepsilon(\mathscr{S}_0) \cup \mathscr{S}_1$ for all $a \in \mathscr{T}_1$. Bicoloured pros and their morphisms form a category $\mathbf{Pro}_{bi}$.

2.2 (Strict monoidal category)**.** A *strict monoidal category* is a 2-category with a single 0-cell.

Comment 2.3. We are using the characterisation of strict monoidal categories in [CG07, Theorem 4.1] as a definition.

2.4. When some object has a single 0-cell, we denote that 0-cell by $\bullet$.

2.5 (Pro)**.** A *pro* is a bicoloured pro with a single 0-cell. We let **Pro** denote the full subcategory of $\mathbf{Pro}_{bi}$ on pros.

Comment 2.6. That is, a pro is a bicoloured pro whose underlying 2-category is a strict monoidal category.

Comment 2.7. Equivalently, a bicoloured pro $(T, \mathscr{T})$ is a 2-category whose 1-cells of type $x \Rightarrow y$ are finite paths from x to y in the graph

$$\mathscr{T}_0 \underset{\partial^-}{\overset{\partial^+}{\leftarrow\!\!\leftarrow}} \mathscr{T}_1.$$

When $x = y$, the path is allowed to be of length 0, in which case it is interpreted as the unit εx.

If T has a single 0-cell, these are the same as finite ordered lists of elements of $\mathscr{T}_1$, that is, elements of the free monoid on $\mathscr{T}_1$. Seeing a pro as the embodiment of a monoidal theory, we interpret the elements of $\mathscr{T}_1$ as *sorts*, and 2-cells

$$\varphi\colon (a_1, \dots, a_n) \Rightarrow (b_1, \dots, b_m)$$

as *operations* taking n inputs of sorts $a_1, \dots, a_n$ and returning m outputs of sorts $b_1, \dots, b_m$.

In particular, if the monoidal theory is one-sorted, then $\mathscr{T}_1$ is a singleton, the 1-cells of T are in bijection with natural numbers, and the type of a 2-cell is fixed by the *arity* of its input and its output. In that case, we may write $\varphi\colon (n) \Rightarrow (m)$ for an operation with n inputs and m outputs.

If we forget the structure of 1-polygraph on a pro, we get a strict monoidal category T, which we may see as a special kind of monoidal category. Given a monoidal category M, a *model* in M of the monoidal theory $(T, \mathscr{T})$ is a strong monoidal functor from T to M.

Remark 2.8. Our definition of a morphism of pros allows a sort to be "collapsed" by mapping it to a unit. We make this choice for technical reasons, even though it seems more common to disallow it, see for example [HR15]. This choice does not have any bearing on the models.

Example 2.9. The monoid $\mathbb{N}$ of natural numbers, seen as a strict monoidal category with no rank-2 cells, is a one-sorted pro. This corresponds to the "trivial" theory of *objects* in a monoidal category.

Example 2.10. There is a one-sorted pro *Mon* whose 1-cell (n) is identified with the finite ordinal $\{0 < \ldots < n-1\}$ for each $n \in \mathbb{N}$, and 2-cells $\varphi\colon (n) \Rightarrow (m)$ are order-preserving maps. The 0-composite of $\varphi\colon (n) \Rightarrow (m)$ and $\psi\colon (p) \Rightarrow (q)$ is given by "concatenation", that is,

$$\varphi \#_0 \psi\colon (n+p) \Rightarrow (m+q), \qquad k \mapsto \begin{cases} \varphi(k) & \text{if } k < n, \\ m + \psi(k-n) & \text{if } k \geq n. \end{cases}$$

This corresponds to the theory of *monoids*.

Example 2.11. If $(T, \mathscr{T})$ is a pro, then $(T^{\mathrm{co}}, \mathscr{T}^{\mathrm{co}})$, obtained by reversing the orientation of all 2-cells of T, is also a pro. For example, Mon^{co} is the theory of *comonoids*.

Example 2.12. Consider the strict monoidal category *Mod* whose 1-cells are injective maps $\imath\colon (k) \hookrightarrow (n)$ in *Mon*, 2-cells $\varphi\colon (\imath\colon (k) \hookrightarrow (n)) \Rightarrow (j\colon (k) \hookrightarrow (m))$ are commutative triangles

$$\begin{array}{ccc} & (k) & \\ {\scriptstyle \imath}\swarrow & & \searrow{\scriptstyle j} \\ (n) & \xrightarrow[\varphi]{} & (m), \end{array}$$

and the 0-composite of a pair of 2-cells

$$\varphi\colon (\imath\colon (k) \hookrightarrow (n)) \Rightarrow (j\colon (k) \hookrightarrow (m)),$$
$$\psi\colon (\imath'\colon (\ell) \hookrightarrow (p)) \Rightarrow (j'\colon (\ell) \hookrightarrow (q))$$

is the commutative triangle

$$\begin{array}{ccc} & (k+\ell) & \\ {\scriptstyle \imath \#_0 \imath'}\swarrow & & \searrow{\scriptstyle j \#_0 j'} \\ (n+p) & \xrightarrow[\varphi \#_0 \psi]{} & (m+q). \end{array}$$

The 1-cells in *Mod* are freely generated under 0-composition by the pair

$$\imath\colon (0) \hookrightarrow (1), \qquad \mathrm{id}_1\colon (1) \to (1),$$

where $\imath$ is the unique inclusion of the empty ordinal, so *Mod* admits the structure of a two-sorted pro.

There are morphisms of pros $Mon \to Mod$ and $\mathbb{N} \to Mod$, sending the generating 1-cell to $\imath$ and id_1, respectively. The models of *Mod* in a monoidal category M are given by an object of M, a monoid in M, and a two-sided action of the monoid on the object, such that the left action commutes with the right action.

2.13. There is an obvious functor $\mathsf{U}\colon \mathbf{Pro}_{bi} \to 2\mathbf{Cat}$ forgetting the structure of 1-polygraph. This functor has a right adjoint $\mathsf{R}\colon 2\mathbf{Cat} \to \mathbf{Pro}_{bi}$, described as follows.

Given a 2-category X, the 1-skeleton of $\mathsf{R}X$ is free on the underlying *reflexive* graph of $\sigma_{\leq 1}X$: that is, the 1-cells in $\mathsf{R}X$ are finite paths $(a_1, \dots, a_n)$ of 1-cells in X quotiented by the equation

$$(a_1, \dots, a_i, \varepsilon x, a_{i+1}, \dots, a_n) = (a_1, \dots, a_n)$$

when $\partial^+ a_i = \partial^- a_{i+1} = x$. The 2-cells of type $(a_1, \dots, a_n) \Rightarrow (b_1, \dots, b_m)$ in $\mathsf{R}X$ are the 2-cells of type $a_1 \#_0 \dots \#_0 a_n \Rightarrow b_1 \#_0 \dots \#_0 b_m$ in X.

Compositions are induced by those of X in the obvious way, and a functor $f\colon X \to Y$ of 2-categories induces a morphism $\mathsf{R}f\colon \mathsf{R}X \to \mathsf{R}Y$ of bicoloured pros by

$$\begin{aligned} &\varphi\colon (a_1, \dots, a_n) \Rightarrow (b_1, \dots, b_m) \\ \mapsto\ &f(\varphi)\colon (f(a_1), \dots, f(a_n)) \Rightarrow (f(b_1), \dots, f(b_m)). \end{aligned}$$

For each 2-cell φ of type $(a_1, \dots, a_n) \Rightarrow (b_1, \dots, b_m)$ in a bicoloured pro $(T, \mathscr{T})$, there is a 2-cell $a_1 \#_0 \dots \#_0 a_n \Rightarrow b_1 \#_0 \dots \#_0 b_m$ in $\mathsf{U}(T, \mathscr{T})$, which in turn induces a 2-cell of type $(a_1, \dots, a_n) \Rightarrow (b_1, \dots, b_m)$ in $\mathsf{RU}(T, \mathscr{T})$; the unit of the adjunction sends φ to this 2-cell.

Conversely, for each 2-category X and 2-cell $\varphi\colon (a_1, \dots, a_n) \Rightarrow (b_1, \dots, b_m)$ in $\mathsf{UR}X$, the counit sends φ to the cell of type $a_1 \#_0 \dots \#_0 a_n \Rightarrow b_1 \#_0 \dots \#_0 b_m$ in X from which it was induced. It is an exercise to show that the unit and counit satisfy the required equations and determine an adjunction.

Lemma 2.14 — *The category* $\mathbf{Pro}_{bi}$ *has equalisers.*

Proof. Let $f, g\colon (T, \mathscr{T}) \to (S, \mathscr{S})$ be parallel morphisms of bicoloured pros. Define T' to be the restriction of T to the cells x that satisfy $f(x) = g(x)$ in S. Then T' is a 2-category.

A 1-cell $(a_1, \dots, a_n)$ in T belongs to T' if and only if $f(a_i) = g(a_i)$ for all $i \in \{1, \dots, n\}$. It follows that $\mathscr{T}' \coloneqq \{a \in \mathscr{T} \mid f(a) = g(a)\}$ gives $\sigma_{\leq 1}T'$ the

structure of a 1-polygraph, so that the inclusion of T' into T is a morphism of bicoloured pros. By a routine argument, it is the equaliser of f and g. ∎

Proposition 2.15 — *The functor* $\mathsf{U}\colon \mathbf{Pro}_{bi} \to 2\mathbf{Cat}$ *is comonadic.*

Proof. We have shown that U has a right adjoint. Moreover, equalisers as constructed in the proof of Lemma 2.14 are evidently created by U.

In order to apply the dual of Beck's monadicity theorem [ML71, §VI.7], it suffices to show that U reflects isomorphisms. Let $f\colon (T, \mathscr{T}) \to (S, \mathscr{S})$ be a morphism of bicoloured pros and suppose that $\mathsf{U}f$ is an isomorphism of 2-categories with inverse g. Then both $\mathsf{U}f$ and g must preserve the rank of all cells.

Let $a \in \mathscr{S}_1$. Then $g(a)$ can be written uniquely as a finite path $(a_1, \dots, a_n)$ with $a_i \in \mathscr{T}_1$ for all $i \in \{1, \dots, n\}$, and

$$a = \mathsf{U}f(g(a)) = (f(a_1), \dots, f(a_n))$$

where $n > 0$ and $f(a_i) \in \mathscr{S}_1$ for all $i \in \{1, \dots, n\}$. Because $\sigma_{\leq 1} S$ is free, this is only possible if $n = 1$ and $f(a_1) = a$. It follows that g sends generators to generators, hence determines a morphism of bicoloured pros, inverse to f in $\mathbf{Pro}_{bi}$. ∎

Corollary 2.16 — *The categories* $\mathbf{Pro}_{bi}$ *and* $\mathbf{Pro}$ *have all small limits and colimits.*

Proof. By the dual of [ML71, Exercise 2, §VI.2], a comonadic functor creates all colimits in its codomain; since $2\mathbf{Cat}$ has all small colimits, so does $\mathbf{Pro}_{bi}$. Moreover, $2\mathbf{Cat}$ has all small limits and $\mathbf{Pro}_{bi}$ has equalisers, so $\mathbf{Pro}_{bi}$ has all small limits by the dual of [Lin69, Corollary 2].

Since $\mathbf{Pro}$ is defined equationally, it is a reflective subcategory of $\mathbf{Pro}_{bi}$. It follows from [Rie17, Proposition 4.5.15] that it also has all small limits and colimits. ∎

2.2. Non-planar monoidal theories

2.17 (Braided strict monoidal category)**.** A *braided strict monoidal category* is a strict monoidal category X together with a family of 2-cells

$$\sigma_{x,y}\colon x \#_0 y \Rightarrow y \#_0 x$$

called *braidings*, indexed by 1-cells x, y, satisfying the following axioms:

1. the braidings are invertible, that is, there are unique 2-cells $\sigma_{x,y}^{-1}$, called *inverse braidings*, such that $\sigma_{x,y} \#_1 \sigma_{x,y}^{-1}$ and $\sigma_{x,y}^{-1} \#_1 \sigma_{x,y}$ are units;

2. they are natural in their parameters, that is, for all 2-cells $\varphi\colon x \Rightarrow x'$ and $\psi\colon y \Rightarrow y'$,

$$(\varphi \#_0 y) \#_1 \sigma_{x',y} = \sigma_{x,y} \#_1 (y \#_0 \varphi),$$
$$(x \#_0 \psi) \#_1 \sigma_{x,y'} = \sigma_{x,y} \#_1 (\psi \#_0 x);$$

3. they are compatible with 0-composition and units, that is,

$$\sigma_{x \#_0 x',y} = (x \#_0 \sigma_{x',y}) \#_1 (\sigma_{x,y} \#_0 x'),$$
$$\sigma_{x,y \#_0 y'} = (\sigma_{x,y} \#_0 y') \#_1 (y \#_0 \sigma_{x,y'}),$$
$$\sigma_{\varepsilon\bullet,y} = \varepsilon y, \qquad \sigma_{x,\varepsilon\bullet} = \varepsilon x,$$

whenever the left-hand side is defined.

A *functor* $f\colon X \to Y$ of braided strict monoidal categories is a functor of the underlying 2-categories that preserves braidings, that is, $f(\sigma_{x,y}) = \sigma_{f(x),f(y)}$ for all 1-cells x, y in X. With their functors, braided strict monoidal categories form a category $\mathbf{BrMonCat}_{str}$.

2.18 (Dual braided structure)**.** Let X be a braided strict monoidal category with braidings $\{\sigma_{x,y}\}$. The family of 2-cells

$$\sigma^*_{x,y} \coloneqq \sigma^{-1}_{y,x}$$

defines a second structure X^* of braided strict monoidal category on the underlying strict monoidal category of X.

If $f\colon X \to Y$ is a functor of braided strict monoidal categories, the same underlying functor of 2-categories determines a functor $f^*\colon X^* \to Y^*$. This defines an involution $-^*$ on $\mathbf{BrMonCat}_{str}$.

2.19 (Symmetric strict monoidal category)**.** A *symmetric strict monoidal category* is a braided strict monoidal category X satisfying $X = X^*$.

2.20 (Prob)**.** A *prob* is a pro together with a structure of braided strict monoidal category on its underlying strict monoidal category.

A *morphism* of probs is a morphism of pros that preserves the braidings. Probs and their morphisms form a category **Prob**.

Comment 2.21. Models of probs live in *braided* monoidal categories M, not necessarily strict. A model of a prob $(T, \mathscr{T})$ in M is a *braided* strong monoidal functor from T to M.

Remark 2.22. To determine a unique structure of braided strict monoidal category on a pro it is, in fact, sufficient to give braidings $\sigma_{a,b}$ for all pairs of generating 1-cells a, b; a morphism of pros that preserves these braidings automatically preserves all braidings. This is a consequence of axiom 3 of braided strict monoidal categories, since every 1-cell in a pro can be decomposed as a composite of generating 1-cells.

Remark 2.23. The duality $-^*$ on $\mathbf{BrMonCat}_{str}$ induces a duality on **Prob**.

2.24 (Prop)**.** A *prop* is a prob whose underlying braided strict monoidal category is symmetric. We let **Prop** denote the full subcategory of **Prob** on props.

2.25. There is an obvious forgetful functor $\mathsf{U}\colon \mathbf{Prob} \to \mathbf{Pro}$ and an inclusion of subcategories $\mathbf{Prop} \hookrightarrow \mathbf{Prob}$. Both of these have left adjoints:

- the left adjoint $\mathsf{F}\colon \mathbf{Pro} \to \mathbf{Prob}$ of U freely adds braidings $\sigma_{x,y}$ and inverse braidings $\sigma_{x,y}^{-1}$ for all pairs of 1-cells x, y of a pro (or just the generating ones, see Remark 2.22), then quotients by the axioms of braided strict monoidal categories;
- the reflector $\mathsf{r}\colon \mathbf{Prob} \to \mathbf{Prop}$ quotients by the equation $\sigma_{x,y} = \sigma_{x,y}^*$ for all pairs of 1-cells x, y of a prob.

Since we have not yet shown that **Prob** has coequalisers, for the moment we interpret the latter as a coequaliser in **Pro**, then observe that the images of the $\sigma_{x,y}$ still form a family of braidings in the quotient.

Example 2.26. The free prob $\mathbb{B} \coloneqq \mathsf{F}\mathbb{N}$ is the *theory of braids*. With 1-composition, 2-cells of type $(n) \Rightarrow (n)$ in $\mathbb{B}$ form the braid group B_n on n strands.

Example 2.27. The prop reflection $\mathbb{S} \coloneqq \mathsf{r}\mathbb{B}$ of the theory of braids is the *theory of permutations*. With 1-composition, 2-cells of type $(n) \Rightarrow (n)$ in $\mathbb{S}$ form the symmetric group S_n on n elements.

Example 2.28. Let *CMon* be defined as *Mon*, but 2-cells of type $(n) \Rightarrow (m)$ are *all* functions from (n) to (m), not just the order-preserving ones. This is a one-sorted prop with braidings generated by

$$\sigma_{1,1}\colon (2) \Rightarrow (2), \qquad 0 \mapsto 1,\ 1 \mapsto 0.$$

It corresponds to the theory of *commutative monoids* in symmetric monoidal categories. Similarly, models of $CMon^{\mathrm{co}}$ are *commutative comonoids*.

Example 2.29. There is a one-sorted pro $Mat_{\mathbb{Z}}$ whose 2-cells $A\colon (n) \Rightarrow (m)$ are $(m \times n)$-matrices of integers, the 1-composite $A \#_1 B$ is the product BA

of matrices, and

$$A \#_0 B \coloneqq \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

This is a prop with braidings generated by

$$\sigma_{1,1} \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There is a morphism of props $\mathit{CMon} \to \mathit{Mat}_{\mathbb{Z}}$ sending $\varphi\colon (n) \Rightarrow (m)$ to the $(m \times n)$-matrix A_φ with entries

$$A_\varphi(j,i) \coloneqq \begin{cases} 1 & \text{if } j = \varphi(i), \\ 0 & \text{otherwise,} \end{cases}$$

and a morphism $\mathit{CMon}^{\mathrm{co}} \to \mathit{Mat}_{\mathbb{Z}}$ sending $\varphi^{\mathrm{co}}\colon (m) \Rightarrow (n)$ to the transpose of A_φ. As a monoidal theory, $\mathit{Mat}_{\mathbb{Z}}$ corresponds to the theory of *commutative and cocommutative Hopf algebras* [BSZ17, Section 7].

2.30 (Gray-category)**.** A *Gray-category* is a 3-precategory G together with a family of 3-cells

$$\chi_{x,y}\colon (x \#_0 \partial^- y) \#_1 (\partial^+ x \#_0 y) \Rightarrow (\partial^- x \#_0 y) \#_1 (x \#_0 \partial^+ y)$$

called *interchangers*, indexed by 2-cells x, y with $\partial_0^+ x = \partial_0^- y$, satisfying the following axioms:

1. the interchangers are invertible, that is, there are unique 3-cells

   $$\chi_{x,y}^{-1}\colon (\partial^- x \#_0 y) \#_1 (x \#_0 \partial^+ y) \Rightarrow (x \#_0 \partial^- y) \#_1 (\partial^+ x \#_0 y)$$

   called *inverse interchangers*, such that $\chi_{x,y} \#_2 \chi_{x,y}^{-1}$ and $\chi_{x,y}^{-1} \#_2 \chi_{x,y}$ are units;

2. the interchangers are natural in their parameters, that is, for all 3-cells $\varphi\colon x \Rightarrow x'$ and $\psi\colon y \Rightarrow y'$ with $\partial_0^+ \varphi = \partial_0^- \psi$,

   $$((\varphi \#_0 \partial^- y) \#_1 (\partial_1^+ \varphi \#_0 y)) \#_2 \chi_{x',y} = \chi_{x,y} \#_2 ((\partial_1^- \varphi \#_0 y) \#_1 (\varphi \#_0 \partial^+ y)),$$
   $$((x \#_0 \partial_1^- \psi) \#_1 (\partial^+ x \#_0 \psi)) \#_2 \chi_{x,y'} = \chi_{x,y} \#_2 ((\partial^- x \#_0 \psi) \#_1 (x \#_0 \partial_1^+ \psi));$$

3. the interchangers are compatible with 1-compositions and units, that is,

   $$\chi_{x \#_1 x',y} = ((x \#_0 \partial^- y) \#_1 \chi_{x',y}) \#_2 (\chi_{x,y} \#_1 (x' \#_0 \partial^+ y)),$$
   $$\chi_{x,y \#_1 y'} = (\chi_{x,y} \#_1 (\partial^+ x \#_0 y')) \#_2 ((\partial^- x \#_0 y) \#_1 \chi_{x,y'}),$$
   $$\chi_{\varepsilon x,y} = \varepsilon(\varepsilon x \#_0 y), \qquad \chi_{x,\varepsilon y} = \varepsilon(x \#_0 \varepsilon y),$$

   whenever the left-hand side is defined;

4. for all pairs of 3-cells φ, ψ with $\partial_1^+\varphi = \partial_1^-\psi$, the equation

$$(\varphi \#_1 \partial_2^-\psi) \#_2 (\partial_2^+\varphi \#_1 \psi) = (\partial_2^-\varphi \#_1 \psi) \#_2 (\varphi \#_1 \partial_2^+\psi)$$

holds in G.

A *functor* $f\colon G \to H$ of Gray-categories is a functor of the underlying 3-precategories that preserves the interchangers, that is, $f(\chi_{x,y}) = \chi_{f(x),f(y)}$ for all suitable 2-cells x, y in G. With their functors, Gray-categories form a category **GrayCat**.

Remark 2.31. Axiom 4 is an instance of (5), allowing us to univocally define the 1-composition $\varphi \#_1 \psi$ of 3-cells with $\partial_1^+\varphi = \partial_1^-\psi$ in a Gray-category.

Comment 2.32. A more concise definition is that a Gray-category is a small category enriched over $2\mathbf{Cat}$ with the "pseudo" Gray tensor product [GPS95, Chapter 5]. As in [Lac11, §1.4], one derives that **GrayCat** is locally finitely presentable, and in particular has all small limits and colimits.

2.33. By Proposition 1.35, every 3-category seen as a 3-precategory has a canonical structure of Gray-category with units as interchangers. This defines an embedding $3\mathbf{Cat} \hookrightarrow \mathbf{GrayCat}$, which makes $3\mathbf{Cat}$ a reflective subcategory of **GrayCat**: the reflector universally turns the interchangers into units.

2.34. Given a braided strict monoidal category X, we define a Gray-category $\mathsf{B}X$ as follows. For all $n \in \mathbb{N}$, we let $\mathsf{B}X_{n+1} \coloneqq X_n$, with the same boundary and unit operators as X between $\mathsf{B}X_{n+2}$ and $\mathsf{B}X_{n+1}$. We let $\mathsf{B}X_0 \coloneqq \{\bullet\}$, with the only possible unit and boundary operators relating it to $\mathsf{B}X_1$. This defines the underlying reflexive 3-graph of $\mathsf{B}X$.

To make $\mathsf{B}X$ a 3-precategory, it suffices to define the principal compositions. Because $\mathsf{B}X$ has no rank-1 cells, the principal compositions are of the form $x \#_k y$ where

- $k = 1$ and $\min\{\mathrm{rk}(x), \mathrm{rk}(y)\} = 2$, or
- $k = 2$ and $\mathrm{rk}(x) = \mathrm{rk}(y) = 3$.

In either case, $x \#_{k-1} y$ is defined in X, and we let $x \#_k y$ be equal to it in $\mathsf{B}X$.

Finally, given 2-cells x, y in $\mathsf{B}X$, we let the interchanger $\chi_{x,y}$ correspond to the braiding $\sigma_{x,y}$ in X. It is an exercise to check that this gives $\mathsf{B}X$ the structure of a Gray-category.

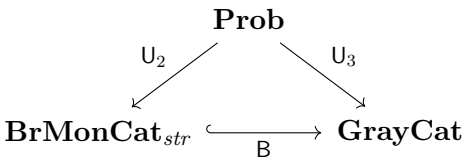
This assignment extends to a functor $\mathsf{B}\colon \mathbf{BrMonCat}_{str} \to \mathbf{GrayCat}$ in the obvious way. By [CG11, Theorem 2.16], this functor is full and faithful, and its essential image consists exactly of those Gray-categories that have a single 0-cell and a single 1-cell. This can be seen as an alternative characterisation of $\mathbf{BrMonCat}_{str}$ as a full subcategory of **GrayCat**.

Remark 2.35. Through B, the duality $-^*$ on $\mathbf{BrMonCat}_{str}$ is the restriction of the duality on **GrayCat** that reverses the orientation of 1-cells.

2.36. If X is a braided strict monoidal category, the structure of a 1-polygraph on $\sigma_{\leq 1}X$ determines a unique structure of 2-prepolygraph on $\sigma_{\leq 2}\mathsf{B}X$, and vice versa. A functor f of braided strict monoidal categories sends generators to generators if and only if $\mathsf{B}f$ does.

Thus, a prob is equivalently defined as a Gray-category T with a single 0-cell, a single 1-cell, and the structure of a 2-prepolygraph $(\sigma_{\leq 2}T, \mathscr{T})$ on its 2-skeleton. A morphism $f\colon (T, \mathscr{T}) \to (S, \mathscr{S})$ of probs is a functor of Gray-categories such that $f(a) \in \{\bullet\} \cup \mathscr{S}_2$ for all $a \in \mathscr{T}_2$.

We have a triangle of functors



that commutes up to natural isomorphism, where U_2 and U_3 are the forgetful functors associated to the two alternative definitions of prob.

Proposition 2.37 — *The categories* $\mathbf{BrMonCat}_{str}$, **Prob**, *and* **Prop** *have all small limits and colimits.*

Sketch of the proof. First of all, the essential image of B is defined equationally in **GrayCat**, so $\mathbf{BrMonCat}_{str}$ is, up to equivalence, a reflective subcategory. Since **GrayCat** has all small limits and colimits, by [Rie17, Proposition 4.5.15] so does $\mathbf{BrMonCat}_{str}$.

Then, we can mimic the proofs of Lemma 2.14 and Proposition 2.15 to show that **Prob** has equalisers and that the functor $\mathsf{U}_2\colon \mathbf{Prob} \to \mathbf{BrMonCat}_{str}$ is comonadic. As in the proof of Corollary 2.16, we deduce that **Prob** has all small limits and colimits, and so does its reflective subcategory **Prop**. ∎

Remark 2.38. The functors $\mathsf{U}\colon \mathbf{Pro}_{bi} \to 2\mathbf{Cat}$, $\mathsf{U}_2\colon \mathbf{Prob} \to \mathbf{BrMonCat}_{str}$ are *pseudomonic*: that is, in addition to being faithful, they reflect and are full on isomorphisms. This captures the fact that a 2-category admits at most one structure of bicoloured pro, a consequence of the general fact, proved by Makkai [Mak05, Section 4, Proposition 8], that an ω-category admits at most one structure of polygraph.

Because the composite of a pseudomonic with a full and faithful functor is pseudomonic, it follows that $\mathsf{U}_3\colon \mathbf{Prob} \to \mathbf{GrayCat}$ is also pseudomonic.

3. Combinatorial results

3.1. In generic dimension

Comment 3.1. In this section, we use results of [Had20a, Section 6]. These results are stated relative to the restricted class of constructible directed complexes, but the proofs do not involve any properties that are not satisfied by all regular directed complexes. Thus all statements hold with *constructible* replaced by *regular*.

3.2. Let U be a closed subset of a regular directed complex. For each $n \geq -1$, the bipartite directed graph $\mathscr{M}_n U$ has

$$\{x \in U \mid \dim(x) \leq n\} + \{x \in U \mid x \text{ is maximal and } \dim(x) > n\}$$

as set of vertices, and an edge $y \to x$ if and only if

- $\dim(y) \leq n$, $\dim(x) > n$, and $y \in \partial_n^- x \setminus \partial_{n-1} x$, or
- $\dim(y) > n$, $\dim(x) \leq n$, and $x \in \partial_n^+ y \setminus \partial_{n-1} y$.

3.3 (Frame dimension)**.** Let U be a closed subset of a regular directed complex. The *frame dimension* of U is the integer

$$\mathrm{frdim}(U) \coloneqq \max\{\dim(\mathrm{cl}\{x\} \cap \mathrm{cl}\{y\}) \mid x, y \text{ maximal in } U,\ x \neq y\}.$$

Remark 3.4. If $\mathrm{frdim}(U) = -1$, then U is a disjoint union of atoms.

3.5 (Frame acyclicity)**.** A regular directed complex P is *frame-acyclic* if, for all molecules U in P, if $\mathrm{frdim}(U) = k$, then $\mathscr{M}_k U$ is acyclic.

Lemma 3.6 — *Let P be a frame-acyclic regular directed complex. Then*

1. *for all molecules U in P, if $U = U_1 \cup U_2$ for some closed subsets U_1, U_2 such that $U_1 \cap U_2 = \partial_k^+ U_1 = \partial_k^- U_2$, then U_1 and U_2 are molecules;*
2. *$(\mathcal{M}o\ell P^*, \{\mathrm{cl}\{x\}\}_{x \in P})$ is a polygraph.*

Proof. A corollary of [Had20a, Proposition 26]. ∎

3.7 (k-Order)**.** Let U be a regular n-molecule. For $k < n$, a *k-order* on U is a linear ordering $(x_1, \ldots, x_m)$ of the set

$$\{x \in U \mid x \text{ is maximal and } \dim(x) > k\}$$

with the property that, if there is a path from x_i to x_j in $\mathscr{M}_k U$, then $i \leq j$.

Proposition 3.8 — *Let U be a regular n-molecule, $k < n$. If $\mathscr{M}_k U$ is acyclic, then U admits a k-order.*

Proof. Every directed acyclic graph admits a topological sorting, that is, a linear order $\preceq$ on its vertices with the property that if there is a path from x to y, then $x \preceq y$. The restriction of a topological sorting of $\mathscr{M}_k U$ to the maximal elements of dimension greater than k is a k-order on U. ∎

Proposition 3.9 — *Let U be a regular n-molecule. Then $\mathscr{M}_{n-1} U$ is acyclic and U admits an $(n-1)$-order.*

Proof. Follows from Proposition 3.8 and [Had20a, Proposition 20]. ∎

Lemma 3.10 — *Let U be a frame-acyclic regular molecule, $k \geq \mathrm{frdim}(U)$, and let $(x_1, \dots, x_m)$ be a k-order on U. Then there exist molecules $V_1, \dots, V_m$, such that*

$$U = V_1 \#_k \dots \#_k V_m$$

and $x_i \in V_j$ if and only if $i = j$ for all $i, j \in \{1, \dots m\}$.

Proof. If $m = 1$ then $V_1 \coloneqq U$ satisfies the statement. Suppose that $m > 1$ and that $k = \mathrm{frdim}(U)$. Then we proceed as in the proof of [Had20a, Proposition 26] to produce $i \in \{1, \dots, m-1\}$ and a decomposition $U = U_1 \cup U_2$ such that

1. U_1 contains $x_1, \dots, x_i$ and U_2 contains $x_{i+1}, \dots, x_m$,
2. $U_1 \cap U_2 = \partial_k^+ U_1 = \partial_k^- U_2$.

By Lemma 3.6, both U_1 and U_2 are molecules. Moreover $(x_1, \dots, x_i)$ and $(x_{i+1}, \dots, x_m)$ are k-orders on U_1 and U_2, respectively. We conclude by the inductive hypothesis applied to U_1 and U_2.

Finally, suppose that $k > \ell \coloneqq \mathrm{frdim}(U)$. By frame acyclicity, we can fix an ℓ-order $(y_1, \dots, y_p)$ on U, and by the first part of the proof we can decompose U as

$$W_1 \#_\ell \dots \#_\ell W_p$$

with $y_i \in W_j$ if and only if $i = j$. Now for each $i \in \{1, \dots, m\}$ there is a unique $j(i) \in \{1, \dots, p\}$ such that $x_i = y_{j(i)}$. Let

$$V_i \coloneqq \partial_k^{\alpha(i,1)} W_1 \#_\ell \dots \#_\ell W_{j(i)} \#_\ell \dots \#_\ell \partial_k^{\alpha(i,p)} W_p, \text{ where}$$

$$\alpha(i,j) \coloneqq \begin{cases} + & \text{if } j = j(i') \text{ for some } i' < i, \\ - & \text{otherwise.} \end{cases}$$

Then $U = V_1 \#_k \dots \#_k V_m$ is the required decomposition. ∎

3.11 (Substitution)**.** Let V and W be regular n-molecules with spherical boundary, let U be a regular n-molecule, and let $V \sqsubseteq U$. Then $U \setminus (V \setminus \partial V)$ is a closed subset of U.

Suppose that $\partial^\alpha V$ is isomorphic to $\partial^\alpha W$ for all $\alpha \in \{+,-\}$. From [Had20b, Lemma 2.2] we obtain a unique isomorphism $\imath\colon \partial U \xhookrightarrow{\sim} \partial V$. We define $U[W/V]$ to be the pushout

$$\begin{array}{ccc} \partial V & \hookrightarrow & U \setminus (V \setminus \partial V) \\ \downarrow & & \downarrow \\ W & \hookrightarrow & U[W/V] \end{array}$$

in $\mathbf{DCpx}^{\mathcal{R}}$, and call it the *substitution* of W for $V \sqsubseteq U$. By [Proposition 2.4, *ibid.*] this is an n-molecule with boundaries isomorphic to those of U, and such that $W \sqsubseteq U[W/V]$.

Remark 3.12. As shown in [Had20b, Lemma 2.5], if U is an n-molecule with a decomposition $U = V_1 \#_{n-1} \dots \#_{n-1} V_m$ as in Lemma 3.10, then $\partial^\alpha V_i$ is isomorphic to $\partial^{-\alpha} V_i[\partial^\alpha x_i / \partial^{-\alpha} x_i]$ for all $\alpha \in \{+,-\}$ and $i \in \{1,\dots,m\}$.

3.2. In low dimension

3.13 (Totally loop-free molecule)**.** Given a regular molecule U, let $\mathscr{H}^o U$ be the directed graph obtained from $\mathscr{H}U$ by reversing all the edges labelled $-$. We say that U is *totally loop-free* if $\mathscr{H}^o U$ is acyclic as a directed graph.

If U is totally loop-free, for all $x, y \in U$, we let $x \preceq y$ if and only if there is a path from x to y in U.

Proposition 3.14 — *Let U be a regular molecule. If $\dim(U) \leq 2$, then U is totally loop-free and $\preceq$ is a linear order on U.*

Proof. If $\dim(U) \in \{0,1\}$ or if U is a 2-dimensional atom, this is easy. Otherwise, decompose U as $V_1 \#_1 \dots \#_1 V_m$ so that each V_i contains a unique 2-dimensional element x_i, and let $V_0 \coloneqq \partial^- U$ and $U' \coloneqq V_0 \#_1 \dots \#_1 V_{m-1}$. Then $U = U' \cup \mathrm{cl}\{x_m\}$, and we may assume inductively that the statement holds for U'.

Suppose that there is a cycle in $\mathscr{H}^o U$. Because both U' and $\mathrm{cl}\{x_m\}$ are totally loop-free, such a cycle must leave U', enter $\mathrm{cl}\{x_m\} \setminus \partial^- x_m$, then return to U'. Such a path either

- enters x_m via $\Delta^- x_m$, then enters $\partial^+ x_m$ and leaves via $\partial_0^+ x_m$, or
- enters $\Delta^+ x_m$ directly via $\partial_0^- x_m$, stays in $\partial^+ x_m$ and leaves via $\partial_0^+ x_m$.

In both cases, the path through $\mathrm{cl}\{x_m\}\setminus\partial^- x_m$ can be replaced with the unique path to $\partial_0^+ x_m$ that stays in $\partial^- x_m \subseteq U'$. Making such replacements, we create a cycle in $\mathscr{H}^o U'$, contradicting the inductive hypothesis.

This proves that U is totally loop-free. To show that $\preceq$ is a linear order, it suffices to compare elements of U' and of $\mathrm{cl}\{x_m\} \setminus \partial^- x_m$. Let $x \in U'$. There are two possible cases:

- $\partial_0^+ x_m \preceq x$ in U'. Then $z \preceq x$ for all elements $z \in \mathrm{cl}\{x_m\}$.
- $x \prec \partial_0^+ x_m$ in U'. Let y be the unique 1-dimensional element of $\Delta^- x_m$ that covers $\partial_0^+ x_m$. Suppose that $y \prec x$ in U', that is, there is a non-trivial path from y to x in $\mathscr{H}^o U'$. Such a path cannot pass through $\partial^+ y = \partial_0^+ x_m$, for otherwise $\partial_0^+ x_m \preceq x$; nor it can enter a 2-dimensional element, because y is not covered by any element of U' with orientation $-$. Therefore $x \preceq y$, so $x \prec x_m$ and $x \prec z$ for all elements $z \in \partial^+ x_m \setminus \partial_0 x_m$.

This proves that $\preceq$ is a linear order on U. ∎

Remark 3.15. If U is a regular molecule with $\dim(U) \leq 2$, by [Ste93, Theorem 2.17] combined with Proposition 3.14, $\mathcal{M}o\ell U^*$ is equal to $\mathcal{M}o\ell U$.

Proposition 3.16 — *Let U be a regular 2-molecule, $k \in \{0,1\}$, and let $x, y \in U$ be maximal elements of dimension $> k$. If there is a path from x to y in $\mathscr{M}_k U$, then $x \preceq y$.*

Proof. Suppose $k = 1$. A path $x = x_0 \to w_0 \to \ldots \to w_{m-1} \to x_m = y$ in $\mathscr{M}_1 U$ is a concatenation of two-step paths $x_i \to w_i \to x_{i+1}$ where $\dim(x_i) = 2$ for all $i \in \{0, \ldots, m\}$ and

$$w_i \in (\partial^+ x_i \setminus \partial_0 x_i) \cap (\partial^- x_{i+1} \setminus \partial_0 x_{i+1}).$$

If $\dim(w_i) = 1$ then $w_i \in \Delta^+ x_i \cap \Delta^- x_{i+1}$, so $x_i \prec w_i \prec x_{i+1}$. Suppose $\dim(w_i) = 0$. Because $\partial^+ x_i$ is pure and 1-dimensional, w_i is covered by some element of $\Delta^+ x_i$, and because $w_i \notin \partial_0 x_i = \Delta(\partial^+ x_i)$, by [Had20b, Lemma 1.16] it is in fact covered by two elements of $\Delta^+ x_i$ with opposite orientations. If $w_i \in \Delta^+ x_i$ covers w_i with orientation $+$, we have $x_i \prec w_i' \prec w_i$.

Dually, we find $w_i'' \in \Delta^- x_{i+1}$ that covers w_i with orientation $-$, so that $w_i \prec w_i'' \prec x_{i+1}$. It follows that $x_i \prec x_{i+1}$ for all $i \in \{0, \ldots, m-1\}$, and we conclude that $x \preceq y$.

Now suppose that $k = 0$. A path from x to y in $\mathscr{M}_0 U$ is a concatenation of two-step paths $x_i \to w_i \to x_{i+1}$ where $\dim(x_i) \in \{1, 2\}$ and w_i is the only element of $\partial_0^+ x_i = \partial_0^- x_{i+1}$. If $\dim(x_i) = 1$ then immediately $x_i \prec w_i$, otherwise there is exactly one element $w_i' \in \Delta^+ x_i$ such that $\partial_0^+ x_i = \partial^+ w_i'$, so $x_i \prec w_i' \prec w_i$. Similarly we find that $w_i \prec x_{i+1}$. ∎

Corollary 3.17 — *If U is a regular 2-molecule, the restriction of $\preceq$ to 2-dimensional elements determines a 1-order on U.*

3.18 (Normal 1-order)**.** Let U be a regular 2-molecule. The *normal 1-order* on U is the 1-order induced by the relation $\preceq$ on the elements of U.

Example 3.19. In the shape of the 2-diagrams



the normal 1-order is indicated by the labels of 2-cells.

In general, a rule-of-thumb for reconstructing the normal 1-order from a string diagram is:

1. if there is an upward path between two 2-cells, then the *lowermost* precedes the *uppermost*;
2. if there is no such path, then the *leftmost* precedes the *rightmost*.

Due to a certain flexibility in the depiction of string diagrams, the second rule may not always strictly hold (but it will hold up to a harmless deformation of the string diagram).

Corollary 3.20 — *Let P be a regular directed complex with $\dim(P) \leq 2$. Then P is frame-acyclic.*

Lemma 3.21 — *Let U be a regular molecule with $\dim(U) \leq 2$ and let $I \subseteq U$ be a 1-molecule with $\partial^- I = \partial_0^- U$ and $\partial^+ I = \partial_0^+ U$. Then*

(a) there is a unique decomposition $U = U_+ \#_1 U_-$ with $\partial_1^+ U_+ = \partial_1^- U_- = I$;

(b) for all $\alpha \in \{+, -\}$, if $V \sqsubseteq U$ is a 2-molecule with spherical boundary and $V \cap I = \partial^\alpha V$, then $V \sqsubseteq U_\alpha$.

Proof. By induction on the number m of 2-dimensional elements of U: if $m = 0$, then necessarily $U = I$ and $U = I \#_1 I$ is the unique decomposition. If $m > 0$, we can write $U = U' \#_1 U_x$ where U_x contains a single 2-dimensional element x. Now either

- $I \subseteq U'$, in which case we have a unique decomposition $U' = U'_+ \#_1 U'_-$ and we can set $U_+ \coloneqq U'_+$ and $U_- \coloneqq U'_- \#_1 U_x$, or
- $\partial^+ x \subseteq I$, since I traces a path in $\mathscr{H}^o U$ through 0-dimensional and 1-dimensional elements, and because $\partial^+ x \subseteq \partial^+ U$ such a path can only enter

$\partial^+ x$ through $\partial_0^- x$, traverse the entire $\partial^+ x$, and leave through $\partial_0^+ x$. Then $I' \coloneqq I[\partial^- x / \partial^+ x]$ is well-defined and a 1-molecule in U'; by the inductive hypothesis, we have a decomposition $U' = U'_+ \#_1 U'_-$ relative to I'. Then setting $U_+ \coloneqq U'_+ \cup \mathrm{cl}\{x\}$ and $U_- \coloneqq U'_-$ produces a decomposition of U relative to I.

Uniqueness is straightforward after observing that removing $x \cup (\partial^+ x \setminus \partial_0 x)$ from a decomposition of U produces a decomposition of U' either relative to I or to I'.

Let $V \sqsubseteq U$ be a 2-molecule with spherical boundary and $V \cap I = \partial^\alpha V$. If V is an atom, then clearly $V \sqsubseteq U_\alpha$. Otherwise, observe that I unaffected by the substitution $U[\langle V \rangle / V]$ and $\partial^\alpha \langle V \rangle = \partial^\alpha V \subseteq I$. Decomposing $U[\langle V \rangle / V]$ as $U'_+ \#_1 U'_-$, by the atom case we have $\langle V \rangle \sqsubseteq U'_\alpha$. Now $U'_\alpha[V/\langle V \rangle]$ and $U'_{-\alpha}$ are factors of a decomposition of U relative to I, so by uniqueness $U_\alpha = U'_\alpha[V/\langle V \rangle]$ and $V \sqsubseteq U_\alpha$. ∎

Proposition 3.22 — *Let U, V, W be regular 2-molecules. Suppose V and W have spherical boundary, $V, W \sqsubseteq U$, and $V \cap W \subseteq \partial V \cup \partial W$. Then $W \sqsubseteq U[\langle V \rangle / V]$ and $V \sqsubseteq U[\langle W \rangle / W]$.*

Proof. Fix $\alpha \in \{+, -\}$ and let $U' \coloneqq U[\langle V \rangle / V]$; by assumption, W is unaffected by this substitution.

We construct a sequence of 1-molecules $I_0, \dots, I_n$ as follows. Let $I_0 \coloneqq \partial^\alpha W$. For $i \geq 0$, if $\partial^- I_i = \partial_0^- U$, then let $k \coloneqq i$ and move to the following cycle, otherwise pick a 1-dimensional element x with $\partial^+ x = \partial^- I_i$ and let $I_{i+1} \coloneqq \mathrm{cl}\{x\} \#_0 I_i$.

For $i \geq k$, if $\partial^+ I_i = \partial_0^+ U$, then let $n \coloneqq i$ and stop, otherwise pick a 1-dimensional element x with $\partial^- x = \partial^+ I_i$ and let $I_{i+1} \coloneqq I_i \#_0 \mathrm{cl}\{x\}$. This process terminates by finiteness of U' and acyclicity of $\mathscr{H}^o U$.

Now $I \coloneqq I_n$ is unaffected by the reverse substitution $U = U'[V/\langle V \rangle]$, has $\partial^- I = \partial_0^- U$ and $\partial^+ I = \partial_0^+ U$, and $W \cap I = \partial^\alpha W$. Consider the unique decomposition $U'_+ \#_1 U'_-$ of U' relative to I given by Lemma 3.21. Clearly $\langle V \rangle \sqsubseteq U'_\beta$ for some $\beta \in \{+, -\}$, so $U_\beta \coloneqq U'_\beta[V/\langle V \rangle]$ and $U_{-\beta} \coloneqq U'_{-\beta}$ produces the unique decomposition of U relative to I.

Now observe that if we decompose relative to $I' \coloneqq I[\partial^{-\alpha} W / \partial^\alpha W]$ instead of I, only the 2-dimensional elements of W "switch sides" in the factorisation, so we can vary α without affecting β. Choosing $\alpha \coloneqq -\beta$, we have

$$V \sqsubseteq U_\beta, \quad W \sqsubseteq U_{-\beta}$$

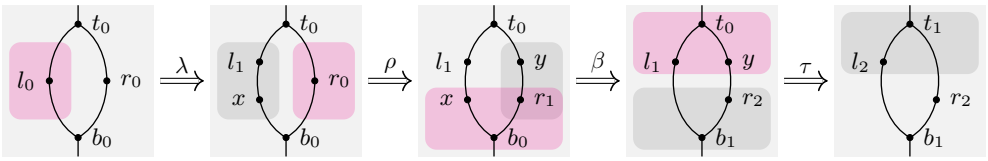
and the substitution of $\langle V \rangle$ for V, or of $\langle W \rangle$ for W, only affects one factor. ∎

Comment 3.23. As a consequence of Proposition 3.22, if V and W are submolecules with spherical boundary of a regular 2-molecule U that only overlap on their boundaries, then they can *both* be substituted in U: if $U[W'/W]$ and $U[V'/V]$ are both defined as 2-molecules, then so are $U[W'/W][V'/V]$ and $U[V'/V][W'/W]$, which are in fact equal. This generalises to an arbitrary number $V_1, \dots, V_n \sqsubseteq U$ of 2-molecules such that $V_i \cap V_j \subseteq \partial V_i \cup \partial V_j$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$.

Dimension 2 is, in fact, the largest dimension in which this result holds. The following is an example of a regular 3-molecule for which the analogous statement fails; it is a simplified version of [Ste93, Section 8], itself based on [Pow91, Example 3.11].

The point in our proof that fails to generalise to higher dimensions is the seemingly innocuous fact that $\partial^+ W$ can always be extended to a 1-molecule I with $\partial I = \partial_0 U$. In the example below, $\partial^+ W$ cannot be extended to any 2-molecule in $U[\langle V \rangle / V]$ whose boundary is equal to $\partial_1 U$.

Example 3.24. Let U be the shape of the 3-diagram



where we use the labels of cells to refer to the corresponding atoms of U. Then both $V \coloneqq \lambda \cup \tau$ and $W \coloneqq \rho \cup \beta$ are submolecules of U, they have spherical boundary, and they do not share any 3-atoms, so they only intersect in the boundary.

However, W is *not* a submolecule of $U[\langle V \rangle / V]$, and V is *not* a submolecule of $U[\langle W \rangle / W]$. Indeed, there are paths

$$\rho \to y \to \langle V \rangle \to x \to \beta, \qquad \lambda \to x \to \langle W \rangle \to y \to \tau$$

in $\mathscr{M}_2 U[\langle V \rangle / V]$ and $\mathscr{M}_2 U[\langle W \rangle / W]$, respectively; note that we are confusing an atom with its greatest element. If $W \sqsubseteq U[\langle V \rangle / V]$ or $V \sqsubseteq U[\langle W \rangle / W]$, then it would be possible to substitute $\langle W \rangle$ for W in $U[\langle V \rangle / V]$, or $\langle V \rangle$ for V in $U[\langle W \rangle / W]$, to obtain a regular 3-molecule U'. These paths would then become cycles in $\mathscr{M}_2 U'$, contradicting Proposition 3.9.

Theorem 3.25 — *Let P be a regular directed complex with $\dim(P) \leq 3$. Then P is frame-acyclic.*

Proof. It suffices to show that for all regular 3-molecules U, if $\mathrm{frdim}(U) = k$, then $\mathscr{M}_k U$ is acyclic. The case $k = 2$ is handled by Proposition 3.9, so suppose $k \in \{0, 1\}$.

By [Had20b, Lemma 2.5] we can decompose U as $V_1 \#_2 \dots \#_2 V_m$, where V_i contains a unique 3-dimensional element x_i and $\partial^\alpha V_i = \partial^{-\alpha} V_i[\partial^\alpha x_i / \partial^{-\alpha} x_i]$ for all $\alpha \in \{+,-\}$ and $i \in \{1,\dots,m\}$.

Since $\mathrm{cl}\{x_i\} \cap \mathrm{cl}\{x_j\}$ has dimension at most 1 when $i \neq j$, we have that

1. $\partial^- x_i \subseteq \partial^- U$ for all $i \in \{1,\dots,m\}$, and
2. by [Had20a, Lemma 18], $\mathrm{cl}\{x_i\} \cap \mathrm{cl}\{x_j\} = \partial^- x_i \cap \partial^- x_j \subseteq \partial_1 x_i \cup \partial_1 x_j$ when $i \neq j$.

Since for all $i \in \{1,\dots,m\}$ we have $\partial^- x_i \sqsubseteq \partial^- V_i$ and

$$\partial^- U = \partial^- V_i[\partial^- x_{i-1} / \partial^+ x_{i-1}] \dots [\partial^- x_1 / \partial^+ x_1],$$

applying Proposition 3.22 repeatedly we find that $\partial^- x_i \sqsubseteq \partial^- U$ and the simultaneous substitution

$$U' \coloneqq \partial^- U[\langle \partial^- x_1 \rangle / \partial^- x_1] \dots [\langle \partial^- x_m \rangle / \partial^- x_m] \tag{6}$$

is defined as a regular 2-molecule with the same frame dimension as U.

Now from every path in $\mathscr{M}_k U$, we construct a path in $\mathscr{M}_k U'$ as follows. The path in $\mathscr{M}_k U$ is a concatenation of two-step paths $y_- \to x \to y_+$, where x is maximal in U, $y_- \in \partial_1^- x$ and $y_+ \in \partial_1^+ x$.

If $\dim(x) < 3$, then this path stays inside $\partial^- U$, and x, y_-, y_+ are unaffected by the substitution (6). If $\dim(x) = 3$, then this path can be replaced by a path $y_- \to \tilde{x} \to y_+$ in $\mathscr{M}_k U'$, where $\tilde{x}$ is the greatest element of $\langle \partial^- x \rangle$.

Assuming there is a cycle in $\mathscr{M}_k U$, with this procedure we construct a cycle in $\mathscr{M}_k U'$, which contradicts Corollary 3.20. Thus $\mathscr{M}_k U$ is acyclic. ∎

4. Pros and diagrammatic sets

4.1. Diagrammatic nerve of a pro

4.1. Given a regular directed complex P and $n \in \mathbb{N}$, let $\sigma_{\leq n} P \subseteq P$ be the closed subset of elements $x \in P$ with $\dim(x) \leq n$. Then $\mathcal{M}o\ell(\sigma_{\leq n} P)^*$ and $\sigma_{\leq n} \mathcal{M}o\ell P^*$ are isomorphic n-categories.

By Lemma 3.6 combined with Theorem 3.25, for $n \leq 3$ the n-category $\sigma_{\leq n} \mathcal{M}o\ell P^*$ admits the structure of a polygraph with $\{\mathrm{cl}\{x\} \mid \dim(x) \leq n\}$ as generating cells. Because, in general, for an ω-category X,

$$\sigma_{\leq k} X = \sigma_{\leq k}(\tau_{\leq n} X) \text{ when } k < n,$$

the 2-category $\tau_{\leq 2} \mathcal{M}o\ell P^*$ has the structure of a bicoloured pro with generators $\{\mathrm{cl}\{x\} \mid \dim(x) \leq 1\}$.

Moreover, if $f\colon P \to Q$ is a morphism in $\mathbf{DCpx}^{\mathcal{R}}$, then $\tau_{\leq 2}\mathcal{M}o\ell f^*$ sends each generator $\mathrm{cl}\{x\}$ to a generator $\mathrm{cl}\{f(x)\}$, so it is compatible with this structure. This defines a functor $\mathsf{P}\colon \mathbf{DCpx}^{\mathcal{R}} \to \mathbf{Pro}_{bi}$ that fits into a commutative square

$$\begin{array}{ccc} \mathbf{DCpx}^{\mathcal{R}} & \xrightarrow{\ \mathsf{P}\ } & \mathbf{Pro}_{bi} \\ {\scriptstyle \mathcal{M}o\ell -^*}\downarrow & & \downarrow{\scriptstyle \mathsf{U}} \\ \omega\mathbf{Cat} & \xrightarrow[\ \tau_{\leq 2}\ ]{} & 2\mathbf{Cat}. \end{array}$$

Because $\mathbf{DCpx}^{\mathcal{R}}$ is small, $\odot\mathbf{Set}$ is locally small, and by Corollary 2.16 $\mathbf{Pro}_{bi}$ has all small colimits, by [Rie17, Corollary 6.2.6] the left Kan extension of $\mathsf{P}\colon \mathbf{DCpx}^{\mathcal{R}} \to \mathbf{Pro}_{bi}$ along the embedding $\mathbf{DCpx}^{\mathcal{R}} \hookrightarrow \odot\mathbf{Set}$ exists. This produces a functor $\mathsf{P}\colon \odot\mathbf{Set} \to \mathbf{Pro}_{bi}$.

Remark 4.2. We may reason as in [Had20b, Proposition 7.10] to show that $\mathsf{P}\colon \mathbf{DCpx}^{\mathcal{R}} \to \mathbf{Pro}_{bi}$ preserves the colimits that are already in $\mathbf{DCpx}^{\mathcal{R}}$, and deduce from [Corollary 1.34, *ibid.*] that the left Kan extension of P along $\mathbf{DCpx}^{\mathcal{R}} \hookrightarrow \odot\mathbf{Set}$ is the same as the left Kan extension of its restriction to $\odot$ along the Yoneda embedding.

4.3 (Diagrammatic nerve of bicoloured pros)**.** The *diagrammatic nerve of bicoloured pros* is the right adjoint

$$\mathsf{N}\colon \mathbf{Pro}_{bi} \to \odot\mathbf{Set}$$

to the functor $\mathsf{P}\colon \odot\mathbf{Set} \to \mathbf{Pro}_{bi}$.

4.4. For all bicoloured pros $(T, \mathscr{T})$,

- morphisms $\mathsf{P}I_n \to (T, \mathscr{T})$ classify 1-cells in T of the form $a_1 \#_0 \dots \#_0 a_n$, where $a_i \in \mathscr{T}$ (including $\varepsilon(\mathscr{T}_0)$) for all $i \in \{1, \dots, n\}$, and
- morphisms $\mathsf{P}U_{n,m} \to (T, \mathscr{T})$ in $\mathbf{Pro}_{bi}$ classify 2-cells of type

  $$a_1 \#_0 \dots \#_0 a_n \Rightarrow b_1 \#_0 \dots \#_0 b_m$$

  in T where $a_i, b_j \in \mathscr{T}$.

These correspond to morphisms $I_n \to \mathsf{N}(T, \mathscr{T})$ and $U_{n,m} \to \mathsf{N}(T, \mathscr{T})$, respectively, in $\odot\mathbf{Set}$, that is, 1-diagrams and 2-cells in $\mathsf{N}(T, \mathscr{T})$.

If U is a 3-atom, a morphism $e\colon U \to \mathsf{N}(T, \mathscr{T})$ restricts, for each $\alpha \in \{+, -\}$, to a 2-diagram $\partial^\alpha e$ of shape $\partial^\alpha U$ in $\mathsf{N}(T, \mathscr{T})$, whose transpose $\widehat{\partial^\alpha e}\colon \mathsf{P}\partial^\alpha U \to T$ is a diagram of 2-cells in T. Because T is a 2-category,

1. the morphism $\widehat{e}\colon \mathsf{P}U \to (T, \mathscr{T})$ exhibits an *equation* between the composites of the diagrams $\widehat{\partial^+ e}$ and $\widehat{\partial^- e}$ in T, and

2. if $e' \colon U \to \mathsf{N}(T, \mathscr{T})$ is another 3-cell with $\partial^\alpha e' = \partial^\alpha e$ for all $\alpha \in \{+, -\}$, then $e = e'$.

More in general, if U is an atom, then a cell $U \to \mathsf{N}(T, \mathscr{T})$ is uniquely determined by its restriction $\sigma_{\leq 2} e$ to $\sigma_{\leq 2} U \subseteq U$.

Lemma 4.5 — *Let X be a diagrammatic set, $(T, \mathscr{T})$ a bicoloured pro, and let $f, g \colon X \to \mathsf{N}(T, \mathscr{T})$ be morphisms of diagrammatic sets. If $f(x) = g(x)$ for all 2-cells x in X, then $f = g$.*

Proof. Let $x \colon U \to X$ be a cell in X with $\dim(U) > 2$. Then $f(x)$ and $g(x)$ are uniquely determined by their restrictions

$$\sigma_{\leq 2}(f(x)) = (\sigma_{\leq 2} x); f, \qquad \sigma_{\leq 2}(g(x)) = (\sigma_{\leq 2} x); g$$

to the directed complex $\sigma_{\leq 2} U$. If f and g agree on 2-cells, these are equal. ∎

Proposition 4.6 — *The functor N is full and faithful.*

Proof. Suppose $\mathsf{N}f = \mathsf{N}g$ for two morphisms $f, g \colon (T, \mathscr{T}) \to (S, \mathscr{S})$ in $\mathbf{Pro}_{bi}$. Given a 2-cell $\varphi \colon (a_1, \dots, a_n) \Rightarrow (b_1, \dots, b_m)$ in T, classified by a morphism $\varphi \colon \mathsf{P}U_{n,m} \to (T, \mathscr{T})$ with transpose $\widehat{\varphi} \colon U_{n,m} \to \mathsf{N}(T, \mathscr{T})$, we have

$$\widehat{\varphi}; \mathsf{N}f = \widehat{\varphi}; \mathsf{N}g$$

in $\odot\mathbf{Set}$. It follows that $\varphi; f = \varphi; g$ in $\mathbf{Pro}_{bi}$, that is, $f(\varphi) = g(\varphi)$. Because f and g agree on all 2-cells, they are equal. This proves that N is faithful.

Let $f' \colon \mathsf{N}(T, \mathscr{T}) \to \mathsf{N}(S, \mathscr{S})$ be a morphism of diagrammatic sets. Given a 2-cell $\varphi \colon (a_1, \dots, a_n) \Rightarrow (b_1, \dots, b_m)$ in T, classified by $\varphi \colon \mathsf{P}U_{n,m} \to (T, \mathscr{T})$ with transpose $\widehat{\varphi} \colon U_{n,m} \to \mathsf{N}(T, \mathscr{T})$, we define $f(\varphi)$ to be the unique 2-cell in S whose classifying morphism $f(\varphi) \colon \mathsf{P}U_{n,m} \to (S, \mathscr{S})$ is the transpose of $\widehat{\varphi}; f' \colon U_{n,m} \to \mathsf{N}(S, \mathscr{S})$.

We want to show that f determines a morphism of bicoloured pros. It is straightforward to verify that f is compatible with all boundaries and with composition and units for 1-cells.

Let x be a 1-cell in T, classified by $x \colon \mathsf{P}I_n \to (T, \mathscr{T})$ with transpose $\hat{x}$; the unit εx is classified by $\varepsilon x \colon \mathsf{P}U_{n,n} \to (T, \mathscr{T})$ with transpose $\widehat{\varepsilon x}$. Let

$$U \coloneqq O(I_n) \Rightarrow U_{n,n},$$

where $O(-)$ is the construction of [Had20b, §2.21]. This is well-defined as a regular 3-atom. There is a unique cell $e \colon U \to \mathsf{N}(T, \mathscr{T})$ such that

1. e is equal to $\varepsilon \widehat{x}$ on $\partial^- U$, see [§4.16, *ibid.*], and

2. e is equal to $\widehat{\varepsilon x}$ on $\partial^+ U$.

Then $e; f' \colon U \to \mathsf{N}(S, \mathscr{S})$ is a 3-cell of type $\varepsilon f'(\widehat{x}) \Rightarrow f'(\widehat{\varepsilon x})$, whose transpose exhibits the equation $\varepsilon f(x) = f(\varepsilon x)$ in S.

Next, suppose φ, ψ are 2-cells in T, classified by morphisms

$$\varphi \colon \mathsf{P}U_{n,m} \to (T, \mathscr{T}), \qquad \psi \colon \mathsf{P}U_{p,\ell} \to (T, \mathscr{T})$$

with transposes $\widehat{\varphi}$, $\widehat{\psi}$. Suppose that $\varphi \#_1 \psi$ is defined; then we may assume $p = m$, and the composite is classified by

$$\varphi \#_1 \psi \colon \mathsf{P}U_{n,\ell} \to (T, \mathscr{T})$$

with transpose $\widehat{\varphi \#_1 \psi}$. Let

$$U \coloneqq (U_{n,m} \#_1 U_{m,\ell}) \Rightarrow U_{n,\ell};$$

this is a regular 3-atom. There is a unique cell $e \colon U \to \mathsf{N}(T, \mathscr{T})$ such that

1. e is equal to $\widehat{\varphi}$ on $U_{n,m} \hookrightarrow \partial^- U$ and to $\widehat{\psi}$ on $U_{m,\ell} \hookrightarrow \partial^- U$, and
2. e is equal to $\widehat{\varphi \#_1 \psi}$ on $\partial^+ U$.

Then $e; f' \colon U \to \mathsf{N}(S, \mathscr{S})$ is a cell of type

$$f'(\widehat{\varphi}) \#_1 f'(\widehat{\psi}) \Rightarrow f'(\widehat{\varphi \#_1 \psi}),$$

whose transpose exhibits an equation $f(\varphi) \#_1 f(\psi) = f(\varphi \#_1 \psi)$ in S.

Finally, suppose that $\varphi \#_0 \psi$ is defined; this composite is classified by

$$\varphi \#_0 \psi \colon \mathsf{P}U_{n+p,m+\ell} \to (T, \mathscr{T})$$

with transpose $\widehat{\varphi \#_0 \psi}$. Let

$$U \coloneqq ((U_{n,m} \#_0 U_{p,\ell}) \#_1 U_{m+\ell,m+\ell}) \Rightarrow U_{n+p,m+\ell}.$$

This is a regular 3-atom and there is a unique cell $e \colon U \to \mathsf{N}(T, \mathscr{T})$ such that

1. e is equal to $\widehat{\varphi}$ on $U_{n,m} \hookrightarrow \partial^- U$ and to $\widehat{\psi}$ on $U_{p,\ell} \hookrightarrow \partial^- U$,
2. e is equal to the transpose of $\varepsilon(\partial^+ \varphi \#_0 \partial^+ \psi)$ on $U_{m+\ell,m+\ell} \hookrightarrow \partial^- U$, and
3. e is equal to $\widehat{\varphi \#_0 \psi}$ on $\partial^+ U$.

Then $e; f' \colon U \to \mathsf{N}(S, \mathscr{S})$ is a cell of type

$$(f'(\widehat{\varphi}) \#_0 f'(\widehat{\psi})) \#_1 f'(\varepsilon(\widehat{\partial^+ \varphi \#_0 \partial^+ \psi})) \Rightarrow f'(\widehat{\varphi \#_0 \psi}).$$

whose transpose exhibits the equation

$$(f(\varphi) \#_0 f(\psi)) \#_1 f(\varepsilon(\partial^+ \varphi \#_0 \partial^+ \psi)) = f(\varphi \#_0 \psi)$$

in S. Because we already know that f is compatible with units, we deduce that $f(\varphi) \#_0 f(\psi) = f(\varphi \#_0 \psi)$.

This proves that $f\colon (T, \mathscr{T}) \to (S, \mathscr{S})$ is a morphism of bicoloured pros. Now $\mathsf{N}f$ and f' are morphisms $\mathsf{N}(T, \mathscr{T}) \to \mathsf{N}(S, \mathscr{S})$ that, by construction, agree on all 2-cells of $\mathsf{N}(T, \mathscr{T})$. It follows from Lemma 4.5 that $\mathsf{N}f = f'$. This proves that N is full. ∎

Comment 4.7. String diagrams are commonly used to depict cells in a pro, usually after an appeal to the Joyal–Street soundness result [JS91]. The diagrammatic nerve construction offers an alternative justification, where diagrams are attributed a combinatorial, rather than topological interpretation.

We avoid mixing the two uses: our string diagrams always represent diagrams in a diagrammatic set. A *caveat* is that, contrary to custom, we are not allowed to have nodes with no input or output wires; instead, we need to explicitly reintroduce *units* and *unitors* [Had20b, §4.17] where necessary.

To distinguish them visually, we draw unit 1-cells as dotted wires, and unitor 2-cells as "dotless nodes": for example, a 2-cell of type $(0) \Rightarrow (1)$ in a one-sorted pro will be depicted as

 as opposed to ,

while a left unitor 2-cell will be depicted as

 as opposed to .

This may seem like unnecessary trouble in dimension 2; the pay-off, on the other hand, is that diagrammatic sets provide sound diagrammatic reasoning in *all* dimensions.

4.2. Realisation of diagrammatic sets in Gray-categories

4.8. Our next goal is to construct a functor $\mathsf{G}\colon \mathbf{DCpx}^{\mathcal{R}} \to \mathbf{GrayCat}$, different from the "obvious" one obtained by composing $\mathcal{M}o\ell{-}^*\colon \mathbf{DCpx}^{\mathcal{R}} \to \omega\mathbf{Cat}$ with $\tau_{\leq 3}\colon \omega\mathbf{Cat} \to 3\mathbf{Cat}$ and then including $3\mathbf{Cat}$ in $\mathbf{GrayCat}$. In particular, $\mathsf{G}P$ will in general have non-trivial interchangers, so it will not be a strict 3-category.

Every regular directed complex is the colimit of the diagram of inclusions of its atoms [Had20b, Corollary 1.34]. We impose that G preserve these colimit diagrams. Then it suffices to define G on atoms of increasing dimension. For each $n \in \mathbb{N} + \{-1\}$, let $\odot_n$ be the full subcategory of $\odot$ on the atoms of dimension $\leq n$.

4.9 (G in dimension ≤ 2)**.** On regular atoms of dimension ≤ 2, we define G to be $\mathcal{M}o\ell - : \dot{\odot}_2 \to \mathbf{3Cat}$ followed by the embedding $\mathbf{3Cat} \hookrightarrow \mathbf{GrayCat}$. We extend G along colimits to all regular directed complexes of dimension ≤ 2.

4.10. Let P be a 2-dimensional regular directed complex. Then $\mathsf{G}(\sigma_{\leq 1}P)$ is equal to (the image under the embedding $\mathbf{3Cat} \hookrightarrow \mathbf{GrayCat}$ of) $\mathcal{M}o\ell\sigma_{\leq 1}P^*$ and has the structure of a 1-(pre)polygraph with the 1-atoms of P as generators. Now, for all 2-atoms $x \in P$,

$$\begin{array}{ccc} \partial O^2 & \hookrightarrow & O^2 \\ \downarrow & & \downarrow{\scriptstyle \mathrm{cl}\{x\}} \\ \mathcal{M}o\ell(\partial x) & \hookrightarrow & \mathcal{M}o\ell(\mathrm{cl}\{x\}) \end{array}$$

is a pushout both in $\omega\mathbf{PreCat}$ and $\mathbf{GrayCat}$. Thus taking the pushout of the span

$$\begin{array}{ccc} \coprod_{\dim(x)=n} \mathcal{M}o\ell(\partial x) & \hookrightarrow & \coprod_{\dim(x)=n} \mathcal{M}o\ell(\mathrm{cl}\{x\}) \\ \downarrow & & \\ \mathcal{M}o\ell\sigma_{\leq 1}P^* & & \end{array}$$

in $\omega\mathbf{PreCat}$ determines a 2-prepolygraph $(\mathsf{G}P)_2$, while in $\mathbf{GrayCat}$, by a standard argument on the commutativity of colimits, it is equivalent to the construction of $\mathsf{G}P$. The results of [FM18, Section 1.6] imply that

1. $\mathsf{G}P$ is obtained from $(\mathsf{G}P)_2$ by freely attaching some 3-cells (interchange generators) indexed by generating cells of $(\mathsf{G}P)_2$, and imposing some equations of 3-cells, so in particular
2. $(\mathsf{G}P)_2$ is the 2-skeleton of $\mathsf{G}P$.

In the terminology of Forest and Mimram, P determines a *presentation* of the 2-precategory $(\mathsf{G}P)_2$, which can be completed to a *Gray presentation* of $\mathsf{G}P$ by freely adding the necessary *structural generators*.

Lemma 4.11 — *Let U be a regular 2-molecule. There is a bijective correspondence between*

1. *cells of rank 2 in $\mathsf{G}U$, and*
2. *2-molecules $V \subseteq U$ together with a 1-order.*

Proof. By the discussion in §4.10, the 2-cells in $\mathsf{G}U$ are the same as the 2-cells in the 2-prepolygraph $(\mathsf{G}U)_2$, so they are freely generated by the atoms of U under principal compositions (§1.32), subject to the axioms of ω-precategories.

Let V be a 2-molecule with a 1-order $(x_1, \dots, x_m)$. By Lemma 3.10, we obtain a decomposition

$$V = V_1 \#_1 \dots \#_1 V_m. \tag{7}$$

Now each V_i has frame dimension 0 or -1, so it has a (clearly unique) decomposition

$$\mathrm{cl}\{y_{i,1}\} \#_0 \dots \#_0 \mathrm{cl}\{y_{i,k}\} \#_0 \mathrm{cl}\{x_i\} \#_0 \mathrm{cl}\{y_{i,k+1}\} \#_0 \dots \#_0 \mathrm{cl}\{y_{i,p}\} \tag{8}$$

where $\dim(y_{i,j}) = 1$ for all $j \in \{1, \dots, p\}$. Replacing the (8) into (7), we obtain a decomposition of U into atoms using only principal compositions, which determines a cell of rank 2 in $(\mathsf{G}U)_2$.

Conversely, by [FM18, Proposition 2], every cell y of rank 2 in $(\mathsf{G}U)_2$ has a unique expression of the form $y_1 \#_1 \dots \#_1 y_m$ where y_i is an expression of the form (8). Now the expression of y is also a valid expression for a 2-cell in $\mathcal{M}o\ell U^*$, which by Remark 3.15 is equal to $\mathcal{M}o\ell U$, so it determines a 2-molecule $V \subseteq U$ together with a decomposition into atoms. From this decomposition we recover uniquely a 1-order $(x_1, \dots, x_m)$ on V. The two constructions are clearly inverse to each other. ∎

Remark 4.12. By Lemma 4.11, every cell of rank 2 in $\mathsf{G}U$ is identified uniquely by a pair $(V, (x_i)_{i=1}^m)$ of a 2-molecule and a 1-order.

More in general, if P is a 2-dimensional regular directed complex, a pair $(V, (x_i)_{i=1}^m)$ of a 2-molecule in P and a 1-order on it determines a unique cell of rank 2 in $\mathsf{G}P$, although these may not exhaust all cells of rank 2 when P is not totally loop-free.

Proposition 4.13 — *Let U and $V \subseteq U$ be regular 2-molecules and let $(x_1, \dots, x_m)$ and $(x'_1, \dots, x'_m)$ be two 1-orders on V. Then in $\mathsf{G}U$ there is a unique 3-cell from $(V, (x_i)_{i=1}^m)$ to $(V, (x'_i)_{i=1}^m)$.*

Proof. For each 1-order $(x_1, \dots, x_m)$ on V, let $\mathsf{w}((x_i)_{i=1}^m)$ be equal to the number of pairs (i, j) such that $i < j$ but $x_j \prec x_i$ in the total order on U. Then

- $0 \leq \mathsf{w}((x_i)_{i=1}^m) \leq \binom{m}{2}$,
- $\mathsf{w}((x_i)_{i=1}^m) = 0$ if and only if $(x_i)_{i=1}^m$ is the normal 1-order, and
- for all non-trivial interchangers $\chi_{x,y} \colon (V, (x_i)_{i=1}^m) \Rightarrow (V, (x'_i)_{i=1}^m)$, we have $\mathsf{w}((x_i)_{i=1}^m) < \mathsf{w}((x'_i)_{i=1}^m)$.

Then $\mathsf{w}(-)$ induces a *termination order* on the 2-cells of $\mathsf{G}U$ in the terminology of [FM18, Section 2.2]. Because the Gray presentation of $\mathsf{G}U$ determined by U as in §4.10 has no non-structural 3-generators, it is always locally confluent,

so [Theorem 11, *ibid.*] applies and $\mathsf{G}U$ has at most one 3-cell between any parallel pair of 2-cells. This proves uniqueness.

For existence, it suffices to observe that, if $\mathsf{w}((x_i)_{i=1}^m) > 0$, then there is a non-trivial inverse interchanger with input $(V, (x_i)_{i=1}^m)$; we leave the proof as an exercise. Applying inverse interchangers repeatedly, we obtain invertible 3-cells of type

$$(V, (x_i)_{i=1}^m) \Rightarrow (V, \text{normal 1-order}), \quad (V, (x_i')_{i=1}^m) \Rightarrow (V, \text{normal 1-order}).$$

Composing the first with the inverse of the second produces a 3-cell of type $(V, (x_i)_{i=1}^m) \Rightarrow (V, (x_i')_{i=1}^m)$. ∎

4.14 (G in dimension 3)**.** Let U be a regular 3-atom. We define $\mathsf{G}U$ to be the pushout

$$\begin{array}{ccc} \partial O^3 & \hookrightarrow & O^3 \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathsf{G}(\partial U) & \hookrightarrow & \mathsf{G}U \end{array}$$

in **GrayCat**, where f sends $\underline{2}^\alpha$ to $(\partial^\alpha U, \text{normal 1-order})$ for each $\alpha \in \{+, -\}$.

Now every map $f\colon U \to V$ in $\odot_3$ determines an assignment of generators of $\mathsf{G}V$ to generators of $\mathsf{G}U$ which is compatible with boundaries, hence extends uniquely to a functor $\mathsf{G}f\colon \mathsf{G}U \to \mathsf{G}V$. This defines $\mathsf{G}\colon \odot_3 \to \mathbf{GrayCat}$. We extend G along colimits to all regular directed complexes of dimension ≤ 3.

4.15. By construction, if P is a regular directed complex of dimension 3, we can associate to each 3-atom U of P a 3-cell

$$[\![U]\!]\colon (\partial^- U, \text{normal 1-order}) \Rightarrow (\partial^+ U, \text{normal 1-order}).$$

in $\mathsf{G}P$. We want to extend this assignment to all 3-molecules in P.

4.16. Suppose that U contains a single 3-dimensional element x. Then $\partial^\alpha x \sqsubseteq \partial^\alpha U$ for all $\alpha \in \{+, -\}$, the substitution $\partial^\alpha U[\langle \partial^\alpha x \rangle / \partial^\alpha x]$ is well-defined, and

$$\partial^- U[\langle \partial^- x \rangle / \partial^- x] = \partial^+ U[\langle \partial^+ x \rangle / \partial^+ x].$$

Pick a 1-order $(x_i)_{i=1}^m$ on $\partial^\alpha U[\langle \partial^\alpha x \rangle / \partial^\alpha x]$; then $\mathrm{cl}\{x_k\} = \langle \partial^\alpha x \rangle$ for a unique $k \in \{1, \ldots, m\}$. Let $y_1, \ldots, y_p$ be the normal 1-order on $\partial^- x$ and let $z_1, \ldots, z_q$ be the normal 1-order on $\partial^+ x$. Then

$$\begin{aligned} (x_i^-)_{i=1}^{m+p-1} &\coloneqq (x_1, \ldots, x_{k-1}, y_1, \ldots, y_p, x_{k+1}, \ldots, x_m), \\ (x_i^+)_{i=1}^{m+q-1} &\coloneqq (x_1, \ldots, x_{k-1}, z_1, \ldots, z_q, x_{k+1}, \ldots, x_m) \end{aligned}$$

are 1-orders on $\partial^- U$ and $\partial^+ U$, respectively.

Now substituting $[\![\mathrm{cl}\{x\}]\!]$ for $\mathrm{cl}\{x_k\}$ in the decomposition of $\partial^\alpha U[\langle \partial^\alpha x\rangle / \partial^\alpha x]$ corresponding to the 1-order $(x_i)_{i=1}^m$ yields a valid expression for a 3-cell

$$c[x] \colon (\partial^- U, (x_i^-)_{i=1}^{m+p-1}) \Rightarrow (\partial^+ U, (x_i^+)_{i=1}^{m+q-1}) \tag{9}$$

in $\mathsf{G}P$. By Proposition 4.13, there are unique 3-cells

$$\chi^- \colon (\partial^- U, \text{normal 1-order}) \Rightarrow (\partial^- U, (x_i^-)_{i=1}^{m+p-1}),$$
$$\chi^+ \colon (\partial^+ U, (x_i^+)_{i=1}^{m+q-1}) \Rightarrow (\partial^+ U, \text{normal 1-order})$$

obtained as composites of interchangers and inverse interchangers. We define $[\![U]\!]$ to be the composite

$$\chi^- \#_2 c[x] \#_2 \chi^+ \colon (\partial^- U, \text{normal 1-order}) \Rightarrow (\partial^+ U, \text{normal 1-order}).$$

We need to show that this is independent of our choice of 1-order $(x_i)_{i=1}^m$. Suppose $(x_i')_{i=1}^m$ is another 1-order on $\partial^\alpha U[\langle \partial^\alpha x\rangle / \partial^\alpha x]$, leading to a potentially different interpretation $\chi'^- \#_2 c'[x] \#_2 \chi'^+$. There are unique 3-cells

$$\psi^- \colon (\partial^- U, (x_i^-)_{i=1}^{m+p-1}) \Rightarrow (\partial^- U, (x_i'^-)_{i=1}^{m+p-1}),$$
$$\psi^+ \colon (\partial^+ U, (x_i^+)_{i=1}^{m+q-1}) \Rightarrow (\partial^+ U, (x_i'^+)_{i=1}^{m+q-1})$$

obtained as composites of interchangers and inverse interchangers, and since they "fix" $\partial^- x$ and $\partial^+ x$, by naturality of interchangers we have

$$\psi^- \#_2 c'[x] = c[x] \#_2 \psi^+$$

hence $c'[x] = (\psi^-)^{-1} \#_2 c[x] \#_2 \psi^+$ and

$$\chi'^- \#_2 c'[x] \#_2 \chi'^+ = \chi'^- \#_2 (\psi^-)^{-1} \#_2 c[x] \#_2 \psi^+ \#_2 \chi'^+.$$

Finally, by Proposition 4.13, $\chi'^- \#_2 (\psi^-)^{-1} = \chi^-$ and $\psi^+ \#_2 \chi'^+ = \chi^+$.

4.17. Let U be any regular 3-molecule in P and fix a 2-order $(x_1, \dots, x_m)$ on U. Then Lemma 3.10 combined with Theorem 3.25 gives a decomposition $U = V_1 \#_2 \dots \#_2 V_m$ where x_i is the only 3-dimensional element of V_i for each $i \in \{1, \dots, m\}$. We let $[\![U]\!]$ in $\mathsf{G}P$ be the composite $[\![V_1]\!] \#_2 \dots \#_2 [\![V_m]\!]$ of the 3-cells

$$[\![V_i]\!] \colon (\partial^- V_i, \text{normal 1-order}) \Rightarrow (\partial^+ V_i, \text{normal 1-order})$$

defined in §4.16.

We need to show that this interpretation is independent of the 2-order chosen on U. Observe that any pair of 2-orders on U is related by a sequence of swaps

of consecutive elements that are not connected by a path in $\mathscr{M}_2U$. Thus it suffices to show that if

$$W_1 \#_2 W_2 = W_1' \#_2 W_2'$$

as 3-molecules, where x is the only 3-dimensional element in W_1 and W_2', while y is the only 3-dimensional element in W_2 and W_1', then

$$[\![W_1]\!] \#_2 [\![W_2]\!] = [\![W_1']\!] \#_2 [\![W_2']\!].$$

The interpretation of W_1 involves a choice of 1-order on $\partial^\alpha W_1[\langle\partial^\alpha x\rangle/\partial^\alpha x]$ but it is independent of this choice. Now $\partial^- y \sqsubseteq \partial^- W_2 = \partial^+ W_1$, and since x and y are not connected by a path in $\mathscr{M}_2U$, necessarily

$$\partial^+ x \cap \partial^- y \subseteq \mathrm{cl}\{x\} \cap \mathrm{cl}\{y\} \subseteq \partial_1 x \cup \partial_1 y,$$

and by Proposition 3.22 $\partial^- y \sqsubseteq \partial^+ W_1[\langle\partial^+ x\rangle/\partial^+ x]$.

Applying the known equalities between the boundaries of W_1, W_2, W_1', W_2', we deduce that the double substitutions

$$\begin{aligned}
&\partial^+ W_1[\langle\partial^+ x\rangle/\partial^+ x][\langle\partial^- y\rangle/\partial^- y],\\
&\partial^- W_2[\langle\partial^- y\rangle/\partial^- y][\langle\partial^+ x\rangle/\partial^+ x],\\
&\partial^+ W_1'[\langle\partial^+ y\rangle/\partial^+ y][\langle\partial^- x\rangle/\partial^- x],\\
&\partial^- W_2'[\langle\partial^- x\rangle/\partial^- x][\langle\partial^+ y\rangle/\partial^+ y]
\end{aligned}$$

are all well-defined and equal to the same regular 2-molecule. Fix a 1-order $(z_i)_{i=1}^p$ on it. Then $\langle\partial^\alpha x\rangle = \mathrm{cl}\{z_k\}$ and $\langle\partial^\beta y\rangle = \mathrm{cl}\{z_\ell\}$ for a unique pair $k, \ell \in \{1, \dots, p\}$. Now

- to interpret W_1, choose the 1-order on $\partial^+ W_1[\langle\partial^+ x\rangle/\partial^+ x]$ obtained by replacing z_ℓ with the normal 1-order on $\partial^- y$ in $(z_i)_{i=1}^p$,
- to interpret W_2, choose the 1-order on $\partial^- W_2[\langle\partial^- y\rangle/\partial^- y]$ obtained by replacing z_k with the normal 1-order on $\partial^+ x$ in $(z_i)_{i=1}^p$,

and similarly for W_1' and W_2'. With the construction of §4.16, any choices of 1-orders lead to expressions

$$\begin{aligned}
&\chi_1^- \#_2 c[x] \#_2 \chi_1^+ \#_2 \chi_2^- \#_2 c[y] \#_2 \chi_2^+,\\
&\chi_1'^- \#_2 c'[y] \#_2 \chi_1'^+ \#_2 \chi_2'^- \#_2 c'[x] \#_2 \chi_2'^+,
\end{aligned}$$

and with the particular choice that we made,

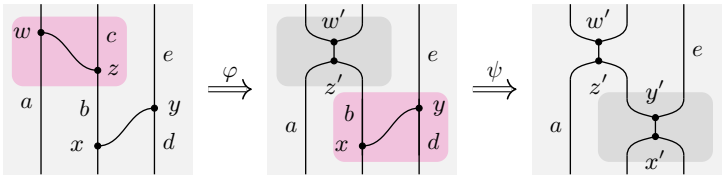
$$\chi_1^- = \chi_1'^-, \qquad \chi_2^+ = \chi_2'^+,$$

while $\chi_1^+ \#_2 \chi_2^-$ and $\chi_1'^+ \#_2 \chi_2'^-$ are units, so they can be eliminated. Finally,

$$\chi_1^- \#_2 c[x] \#_2 c[y] \#_2 \chi_2^+ = \chi_1^- \#_2 c'[y] \#_2 c'[x] \#_2 \chi_2^+$$

is a consequence of axiom 4 of Gray-categories. This proves that $[\![W_1]\!] \#_2 [\![W_2]\!]$ is equal to $[\![W_1']\!] \#_2 [\![W_2']\!]$, and we conclude that $[\![U]\!]$ is independent of the choice of 2-order.

Example 4.18. Let U be the shape of diagram (3). We introduce names for some atoms of U as follows:



The 3-atoms φ and ψ are interpreted in $\mathsf{G}U$ as 3-cells

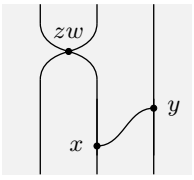
$$[\![\varphi]\!] \colon (a \#_0 z) \#_1 (w \#_0 c) \Rightarrow z' \#_1 w',$$
$$[\![\psi]\!] \colon (x \#_0 d) \#_1 (b \#_0 y) \Rightarrow x' \#_1 y';$$

notice that in this case both $\partial^\alpha \varphi$ and $\partial^\alpha \psi$ admit a single 1-order, which is necessarily the normal 1-order.

We pick the 2-order (φ, ψ) on U, which determines the decomposition

$$U = V_1 \#_1 V_2, \quad V_1 \coloneqq \varphi \cup \partial^- \psi, \quad V_2 \coloneqq \psi \cup \partial^+ \varphi.$$

To interpret V_1 in $\mathsf{G}U$, first we need to consider $V_1[\langle \partial^- \varphi \rangle / \partial^- \varphi]$. This is the shape of the diagram



on which we pick the 1-order (x, y, zw). The 3-cell $c[\varphi]$ corresponding to this 1-order, defined as in (9), is

$$(a \#_0 x \#_0 d) \#_1 (a \#_0 b \#_0 y) \#_1 ([\![\varphi]\!] \#_0 e)$$

of type $(\partial^- V_1, (x, y, z, w)) \Rightarrow (\partial^+ V_1, (x, y, z', w'))$ in $\mathsf{G}U$.

The normal 1-order on $\partial^- V_1$ is in fact (x, z, w, y). Applying a pair of interchangers to first move w after y, then z after y, we obtain a 3-cell

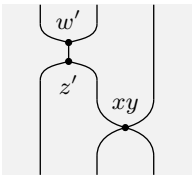
$$\chi_1^- \colon (\partial^- V_1, (x, z, w, y)) \Rightarrow (\partial^- V_1, (x, y, z, w)).$$

Similarly, the normal 1-order on $\partial^+ V_1$ is (x, z', w', y), so we apply a pair of inverse interchangers to move y after z', then y after w', producing a 3-cell

$$\chi_1^+ \colon (\partial^+ V_1, (x, y, z', w')) \Rightarrow (\partial^+ V_1, (x, z', w', y)).$$

Then $[\![V_1]\!]$ is defined to be $\chi_1^- \#_2 c[\varphi] \#_2 \chi_1^+$.

Next, consider $V_2[\langle \partial^- \psi \rangle / \partial^- \psi]$. This is the shape of the diagram



which admits only the 1-order (xy, z', w'). Correspondingly, we construct the 3-cell

$$c[\psi] \coloneqq (a \#_0 [\![\psi]\!]) \#_1 (z' \#_0 e) \#_1 (w' \#_0 e)$$

which is of type $(\partial^- V_2, (x, y, z', w')) \Rightarrow (\partial^+ V_2, (x', y', z', w'))$. The normal 1-order on $\partial^- V_2$ is (x, z', w', y), and we have a composite of interchangers

$$\chi_2^- \colon (\partial^- V_1, (x, z', w', y))(\partial^- V_2, (x, y, z', w')),$$

which is in fact the inverse of χ_1^+. On the other hand, (x', y', z', w') is already the normal 1-order on $\partial^+ V_2$, so $[\![V_2]\!]$ is just $\chi_2^- \#_2 c[\psi]$. Overall, $[\![U]\!]$ is

$$\chi_1^- \#_2 c[\varphi] \#_2 c[\psi] \colon (\partial^- U, (x, z, w, y)) \Rightarrow (\partial^+ U, (x', y', z', w')).$$

If we had picked the 2-order (ψ, φ), we would have instead ended up with the expression $\chi_1^- \#_2 c'[\psi] \#_2 c'[\varphi]$ where

$$\begin{aligned} c'[\psi] &\coloneqq (a \#_0 [\![\psi]\!]) \#_1 (a \#_0 z \#_0 e) \#_1 (w \#_0 c \#_0 e), \\ c'[\varphi] &\coloneqq (a \#_0 x') \#_1 (a \#_0 y') \#_1 ([\![\varphi]\!] \#_0 e). \end{aligned}$$

It follows from axiom 4 of Gray-categories that $c[\varphi] \#_2 c[\psi] = c'[\psi] \#_2 c'[\varphi]$, confirming that $[\![U]\!]$ is independent of the 2-order on U.

4.19 (G in dimension ≥ 4)**.** Let U be a regular 4-atom. We define $\mathsf{G}U$ to be the quotient of $\mathsf{G}(\partial U)$ by the equation

$$[\![\partial^- U]\!] = [\![\partial^+ U]\!],$$

where the 3-molecules $\partial^\alpha U$ are interpreted in $\mathsf{G}(\partial U)$ as by §4.17.

If $f \colon U \to V$ is a map in $\odot_4$, its restriction to ∂U determines a functor $\mathsf{G}(\partial f) \colon \mathsf{G}(\partial U) \to \mathsf{G}V$. If U is a 4-atom, then either $\dim(f(U)) < 4$ and

$f(\partial^- U) = f(\partial^+ U)$, or $\dim(f(U)) = 4$, $f(U) = V$ and $f(\partial^\alpha U) = \partial^\alpha V$ for each $\alpha \in \{+, -\}$. In either case, $\mathsf{G}(\partial f)$ is compatible with the equation $[\![\partial^- U]\!] = [\![\partial^+ U]\!]$, so it factors uniquely through a functor $\mathsf{G}f \colon \mathsf{G}U \to \mathsf{G}V$.

This defines $\mathsf{G} \colon \odot_4 \to \mathbf{GrayCat}$, and we extend it along colimits to all regular directed complexes of dimension ≤ 4.

Finally, if $f \colon P \to Q$ is a map of regular directed complexes of arbitrary dimension, it restricts to a map $\sigma_{\leq 4} f \colon \sigma_{\leq 4} P \to \sigma_{\leq 4} Q$, and we let $\mathsf{G}f$ be equal to $\mathsf{G}(\sigma_{\leq 4} f)$. This defines $\mathsf{G} \colon \mathbf{DCpx}^{\mathcal{R}} \to \mathbf{GrayCat}$.

Comment 4.20. By construction, G ignores any elements of dimension > 4. The idea is that, while 4-atoms can contribute non-trivial equations of 3-cells in a Gray-category, higher-dimensional atoms can only contribute trivial "equations of equations" with no visible effect.

4.21. Because **GrayCat** has all small colimits, we are in the conditions of [Rie17, Corollary 6.2.6] and we can define a functor

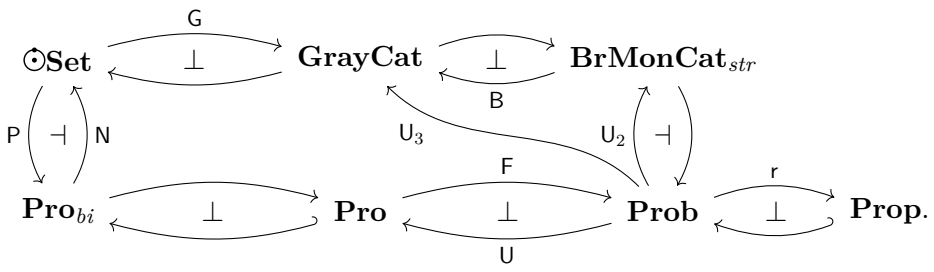
$$\mathsf{G} \colon \dot{\odot}\mathbf{Set} \to \mathbf{GrayCat}$$

as the left Kan extension of $\mathsf{G} \colon \mathbf{DCpx}^{\mathcal{R}} \to \mathbf{GrayCat}$ along the embedding $\mathbf{DCpx}^{\mathcal{R}} \hookrightarrow \dot{\odot}\mathbf{Set}$.

Remark 4.22. Since we made sure at every step that G preserve the colimits in $\mathbf{DCpx}^{\mathcal{R}}$, this is in fact equal to the left Kan extension of the restriction of G to $\dot{\odot}$ along the Yoneda embedding.

Remark 4.23. For the usual reasons, G has a right adjoint, of which we will not make use. Unlike the diagrammatic nerve of pros, it is not full; see [Had20b, Remark 7.20] for a counterexample that also applies to the present case.

Comment 4.24. The following (generally non-commutative) diagram of functors recaps the adjunctions that we have established:



5. The smash product

5.1. The tensor product of pros

We reconstruct the tensor product of props, as defined by Hackney and Robertson, as a reflection of an "external" tensor product of pros, whose combinatorics are only slightly more involved.

Lemma 5.1 — *Let s be a permutation on the set $\{1,\dots,n\}$. Then s is either the identity or admits a unique decomposition*

$$s = s_1;\dots;s_p$$

with the following properties. For each $i \in \{1,\dots,p\}$, let $s^{(i)} \coloneqq s_i;\dots;s_p$. Then

1. *s_i is an elementary transposition $(k\ k+1)$ of two consecutive elements, and*
2. *k is the least element of $\{1,\dots,n\}$ such that $s^{(i)}(k+1) < s^{(i)}(k)$.*

Proof. We construct, step by step, decompositions $s = s_1;\dots;s_{i-1};s^{(i)}$. For $i = 1$, we let $s = s^{(1)}$ trivially. For each $i \geq 1$, if $s^{(i)}$ is the identity, we let $p \coloneqq i-1$ and we stop.

Otherwise, there exists a least k such that $s^{(i)}(k+1) < s^{(i)}(k)$. We let $s_i \coloneqq (k\ k+1)$ and $s^{(i+1)} \coloneqq s_i^{-1};s^{(i)}$. Then $s = s_1;\dots;s_i;s^{(i+1)}$.

At each step, the number of pairs $j, j' \in \{1,\dots,n\}$ such that $j < j'$ but $s^{(i)}(j') < s^{(i)}(j)$ strictly decreases, and it is equal to 0 if and only if $s^{(i)}$ is the identity. It follows that the algorithm terminates after a finite number of steps, producing a decomposition with the desired properties.

Uniqueness is immediate, since the conditions fix the factor s_i uniquely at each step. ∎

5.2. Let s be a permutation on the set $\{1,\dots,n\}$. For all 1-cells $(a_1,\dots,a_n)$ in a prob $(T,\mathscr{T})$, we define an invertible 2-cell

$$\sigma(s)\colon (a_1,\dots,a_n) \Rightarrow (a_{s(1)},\dots,a_{s(n)})$$

in T; the dependence of $\sigma(s)$ on $(a_1,\dots,a_n)$ is left implicit.

- If s is the identity, we let $\sigma(s)$ be the unit on $= (a_1,\dots,a_n)$.
- If s is an elementary transposition $(k\ k+1)$ of two consecutive elements, we let

$$\sigma(s) \coloneqq a_1 \#_0 \dots \#_0 a_{k-1} \#_0 \sigma_{a_k,a_{k+1}} \#_0 a_{k+2} \dots \#_0 a_n.$$

- In general, if $s = s_1; \dots ; s_p$ is the decomposition of s given by Lemma 5.1, we let
$$\sigma(s) \coloneqq \sigma(s_1) \#_1 \dots \#_1 \sigma(s_p).$$

We also define a second invertible 2-cell

$$\sigma^*(s) \colon (a_1, \dots, a_n) \Rightarrow (a_{s(1)}, \dots, a_{s(n)})$$

by $\sigma^*(s) \coloneqq (\sigma(s^{-1}))^{-1}$.

Remark 5.3. If $(T, \mathscr{T})$ is a prop, then $\sigma(s) = \sigma^*(s)$ for all permutations s.

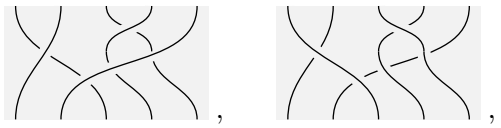
Example 5.4. Let s be the permutation $(1, 2, 3, 4, 5) \mapsto (3, 1, 5, 4, 2)$. The decomposition of s given by Lemma 5.1 is

$$s = (2\ 3); (1\ 2); (3\ 4); (4\ 5); (3\ 4).$$

We use the graphical notation



for the braiding $\sigma_{a,b}$ and the inverse braiding $\sigma^*_{a,b}$, respectively, in a prob. The 2-cells $\sigma(s)$ and $\sigma^*(s)$ of type $(a_1, a_2, a_3, a_4, a_5) \Rightarrow (a_3, a_1, a_5, a_4, a_2)$ can be pictured as



respectively. In a prop, these are identical and may both be pictured as their "shadow"



5.5. Let $(T, \mathscr{T})$ be a prob and let $\{a_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ be a doubly indexed collection of 1-cells in $\mathscr{T}$. We denote by $((a_{i,j})_{i=1}^n)_{j=1}^m$ the 1-cell

$$(a_{1,1}, \dots, a_{n,1}, a_{1,2}, \dots, a_{n,2}, \dots\dots, a_{1,m}, \dots, a_{n,m}) \tag{10}$$

and by $((a_{i,j})_{j=1}^m)_{i=1}^n$ the 1-cell

$$(a_{1,1}, \dots, a_{1,m}, a_{2,1}, \dots, a_{2,m}, \dots\dots, a_{n,1}, \dots, a_{n,m}). \tag{11}$$

We let

$$\sigma\colon ((a_{i,j})_{j=1}^{m})_{i=1}^{n} \Rightarrow ((a_{i,j})_{i=1}^{n})_{j=1}^{m},$$
$$\sigma^*\colon ((a_{i,j})_{i=1}^{n})_{j=1}^{m} \Rightarrow ((a_{i,j})_{j=1}^{m})_{i=1}^{n}$$

be equal to $\sigma(s^{-1})$ and its inverse $\sigma^*(s)$, respectively, for the permutation s implied by the reordering of (10) into (11).

Example 5.6. The 2-cells

$$\sigma\colon (a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3}) \Rightarrow (a_{1,1}, a_{2,1}, a_{1,2}, a_{2,2}, a_{1,3}, a_{2,3}),$$
$$\sigma^*\colon (a_{1,1}, a_{2,1}, a_{1,2}, a_{2,2}, a_{1,3}, a_{2,3}) \Rightarrow (a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3})$$

can be pictured as



respectively.

5.7 (Tensor product of pros)**.** The *tensor product* $(T, \mathscr{T}) \otimes (S, \mathscr{S})$ of two pros $(T, \mathscr{T})$ and $(S, \mathscr{S})$ is the prob $(T \otimes S, \mathscr{T} \otimes \mathscr{S})$ constructed as follows.

1. Let $(\mathscr{T} \otimes \mathscr{S})_0 \coloneqq \{\bullet\}$ and $(\mathscr{T} \otimes \mathscr{S})_1 \coloneqq \{a \otimes c\colon \bullet \Rightarrow \bullet \mid a \in \mathscr{T}_1, c \in \mathscr{S}_1\}$. This determines $\sigma_{\leq 1}(T \otimes S)$ together with its 1-polygraph structure, which makes it a pro.
2. Construct the coproducts
$$\coprod_{c \in \mathscr{S}_1} (T, \mathscr{T}), \qquad \coprod_{a \in \mathscr{T}_1} (S, \mathscr{S}) \tag{12}$$
in **Pro**. Denote by
$$- \otimes d\colon (T, \mathscr{T}) \hookrightarrow \coprod_{c \in \mathscr{S}_1} (T, \mathscr{T}), \qquad b \otimes -\colon (S, \mathscr{S}) \hookrightarrow \coprod_{a \in \mathscr{T}_1} (S, \mathscr{S})$$
the inclusions into the d-indexed and b-indexed summand, respectively. There are morphisms
$$\sigma_{\leq 1}(T \otimes S) \to \coprod_{c \in \mathscr{S}_1} (T, \mathscr{T}), \qquad \sigma_{\leq 1}(T \otimes S) \to \coprod_{a \in \mathscr{T}_1} (S, \mathscr{S})$$
uniquely determined by the "tautologous" assignments $a \otimes c \mapsto a \otimes c$. Construct the pushout
$$\begin{array}{ccc} \sigma_{\leq 1}(T \otimes S) & \longrightarrow & \coprod_{c \in \mathscr{S}_1} (T, \mathscr{T}) \\ \downarrow & & \downarrow \\ \coprod_{a \in \mathscr{T}_1} (S, \mathscr{S}) & \longrightarrow & T \square S \end{array} \tag{13}$$
in **Pro**.

3. Construct the free prob $\mathsf{F}(T \square S)$ and quotient it by the following equations: for all 2-cells

$$\varphi \colon (a_1, \dots, a_n) \Rightarrow (b_1, \dots, b_m) \text{ in } T,\ \psi \colon (c_1, \dots, c_p) \Rightarrow (d_1, \dots, d_q) \text{ in } S,$$

the 1-composite of

$$\begin{aligned}
(a_1 \otimes \psi) \mathbin{\#_0} \dots \mathbin{\#_0} (a_n \otimes \psi) &\colon ((a_i \otimes c_k)_{k=1}^{p})_{i=1}^{n} \Rightarrow ((a_i \otimes d_\ell)_{\ell=1}^{q})_{i=1}^{n}, \\
\sigma &\colon ((a_i \otimes d_\ell)_{\ell=1}^{q})_{i=1}^{n} \Rightarrow ((a_i \otimes d_\ell)_{i=1}^{n})_{\ell=1}^{q}, \\
(\varphi \otimes d_1) \mathbin{\#_0} \dots \mathbin{\#_0} (\varphi \otimes d_q) &\colon ((a_i \otimes d_\ell)_{i=1}^{n})_{\ell=1}^{q} \Rightarrow ((b_j \otimes d_\ell)_{j=1}^{m})_{\ell=1}^{q}, \\
\sigma^* &\colon ((b_j \otimes d_\ell)_{j=1}^{m})_{\ell=1}^{q} \Rightarrow ((b_j \otimes d_\ell)_{\ell=1}^{q})_{j=1}^{m}
\end{aligned}$$

is equal to the 1-composite of

$$\begin{aligned}
\sigma &\colon ((a_i \otimes c_k)_{k=1}^{p})_{i=1}^{n} \Rightarrow ((a_i \otimes c_k)_{i=1}^{n})_{k=1}^{p}, \\
(\varphi \otimes c_1) \mathbin{\#_0} \dots \mathbin{\#_0} (\varphi \otimes c_p) &\colon ((a_i \otimes c_k)_{i=1}^{n})_{k=1}^{p} \Rightarrow ((b_j \otimes c_k)_{j=1}^{m})_{k=1}^{p}, \\
\sigma^* &\colon ((b_j \otimes c_k)_{j=1}^{m})_{k=1}^{p} \Rightarrow ((b_j \otimes c_k)_{k=1}^{p})_{j=1}^{m}, \\
(b_1 \otimes \psi) \mathbin{\#_0} \dots \mathbin{\#_0} (b_m \otimes \psi) &\colon ((b_j \otimes c_k)_{k=1}^{p})_{j=1}^{m} \Rightarrow ((b_j \otimes d_\ell)_{\ell=1}^{q})_{j=1}^{m}.
\end{aligned}$$

We label this equation $\varphi \otimes \psi$.

Note that any composite indexed by an empty list must be interpreted as a unit on $\bullet$ of the appropriate dimension.

If $f \colon (T, \mathscr{T}) \to (T', \mathscr{T}')$ and $g \colon (S, \mathscr{S}) \to (S', \mathscr{S}')$ are morphisms of pros, we can define morphisms

$$\coprod_{c \in \mathscr{S}_1} (T, \mathscr{T}) \to \mathsf{U}(T' \otimes S', \mathscr{T}' \otimes \mathscr{S}'), \qquad \coprod_{a \in \mathscr{T}_1} (S, \mathscr{S}) \to \mathsf{U}(T' \otimes S', \mathscr{T}' \otimes \mathscr{S}'),$$

$$x \otimes c \mapsto f(x) \otimes g(c), \qquad a \otimes y \mapsto f(a) \otimes g(y).$$

Taking the transpose morphisms in **Prob**, and using the universal property of the pushout (13) which is preserved by F, we obtain a unique morphism $\mathsf{F}(T \square S) \to (T' \otimes S', \mathscr{T}' \otimes \mathscr{S}')$ of probs which is compatible with the $\varphi \otimes \psi$ equations, hence factors uniquely through a morphism

$$f \otimes g \colon (T \otimes S, \mathscr{T} \otimes \mathscr{S}) \to (T' \otimes S', \mathscr{T}' \otimes \mathscr{S}').$$

This defines a functor $- \otimes - \colon \mathbf{Pro} \times \mathbf{Pro} \to \mathbf{Prob}$.

Remark 5.8. When either φ or ψ is a unit, the equation $\varphi \otimes \psi$ holds automatically by the axioms of braidings. So $\varphi \otimes \psi$ is only non-trivial when both cells have rank 2.

One can derive, as a consequence, that the monoid $\mathbb{N}$ is a "relative unit" for the tensor product, in the sense that the functors $\mathbb{N} \otimes -$ and $- \otimes \mathbb{N}$ are naturally isomorphic to $\mathsf{F} \colon \mathbf{Pro} \to \mathbf{Prob}$.

Example 5.9. We compute the tensor product $\mathit{Bialg} \coloneqq \mathit{Mon} \otimes \mathit{Mon}^{\mathrm{co}}$ of the theories of monoids and comonoids. Both Mon and $\mathit{Mon}^{\mathrm{co}}$ are one-sorted, so Bialg is also one-sorted.

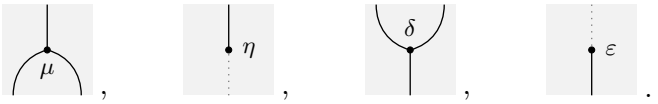
In fact, the indexed coproducts (12) are equal to Mon and $\mathit{Mon}^{\mathrm{co}}$, respectively, while $\sigma_{\leq 1}(\mathit{Bialg})$ is isomorphic to $\mathbb{N}$, so the pushout (13) can be computed as

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathit{Mon} \\ \downarrow & & \downarrow \\ \mathit{Mon}^{\mathrm{co}} & \longrightarrow & \mathit{Mon} \uplus \mathit{Mon}^{\mathrm{co}} \end{array}$$

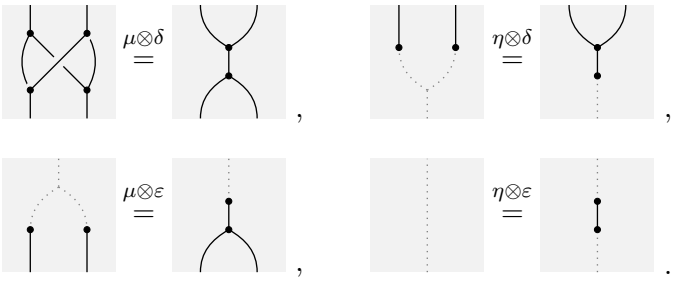
in **Pro**. The 2-cells in $\mathit{Mon} \uplus \mathit{Mon}^{\mathrm{co}}$ are freely generated by those of Mon and $\mathit{Mon}^{\mathrm{co}}$, modulo any equations that hold in the two factors, separately: a model of $\mathit{Mon} \uplus \mathit{Mon}^{\mathrm{co}}$ is a pair of a monoid and a comonoid structure on the same object.

Finally, to obtain Bialg, we quotient $\mathsf{F}(\mathit{Mon} \uplus \mathit{Mon}^{\mathrm{co}})$ by the $\varphi \otimes \psi$ equations. It suffices to let φ and ψ range over 2-cells that generate Mon and $\mathit{Mon}^{\mathrm{co}}$, respectively, under composition.

An obvious choice is to take the unique maps $\mu\colon (2) \Rightarrow (1)$ and $\eta\colon (0) \Rightarrow (1)$ as generators of Mon, and their duals $\delta\colon (1) \Rightarrow (2)$ and $\varepsilon\colon (1) \Rightarrow (0)$ as generators of $\mathit{Mon}^{\mathrm{co}}$. We may picture these as



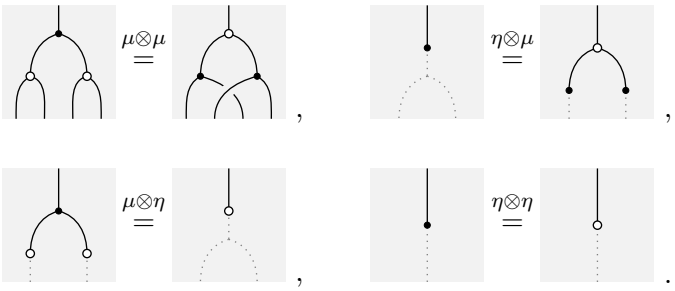
The four corresponding equations are



In a symmetric monoidal category, a pair of a monoid and a comonoid satisfying these equations is a *bialgebra* [Pir02]. In a braided monoidal category, this notion forks into two variants, distinguished by the use of braidings or inverse braidings, classified by Bialg and by $\mathit{Bialg}^* = \mathit{Mon}^{\mathrm{co}} \otimes \mathit{Mon}$, respectively.

Example 5.10. We compute the tensor product $\mathit{BrCMon} = \mathit{Mon} \otimes \mathit{Mon}$. We proceed as in Example 5.9 to derive that BrCMon is the quotient of $\mathsf{F}(\mathit{Mon} \uplus \mathit{Mon})$ by the equations $\varphi \otimes \psi$ where φ, ψ range over $\{\mu, \eta\}$.

Using different colours to distinguish cells from each copy of *Mon*, these can be pictured as



A model of $Mon \uplus Mon$ is a pair of monoid structures on the same object. It is a consequence of the Eckmann–Hilton argument, valid in every braided monoidal category, that a pair of monoid structures satisfying the equations $\mu \otimes \mu$ and $\eta \otimes \eta$ coincide with a single commutative monoid structure. The equations $\eta \otimes \mu$ and $\mu \otimes \eta$ are derivable from the rest.

We conclude that *BrCMon* is the *braided* monoidal theory of commutative monoids, whose reflection $\mathsf{r}(BrCMon)$ is isomorphic to *CMon*. Dually, $Mon^{\mathrm{co}} \otimes Mon^{\mathrm{co}}$ is the braided monoidal theory of commutative comonoids.

5.11 (Tensor product of props)**.** The *tensor product* $(T, \mathscr{T}) \otimes_{\mathbb{S}} (S, \mathscr{S})$ of two props $(T, \mathscr{T})$ and $(S, \mathscr{S})$ is the quotient of $\mathsf{r}(\mathsf{U}(T, \mathscr{T}) \otimes \mathsf{U}(S, \mathscr{S}))$ by the equations

$$\sigma_{a,b} \otimes c = \sigma_{a \otimes c, b \otimes c}, \quad a \otimes \sigma_{c,d} = \sigma_{a \otimes c, a \otimes d} \tag{14}$$

for all $a, b \in \mathscr{T}_1$ and $c, d \in \mathscr{S}_1$, where $\sigma_{a,b}$ and $\sigma_{c,d}$ are the original braidings of T and S.

As shown in [HR15, Section 3], the tensor product of props is part of a symmetric monoidal closed structure on **Prop**, whose unit is the theory of permutations $\mathbb{S}$.

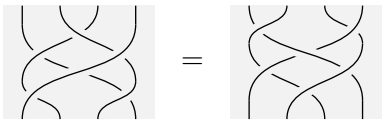
Example 5.12. Given a prop $(T, \mathscr{T})$, the tensor product $(T, \mathscr{T}) \otimes_{\mathbb{S}} CMon^{\mathrm{co}}$ is a *cartesian* prop, also known as a Lawvere theory. It is in fact the free cartesian prop on $(T, \mathscr{T})$ [Bae06].

Comment 5.13. The tensor product of props is compatible with the tensor product of pros in the sense that the diagram of functors

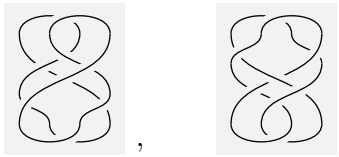
$$\begin{array}{ccc} \mathbf{Pro} \times \mathbf{Pro} & \xrightarrow{\otimes} & \mathbf{Prob} \\ {\scriptstyle \mathsf{rF} \times \mathsf{rF}} \downarrow & & \downarrow {\scriptstyle \mathsf{r}} \\ \mathbf{Prop} \times \mathbf{Prop} & \xrightarrow[\otimes_{\mathbb{S}}]{} & \mathbf{Prop} \end{array} \tag{15}$$

commutes up to natural isomorphism. The reason why this works is that, when φ or ψ is a braiding $\sigma_{a,b}$, the equation $\varphi \otimes \psi$ combined with (14) holds automatically in a prop. It follows that, while $\mathsf{UrF}(T, \mathscr{T}) \otimes \mathsf{UrF}(S, \mathscr{S})$ has additional generators and equations compared to $(T, \mathscr{T}) \otimes (S, \mathscr{S})$, these are all trivialised by the combined action of r and (14).

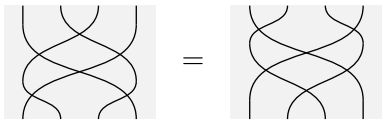
This fact is specific to props and does *not* generalise to probs: the quotient of $\mathsf{UF}(T, \mathscr{T}) \otimes \mathsf{UF}(S, \mathscr{S})$ by (14) in **Prob** is not in general isomorphic to $(T, \mathscr{T}) \otimes (S, \mathscr{S})$. For example, the quotient of $\mathsf{U}\mathbb{B} \otimes \mathsf{U}\mathbb{B}$ by (14) is not isomorphic to $\mathbb{N} \otimes \mathbb{N} \simeq \mathbb{B}$. Indeed, if $\sigma_{1,1} \colon (2) \Rightarrow (2)$ is the generating braiding in $\mathbb{B}$, the equation $\sigma_{1,1} \otimes \sigma_{1,1}$ becomes



which does not hold in the braid group on 4 strands. This can be checked by considering the link diagrams



and observing that the first is an unlink while the second is not. On the other hand, the reflected equation



is valid in the theory of permutations.

As a consequence, there does not seem to be an interesting monoidal structure on **Prob** that generalises the one on **Prop**.

Remark 5.14. As shown in [HR15, Proposition 40], the tensor product of props extends the Boardman-Vogt product of symmetric operads [BV06], in the sense that there is an embedding of the category of symmetric operads into the category of props which is strong monoidal with respect to the two monoidal structures.

Remark 5.15. The tensor product of pros is *not* symmetric. Up to the definition of $T \square S$ as the pushout (13), the construction of $(T, \mathscr{T}) \otimes (S, \mathscr{S})$ and $(S, \mathscr{S}) \otimes (T, \mathscr{T})$ is, indeed, identical up to a change of notation. However, in

the final quotient, the roles of σ and σ^*, or braidings and inverse braidings, are reversed.

Nevertheless, this argument reveals a natural isomorphism between

$$(T, \mathscr{T}) \otimes (S, \mathscr{S}) \quad \text{and} \quad ((S, \mathscr{S}) \otimes (T, \mathscr{T}))^*,$$

where $-^*$ is the duality defined in §2.18. From this we can recover a symmetry for the tensor product of props.

5.2. The smash product of pointed diagrammatic sets

5.16 (Gray product)**.** Let P, Q be regular directed complexes. The *Gray product* $P \otimes Q$ of P and Q is the cartesian product $P \times Q$ of their underlying posets with the following orientation. Write $x \otimes y$ for a generic element of $P \otimes Q$. For all x' covered by x in P and all y' covered by y in Q,

$$o(x \otimes y \to x' \otimes y) \coloneqq o_P(x \to x'),$$
$$o(x \otimes y \to x \otimes y') \coloneqq (-)^{\dim(x)} o_Q(y \to y'),$$

where o_P and o_Q are the orientations of P and Q, respectively.

As shown in [Had20b, Section 2.2], $P \otimes Q$ is a regular directed complex. If $f\colon P \to P'$ and $g\colon Q \to Q'$ are maps of regular directed complexes, let $f \otimes g\colon P \otimes Q \to P' \otimes Q'$ have the cartesian product of f and g as underlying function. Then $f \otimes g$ is a map of regular directed complexes.

Gray products determine a monoidal structure on $\mathbf{DCpx}^{\mathcal{R}}$ whose unit is the terminal object 1.

5.17. The monoidal structure on $\mathbf{DCpx}^{\mathcal{R}}$ restricts to a monoidal structure on $\odot$, which, by Day's theory [Day70], extends along the Yoneda embedding to a monoidal biclosed structure on $\odot\mathbf{Set}$.

Explicitly, let X and Y be diagrammatic sets. The *Gray product* $X \otimes Y$ of X and Y is the colimit in $\odot\mathbf{Set}$ of the diagram

$$\odot/_X \times \odot/_Y \xrightarrow{\text{dom} \times \text{dom}} \odot \times \odot \xrightarrow{\otimes} \odot \hookrightarrow \odot\mathbf{Set}, \tag{16}$$

where $\odot/_X$ is the category whose objects are cells $x\colon U \to X$ and morphisms from $x\colon U \to X$ to $y\colon V \to X$ are commutative triangles

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & V \\ & {\scriptstyle x}\searrow \quad \swarrow{\scriptstyle y} & \\ & X & \end{array},$$

while dom sends such a triangle to the map $f\colon U \to V$ in $\odot$.

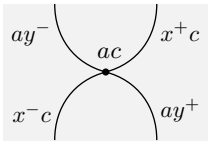
In particular, for each pair of cells $x\colon U \to X$ and $y\colon V \to Y$, the image of the pair (x, y) through the diagram (16) is $U \otimes V$, so we obtain a morphism $U \otimes V \to X \otimes Y$ to the colimit, that is, a cell of shape $U \otimes V$ in $X \otimes Y$. This is the cell $x \otimes y$ obtained as the Gray product of x and y in $\odot\mathbf{Set}$.

Remark 5.18. The dimensions of cells add under the Gray product, that is, if x is an n-cell and y is an m-cell, then $x \otimes y$ is an $(n + m)$-cell.

Remark 5.19. The Gray product is not the cartesian product in $\odot\mathbf{Set}$. However, the monoidal unit is the terminal object, which gives us "projection" morphisms $X \otimes Y \to X$ and $X \otimes Y \to Y$. These send a cell $x \otimes y$ of shape $U \otimes V$ to $p_1; x$ and $p_2; y$, respectively, where $p_1 : U \otimes V \twoheadrightarrow U$ and $p_2 : U \otimes V \twoheadrightarrow V$ are projections in $\odot$.

Comment 5.20. We use string diagrams to give some intuition about cells $x \otimes y$ of shape $U \otimes V$ in low dimension; in the pictures, we write xy for $x \otimes y$. First of all, if x or y is a 0-cell, then $U \otimes V$ is isomorphic to V or U, respectively, and $x \otimes y$ has the same dimension and shape as y or x.

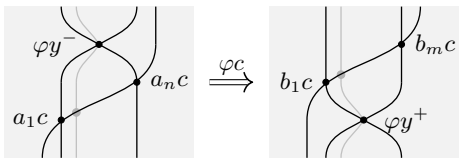
Let $a\colon x^- \Rightarrow x^+$ be a 1-cell in X and $c\colon y^- \Rightarrow y^+$ a 1-cell in Y. Then $a \otimes c$ is a 2-cell of the form



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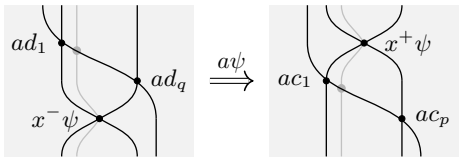
that is, it is of type $(x^- \otimes c) \#_0 (a \otimes y^+) \Rightarrow (a \otimes y^-) \#_0 (x^+ \otimes c)$ in $X \otimes Y$.

Next, let $\varphi\colon a_1 \#_0 \ldots \#_0 a_n \Rightarrow b_1 \#_0 \ldots \#_0 b_m$ be a 2-cell in X, and let $\psi\colon c_1 \#_0 \ldots \#_0 c_p \Rightarrow d_1 \#_0 \ldots \#_0 d_q$ be a 2-cell in Y. Then $\varphi \otimes c$ is a 3-cell of the form



(17)
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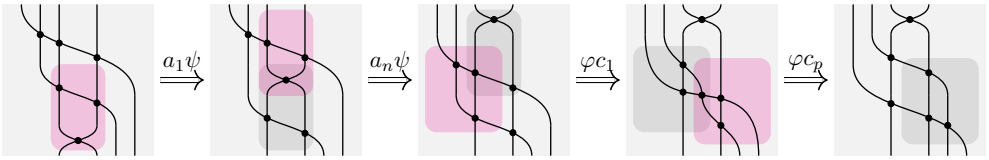
while $a \otimes \psi$ is a 3-cell of the form



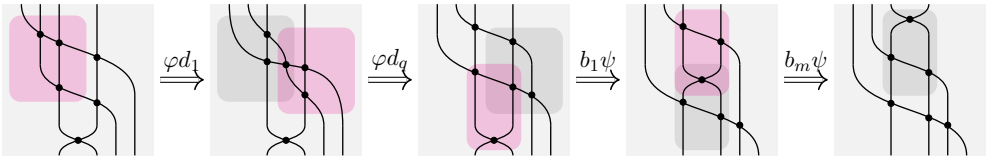
(18)

in $X \otimes Y$. It is useful to think of these as *sliding moves*: $\varphi \otimes c$ slides a 2-cell in the fibre of φ left-to-right, top-to-bottom past a 1-cell in the fibre of c, while $a \otimes \psi$ slides a 2-cell in the fibre of ψ left-to-right, bottom-to-top past a 1-cell in the fibre of a.

Next, we consider the 4-cell $\varphi \otimes \psi$; to simplify, we depict φ and ψ as if they had only 2 inputs and 2 outputs each. Then $\partial^-(\varphi \otimes \psi)$ is the 3-diagram

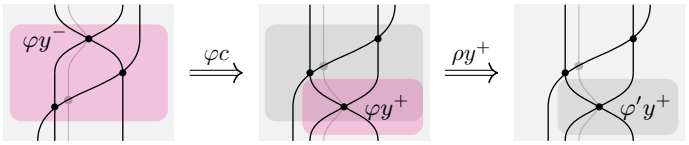


where the sequence of sliding moves $a_1 \otimes \psi, \dots, a_n \otimes \psi$ is followed by the sequence $\varphi \otimes c_1, \dots, \varphi \otimes c_p$, while $\partial^+(\varphi \otimes \psi)$ is the 3-diagram

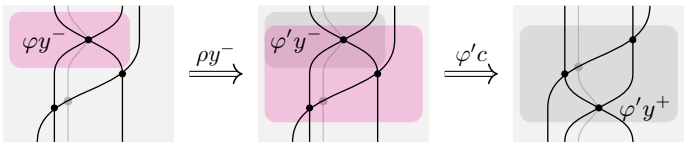


where the sequence of sliding moves $\varphi \otimes d_1, \dots, \varphi \otimes d_q$ is followed by the sequence $b_1 \otimes \psi, \dots, b_m \otimes \psi$. In the case $n, m, p, q = 2$, one can recognise the two sides of the Zamolodchikov tetrahedron equation [KV94].

Next, let ρ be a 3-cell in X and consider the 4-cell $\rho \otimes c$. To simplify, we depict ρ as if it were of type $\varphi \Rightarrow \varphi'$ where φ and φ' are both 2-cells. Then $\partial^-(\rho \otimes c)$ has the form

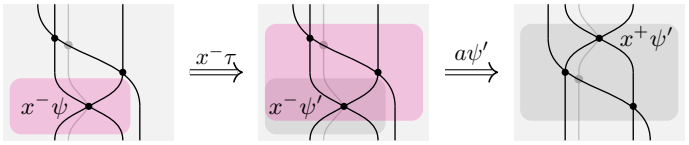


while $\partial^+(\rho \otimes c)$ has the form

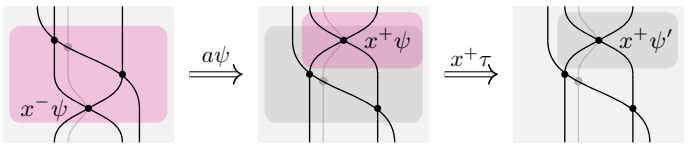


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Dually, if $\tau \colon \psi \Rightarrow \psi'$ is a 3-cell in Y, $\partial^-(a \otimes \tau)$ has the form



while $\partial^+(a \otimes \tau)$ has the form



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5.21 (Pointed diagrammatic set)**.** A *pointed diagrammatic set* is a diagrammatic set X together with a distinguished 0-cell $\bullet\colon 1 \to X$, the *basepoint*.

A morphism $f\colon (X, \bullet_X) \to (Y, \bullet_Y)$ of pointed diagrammatic sets is a morphism $f\colon X \to Y$ such that $f(\bullet_X) = \bullet_Y$. With their morphisms, pointed diagrammatic sets form a category $\odot\mathbf{Set}_\bullet$.

5.22. The obvious forgetful functor $\odot\mathbf{Set}_\bullet \to \odot\mathbf{Set}$ has a left adjoint sending a diagrammatic set X to the coproduct $X + 1$, pointed with the inclusion of 1 into the coproduct.

The terminal object 1 of $\odot\mathbf{Set}$, pointed with its only 0-cell, is a zero object in $\odot\mathbf{Set}_\bullet$, both terminal and initial.

5.23 (Wedge sum)**.** The *wedge sum* of two pointed diagrammatic sets $(X, \bullet_X)$ and $(Y, \bullet_Y)$ is the pointed diagrammatic set $(X \vee Y, \bullet)$ where

1. $X \vee Y$ is the quotient of $X + Y$ by the equation $\bullet_X = \bullet_Y$, and
2. $\bullet$ is the result of the identification of $\bullet_X$ and $\bullet_Y$.

5.24 (Smash product)**.** Let $(X, \bullet_X)$ and $(Y, \bullet_Y)$ be pointed diagrammatic sets. There is an inclusion $X \vee Y \hookrightarrow X \otimes Y$ defined by

$$x \mapsto x \otimes \bullet_Y, \qquad y \mapsto \bullet_X \otimes y$$

on cells in X and Y, respectively.

The *smash product* of $(X, \bullet_X)$ and $(Y, \bullet_Y)$ is the pointed diagrammatic set $(X \owedge Y, \bullet)$ obtained from the pushout diagram

$$\begin{array}{ccc} X \vee Y & \hookrightarrow & X \otimes Y \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[\bullet]{} & X \owedge Y \end{array}$$

in $\odot\mathbf{Set}$ (the "quotient of $X \otimes Y$ by the subspace $X \vee Y$").

The smash product is part of a monoidal structure on $\odot\mathbf{Set}_\bullet$, whose unit is the diagrammatic set $1 + 1$, pointed with one of the coproduct inclusions, and all structural isomorphisms are derived from those of the Gray product.

Comment 5.25. The smash product of pointed diagrammatic sets is a "directed" counterpart to the smash product of pointed topological spaces, with the Gray product playing the role of the cartesian product of spaces.

The formal correspondence between definitions is made concrete through the geometric realisation of diagrammatic sets [Had20b, §8.38]. This functor

$|-|\colon \odot\mathbf{Set} \to \mathbf{cgHaus}$ sends 0-cells in a diagrammatic set to points in a space, so it lifts to a functor

$$|-|\colon \odot\mathbf{Set}_\bullet \to \mathbf{cgHaus}_\bullet$$

to the category of pointed compactly generated Hausdorff spaces and continuous maps that preserve the basepoint.

We claim that this functor sends smash products in $\odot\mathbf{Set}_\bullet$ to smash products in $\mathbf{cgHaus}_\bullet$, that is, it is strong monoidal with respect to the two monoidal structures.

Proof. On regular atoms, $|-|$ is defined as the forgetful functor from $\odot$ to the category of posets and order-preserving maps, followed by the simplicial nerve of posets, followed by the geometric realisation of simplicial sets. The first sends Gray products to cartesian products and the other two preserve finite products. Thus $|U \otimes V| \simeq |U| \times |V|$ naturally in U and V.

Both Gray products in $\odot\mathbf{Set}$ and products in $\mathbf{cgHaus}$ are part of a biclosed monoidal structure, so they preserve colimits separately in each variable. Since $|-|$, a left adjoint functor, also preserves colimits, we can extend to an isomorphism $|X \otimes Y| \simeq |X| \times |Y|$ natural in the diagrammatic sets X and Y.

Finally, $|-|$ also preserves the terminal object, so it sends the colimit diagrams that define wedge sums and smash products in $\odot\mathbf{Set}_\bullet$ to the colimit diagrams that define them in $\mathbf{cgHaus}_\bullet$. ∎

5.26. Like the smash product of pointed spaces, the smash product of pointed diagrammatic sets is part of a biclosed structure on $\odot\mathbf{Set}_\bullet$.

Left homs and right homs can be computed by a formal argument. If $(X, \bullet_X) \multimap (Y, \bullet_Y)$ is a right hom in $(\odot\mathbf{Set}_\bullet, \owedge, 1 + 1)$, cells of shape U in its underlying diagrammatic set correspond to pointed morphisms from $U + 1$ to $(X, \bullet_X) \multimap (Y, \bullet_Y)$, which correspond to pointed morphisms from $(X, \bullet_X) \owedge (U + 1)$ to $(Y, \bullet_Y)$.

Now $X \owedge (U + 1)$ is isomorphic to the quotient of $X \otimes U$ by the subspace $\{\bullet_X\} \otimes U$. By the universal property of this quotient, we conclude that there is a bijection between

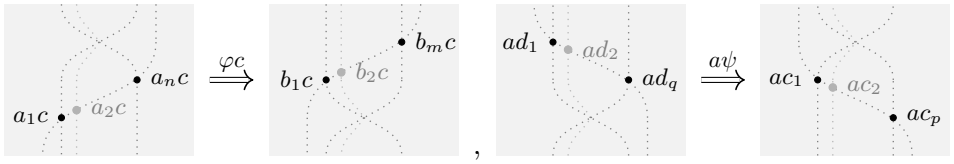
1. cells of shape U in $(X, \bullet_X) \multimap (Y, \bullet_Y)$ and
2. morphisms $X \otimes U \to Y$ which send $\{\bullet_X\} \otimes U$ to $\{\bullet_Y\}$.

Similarly, cells of shape U in the left hom $(Y, \bullet_Y) \multimap (X, \bullet_X)$ correspond bijectively to morphisms $U \otimes X \to Y$ sending $U \otimes \{\bullet_X\}$ to $\{\bullet_Y\}$.

In particular, the 0-cells in both the left and the right hom are the pointed morphisms from $(X, \bullet_X)$ to $(Y, \bullet_Y)$. The basepoint, classified by the only morphism from the zero object, is the constant morphism $X \mapsto \bullet_Y$.

Comment 5.27. From the string diagram of a cell in $X \otimes Y$, it is easy to obtain a picture of the same cell in $X \mathbin{⍉} Y$: we simply need to identify every cell of the form $x \otimes \bullet_Y$ or $\bullet_X \otimes y$ and shape U with the cell $!;\bullet$ of the same shape, where $! \colon U \twoheadrightarrow 1$ is the unique map to the terminal object.

We will depict all such 1-cells as dotted wires, and all such 2-cells as dotless nodes, which is consistent with our convention for units and unitors in the nerve of a pro. For example, if X and Y have a single 0-cell, then any 3-cell of the form $\varphi \otimes c$ as in (17) or $a \otimes \psi$ as in (18) becomes



respectively, in $X \mathbin{⍉} Y$.

5.3. Comparison of the constructions

We are ready to state our main theorem.

5.28. Let $(T, \mathscr{T})$ be a pro. Its diagrammatic nerve $\mathsf{N}T$ has a single 0-cell, so it is canonically pointed, and every morphism in the image of N trivially preserves the basepoint. Thus N, restricted to **Pro**, lifts uniquely to a functor $\mathsf{N} \colon \mathbf{Pro} \to \dot{\odot}\mathbf{Set}_\bullet$.

Theorem 5.29 — *The diagram of functors*

$$\begin{array}{ccccc} \mathbf{Pro} \times \mathbf{Pro} & & \xrightarrow{\quad - \otimes - \quad} & & \mathbf{Prob} \\ \Big\downarrow{\scriptstyle \mathsf{N} \times \mathsf{N}} & & & \searrow^{\mathsf{U}_3} & \\ & & & & \mathbf{GrayCat} \\ \dot{\odot}\mathbf{Set}_\bullet \times \dot{\odot}\mathbf{Set}_\bullet & & \xrightarrow[- \mathbin{⍉} (-)^\circ]{\quad} & \nearrow_{\mathsf{G}} & \dot{\odot}\mathbf{Set}_\bullet \end{array}$$

commutes up to natural isomorphism.

Comment 5.30. The form of Theorem 5.29 does not suggest, at first sight, that the smash product of pointed diagrammatic sets *subsumes* and *generalises* the tensor product of props. Nevertheless, we argue that this is essentially the case.

First of all, since U_3 is pseudomonic by Remark 2.38, if $\mathsf{G}X$ is isomorphic to $\mathsf{U}_3(T, \mathscr{T})$ for some prob $(T, \mathscr{T})$, then this prob is essentially unique. It follows that, on the image of $\mathsf{N} - \mathbin{⍉} (\mathsf{N}-)^\circ$, we can lift G to a functor with codomain **Prob**, and compute the tensor product of two pros through the lower leg of

the diagram. In this sense, the smash product on $\odot\mathbf{Set}_\bullet$ strictly subsumes the "external" tensor product of pros.

From the tensor product of pros, we can recover the tensor product of props via a universal characterisation in **Prop**, independent of the specific construction. If $(T, \mathscr{T})$ and $(S, \mathscr{S})$ are two props, we have families of morphisms

$$\begin{aligned} \mathrm{id}_T \otimes c &\colon \mathsf{FU}(T, \mathscr{T}) \to \mathsf{U}(T, \mathscr{T}) \otimes \mathsf{U}(S, \mathscr{S}), \\ a \otimes \mathrm{id}_S &\colon \mathsf{FU}(S, \mathscr{S}) \to \mathsf{U}(T, \mathscr{T}) \otimes \mathsf{U}(S, \mathscr{S}) \end{aligned}$$

in **Prob** indexed by $c \in \mathscr{S}_1$ and $a \in \mathscr{T}_1$, where $c \colon \mathbb{N} \to \mathsf{U}(S, \mathscr{S})$ and $a \colon \mathbb{N} \to \mathsf{U}(T, \mathscr{T})$ send the generating 1-cell of $\mathbb{N}$ to c and a, respectively; here we use the fact that $- \otimes \mathbb{N}$ and $\mathbb{N} \otimes -$ are naturally isomorphic to F. Then $(T, \mathscr{T}) \otimes_{\mathbb{S}} (S, \mathscr{S})$ is the pushout

$$\begin{array}{ccc} \coprod_{c \in \mathscr{S}_1} \mathsf{rFU}(T, \mathscr{T}) + \coprod_{a \in \mathscr{T}_1} \mathsf{rFU}(S, \mathscr{S}) & \longrightarrow & \mathsf{r}(\mathsf{U}(T, \mathscr{T}) \otimes \mathsf{U}(S, \mathscr{S})) \\ \downarrow & & \downarrow \\ \coprod_{c \in \mathscr{S}_1} (T, \mathscr{T}) + \coprod_{a \in \mathscr{T}_1} (S, \mathscr{S}) & \longrightarrow & (T, \mathscr{T}) \otimes_{\mathbb{S}} (S, \mathscr{S}) \end{array}$$

in **Prop**, where the top leg is obtained universally from the family of morphisms $\{\mathsf{r}(\mathrm{id}_T \otimes c), \mathsf{r}(a \otimes \mathrm{id}_S) \mid c \in \mathscr{S}_1, a \in \mathscr{T}_1\}$ and the left leg from the counit of the adjunction between rF and U.

5.31 (Skeleta of diagrammatic sets)**.** For each $n \in \mathbb{N} + \{-1\}$, the restriction functor $\odot\mathbf{Set} \to \mathrm{PSh}(\odot_n)$ has a left adjoint; let $\sigma_{\leq n}$ be the comonad induced by this adjunction.

The *n-skeleton* of a diagrammatic set X is the counit $\sigma_{\leq n} X \to X$. For all $k \leq n$, the k-skeleton factors uniquely through the n-skeleton of X. By a standard argument X is the colimit of the sequence of its skeleta.

Proof of Theorem 5.29. Let $(T, \mathscr{T})$ and $(S, \mathscr{S})$ be two pros and let $(X, \bullet_X)$ and $(Y, \bullet_Y)$ be equal to $\mathsf{N}(T, \mathscr{T})$ and $(\mathsf{N}(S, \mathscr{S}))^\circ$, respectively.

As seen in §5.17, $X \otimes Y$ is the colimit of a diagram of atoms $U \otimes V$ indexed by pairs of cells $x \colon U \to X$ and $y \colon V \to Y$, which are transposes of morphisms $x \colon \mathsf{P}U \to (T, \mathscr{T})$ and $y \colon \mathsf{P}V^\circ \to (S, \mathscr{S})$ in **Pro**. The smash product $X \mathbin{⍉} Y$ is then the colimit of this diagram extended with a morphism $U \otimes V \twoheadrightarrow 1$ for all atoms $U \otimes V$ indexed by $(x, \bullet_Y)$ or by $(\bullet_X, y)$.

Each diagrammatic set is the colimit of the sequence of its skeleta, and this colimit is preserved by smash products separately in each variable. Because colimits commute with colimits, we can compute $X \mathbin{⍉} Y$ in steps, increasing i and j separately in $\sigma_{\leq i} X \mathbin{⍉} \sigma_{\leq j} Y$. This corresponds to restricting the indexing category to pairs (x, y) with $\dim(x) \leq i$ and $\dim(y) \leq j$.

The functor $\mathsf{G}\colon \odot\mathbf{Set} \to \mathbf{GrayCat}$, a left adjoint, preserves colimits, and we know how to explicitly compute G on atoms. We will use this to compute $\mathsf{G}(X \owedge Y)$.

- Let $i = 0$ or $j = 0$. Since the only 0-cell of X and Y is their basepoint, both $\sigma_{\leq 0}X \owedge Y$ and $X \owedge \sigma_{\leq 0}Y$ are isomorphic to the terminal diagrammatic set. Their image through G is the terminal Gray-category with one 0-cell and no cells of higher rank.
- Let $i = j = 1$. The 1-cells in X that do not factor through $\sigma_{\leq 0}X$ correspond bijectively to generating 1-cells $a \in \mathscr{T}_1$, and the 1-cells in Y that do not factor through $\sigma_{\leq 0}Y$ to generating 1-cells $c \in \mathscr{S}_1$.
  The boundary of $a \otimes c$ contains only 1-cells of the form $!;\bullet$, which G sends to units on $\bullet$. Through G, then, $a \otimes c$ becomes a 2-cell of type $\varepsilon\bullet \Rightarrow \varepsilon\bullet$. Thus $\mathsf{G}(\sigma_{\leq 1}X \owedge \sigma_{\leq 1}Y)$ has a single 0-cell, a single 1-cell, and its 2-cells are freely generated by the $a \otimes c$: this makes it a prob in the sense of §2.36, isomorphic to $\mathsf{F}(\sigma_{\leq 1}(T \otimes S))$. This structure of prob will be inherited by $\mathsf{G}(\sigma_{\leq i}X \owedge \sigma_{\leq j}Y)$ for all higher i, j.
- We fix $j = 1$ and increase i; observe that we can stop at $i = 3$, since for $i > 3$ we only include cells of dimension > 4, whose contribution through G is trivial.
  Let $c \in \mathscr{S}_1$. Each 2-cell $\varphi\colon (a_1, \dots, a_n) \Rightarrow (b_1, \dots, b_m)$ in T contributes a 3-cell $\varphi \otimes c$ in $\sigma_{\leq 2}X \owedge \sigma_{\leq 1}Y$; any other 2-cell in X factors through one of this form, so it does not give a contribution. We can read the form of $\varphi \otimes c$ from Comment 5.27: unravelling the definition of G on 3-atoms, we see that it sends $\varphi \otimes c$ to a 3-cell of type

  $$(a_1 \otimes c) \#_1 \dots \#_1 (a_n \otimes c) \Rightarrow (b_1 \otimes c) \#_1 \dots \#_1 (b_m \otimes c).$$

  Extending along colimits, a 2-diagram $x \#_k y$ in X induces a diagram $(x \otimes c) \#_{k+1} (y \otimes c)$ in $\mathsf{G}(\sigma_{\leq 2}X \owedge \sigma_{\leq 1}Y)$ for each $k \in \{0, 1\}$.
  From Proposition 4.6, we know that 3-cells ρ in X exhibit all and only the equations of diagrams $x = y$ that hold in T. Reading the form of $\rho \otimes c$ from Comment 5.20, we see that the only part surviving both the smash product quotient and G is an equation between the composites of $x \otimes c$ and $y \otimes c$. For each $c \in \mathscr{S}_1$, then, we can define a morphism of probs

  $$- \otimes c\colon \mathsf{F}(T, \mathscr{T}) \to \mathsf{G}(X \owedge \sigma_{\leq 1}Y)$$

  by $x \mapsto x \otimes c$ on cells in T, extending universally to the free prob, and prove that it is injective. Moreover, the family of the $- \otimes c$ is jointly surjective and only overlaps on $\bullet$. We conclude that there is an isomorphism

between $\mathsf{G}(X \obslash \sigma_{\leq 1} Y)$ and

$$\coprod_{c \in \mathscr{S}_1} \mathsf{F}(T, \mathscr{T}) \simeq \mathsf{F}(\coprod_{c \in \mathscr{S}_1} (T, \mathscr{T})),$$

the coproducts being in **Prob** and **Pro**, respectively.

- The case where we fix $i = 1$ and increase j is dual, with a subtlety due to the way Gray products change orientations in their second factor depending on the dimension of the first factor. Since we defined Y to be the *dual* of $\mathsf{N}(S, \mathscr{S})$, each 2-cell $\psi \colon (c_1, \dots, c_p) \Rightarrow (d_1, \dots, d_q)$ corresponds to a 2-cell

  $$\psi \colon d_q \#_0 \dots \#_0 d_1 \Rightarrow c_p \#_0 \dots \#_0 c_1$$

  in Y, which for each $a \in \mathscr{T}_1$ contributes a 3-cell $a \otimes \psi$ in $\sigma_{\leq 1} X \obslash \sigma_{\leq 2} Y$. By inspection of the shape of $a \otimes \psi$ in Comment 5.27, we see that G sends it to a 3-cell of type

  $$(a \otimes c_1) \#_1 \dots \#_1 (a \otimes c_p) \Rightarrow (a \otimes d_1) \#_1 \dots \#_1 (a \otimes d_q),$$

  which matches the original orientation of ψ in S. Proceeding as before, then, we construct an isomorphism between $\mathsf{G}(\sigma_{\leq 1} X \obslash Y)$ and

  $$\coprod_{a \in \mathscr{T}_1} \mathsf{F}(S, \mathscr{S}) \simeq \mathsf{F}(\coprod_{a \in \mathscr{T}_1} (S, \mathscr{S})).$$

  It follows that $\mathsf{G}(X \obslash \sigma_{\leq 1} Y \cup \sigma_{\leq 1} X \obslash Y)$ can be computed as the pushout

  $$\begin{array}{ccc} \mathsf{F}(\sigma_{\leq 1}(T \otimes S)) & \longrightarrow & \mathsf{F}(\coprod_{c \in \mathscr{S}_1} (T, \mathscr{T})) \\ \downarrow & & \ulcorner \quad \downarrow \\ \mathsf{F}(\coprod_{a \in \mathscr{T}_1} (S, \mathscr{S})) & \longrightarrow & \mathsf{G}(X \obslash \sigma_{\leq 1} Y \cup \sigma_{\leq 1} X \obslash Y) \end{array}$$

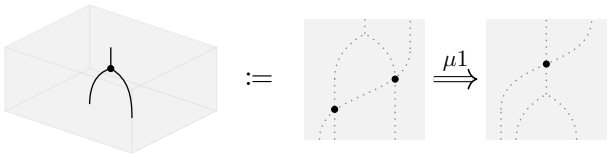
  in **Prob**, isomorphic to $\mathsf{F}(T \square S)$.

- Finally, let $i = j = 2$; increasing either i or j beyond 2 only includes cells of dimension > 4, whose contribution is trivial.
  Each pair of a 2-cell $\varphi \colon (a_1, \dots, a_n) \Rightarrow (b_1, \dots, b_m)$ in T and a 2-cell $\psi \colon (c_1, \dots, c_p) \Rightarrow (d_1, \dots, d_q)$ in S contributes a 4-cell $\varphi \otimes \psi$ to $X \obslash Y$, and any other pair factors through one of this form.
  Remember that the orientation of ψ is reversed in Y. Reading the form of $\varphi \otimes \psi$ from Comment 5.20 and unravelling the definition of G on 4-atoms, we find that the boundaries of $\varphi \otimes \psi$ are mapped by G to the two sides of the $\varphi \otimes \psi$ equation in $\mathsf{F}(T \square S)$.

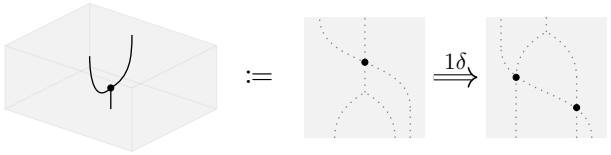
We conclude that $\mathsf{G}(X \obslash Y)$ with its unique prob structure is isomorphic to $(T, \mathscr{T}) \otimes (S, \mathscr{S})$. It is straightforward to verify that the isomorphism is natural in $(T, \mathscr{T})$ and $(S, \mathscr{S})$. ∎

Example 5.32. We compute an equation of the prob $Bialg = Mon \otimes Mon^{\mathrm{co}}$ through $\dot{\odot}\mathbf{Set}_\bullet$, to illustrate how it arises from a 4-cell in the smash product of $X \coloneqq \mathsf{N}(Mon)$ and $Y \coloneqq \mathsf{N}(Mon^{\mathrm{co}})^\circ$.

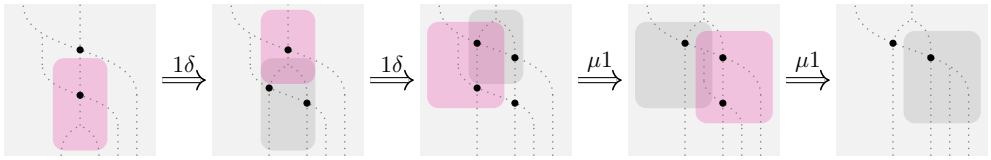
The two generating 1-cells of Mon and Mon^{co} produce a 2-cell $1 \otimes 1$ in $X \mathbin{⦶} Y$. From the generator $\mu\colon (2) \Rightarrow (1)$ of Mon, we obtain a 3-cell $\mu \otimes 1$ in $X \mathbin{⦶} Y$. Because $1 \otimes 1$ is the only non-degenerate 2-cell appearing in the boundary of $\mu \otimes 1$, we may informally picture this 3-cell as a string diagram in 3-dimensional space, "tracing the history" of the various copies of $1 \otimes 1$:



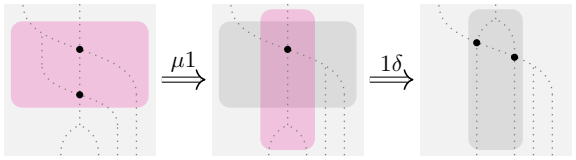
Similarly, from the generator $\delta\colon (1) \Rightarrow (2)$ of Mon^{co}, we obtain a 3-cell $1 \otimes \delta$ of the form



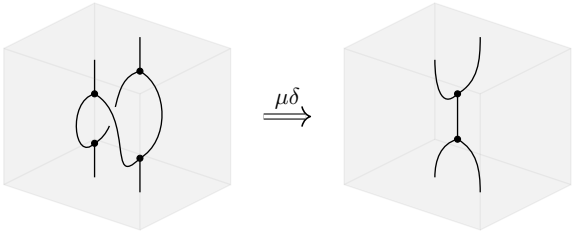
Now $\partial^-(\mu \otimes \delta)$ has the form



while $\partial^+(\mu \otimes \delta)$ has the form



In 3-dimensional string diagrams, we may picture $\mu \otimes \delta$ as



(19)

If we also "trace the history" of the dotted wires to produce *surface diagrams* in the style of [VD19], we recover, up to a deformation, the "intersecting surfaces" picture (2).

The picture of equation $\mu \otimes \delta$ in Example 5.9 can be interpreted as a planar projection of (19). Technically, the single instance of a braiding in this equation arises, by definition of G, from the fact that the input 2-cells of the first instance of $\mu \otimes 1$ in $\partial^-(\mu \otimes \delta)$ are not consecutive in the normal 1-order on the overall 2-diagram.

6. Higher-dimensional cells

Opening blurb...

6.1 (Diagrammatic complex)**.** A *diagrammatic complex* is a diagrammatic set X together with a set $\mathscr{X} = \sum_{n\in\mathbb{N}} \mathscr{X}_n$ of cells such that, for all $n \in \mathbb{N}$,

$$\begin{array}{ccc} \coprod_{x\in\mathscr{X}_n} \partial U(x) & \hookrightarrow & \coprod_{x\in\mathscr{X}_n} U(x) \\ \downarrow & & \downarrow{\scriptstyle (x)_{x\in\mathscr{X}_n(X)}} \\ \sigma_{\leq n-1}X & \hookrightarrow & \sigma_{\leq n}X \end{array} \tag{20}$$

is a pushout in $\hat{\odot}\mathbf{Set}$. Here $U(x)$ denotes the shape of the cell x.

Remark 6.2. It follows from the results of [Had20b, Section 8.3] that the geometric realisation of diagrammatic sets sends a diagrammatic complex $(X, \mathscr{X})$ to a CW complex with one cell for each cell in $\mathscr{X}$.

6.3 (Diagrammatic presentation)**.** Let $(T, \mathscr{T})$ be a bicoloured pro. A *diagrammatic presentation* of $(T, \mathscr{T})$ is a diagrammatic complex $(X, \mathscr{X})$ such that $\mathsf{P}X$ is isomorphic to $(T, \mathscr{T})$.

Talk about generalisations of pros, the iteration of the smash product and the periodic table of higher categories, models in weak higher categories (but also strict through the nerve construction)...

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References

[AM20] D. Ara and G. Maltsiniotis. Joint et tranches pour les ∞-catégories strictes. *Mémoires de la Société Mathématique de France*, 165, 2020.

[AR94] J. Adámek and J. Rosický. *Locally presentable and accessible categories*, volume 189. Cambridge University Press, 1994.

[Bae06] J.C. Baez. Universal algebra and diagrammatic reasoning. Lectures at *Geometry of Computation* meeting. Transparencies available at `https://math.ucr.edu/home/baez/universal/`, 2006.

[BD95] J.C. Baez and J. Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36(11):6073–6105, 1995.

[BSZ14] F. Bonchi, P. Sobociński, and F. Zanasi. Interacting bialgebras are Frobenius. In *International Conference on Foundations of Software Science and Computation Structures*, pages 351–365. Springer, 2014.

[BSZ17] F. Bonchi, P. Sobociński, and F. Zanasi. Interacting Hopf algebras. *Journal of Pure and Applied Algebra*, 221(1):144–184, 2017.

[BSZ18] F. Bonchi, P. Sobociński, and F. Zanasi. Deconstructing Lawvere with distributive laws. *Journal of Logical and Algebraic Methods in Programming*, 95:128–146, 2018.

[Bur93] A. Burroni. Higher-dimensional word problems with applications to equational logic. *Theoretical Computer Science*, 115(1):43–62, 1993.

[BV06] J.M. Boardman and R.M. Vogt. *Homotopy invariant algebraic structures on topological spaces*, volume 347. Springer, 2006.

[Car18] J.S. Carter. Graphs, foams, tensors, polytopes, and homology. *Journal of Knot Theory and Its Ramifications*, 27(11):1843015, 2018.

[CG07] E. Cheng and N. Gurski. The periodic table of n-categories for low dimensions I: degenerate categories and degenerate bicategories. *Contemporary Mathematics*, 431:143–164, 2007.

[CG11] E. Cheng and N. Gurski. The periodic table of n-categories II: degenerate tricategories. *Cahiers de topologie et géométrie différentielle catégoriques*, 52(2):82–125, 2011.

[Day70] B. Day. On closed categories of functors. In *Lecture Notes in Mathematics*, pages 1–38. Springer Berlin Heidelberg, 1970.

[FM18] S. Forest and S. Mimram. Coherence of Gray categories via rewriting. In *3rd International Conference on Formal Structures for Computation and Deduction (FSCD 2018)*, volume 108, 2018.

[Fox76] T. Fox. Coalgebras and cartesian categories. *Communications in Algebra*, 4(7):665–667, 1976.

[GPS95] R. Gordon, J. Power, and R. Street. *Coherence for tricategories*, volume 558. American Mathematical Soc., 1995.

[Had17] A. Hadzihasanovic. *The algebra of entanglement and the geometry of composition*. PhD thesis, University of Oxford, 2017.

[Had20a] A. Hadzihasanovic. A combinatorial-topological shape category for polygraphs. *Applied Categorical Structures*, 28(3):419–476, 2020.

[Had20b] A. Hadzihasanovic. Diagrammatic sets and rewriting in weak higher categories. *arXiv preprint arXiv:2007.14505*, 2020.

[HR15] P. Hackney and M. Robertson. On the category of props. *Applied Categorical Structures*, 23(4):543–573, 2015.

[JS91] A. Joyal and R. Street. The geometry of tensor calculus, I. *Advances in mathematics*, 88(1):55–112, 1991.

[KV94] M.M. Kapranov and V.A. Voevodsky. 2-categories and Zamolodchikov tetrahedra equations. In *Proc. Symp. Pure Math*, volume 56, pages 177–260, 1994.

[Lac04] S. Lack. Composing PROPs. *Theory and Applications of Categories*, 13(9):147–163, 2004.

[Lac11] S. Lack. A Quillen model structure for Gray-categories. *Journal of K-Theory*, 8(2):183–221, 2011.

[Law63] F.W. Lawvere. Functorial semantics of algebraic theories. *Proceedings of the National Academy of Sciences of the United States of America*, 50(5):869, 1963.

[Lin69] F.E.J. Linton. Coequalizers in categories of algebras. In *Seminar on triples and categorical homology theory*, pages 75–90. Springer, 1969.

[Mak05] M. Makkai. The word problem for computads. Available on the author's web page `http://www.math.mcgill.ca/makkai`, 2005.

[Mim14] S. Mimram. Towards 3-dimensional rewriting theory. *Logical Methods in Computer Science*, 10(1):1–47, 2014.

[ML65] S. Mac Lane. Categorical algebra. *Bulletin of the American Mathematical Society*, 71(1):40–106, 1965.

[ML71] S. Mac Lane. *Categories for the Working Mathematician*. Springer New York, 1971.

[Pir02] T. Pirashvili. On the PROP corresponding to bialgebras. *Cahiers de topologie et géométrie différentielle catégoriques*, 43(3):221–239, 2002.

[Pow91] J. Power. An n-categorical pasting theorem. In *Lecture Notes in Mathematics*, pages 326–358. Springer Nature, 1991.

[Rie17] E. Riehl. *Category theory in context*. Courier Dover Publications, 2017.

[Sel10] P. Selinger. A survey of graphical languages for monoidal categories. In *New Structures for Physics*, pages 289–355. Springer, 2010.

[Sim09] C. Simpson. *Homotopy Theory of Higher Categories*. Cambridge University Press, 2009.

[Ste93] R. Steiner. The algebra of directed complexes. *Applied Categorical Structures*, 1(3):247–284, 1993.

[Ste04] R. Steiner. Omega-categories and chain complexes. *Homology, Homotopy and Applications*, 6(1):175–200, 2004.

[VD19] J. Vicary and L. Dunn. Coherence for Frobenius pseudomonoids and the geometry of linear proofs. *Logical Methods in Computer Science*, 15(3), 2019.

[Zaw17] M. Zawadowski. Positive opetopes with contractions form a test category. *arXiv preprint arXiv:1712.06033*, 2017.