

Infinite Analysis
Author(s): IAN HACKING
Source: *Studia Leibnitiana*, Bd. 6, H. 1 (1974), pp. 126-130
Published by: Franz Steiner Verlag
Stable URL: https://www.jstor.org/stable/40693729
Accessed: 18-06-2019 12:11 UTC

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at https://about.jstor.org/terms



Franz Steiner Verlag is collaborating with JSTOR to digitize, preserve and extend access to *Studia Leibnitiana*

This content downloaded from 128.40.255.154 on Tue, 18 Jun 2019 12:11:56 UTC
All use subject to https://about.jstor.org/terms

is open to easy criticism, and its very foundation, as we have seen, is not thoroughly established by its author, in an explicit and conscious way at least.

The limits of this method, too, are not clearly defined; obviously, more particular knowledge is empirical not only in as far as it proceeds from the senses, but also in that it can reach probabilistic generality but not universal validity for man. Only, I did not find Changeux explaining at which level this kind of knowledge begins. It is not quite clear whether the laws of motion are empirical or not; Changeux expounds Maupertuis' principle of the least action (which was considered by Maupertuis himself, and by other scientist of that time, as the foundation of the laws of motion), and he describes it as a "geometrical truth" (p. 256–261), which suggests that he considers that law to be absolutely necessary, i.e., above empirical generality. In fact, D'Alembert held the laws of motion for absolutely necessary, but Maupertuis, while basically inclining towards the same position, had finally rejected it[74]. If Changeux was of the same opinion as D'Alembert, then the laws of nature whose necessity cannot be proven belong to a level lower (more particular) than that of the laws of motion — but it is not clear where this level can be established, as Changeux applies the method of the extremes, i.e., his meta-empirical procedure, to notions such as generation and corruption (I, 309ff.), solidity and liquidity (I, 316ff.), physical pleasure and pain (I, 389–390), so that there seem to be other principles of nature, belonging to physics, biology, physiology, which are capable of an absolute foundation. Changeux certainly opens many more problems than he is capable of solving.

Infinite Analysis

By

IAN HACKING (CAMBRIDGE, ENGLAND)

"Existential propositions, when true, are not necessary, for they cannot be proved unless an infinity of propositions is used, i.e. unless an analysis is carried out to infinity."[1] Infinite proof is central to Leibniz's system because it provides the distinction between necessity and contingency and creates the space essential for human freedom. Is there a

[74] See G. TONELLI, *La nécessité des lois de la nature au XVIIIe siècle et chez Kant en 1762*, in: *Revue d'histoire des sciences et de leurs applications*, XII, 1959.

[1] *Generales Inquisitiones de Analysi Notionum et Veritatum*, 74, LEIBNIZ, *Opuscules et fragments inédits*, ed. L. Couturat, Paris, 1901, p. 374.

This content downloaded from 128.40.255.154 on Tue, 18 Jun 2019 12:11:56 UTC
All use subject to https://about.jstor.org/terms

coherent concept of infinite proof? Few readers have been able to make much out of it. This note will show that there is a completely coherent concept of infinite proof that exactly fits Leibniz's own descriptions. There is nothing novel in this concept. I shall employ a theory due to K. Schütte, (1951, 1960), which is a version of ideas developed by G. Gentzen (1935). I am not contending that Leibniz "anticipated" these developments in mathematical logic, nor do I claim that these ideas are likely to do as much for the concept of contingency as Leibniz hoped. I note only that his notion of infinite proof turns out to be coherent.

Logicians have from time to time tried to explicate Leibniz's remarks on infinite proof. Most notable perhaps is J. Hintikka (1969). Hintikka's theory may have particular value in explaining how monads may mirror the rest of the world, but in one respect it does not seem faithful to Leibniz, who insists that infinite proofs *converge*. As he often expresses it, "one can never arrive at a perfect proof, but one approaches it more and more so that *the difference is less than any given difference*"[2]. These final words tell us to seek a model of convergence, motivated originally by the infinitesimal calculus.

Infinite proof, then, is modelled on a convergent series, as for example the series of sums $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots \frac{2^n-1}{2^n}$, which converges on unity[3]. Hintikka does indeed give us an infinite sequence of sentences, arranged in order of the number of nested quantifiers, but there appears to be no term on which his sequence converges. Many modern readers probably suppose, following the analogy with finite proof, that the initial terms of the sequence would be axioms — ideally identities and contra-identities — and that the theorem to be proved corresponds to unity in the above sequence. That is, the proof is converging on the theorem proved. But this cannot be right. We must recall the old distinction between analysis and synthesis. Proofs of our standard sort from Euclid on are called *synthetic* because they commence with axioms and build up to a conclusion. It was long supposed that the ancients possessed a wonderful device called analysis, which, starting with the theorem, worked out its proof. Synthesis, roughly speaking, proceeds downwards, from axioms to theorem, while analysis works up, from theorem to axioms. Descartes was one of many who had little use for the synthetic method. He believed that both his geometry and his *Meditations* were analytic. Leibniz

[2] Op. cit. n. 1, section 66. Cf. LEIBNIZ, *Math. Schrift.*, ed. C. I. GERHARDT, V, p. 350.

[3] Leibniz's own preferred example is that of the difference between commensurable and incommensurable lengths. If A and B are commensurable, then there are finite numbers k and n such that by extending A k times, and B n times, two coincident lengths are produced. This is like finite proof. If A and B are incommensurable, one can constantly reduce the lack of coincidence, but never, in a finite number of steps, produce coincident lengths.

This content downloaded from 128.40.255.154 on Tue, 18 Jun 2019 12:11:56 UTC
All use subject to https://about.jstor.org/terms

thought that in the case of finite proofs, it makes little difference, once the proof is available, whether one presents it synthetically or analytically. But for infinite proof the difference is crucial.

Infinite proof is infinite analysis. That is, it starts with the theorem proved, and proceeds in the direction of less and less complexity. The end product, on which it converges but never reaches, is a set of identities and contra-identities, which we may call axioms. Thus in the above sequence it is not the "$\frac{1}{2}, \frac{3}{4}$" and so forth that compare to identities, while the theorem matches unity. It is the other way about.

A finite proof, as most logic textbooks now tell us, is a finite sequence of sentences each of which is either an axiom or is derived from preceding members of the sequence by one application of a rule of inference; the final sentence is the theorem proved. (We express the direction in this way, but of course that is immaterial to the structure of the proof: we could write proofs "backwards.") *An infinite proof is an infinite sequence of sentences each of which is either an axiom or is derived from subsequent members of the sequence by one application of a rule of inference; the initial sentence is the theorem proved.*

It is necessary to impose certain technical constraints on this idea. First, *the set of sentences in the proof should be countable* (can be placed in 1—1 correlation with the natural numbers). Second, *the order type of the sequence must be specified.* We now have strong reasons for supposing that the minimum order type that will give results of interest is ε_0. (The order type of the sequence of integers is called ω; that of an ω sequence of ω sequences is ω^2; so by a procedure of generalization one obtains ω^ω; then ω exponentiations by ω lead to ε_0). Thirdly in order to conform to the notion that proofs should be perspicuous, one would doubtless insist that *the proof is given by a recursive function* f *from the set of sentences into the ordinals of type less than* ε_0, *where if* $f(s) = \alpha$ *then the sentence* s *occurs after all* s *such that* $f(s') = \alpha'$ *and* $\alpha' < \alpha$. (If $f(s) = 0$, then s does not occur in the proof). Finally, to capture the specific notion of Leibnizian analysis, we may insist that the sequence of sentences becomes decreasingly complex. The complexity of a sentence may be measured by the number of logical constants and quantifiers that it contains. We insist that *if* $f(s) = \alpha$ *and* $f(s') = \alpha'$ *and* $'\alpha < \alpha$, *so that* s *occurs after* s', *then* s *is never more complex than* s'.

Although I have followed the common fiction which supposes that proofs are linear we know that proofs are typically trees, in which we have a large number of branches leading down to the theorem proved. It would be more realistic to alter the above definition to speak not of a single sequence but of a collection of sequences, each corresponding to one branch on the tree. For a formalism of this sort, see the appendix to E. Mendelson (1964). In our linear definition, sentences are ordered by complexity to a tree, whose branches always decrease in complexity as

This content downloaded from 128.40.255.154 on Tue, 18 Jun 2019 12:11:56 UTC
All use subject to https://about.jstor.org/terms

they ascend, can always be collapsed into a sequence by letting the complexity determine the ordering of sentences from different branches. (Care must be taken to ensure that the order type is not disturbed too much by this operation).

Now we turn to an actual system of infinite proof. It concerns arithmetic, formulated in a language that enables us to express all the numerals, addition and multiplication, and first order logic. The axioms are the true arithmetical identities (e.g. $9+5=2\times 7$) and the true arithmetical contra-identities (e.g. the negation of $9+4=2\times 6$). The rules of logic are of the sort that Gentzen first imagined for what he called his sequence calculus. These have the remarkable feature that they are in a certain sense "analytic." The result of applying a rule is always more complex than the premises (I disregard rules which merely rearrange premises and the like). Thus we infer $A\,\&\,B$ only from the premises A and B; we infer $(\exists x)\,(Fx)$ only from a sentence of the form Ft. More precisely, the rules all satisfy what D. Prawitz (1965 p. 43) calls the subformula principle, that sentences got by the application of one rule of inference are obtained from premises which are their subformulae. In the system of Schütte, however, we have an unusual rule for the universal quantifier, which corresponds to arithmetical induction. If we have a proof of each of $F(1)$, $F(2)$, $F(3)\ldots\ldots$, we are allowed to conclude $(\forall n)\,(Fn)$. It is this rule and only this rule that makes our proofs infinite. Notice that the rule is not just, as it were, infinitely wide: we have an infinite sequence of sentences as premises from which we infer a single sentence. It may also be the case that, because of this rule, proofs become infinitely "high" as well. For example, the proof of $F(1)$ might be just two steps long; of $F(2)$, four steps, of $F(n)$, $2n$ steps, so there would be no finite bound to the height of proofs above the application of this induction rule.

There are many ways of presenting such a calculus, for two of which, see the cited references to Mendelson and Schütte. In any case the fundamental theorem to be proved is what is usually called Gentzen's *Hauptsatz* which assures us that the rule *modus ponens* is valid in the system. Since all our rules have the "analytic" character of always increasing complexity, the rule "A, $A\supset B$, so B", cannot be a primitive rule; we must show that if A and $A\supset B$ are derivable, then B can also be derived. This task is not trivial. The proof is inductive, and Gentzen (1943) indicates that the order type of proofs required must include all ordinals less than ε_0. It is this rather profound analysis that leads us to suppose ε_0 is the minimum order type for a productive concept of infinite proof.

Leibniz of course did not know about transfinite ordinals. When he spoke of infinite proof, he surely had in mind proofs of order type ω only. Then it would certainly be true to say that although the proof cannot be completely written down, for any sentence s of given complexity c that has occurred in the analysis of the theorem, we can always

This content downloaded from 128.40.255.154 on Tue, 18 Jun 2019 12:11:56 UTC
All use subject to https://about.jstor.org/terms

exhibit a sentence, later in the proof, that has complexity less than c. Thus "the difference is less than any given difference." If the order type of a proof were ω, then any sentence in the proof will, sooner or later, be written down in its proper place by a proof-writing machine. But when the order type exceeds ω, this is not true. Yet Leibniz's basic intuition is entirely vindicated. In Schütte's proof trees, for example, we have at the top identities of complexity 0, and each successive sentence in a tree, starting with the theorem at the bottom, will be of greater (or equal) complexity than every sentence in every branch above it. So that for any sentence in the proof of a sentence of complexity $c > 0$ there will be sentences of complexity lower than c above that sentence. Moreover the Leibnizian will find it agreeable that the only axioms are identities and contra-identities!

Although Leibniz regularly says that some infinite analyses will never be kown to us, and that only God will see the nexus of terms, he also insists that some infinite proofs may be available to us. That is, we may see that the conditions on an infinite proof are in fact satisfied. This is exactly what happens in the Schütte theory. In Hacking (1973) it is urged that Leibniz is the first philosopher of modern times to have a clear conception of finite proof. It is perhaps unfortunate that his concept of finite proof became the standard one while his idea of infinite proof remained obscure. The celebrated result of K. Gödel, about the incompleteness of arithmetic, shows that there can be no axiomatization of recursive arithmetic that is complete *so long as one restricts oneself to the domain of finite proof*. If we had always been at home with Leibniz's notion of infinite proof, this result might have seemed less important. Schütte has given us a theory of arithmetic which is complete just because it allows proof trees of order type ε_0.

References

Gentzen, G. (1934). *Untersuchungen über das logische Schließen.* In: *Math. Zeit.* 39, 176—210, 405—431.

Gentzen, G. (1943). *Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie.* In: *Math. Ann. 119,* 140—161.

Hacking, I. (1973). *Leibniz and Descartes: Proof and Eternal Truths,* Dawes Hicks lecture on Philosophy, British Academy, 16pp. To appear in *The Proceedings of the British Academy,* 59.

Hintikka, J. (1969). *Leibniz on Plenitude, relations and the reign of law.* In: *Ajatus,* 21, pp. 28. Extended in *Leibniz,* ed. H. Frankfurt, New York, 1972, 155—190.

Ishiguro, H. (1972). *Leibniz's Philosophy of Logic and Language,* London: Duckworth.

Mendelson, E. (1961). *Mathematical Logic,* Princeton: Van Nostrand.

Prawitz, D. (1965). *Natural Deduction,* Stockholm: Aksell & Wiksell.

Schütte, K. (1951). *Beweistheoretische Erfassung der unendlichen Induktion in der Zahlentheorie.* In: *Math. Ann.* 122, 369—389.

Schütte, K. (1960). *Beweistheorie,* Berlin–Göttingen–Heidelberg: Springer.

This content downloaded from 128.40.255.154 on Tue, 18 Jun 2019 12:11:56 UTC
All use subject to https://about.jstor.org/terms