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Intrinsic Linear Programming*

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0. INTRODUCTION

In the usual terminology, a linear program is an extremal problem in which one seeks to maximize an expression cx subject to the condition $Ax \leq b$. Here A is a given $m \times n$ matrix, x is an unknown $n \times 1$ vector, b is a given $m \times 1$ vector and c is a given $1 \times n$ vector. The dual program is to minimize $b^t z$ subject to $A^t z \geq c^t$, where $(\ )^t$ is the transpose operation and z is an unknown $m \times 1$ vector. Alternatively, if $w = z^t$ this can be expressed as minimize wb (or maximize $-wb$) subject to $wA \geq c$. The standard duality theorem (see, e.g., [4, 11]) asserts that exactly one of the following occurs: (i) both the program and its dual have solutions and $\max cx = \min wb$ (or $\max(cx) + \max(-wb) = 0$); (ii) neither has a solution; (iii) one has no solution and the other is unbounded.

In this paper we give an alternative description of linear programs in terms of diagrams of vector spaces and linear transformations in such a way that the duality theorem can be proved in a completely intrinsic fashion. The theory therefore applies to infinite dimensional situations as well as the usual finite dimensional case. However, in the infinite dimensional case certain topological hypotheses are required. We develop two sets of conditions. Under both sets, duality holds but if one set is relaxed then duality can fail by the program or its dual failing to have a solution but the other cannot be shown to be unbounded. If the other set is relaxed, then duality can fail by both of the programs having solutions but $\max cx \neq \min wb$ or in the same way as above. In both cases this duality gap is repaired by passing to the dual program.

In a subsequent paper [5] the categorical aspects of this situation and the functorial nature of the constructions that are introduced will be discussed. In particular, the closed category considerations that lie behind this whole theory and are responsible for its discovery will be treated there.

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1. Basic Concepts

The basic data of a linear program **P** can be described by a diagram

$$\mathbf{P}\colon R \overset{c}{\leftarrow} V_1 \overset{A}{\rightarrow} V_2 \overset{b}{\leftarrow} R.$$

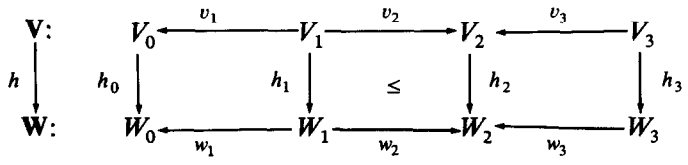
Here R denotes the real numbers with their usual order structure and V_1 and V_2 are preordered vector spaces; i.e., with respect to the product preorder, $+\colon V \times V \to V$ and $\cdot\colon R \times V \to V$ are order preserving functions. A, b, and c are linear transformations which need not be order preserving. The only changes from the usual description are that b, instead of being thought of as a vector in V_2, is regarded as the unique linear transformation from R to V_2 taking $1 \in R$ to that vector, and that c is regarded as a linear functional on V_1 (which it was all along anyway). We shall use bold face **P** to refer to such a diagram and call it a *linear program*. Generalizing slightly, we call a diagram **V** of the form

$$\mathbf{V}\colon V_0 \overset{v_1}{\leftarrow} V_1 \overset{v_2}{\rightarrow} V_2 \overset{v_3}{\leftarrow} V_3$$

a *generalized linear program*. Here each V_i is a preordered vector space and each v_i is a linear transformation. Some of our constructions can be carried out more conveniently for generalized linear programs, but the final results will refer only to linear programs.

Next, we turn to the proper notions of homomorphisms for linear programs. There are two basic types with two sorts of modifications for each. We have been inspired by Harper [6] here, but our definitions differ slightly, albeit significantly, from his.

1.1. Definition. (i) A *homomorphism* $h\colon \mathbf{V} \to \mathbf{W}$ of generalized linear programs is a sequence of linear transformations $h_i\colon V_i \to W_i$, $i = 0, \ldots, 3$, such that in the diagram



the first and third squares commute, while the second square satisfies an inequality relation; i.e., $w_1h_1 = h_0v_1$, $w_2h_1 \leq h_2v_2$, $w_3h_3 = h_2v_3$. (To explain the inequality, let $V^+ = \{x \in V | x \geq 0\}$; i.e., V^+ is the *positive* cone of V. If $f, g\colon V \to W$ are linear transformations, then $f \leq g$ means that for all

$x \in V^+$, $f(x) \leq g(x)$, cf. Schaefer [10]. If $0 \leq f$, then f is called *positive* or *order preserving*.)

(ii) A homomorphism h: $\mathbf{V} \to \mathbf{W}$ in which the second square also commutes, i.e., $w_2 h_1 = h_2 v_2$, is called a *strict homomorphism*.

Strict homomorphisms h: $\mathbf{V} \to \mathbf{W}$, k: $\mathbf{W} \to \mathbf{U}$ can be composed by the rule $(kh)_i = k_i h_i$. However, for arbitrary homomorphisms h: $\mathbf{V} \to \mathbf{W}$, k: $\mathbf{W} \to \mathbf{U}$ there is a problem in that the composition may not satisfy the required inequality. If we try to derive the inequality by writing

$$u_2 k_1 h_1 \leq k_2 w_2 h_1 \leq k_2 h_2 v_2 \tag{1}$$

we see that the first inequality here holds in general only if $h_1 \geq 0$ while the second requires $k_2 \geq 0$. This leads us to the following definition:

1.2. DEFINITION. (i) h: $\mathbf{V} \to \mathbf{W}$ is called *composable* (or *feasible*) if $h_1 \geq 0$.

(ii) k: $\mathbf{W} \to \mathbf{U}$ is called *cocomposable* (or *dual feasible*) if $k_2 \geq 0$.

1.3. PROPOSITION. *The composition*

$$\mathbf{V} \overset{h}{\to} \mathbf{W} \overset{k}{\to} \mathbf{U}$$

is defined in the following circumstances:

(o) *h and k both strict*,

(i) *h composable and k cocomposable*,

(ii) *h strict composable and k arbitrary*,

(iii) *h arbitrary and k strict cocomposable*.

Proof. Case (i) is the situation treated in (1). Case (ii) follows because in (1) the second inequality is an equality. Case (iii) is similar.

As might be expected from the name, composability (feasibility) will play a central role in what follows. Before getting down to business we need one more pair of adjectives.

1.4. DEFINITION. (i) h: $\mathbf{V} \to \mathbf{W}$ is called *proper* if $h_3 = \mathrm{id}$. (id stands for the identity linear transformation of a vector space to itself.)

(ii) k: $\mathbf{W} \to \mathbf{U}$ is called *coproper* if $k_0 = \mathrm{id}$.

If h: $\mathbf{P} \to \mathbf{P}'$ is a homomorphism between linear programs then its zeroth and third components are linear maps from R to R. Such a map can be identified with its value at 1 and considered as an element of R.

1.5. DEFINITION. If $\mathbf{P}$ and $\mathbf{P}'$ are linear programs and $h\colon \mathbf{P} \to \mathbf{P}'$ is a homomorphism then $\mu(h) = h_0 \in R$ and $\nu(h) = h_3 \in R$. Thus μ and ν are real-valued functions defined on the set of all homomorphisms between linear programs.

1.6. PROPOSITION.

(i) $\mu(kh) = \mu(k)\mu(h)$
$\nu(kh) = \nu(k)\nu(h)$.

(ii) $h\colon \mathbf{P} \to \mathbf{P}'$ *is proper* (*resp.*, *coproper*) *if and only if* $\nu(h) = 1$ (*resp.*, $\mu(h) = 1$)).

In the final results, some restrictions will have to be placed on the vector spaces and linear transformations that occur. We will treat two general models in what follows:

Model 1. All vector spaces are finite dimensional; there are no restrictions on the linear transformations.

Model 2. The vector spaces are equipped with topologies such that addition and scalar multiplication are continuous. Linear transformations are continuous.

2. POINTS AND COPOINTS

Consider the very simple linear program

$$\mathbf{1}\colon R \xleftarrow{\text{id}} R \xrightarrow{\text{id}} R \xleftarrow{\text{id}} R.$$

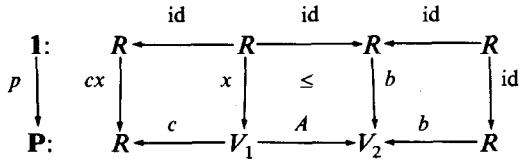
2.1. DEFINITION. Let $\mathbf{P}$ be a linear program.

(i) A composable proper homomorphism $p\colon \mathbf{1} \to \mathbf{P}$ is called a *point* of $\mathbf{P}$. If p is strict it is called a *strict point* of P. If p is not necessarily composable it is called an *arbitrary point* (or *arbitrary strict point*).

(ii) $\mathrm{Pt}(\mathbf{P})$ (resp., $\mathrm{sPt}(\mathbf{P})$) denotes the set of points (resp., strict points) of $\mathbf{P}$. In either case a subscript "a" denotes the set of arbitrary (strict) points. Thus there are inclusions

$$\begin{array}{ccc} \mathrm{sPt}(\mathbf{P}) & \subseteq & \mathrm{Pt}(\mathbf{P}) \\ \cap & & \cap \\ \mathrm{sPt}_a(\mathbf{P}) & \subseteq & \mathrm{Pt}_a(\mathbf{P}). \end{array}$$

To understand these definitions, consider an arbitrary point p: $\mathbf{1} \to \mathbf{P}$. It looks like the diagram



Properness says that $p_3 = \mathrm{id}$ and commutativity of the third square forces $p_2 = b$. We denote p_1 by x: $R \to V_1$ since it corresponds to an element of V_1. It satisfies $Ax \leq b$. If p is a point (rather than an arbitrary point) then $x \geq 0$. Finally, commutativity of the first square forces $p_0 = cx$, which we can regard either as a linear transformation of R into R, or (by identifying it with its value at $1 \in R$) as an element of R. *Thus a point is precisely a feasible solution of a canonical linear program.* Similarly, a strict point is a feasible solution of a standard linear program. Since a point is determined by x: $R \to V_1$, we shall denote it by $p = \bar{x}$ from now on.

An arbitrary point is a homomorphism $\bar{x}$: $\mathbf{1} \to \mathbf{P}$ with $\nu(\bar{x}) = 1$ (cf. 1.5). On the other hand $\mu(\bar{x})$ plays a crucial role and we shall regard it as giving a real-valued function

$$\mu\colon \mathrm{Pt}_a(\mathbf{P}) \to R\colon \mu(\bar{x}) = cx$$

called the *valuation*. Its restrictions to points and strict (arbitrary) points are also denoted by μ. In particular, the *value* of $\mathbf{P}$ is defined by

$$\mu(\mathbf{P}) = \sup\{\mu(\bar{x}) | \bar{x} \in \mathrm{Pt}(\mathbf{P})\}.$$

The arbitrary value $\mu_a(\mathbf{P})$, the strict value $\mu_s(\mathbf{P})$ and the arbitrary strict value $\mu_{as}(\mathbf{P})$ are defined analogously. By convention, these values are taken to be $-\infty$ if there are no points of the appropriate kind.

Finally we set

$$^{+}\mathrm{Pt}(\mathbf{P}) = \{\bar{x} \in \mathrm{Pt}(\mathbf{P}) | \mu(\bar{x}) \geq 0\} = \mu^{-1}(R^{+})$$

$$^{++}\mathrm{Pt}(\mathbf{P}) = \{\bar{x} \in \mathrm{Pt}(\mathbf{P}) | \mu(\bar{x}) > 0\} = \mu^{-1}(R^{++})$$

with similar conventions for other kinds of points.

To describe copoints, we need the first of eight constructions that play a role in our version of the duality theorem.

2.2 DEFINITION. The *dual* of a generalized linear program

$$\mathbf{V}\colon V_0 \overset{v_1}{\leftarrow} V_1 \overset{v_2}{\rightarrow} V_2 \overset{v_3}{\leftarrow} V_3$$

is the generalized linear program

$$\mathbf{V}^*\colon V_3^* \overset{-v_3^*}{\leftarrow} V_2^* \overset{-v_2^*}{\rightarrow} V_1^* \overset{-v_1^*}{\leftarrow} V_0^*.$$

The *dual* of a homomorphism h from $\mathbf{V}$ to $\mathbf{W}$ is the homomorphism h^* from W^* to V^* given by

$$(h^*)_i = (h_{3-i})^* \qquad i = 0, \ldots, 3.$$

In Model 1, there is nothing more to be said about the dual construction except to point out that the preorder on the algebraic dual V^* is given by the order described in Definition 1.1. Also recall that if bases are chosen for V and W yielding a matrix $M(f)$ for a linear transformation $f\colon V \to W$ then, with respect to the dual bases for V^* and W^*, one has $M(f^*) = M(f)^t$, where t denotes the transposed matrix.

In Model 2, with topological vector spaces and continuous linear transformations, V^* means the topological dual; i.e., its elements are continuous linear functionals $\phi\colon V \to R$. In this case, V^* is topologized with the weak-$*$-topology.

2.3. PROPOSITION. (o) *The dual of a linear program is a linear program.*

(i) *The dual of a (resp., strict) homomorphism is a (resp., strict) homomorphism.*

(ii) *Duality interchanges composable and cocomposable homomorphisms.*

(iii) *Duality interchanges proper and coproper homomorphisms.*

Proof. (o) The dual of

$$\mathbf{P}\colon R \overset{c}{\leftarrow} V_1 \overset{A}{\rightarrow} V_2 \overset{b}{\leftarrow} R$$

is

$$\mathbf{P}^*\colon R \overset{-b^*}{\leftarrow} V_2^* \overset{-A^*}{\rightarrow} V_1^* \overset{-c^*}{\leftarrow} R$$

where we identify R^* with R as usual.

(i) If $f, g\colon V \to W$ with $f \leq g$ then $f^* \leq g^*$ since for any positive $\phi\colon W \to R$, $f^*(\phi) = \phi f \leq \phi g = g^*(\phi)$. Hence if $h\colon \mathbf{V} \to \mathbf{W}$ then $h^*\colon \mathbf{W}^* \to$

$\mathbf{V}^*$, since $w_2h_1 \leq h_2v_2$ implies $(-v_2^*)h_2^* \leq h_1^*(-w_2^*)$. This is one main reason for the minus sign in the definition of $\mathbf{V}^*$.

(ii) If $h_1 \geq 0$ then $(h^*)_2 = (h_1)^* \geq 0$.

(iii) If $h_3 = \mathrm{id}$ then $(h^*)_0 = (h_3)^* = \mathrm{id}$.

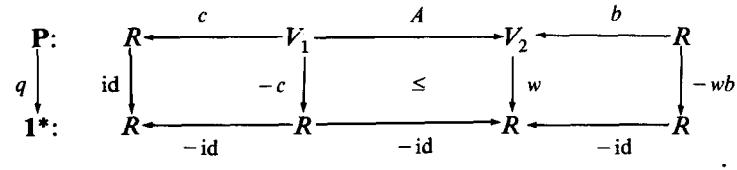
2.4. DEFINITION. Let $\mathbf{P}$ be a linear program.

(i) A cocomposable coproper homomorphism q: $\mathbf{P} \to \mathbf{1}^*$ is called a *copoint* of $\mathbf{P}$. If q is strict it is called a *strict copoint* of $\mathbf{P}$. If q is not necessarily cocomposable it is called an *arbitrary copoint* (or *arbitrary strict copoint*).

(ii) $\mathrm{coPt}(\mathbf{P})$ (resp., $\mathrm{cosPt}(\mathbf{P})$) denotes the set of copoints (resp., strict copoints) of $\mathbf{P}$. In either case a subscript "a" denotes the set of arbitrary (strict) copoints. Thus there are inclusions

$$\begin{array}{ccc} \mathrm{cosPt}(\mathbf{P}) & \subseteq & \mathrm{coPt}(\mathbf{P}) \\ \cap & & \cap \\ \mathrm{cosPt}_a(\mathbf{P}) & \subseteq & \mathrm{coPt}_a(\mathbf{P}). \end{array}$$

As with points, an arbitrary copoint q: $\mathbf{P} \to \mathbf{1}^*$ looks like the diagram



Coproperness says that $q_0 = \mathrm{id}$ and commutativity of the first square forces $q_1 = -c$. We denote q_2 by w: $V_2 \to R$. It is some arbitrary linear functional on V_2 satisfying $c \leq wA$. If q is a copoint (rather than an arbitrary copoint) then $w \geq 0$. Finally commutativity of the third square forces $q_3 = -wb$. *Thus a copoint is precisely a dual feasible solution of one way of writing the canonical dual program.* Since a copoint is determined by w: $V_2 \to R$ we shall denote it by $q = \overline{w}$ from now on.

An arbitrary copoint is a homomorphism $\overline{w}$: $\mathbf{P} \to \mathbf{1}^*$ with $\mu(\overline{w}) = 1$. On the other hand $\nu(\overline{w})$ plays a crucial role and we shall regard it as giving a real-valued function

$$\nu\colon \mathrm{coPt}_a(\mathbf{P}) \to R\colon \nu(\overline{w}) = -wb$$

called the *covaluation*. Its restrictions to copoints and strict (arbitrary) copoints are also denoted by ν. In particular, the *covalue* of $\mathbf{P}$ is defined by

$$\nu(\mathbf{P}) = \sup\{\nu(\overline{w}) | \overline{w} \in \mathrm{coPt}(\mathbf{P})\}.$$

The arbitrary covalue $\nu_a(\mathbf{P})$, the strict covalue $\nu_s(\mathbf{P})$, and the arbitrary strict covalue $\nu_{as}(\mathbf{P})$ are defined analogously. By convention, these are also taken to be $-\infty$ if there are no copoints of the appropriate kind.

2.5. PROPOSITION. *Duality yields an isomorphism over R:* $\mathrm{coPt}_a(\mathbf{P}) \simeq \mathrm{Pt}_a(\mathbf{P}^*)$ *which takes copoints to points and preserves strictness.*

Proof. Here and elsewhere, "over R" means the diagram

$$\begin{array}{ccc} \mathrm{coPt}(\mathbf{P}) & \simeq & \mathrm{Pt}(\mathbf{P}^*) \\ & {\scriptstyle\nu}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & R & \end{array}$$

commutes.

The dual of an arbitrary copoint is clearly an arbitrary point since $\mathbf{1}^{**} = \mathbf{1}$ and duality takes coproper homomorphisms to proper homomorphisms. The isomorphism is over R since if $\overline{w}\colon \mathbf{P} \to \mathbf{1}^*$ then $\nu(\overline{w}) = -wb$ while $\mu(\overline{w}^*) = -b^*w^* = -(wb)^*$. But, following our usual convention of identifying R^* with R, $-(wb)^* = -wb$ since the (linear) endomorphism ring of R is commutative.

2.6. COROLLARY. $\nu_a(\mathbf{P}) = \mu_a(\mathbf{P}^*)$ *and* $\nu(\mathbf{P}) = \mu(\mathbf{P}^*)$.

There is no guarantee in general that $\mathbf{P}$ and $\mathbf{P}^*$ can be interchanged in the preceding proposition, but they will have to be so interchanged in one step of the duality theorem. Hence we need the following definition.

2.7. DEFINITION. A linear program $\mathbf{P}$ is called *reflexive* if the inclusion homomorphism $\mathbf{P} \hookrightarrow \mathbf{P}^{**}$ induces an isomorphism over R

$$\mathrm{Pt}(\mathbf{P}) \backsimeq \mathrm{Pt}(\mathbf{P}^{**}).$$

2.8. PROPOSITION. *If $\mathbf{P}$ is reflexive then there is an isomorphism over R*

$$\mathrm{coPt}(\mathbf{P}^*) \backsimeq \mathrm{Pt}(\mathbf{P}).$$

Proof. Immediate.

In the models, all linear programs are reflexive since the vector spaces in the programs, being either finite dimensional or having the weak-$*$-topology on the dual, are themselves reflexive. However, in Sections 5 and 6 we want to consider what happens if the dual space has some other topology. It can happen that a linear program satisfies a reflexivity condition without its component vector spaces being reflexive; for instance, by an appropriate set of points being empty.

2.9. DEFINITION. For a linear program

$$\mathbf{P}\colon R \overset{c}{\leftarrow} V_1 \overset{A}{\rightarrow} V_2 \overset{b}{\leftarrow} R,$$

if c (resp. b) is replaced by 0, we denote the resulting program by

$$Z\mathbf{P}\colon R \overset{0}{\leftarrow} V_1 \overset{A}{\rightarrow} V_2 \overset{b}{\leftarrow} R$$

(resp.,

$$\mathbf{P}Z\colon R \overset{c}{\leftarrow} V_1 \overset{A}{\rightarrow} V_2 \overset{0}{\leftarrow} R\Big).$$

2.10. PROPOSITION. (i) *If* $\mathbf{P} = Z\mathbf{P}$ *then*

(I) *at least one of* $\mathrm{sPt}(\mathbf{P})$ *and* $^{++}\mathrm{coPt}_a(\mathbf{P})$ *is empty*;

(II) *at least one of* $\mathrm{Pt}(\mathbf{P})$ *and* $^{++}\mathrm{coPt}(\mathbf{P})$ *is empty*.

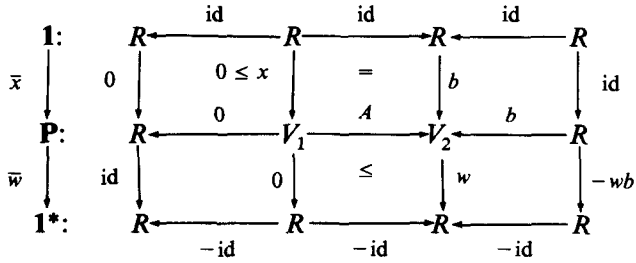
(ii) *If* $\mathbf{P} = \mathbf{P}Z$ *then*

(I^{d}) *at least one of* $\mathrm{cosPt}(\mathbf{P})$ *and* $^{++}\mathrm{Pt}_a(\mathbf{P})$ *is empty*;

(II^{d}) *at least one of* $\mathrm{coPt}(\mathbf{P})$ *and* $^{++}\mathrm{Pt}(\mathbf{P})$ *is empty*.

Proof. Note that in statements I and I^{d}, one entity is strict and the other is arbitrary, while in II and II^{d} no such qualifications are present. These correspond to the composability criteria in 1.3.

Writing out the situation in case I gives a diagram



where $\bar{x} \in \mathrm{sPt}(\mathbf{P})$ and $\bar{w} \in \mathrm{coPt}_a(\mathbf{P})$. By 1.3, $\bar{w}$ and $\bar{x}$ can be composed so $0 \leq wAx = wb$. Thus $\nu(\bar{w}) = -wb > 0$ is impossible so $\bar{w} \notin {}^{++}\mathrm{coPt}_a(\mathbf{P})$. The other cases are treated similarly.

We would like to assert that in each case exactly one of the pairs of sets is empty. (Note that for sets X and Y, the statement $X \times Y = \varnothing$ says that at least one of X and Y is empty, while the additional condition $X + Y \neq \varnothing$ says that exactly one of them is empty.)

For instance, in case I, suppose $\mathrm{sPt}(\mathbf{P}) = \varnothing$; i.e., for all $x \geq 0$, $Ax \neq b$, or

$$b \notin \{Ax | x \geq 0\} = A(V_1^+).$$

An element $\overline{w} \in {}^{++}\mathrm{coPt}(\mathbf{P})$ is a "separating hyperplane" for b and the cone $A(V_1^+)$; i.e., a linear functional w: $V_2 \to R$ such that

$$wb < \{wAx | x \geq 0\}.$$

Taking $x = 0$ gives $wb < 0$ or $-wb > 0$, as required. If such separating hyperplanes exist, then $\mathrm{sPt}(\mathbf{P}) = \varnothing$ implies ${}^{++}\mathrm{coPt}(\mathbf{P}) \neq \varnothing$, and thus, by the excluded middle, ${}^{++}\mathrm{coPt}(\mathbf{P}) = \varnothing$ implies $\mathrm{sPt}(\mathbf{P}) \neq \varnothing$.

In case I^{d}, with $\mathbf{P} = \mathbf{P}Z$, either there is a $w \geq 0$ with $wA = c$ or an x with $Ax \leq 0$, $cx > 0$. This is of course the same as case I for $\mathbf{P}^*$ except that the separating hyperplane instead of just being an element of V_1^{**} is required to be an element of V_1 itself.

In the models such separating hyperplanes need not exist even in the finite dimensional case, since we deal with preorders. A precise sufficient condition to apply the geometric Hahn–Banach theorem for the existence of such hyperplanes is that $A(V_1^+)$ is closed and V_2 is locally convex. For our purposes we shall treat the situation in which exactly one of the pairs of sets is empty as an axiom, called the Farkas Alternative, which a program may or may not satisfy.

2.11. Axiom (The Farkas Alternatives). (i) *If* $\mathbf{P} = Z\mathbf{P}$ *then*

(FI) *at most one of* $\mathrm{sPt}(\mathbf{P})$ *and* ${}^{++}\mathrm{coPt}_a(\mathbf{P})$ *is empty*;

(FII) *at most one of* $\mathrm{Pt}(\mathbf{P})$ *and* ${}^{++}\mathrm{coPt}(\mathbf{P})$ *is empty*.

(ii) *If* $\mathbf{P} = \mathbf{P}Z$ *then*

(FI^{d}) *at most one of* $\mathrm{cosPt}(\mathbf{P})$ *and* ${}^{++}\mathrm{Pt}_a(\mathbf{P})$ *is empty*;

($\mathrm{FII}^{\mathrm{d}}$) *at most one of* $\mathrm{coPt}(\mathbf{P})$ *and* ${}^{++}\mathrm{Pt}(\mathbf{P})$ *is empty*.

Clearly (FII) is equivalent in the presence of 2.10 to requiring that exactly one of $\mathrm{Pt}(\mathbf{P})$ and ${}^{++}\mathrm{coPt}(\mathbf{P})$ is empty, or, as it is frequently phrased in the literature, either there is a point or there is a copoint with positive covalue but not both. Analogous reformulations hold for the other statements.

3. Strictification, Dual Strictification, and Sums

Three more constructions are treated here together with the first consequence of the Farkas Alternative. The simplest of these is the sum construc-

tion. It is neither a product nor a coproduct of linear programs but with respect to points and copoints it is both at the same time.

3.1. DEFINITION. Given linear programs

$$\mathbf{P}\colon R \overset{c}{\leftarrow} V_1 \overset{A}{\rightarrow} V_2 \overset{b}{\leftarrow} R$$

$$\mathbf{P}'\colon R \overset{c'}{\leftarrow} V_1' \overset{A'}{\rightarrow} V_2' \overset{b'}{\leftarrow} R,$$

then $\mathbf{P} \boxplus \mathbf{P}'$ denotes the linear program

$$\mathbf{P} \boxplus \mathbf{P}'\colon R \overset{\langle c, c'\rangle}{\leftarrow} V_1 \times V_1' \overset{A \times A'}{\rightarrow} V_2 \times V_2' \overset{[b, b']}{\leftarrow} R.$$

Here we write $V \times W$ for the product preordered vector space. Linear maps U into $V \times W$ are determined by pairs of linear maps $f\colon U \to V$, $g\colon U \to W$ and we write $[f, g]\colon U \to V \times W$ for the linear map with $[f, g](u) = (f(u), g(u))$. Now $V \times W$ is at the same time the coproduct of V and W so linear maps from $V \times W$ to U are determined by pairs of linear maps $h\colon V \to U$, $k\colon W \to U$. We write $\langle h, k\rangle\colon V \times W \to U$ for the linear map with $\langle h, k\rangle(v, w) = h(v) + k(w)$. Note that $\langle h, k\rangle[f, g] = hf + kg$.

3.2. PROPOSITION. *There exist isomorphisms giving commutative diagrams*

$$\begin{array}{ccc}
\mathrm{Pt}(\mathbf{P}) \times \mathrm{Pt}(\mathbf{P}') & \xrightarrow{\sim} & \mathrm{Pt}(\mathbf{P} \boxplus \mathbf{P}') \\
{\scriptstyle \mu \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
R \times R & \xrightarrow{+} & R
\end{array}$$

$$\begin{array}{ccc}
\mathrm{coPt}(\mathbf{P}) \times \mathrm{coPt}(\mathbf{P}') & \xrightarrow{\sim} & \mathrm{coPt}(\mathbf{P} \boxplus \mathbf{P}') \\
{\scriptstyle \nu \times \nu}\downarrow & & \downarrow{\scriptstyle \nu} \\
R \times R & \xrightarrow{+} & R
\end{array}.$$

These isomorphisms, which also hold for arbitrary points and copoints, preserve strictness.

Proof. The first part follows from the diagram

$$\begin{array}{ccccccccc}
\mathbf{1} & : & R & \xleftarrow{\mathrm{id}} & R & \xrightarrow{\mathrm{id}} & R & \xleftarrow{\mathrm{id}} & R \\
\downarrow & & {\scriptstyle cx + c'x'}\downarrow & & {\scriptstyle [x, x']}\downarrow & & {\scriptstyle [b, b']}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
P \boxplus \mathbf{P}' & : & R & \xleftarrow[\langle c, c'\rangle]{} & V_1 \times V_1' & \xrightarrow[A \times A']{} & V_2 \times V_2' & \xleftarrow[[b, b']]{} & R
\end{array}$$

and the fact that

$$(A \times A')[x, x'] = [Ax, Ax'] \leq [b, b']$$

if and only if $Ax \leq b$ and $A'x' \leq b'$. The second part is proved analogously using

$$\langle w, w' \rangle (A \times A') = \langle wA, w'A' \rangle \geq \langle c, c' \rangle$$

if and only if $wA \geq c$ and $w'A' \geq c'$.

Note that this result says that as far as points are concerned, $\mathbf{P} \boxplus \mathbf{P}'$ looks like the (non-existent) product of $\mathbf{P}$ and $\mathbf{P}'$ while, as far as copoints are concerned, it looks like their (non-existent) coproduct.

3.3. COROLLARY.

$$\mu(\mathbf{P} \boxplus \mathbf{P}') = \mu(\mathbf{P}) + \mu(\mathbf{P}')$$
$$\nu(\mathbf{P} \boxplus \mathbf{P}') = \nu(\mathbf{P}) + \nu(\mathbf{P}')$$

with analogous equalities for the other three cases each of values and covalues.

The next two constructions can be carried out most conveniently for generalized linear programs. The first form, strictification, reflects the operation of adjoining *slack* variables to a linear program on the one hand, and on the other is a special case of constructing lax limits in order to form the prolongation of a diagram in a 2-category. The second form, dual strictification, has no usual role in linear programming but is a special case of a lax colimit.

3.4. DEFINITION. (i) Given a generalized linear program

$$\mathbf{V}\colon V_0 \overset{v_1}{\leftarrow} V_1 \overset{v_2}{\rightarrow} V_2 \overset{V_3}{\leftarrow} V_3$$

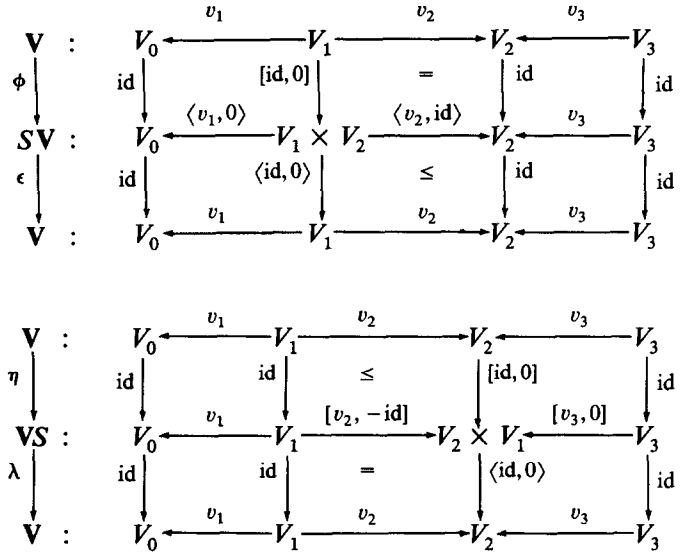
then the *strictification* of $\mathbf{V}$ is the generalized linear program

$$S\mathbf{V}\colon V_0 \overset{\langle v_1, 0 \rangle}{\leftarrow} V_1 \times V_2 \overset{\langle v_2, \mathrm{id} \rangle}{\rightarrow} V_2 \overset{v_3}{\leftarrow} V_3$$

and the *dual strictification* of $\mathbf{V}$ is the generalized linear program

$$\mathbf{V}S\colon V_0 \overset{v_1}{\leftarrow} V_1 \overset{[v_2, -\mathrm{id}]}{\rightarrow} V_2 \times V_1 \overset{[v_3, 0]}{\leftarrow} V_3.$$

(ii) Homomorphisms involving $S\mathbf{V}$ and $\mathbf{V}S$ are defined by the diagrams

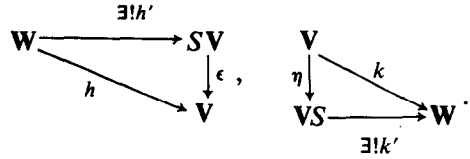


3.5. PROPOSITION. (i) ϕ, ϵ, λ, *and* η *are proper, coproper, composable, and cocomposable, and* ϕ *and* λ *are strict.*

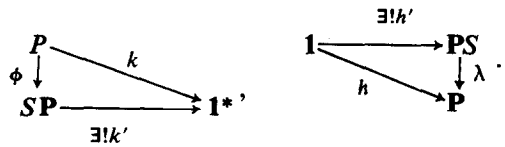
(ii) $\epsilon\phi = \mathrm{id}_{\mathbf{V}}$ *and* $\lambda\eta = \mathrm{id}_{\mathbf{V}}$ *so* $\mathbf{V}$ *is a retract of* $S\mathbf{V}$ *and a coretract of* $\mathbf{V}S$.

(iii) *If* $\mathbf{P}$ *is a linear program then so are* $S\mathbf{P}$ *and* $\mathbf{P}S$.

(iv) *Given a composable (resp., cocomposable) homomorphism* $h\colon \mathbf{W} \to \mathbf{V}$ (*resp.*, $k\colon \mathbf{V} \to \mathbf{W}$) *there exists a unique composable (resp., cocomposable) strict homomorphism* $h'\colon \mathbf{W} \to S\mathbf{V}$ (*resp.*, $k'\colon \mathbf{V}S \to \mathbf{W}$) *such that* $\epsilon h' = h$ (*resp.*, $k'\eta = k$):



(v) *If* $\mathbf{P}$ *is a linear program, then given a cocomposable (resp., composable) homomorphism* $k\colon \mathbf{P} \to \mathbf{1}^*$ (*resp.*, $h\colon \mathbf{1} \to \mathbf{P}$), *there exists a unique homomorphism* $k'\colon S\mathbf{P} \to \mathbf{1}'$ (*resp.*, $h'\colon \mathbf{1} \to \mathbf{P}S$) *such that* $k'\phi = k$ (*resp.*, $\lambda h' = h$):



(vi) *For a linear program* $\mathbf{P}$, *compositions with* ϵ *and* ϕ, *respectively, yield isomorphisms over* R

$$\mathrm{sPt}(S\mathbf{P}) \tilde{\to} \mathrm{Pt}(\mathbf{P})$$

$$\mathrm{coPt}_a(S\mathbf{P}) \tilde{\to} \mathrm{coPt}(\mathbf{P})$$

while compositions with η *and* λ *respectively yield isomorphisms over* R

$$\mathrm{cosPt}(\mathbf{P}S) \tilde{\to} \mathrm{coPt}(\mathbf{P})$$

$$\mathrm{Pt}_a(\mathbf{P}S) \tilde{\to} \mathrm{Pt}(\mathbf{P}).$$

Proof. Parts (i), (ii), and (iii) are evident. To prove (iv), let h: $\mathbf{W} \to \mathbf{V}$ be given. From the form of ϵ it is clear that the equation $\epsilon h' = h$ implies that $h'_0 = h_0$, $h'_2 = h_2$, and $h'_3 = h_3$. Furthermore, h'_1 is of the form $[m, n]$: $W_1 \to V_1 \times V_2$ and satisfies three equations

(a) $\langle \mathrm{id}, 0\rangle[m, n] = h_1$

(b) $\langle v_1, 0\rangle[m, n] = h_0 w_1$

(c) $\langle v_2, \mathrm{id}\rangle[m, n] = h_2 w_2$.

The first says $m = h_1$, the second says $v_1 h_1 = h_0 w_1$ (which holds), and the third says $v_2 h_1 + n = h_2 w_2$ or $n = h_2 w_2 - v_2 h_1 \geq 0$. Thus $h'_1 = [h_1, h_2 w_2 - v_2 h_1]$ and h' is composable if and only if h is.

(v) If k: $\mathbf{P} \to \mathbf{1}^*$ is a homomorphism with components $(r, -rc, w, -wb)$, then k' should be $(r, \langle -rc, 0\rangle, w, -wb)$. But this describes a homomorphism from $S\mathbf{P}$ to $\mathbf{1}^*$ if and only if $\langle rc, 0\rangle \leq w\langle A, \mathrm{id}\rangle$ which holds if and only if $rc \leq wA$ and $0 \leq w$, thus if and only if k is cocomposable.

(vi) Specializing (iv) to linear programs and $\mathbf{W} = \mathbf{1}$ yields the first isomorphism. It is over R since $\epsilon_0 = \mathrm{id}$ implies $\mu(\bar{x}) = \mu(\epsilon\bar{x})$. The second isomorphism follows by specializing (v) to coproper homomorphisms.

3.6. COROLLARY.

$$\mu(\mathbf{P}) = \mu_s(S\mathbf{P}) = \mu_a(\mathbf{P}S)$$
$$\nu(\mathbf{P}) = \nu_s(\mathbf{P}S) = \nu_a(S\mathbf{P})$$

3.7. PROPOSITION. *The following relations hold*:

(1) $\boxplus$ *is associative with a unit*;

(2) $Z(\mathbf{P} \boxplus \mathbf{P}') = Z\mathbf{P} \boxplus Z\mathbf{P}'$,
$(\mathbf{P} \boxplus \mathbf{P}')Z = \mathbf{P}Z \boxplus \mathbf{P}'Z$;

(3) $ZS\mathbf{P} = SZ\mathbf{P}$, $\mathbf{P}ZS = \mathbf{P}SZ$,
$Z(\mathbf{P}S) = (Z\mathbf{P})S$, $S(\mathbf{P}Z) = (S\mathbf{P})Z$;

(4) $S(\mathbf{P} \boxplus \mathbf{P}') = S\mathbf{P} \boxplus S\mathbf{P}'$,
$(\mathbf{P} \boxplus \mathbf{P}')S = \mathbf{P}S \boxplus \mathbf{P}'S$;

(5) $(Z\mathbf{P})^* = (\mathbf{P}^*)Z$,
$Z(\mathbf{P}^*) = (\mathbf{P}Z)^*$;

(6) $(S\mathbf{V})^* = (\mathbf{V}^*)S$,
$S(\mathbf{V}^*) = (\mathbf{V}S)^*$;

(7) $S(\mathbf{P} \boxplus \mathbf{P}^*) = S\mathbf{P} \boxplus (\mathbf{P}S)^*$,
$(\mathbf{P} \boxplus \mathbf{P}^*)S = \mathbf{P}S \boxplus (S\mathbf{P})^*$.

Proof. (1) Associativity is immediate. The unit for $\boxplus$ is the program

$$\mathbf{0}\colon R \overset{0}{\leftarrow} 0 \overset{0}{\rightarrow} 0 \overset{0}{\leftarrow} R.$$

Relations (2) and (3) are clear. Relation (4) uses the "middle-four interchange"

$$V_1 \times V_1' \times V_2 \times V_2' \simeq V_1 \times V_2 \times V_1' \times V_2'.$$

Relation (5) is clear. Relation (6) uses the isomorphism $(V \times W)^* \simeq V^* \times W^*$ and the equations

$$[f, g]^* = \langle f^*, g^* \rangle \qquad \text{and} \qquad \langle h, k \rangle^* = [h^*, k^*].$$

Relation (7) follows from the preceding results. Note that (6) can be used to prove the dual assertions in 3.5.

3.8. THEOREM. *If* $\mathbf{P} = Z\mathbf{P}$ *and if* $S\mathbf{P}$ *satisfies* (FI) (*cf.* 2.11) *then* $\mathbf{P}$ *satisfies* (FII). *Dually, if* $\mathbf{P} = \mathbf{P}Z$ *and if* $\mathbf{P}S$ *satisfies* (FI^{d}) *then* $\mathbf{P}$ *satisfies* ($\mathrm{FII}^{\mathrm{d}}$).

Proof. By Proposition 3.7, $S\mathbf{P} = ZS\mathbf{P}$ so (FI) applies to $S\mathbf{P}$. If it is satisfied then exactly one of

$$\mathrm{sPt}(S\mathbf{P}) \simeq \mathrm{Pt}(\mathbf{P})$$

and

$$^{++}\mathrm{coPt}_{\mathrm{a}}(S\mathbf{P}) \simeq {}^{++}\mathrm{coPt}(\mathbf{P})$$

is empty, the isomorphisms by 3.5.

In the models, the conditions given after 2.10 for a program to satisfy (FI) translate, via $S\mathbf{P}$, into conditions for a program to satisfy (FII); namely,

$A(V_1^+) + V_2^+$ is closed and V_2 is locally convex. If A is positive then the first condition is satisfied provided V_2^+ is closed. We do not know if $A(V_1^+)$ and V_2^+ closed imply $A(V_1^+) + V_2^+$ closed, even for partially ordered vector spaces.

3.9. PROPOSITION. *If* $\mathbf{P} = Z\mathbf{P}$, $\mathbf{P}' = Z\mathbf{P}'$, *and both* $\mathbf{P}$ *and* $\mathbf{P}'$ *satisfy Axiom* (FI) (*resp.* (FII)), *then so does* $\mathbf{P} \boxplus \mathbf{P}'$. *Dually if* $\mathbf{P} = \mathbf{P}Z$ *and* $\mathbf{P}' = \mathbf{P}'Z$ *and both* $\mathbf{P}$ *and* $\mathbf{P}'$ *satisfy Axiom* (FI^{d}) (*resp.* ($\mathrm{FII}^{\mathrm{d}}$)), *then so does* $\mathbf{P} \boxplus \mathbf{P}'$.

Proof. By 3.7, $\mathbf{P} \boxplus \mathbf{P}' = Z(\mathbf{P} \boxplus \mathbf{P}')$. Suppose both satisfy (FI). If they both satisfy the same alternative then so does $\mathbf{P} \boxplus \mathbf{P}'$ by 3.2. But suppose $\bar{x} \in \mathrm{sPt}(\mathbf{P})$ and $\bar{w}' \in {}^{++}\mathrm{coPt}_{\mathrm{a}}(\mathbf{P}')$. Since $\mathrm{sPt}(\mathbf{P}') = \varnothing$, one has $\mathrm{sPt}(\mathbf{P} \boxplus \mathbf{P}') = \varnothing$ by 3.2. But $\langle 0, w' \rangle \colon V_2 \times V_2' \to R$ satisfies

$$\langle 0, w' \rangle \circ (A \times A') = w'A' \geq 0$$
$$-\langle 0, w' \rangle \circ [b, b'] = -w'b' > 0$$

so ${}^{++}\mathrm{coPt}_{\mathrm{a}}(\mathbf{P} \boxplus \mathbf{P}') \neq \varnothing$. The other cases are handled similarly.

Remark. There is also a tensor product of linear programs defined by

$$\mathbf{P} \otimes \mathbf{P}' \colon R \overset{c \otimes c'}{\leftarrow} V_1 \otimes V_1' \overset{A \otimes A'}{\rightarrow} V_2 \otimes V_2' \overset{b \otimes b'}{\leftarrow} R$$

where we have identified $R \otimes R \simeq R$. This operation is associative, has the program **1** as a unit and has an associated internal hom construction given by

$$\mathbf{Hom}(\mathbf{P}, \mathbf{P}') \colon R \overset{\mathrm{Hom}(b, c')}{\leftarrow} \mathrm{Hom}(V_2, V_1') \overset{\mathrm{Hom}(A, A')}{\rightarrow} \mathrm{Hom}(V_1, V_2') \overset{\mathrm{Hom}(c, b')}{\leftarrow} R$$

where we have identified $\mathrm{Hom}(R, R) \simeq R$. It satisfies

$$\mathbf{Hom}(\mathbf{Q}, \mathbf{Hom}(\mathbf{P}, \mathbf{P}')) \simeq \mathbf{Hom}(\mathbf{Q} \otimes \mathbf{P}, \mathbf{P}').$$

However, this plays no role in the theory discussed here and so will not be considered further.

4. THE LAST CONSTRUCTION

So far we can only treat linear programs satisfying $c = 0$ or $b = 0$. Such programs can be expected to satisfy one of the Farkas Alternatives. However, programs with both b and c different from zero do not satisfy these alternatives. Instead they satisfy the Duality Theorem given below. In order

to apply the Farkas Alternatives to prove the Duality Theorem, we need constructions turning a linear program with arbitrary values of b and c into one with either $b = 0$ or $c = 0$. Now $Z\mathbf{P}$ and $\mathbf{P}Z$ are such constructions and there are obvious isomorphisms

$$\mathrm{Pt}(Z\mathbf{P}) \simeq \mathrm{Pt}(\mathbf{P})$$
$$\mathrm{coPt}(\mathbf{P}Z) \simeq \mathrm{coPt}(\mathbf{P})$$

which are of course not over R since for $Z\mathbf{P}$ one has $\mu \equiv 0$ while for $\mathbf{P}Z$ one has $\nu \equiv 0$. However, the situation for copoints of $Z\mathbf{P}$ and points of $\mathbf{P}Z$ is not so trivial and plays a role in analyzing the main constructions of this section. We have no names for these two constructions and just denote them by $R\mathbf{P}$ and $\mathbf{P}R$.

4.1. DEFINITION. Given a linear program

$$\mathbf{P}\colon R \overset{c}{\leftarrow} V_1 \overset{A}{\rightarrow} V_2 \overset{b}{\leftarrow} R$$

then $R\mathbf{P}$ and $\mathbf{P}R$ are the linear programs defined by

$$R\mathbf{P}\colon R \overset{0}{\leftarrow} V_1 \overset{[A,-c]}{\rightarrow} V_2 \times R \overset{[b,0]}{\leftarrow} R$$

$$\mathbf{P}R\colon R \overset{\langle c,0\rangle}{\leftarrow} V_1 \times R \overset{\langle A,-b\rangle}{\rightarrow} R \overset{0}{\leftarrow} R.$$

4.2. PROPOSITION. *There are isomorphisms*

$$\mathrm{Pt}(R\mathbf{P}) \simeq {}^{+}\mathrm{Pt}(\mathbf{P})$$
$$\mathrm{coPt}(\mathbf{P}R) \simeq {}^{+}\mathrm{coPt}(\mathbf{P}).$$

Proof. A point $\bar{x}\colon \mathbf{1} \to R\mathbf{P}$ satisfies $Ax \leq b$, $-cx \leq 0$, and $x \geq 0$. Hence it corresponds to a point $\bar{x}\colon \mathbf{1} \to \mathbf{P}$ with $\mu(\bar{x}) = cx \geq 0$. Note that this isomorphism is not over R since for $R\mathbf{P}$, $\mu \equiv 0$.

4.3. PROPOSITION. *The following relations hold*:

(1) $R\mathbf{P} = ZR\mathbf{P}$, $\mathbf{P}R = \mathbf{P}RZ$;

(2) $(R\mathbf{P})Z = R(\mathbf{P}Z)$, $(Z\mathbf{P})R = Z(\mathbf{P}R)$;

(3) $(S\mathbf{P})R = S(\mathbf{P}R)$, $(R\mathbf{P})S = R(\mathbf{P}S)$;

(4) $(R\mathbf{P})^* = (\mathbf{P}^*)R$, $(\mathbf{P}R)^* = R(\mathbf{P}^*)$.

Proof. These are simple calculations.

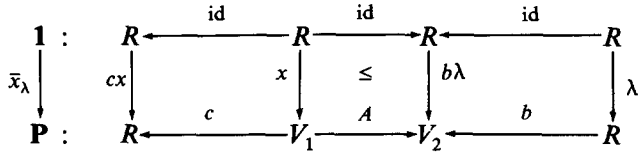
However, certain things do not hold; thus $SR\mathbf{P} \neq RS\mathbf{P}$ and $\mathbf{P}SR \neq \mathbf{P}RS$. Also we don't have any equations for $R(\mathbf{P} \boxplus \mathbf{P}')$ or $(\mathbf{P} \boxplus \mathbf{P}')R$. We want to

calculate Pt(**P***Z*), Pt(**P***R*), coPt(*Z***P**), and coPt(*R***P**); especially, we need Pt((**P** ⊞ **P**′)*R*) and coPt(*R*(**P** ⊞ **P**′)) as will be evident from the proof of the Duality Theorem. In order to make these calculations we have to consider all homomorphisms from **1** to **P** (resp., from **P** to **1***), not just the proper (resp., coproper) ones we have called points (resp., copoints). In this context it is best to sort them out by their zeroth (resp., third) component.

4.4. DEFINITION. (i) A composable homomorphism h: **1** → **R** with $\nu(h) = h_3 = \lambda$: $R \to R$ is called a λ-*point* of **P**. We write λ-Pt(**P**) for the set of λ-points of **P**. Clearly, 1-points are just ordinary points.

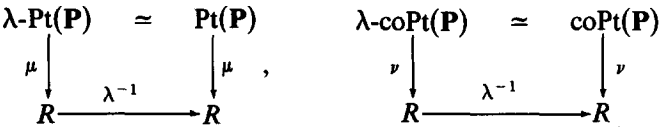
(ii) A cocomposable homomorphism k: **P** → **1*** with $\mu(k) = k_0 = \lambda$: $R \to R$ is called a λ-*copoint* of **P**. We write λ-coPt(**P**) for the set of λ-copoints of **P**.

Our notation for a λ-point of **P** is



4.5. PROPOSITION. *Let* **P** *be a linear program.*

(i) *If* λ > 0 *then there are isomorphisms giving commutative diagrams*



(ii) *If* λ = 0 *then there are isomorphisms over R*

$$0\text{-Pt}(\mathbf{P}) \simeq \text{Pt}(\mathbf{P}Z), \qquad 0\text{-coPt}(\mathbf{P}) \simeq \text{coPt}(Z\mathbf{P}).$$

Proof. (i) The isomorphisms are given by multiplying all four components of $\bar{x}_\lambda$: **1** → **P** (resp., $\bar{w}_\lambda$: **P** → **1***) by $\lambda^{-1} > 0$.

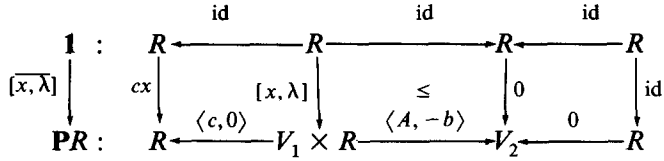
(ii) If λ = 0, then $\bar{x}_0$: **1** → **P** satisfies $Ax \leq 0$ which is exactly the same condition satisfied by a point of **P***Z*.

4.6. PROPOSITION. *Let* **P** *be a linear program. Then there are isomorphisms over R*:

$$\text{Pt}(\mathbf{P}R) \simeq \bigcup_{\lambda \geq 0} \lambda\text{-Pt}(\mathbf{P})$$

$$\text{coPt}(R\mathbf{P}) \simeq \bigcup_{\lambda \geq 0} \lambda\text{-coPt}(\mathbf{P}).$$

Proof. A point of $\mathbf{P}R$ looks like



and satisfies

$$\langle A, -b\rangle[x, \lambda] = Ax - b\lambda \leq 0$$

or $Ax \leq b\lambda$ which is exactly the condition satisfied by the λ-point $\bar{x}_\lambda$: $\mathbf{1} \to \mathbf{P}$. Note that $[\overline{x, \lambda}]$ is composable iff $x \geq 0$ and $\lambda \geq 0$.

4.7. DEFINITION. (i) p: $\mathrm{Pt}(\mathbf{P}R) \to R$ denotes the function given by $p([\overline{x, \lambda}]) = \lambda$. Similarly, q: $\mathrm{coPt}(R\mathbf{P}) \to R$ denotes the function given by $q(\langle \overline{w, \lambda} \rangle) = \lambda$.

(ii) In the following proposition $\mathrm{Pt}(\mathbf{P}R) \times_R \mathrm{Pt}(\mathbf{P}'R)$ denotes the *pullback* over R of these two sets; i.e., pairs $([\overline{x, \lambda}], [\overline{x', \lambda'}]) \in \mathrm{Pt}(\mathbf{P}R) \times \mathrm{Pt}(\mathbf{P}'R)$ such that

$$p\left([\overline{x, \lambda}]\right) = \lambda = \lambda' = p\left([\overline{x', \lambda'}]\right).$$

The pullback expression for copoints is defined analogously using q.

4.8. PROPOSITION. *Let* $\mathbf{P}$ *and* $\mathbf{P}'$ *be linear programs. Then there are isomorphisms giving commutative diagrams*

$$\begin{array}{ccc} \mathrm{Pt}(\mathbf{P}R) \times_R \mathrm{Pt}(\mathbf{P}'R) & \xrightarrow{\sim} & \mathrm{Pt}((\mathbf{P} \boxplus \mathbf{P}')R) \\ {\scriptstyle \mu \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ R \times R & \xrightarrow{+} & R \end{array}$$

$$\begin{array}{ccc} \mathrm{coPt}(R\mathbf{P}) \times_R \mathrm{coPt}(R\mathbf{P}') & \xrightarrow{\sim} & \mathrm{coPt}(R(\mathbf{P} \boxplus \mathbf{P}')) \\ {\scriptstyle \nu \times \nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ R \times R & \xrightarrow{+} & R. \end{array}$$

Proof. There is a sequence of isomorphisms

$$\begin{aligned} \mathrm{Pt}(\mathbf{P}R) \times_R \mathrm{Pt}(\mathbf{P}'R) &\simeq \bigcup_{\lambda \geq 0} \lambda\text{-}\mathrm{Pt}(\mathbf{P}) \times \lambda\text{-}\mathrm{Pt}(\mathbf{P}') \\ &\simeq \bigcup_{\lambda \geq 0} \lambda\text{-}\mathrm{Pt}(\mathbf{P} \boxplus \mathbf{P}') \\ &\simeq \mathrm{Pt}((\mathbf{P} \boxplus \mathbf{P}')R) \end{aligned}$$

the first by the definition of the pullback and 4.6, the second by a proof analogous to that of 3.2 and the third by 4.6 again.

Note that, as far as points are concerned, $(\mathbf{P} \boxplus \mathbf{P}')R$ looks like the (non-existent) pullback of $\mathbf{P}R$ and $\mathbf{P}'R$ over $\mathbf{R}$ where $\mathbf{P}R$ maps to the linear program

$$\mathbf{R}\colon R \overset{0}{\leftarrow} R \overset{0}{\rightarrow} 0 \overset{0}{\leftarrow} R$$

by the homomorphism $\tilde{p}\colon \mathbf{P}R \to \mathbf{R}$ given by $\tilde{p} = (0, pr_R, 0, \mathrm{id})$. It is immediate that $\mathrm{Pt}(\mathbf{R}) \simeq R$ and the map $\mathrm{Pt}(\mathbf{P}R) \to \mathrm{Pt}(\mathbf{R})$ induced by $\tilde{p}$ is just p. Analogous comments apply to $R(\mathbf{P} \boxplus \mathbf{P}')$.

4.9. Discussion. At this point one would like analogues of 3.8 and 3.9 giving conditions for $R\mathbf{P}$ to satisfy (FI) or (FII) and for $\mathbf{P}R$ to satisfy (FI^{d}) or ($\mathrm{FII}^{\mathrm{d}}$). Unfortunately, all we can do is adapt the conditions given after 2.10 and 3.8. Thus, assume V_2 is locally convex. Then

(i) $R\mathbf{P}$ satisfies (FI) if $[A, -c](V_1^+)$ is closed,

(ii) $R\mathbf{P}$ satisfies (FII) if $[A, -c](V_1^+) + (V_2^+ \times R^+)$ is closed.

5. The Duality Theorem

We are now ready to deal with a general linear program $\mathbf{P}$. Such a program need not satisfy the Farkas Alternatives. Specifically, we shall call $\mathbf{P}$ *vacuous* if $\mathrm{Pt}(\mathbf{P}) = \varnothing = \mathrm{coPt}(\mathbf{P})$; i.e., if $\mathbf{P}$ has neither points nor copoints. We are only interested in non-vacuous linear programs, and we want to calculate the sum $\mu(\mathbf{P}) + \nu(\mathbf{P})$ or, equivalently by 2.6, the sum $\mu(\mathbf{P}) + \mu(\mathbf{P}^*)$. Recall our conventions following 2.1 and 2.4 that $\mu(\mathbf{P}) = -\infty$ (resp., $\nu(\mathbf{P}) = -\infty$) if $\mathbf{P}$ has no points (resp., copoints). In calculating this sum we make the further convention that

$$\begin{aligned} -\infty + \infty &= 0 = \infty - \infty \\ -\infty + n &= -\infty = n - \infty \\ +\infty + n &= +\infty = n + \infty. \end{aligned}$$

First a simple result that requires no assumptions about $\mathbf{P}$.

5.1. Proposition. *If $\mathbf{P}$ is a linear program then $\mu(\mathbf{P}) + \nu(\mathbf{P}) \leq 0$ or, equivalently, $\mu(\mathbf{P}) + \mu(\mathbf{P}^*) \leq 0$.*

Proof. If $\mathbf{P}$ has no points or $\mathbf{P}$ has no copoints then the result follows from the above convention. If $\mathbf{P}$ has both a point $\bar{x}\colon \mathbf{1} \to \mathbf{P}$ and a copoint $\bar{w}\colon \mathbf{P} \to \mathbf{1}^*$ then by Definitions 2.1 and 2.4 and Proposition 1.3, $\bar{w}\bar{x}\colon \mathbf{1} \to \mathbf{1}^*$

is defined and its first and second components satisfy $cx \leq wb$. Hence

$$\begin{aligned}\mu(\mathbf{P}) &= \max\{\mu(\bar{x})|\bar{x} \in \mathrm{Pt}(\mathbf{P})\}\\ &= \max\{cx|\bar{x} \in \mathrm{Pt}(\mathbf{P})\}\\ &\leq \min\{wb|\bar{w} \in \mathrm{coPt}(\mathbf{P})\}\\ &= -\max\{-wb|\bar{w} \in \mathrm{coPt}(\mathbf{P})\}\\ &= -\max\{\nu(\bar{w})|\bar{w} \in \mathrm{coPt}(\mathbf{P})\}\\ &= -\nu(\mathbf{P})\end{aligned}$$

so $\mu(\mathbf{P}) + \nu(\mathbf{P}) \leq 0$.

5.2. DEFINITION. (i) $\mathbf{P}$ has a *finite duality gap* if $\mu(\mathbf{P})$ and $\nu(\mathbf{P})$ are both finite but $\mu(\mathbf{P}) + \nu(\mathbf{P}) \neq 0$.

(ii) $\mathbf{P}$ has an *infinite duality gap* if one of $\mu(\mathbf{P})$ and $\nu(\mathbf{P})$ is $-\infty$ but the other is finite. It is called a *no-point* infinite gap if $\mu(\mathbf{P}) = -\infty$ and a *no-copoint* infinite gap if $\nu(\mathbf{P}) = -\infty$.

We would like to characterize those linear programs which have no duality gap either finite or infinite, which by the above conventions means that $\mu(\mathbf{P}) + \nu(\mathbf{P}) = 0$. Unfortunately all we can do is give sufficient conditions for this to happen. If one of the conditions (reflexivity) is omitted we can also identify which kind of gap (finite or infinite) is possible and eliminate it by passing to the dual program.

5.3. DEFINITION. Let $\mathbf{P}$ be a linear program.

(i) $\mathbf{P}$ is called *regular* if $SR(\mathbf{P} \boxplus \mathbf{P}^*)$ satisfies (FI) (cf. 2.11, 3.1, 3.4, and 4.1).

(ii) $\mathbf{P}$ is called *dual regular* if $(\mathbf{P} \boxplus \mathbf{P}^*)RS$ satisfies (FI^d).

5.4. DUALITY THEOREM. *Let* $\mathbf{P}$ *be a non-vacuous linear program.*

(i) *Suppose* $\mathbf{P}$ *is regular. Then*

(a) *if* $\mathbf{PZ}$ *is reflexive, then* $\mathbf{P}$ *does not have a duality gap.*

(b) *If* $\mathbf{PZ}$ *is not reflexive, then* $\mathbf{P}$ *either does not have a duality gap or has a no-copoint infinite duality gap.*

(ii) *Suppose* $\mathbf{P}$ *is dual regular. Then*

(a) *if* $\mathbf{P}$ *is reflexive, then* $\mathbf{P}$ *does not have a duality gap.*

(b) *If* $\mathbf{P}$ *is not reflexive, then* $\mathbf{P}$ *either does not have a duality gap or has a duality gap which can be either finite or no-point infinite.*

(iii) *If* $\mathbf{P}$ *is either regular or dual regular then* $\mathbf{P}^*$ *does not have a duality gap.*

Proof (cf. Franklin [4]). (i) If **P** is regular then, by 3.8, $R(\mathbf{P} \boxplus \mathbf{P}^*)$ satisfies (FII) (cf. 2.11).

Case 1. $\mathrm{Pt}(R(\mathbf{P} \boxplus \mathbf{P}^*)) \neq \varnothing$. But

$$\begin{aligned}\mathrm{Pt}(R(\mathbf{P} \boxplus \mathbf{P}^*)) &\simeq {}^{+}\mathrm{Pt}(\mathbf{P} \boxplus \mathbf{P}^*) \\ &\simeq {}^{+} (\mathrm{Pt}(\mathbf{P}) \times \mathrm{Pt}(\mathbf{P}^*)),\end{aligned}$$

the first isomorphism by 4.2 and the second by 3.2. Using the additivity of valuation 3.3, it follows that $\mu(\mathbf{P}) + \mu(\mathbf{P}^*) \geq 0$. Hence by 5.1, $\mu(\mathbf{P}) + \mu(\mathbf{P}^*) = 0$. Both programs have points so neither value is $-\infty$. Therefore both are finite and $\mu(\mathbf{P}) = -\mu(\mathbf{P}^*)$.

Case 2. ${}^{++}\mathrm{coPt}(R(\mathbf{P} \boxplus \mathbf{P}^*) \neq \varnothing$. But

$$\begin{aligned}{}^{++}\mathrm{coPt}(R(\mathbf{P} \boxplus \mathbf{P}^*)) &\simeq {}^{++}\left(\bigcup_{\lambda \geq 0} \lambda\text{-coPt}(\mathbf{P} \boxplus \mathbf{P}^*)\right) \\ &\simeq {}^{++}\left(\bigcup_{\lambda \geq 0} \lambda\text{-coPt}(\mathbf{P}) \times \lambda\text{-coPt}(\mathbf{P}^*)\right)\end{aligned}$$

by 4.8.

Case 2.1. Suppose $\lambda > 0$ occurs. Then by 4.3

$${}^{++}(\lambda\text{-coPt}(\mathbf{P}) \times \lambda\text{-coPt}(\mathbf{P}^*)) \simeq {}^{++} (\mathrm{coPt}(\mathbf{P}) \times \mathrm{coPt}(\mathbf{P}^*))$$

so there are copoints $\bar{w}$: $\mathbf{P} \to \mathbf{1}^*$ and $\bar{z}$: $\mathbf{P}^* \to \mathbf{1}^*$ with $\nu(\bar{w}) + \nu(\bar{z}) > 0$. Now $\bar{w}$ corresponds by duality to a point $\bar{w}'$: $\mathbf{1} \to \mathbf{P}^*$ (by 2.5) with $\mu(\bar{w}') = \nu(\bar{w})$. Hence $\mu(\bar{w}') + \nu(\bar{z}) > 0$ which contradicts 5.1 for $\mathbf{P}^*$. Thus $\lambda = 0$.

Case 2.2. We have therefore that

$$\begin{aligned}{}^{++}\mathrm{coPt}(R(\mathbf{P} \boxplus \mathbf{P}^*)) &\simeq {}^{++} (0\text{-coPt}(\mathbf{P} \boxplus \mathbf{P}^*)) \\ &\simeq {}^{++} (\mathrm{coPt}(Z(\mathbf{P} \boxplus \mathbf{P}^*)) \\ &\simeq {}^{++} (\mathrm{coPt}(Z\mathbf{P} \boxplus Z\mathbf{P}^*) \\ &\simeq {}^{++} (\mathrm{coPt}(Z\mathbf{P}) \times \mathrm{coPt}(Z\mathbf{P}^*)) \neq \varnothing\end{aligned}$$

by Case 2.1, 4.5, 4.6, and 3.2. Thus there are copoints $\bar{w}$: $Z\mathbf{P} \to \mathbf{1}^*$ and $\bar{z}$: $Z\mathbf{P}^* \to \mathbf{1}^*$ with $\nu(\bar{w}) + \nu(\bar{z}) > 0$. Since this sum is greater than zero at least one summand must be greater than zero.

Case 2.2.1. If $\nu(\bar{w}) > 0$ then $\bar{w} \in {}^{++}\mathrm{coPt}(Z\mathbf{P})$ so by 2.10 (which applies because $Z(-)$ is idempotent) $\mathrm{Pt}(Z\mathbf{P}) \simeq \mathrm{Pt}(\mathbf{P}) = \varnothing$, the isomorphism by the remark at the beginning of Section 4. Thus **P** has no points so $\mu(\mathbf{P}) = -\infty$.

Since $\mathbf{P}$ is non-vacuous, it has a copoint $\bar{w}_0$: $\mathbf{P} \to \mathbf{1}^*$ satisfying $w_0 \geq 0$ and $w_0 A \geq c$. Now define $\bar{w}_n = \bar{w}_0 + n\bar{w}$. Then

$$w_n A = w_0 A + nwA \geq c + 0 = c$$

and $w_n \geq 0$ so $\bar{w}_n \in \mathrm{coPt}(\mathbf{P})$. Furthermore

$$\nu(\bar{w}_n) = \nu(\bar{w}_0) + n\nu(\bar{w})$$

which approaches $+\infty$ as n increases since $\nu(\bar{w}) > 0$. Hence $\nu(\mathbf{P}) = \mu(\mathbf{P}^*) = +\infty$. Thus $\mathbf{P}$ has no duality gap in this case since $\mu(\mathbf{P}) = -\infty$ and $\nu(\mathbf{P}) = +\infty$.

Case 2.2.2. If $\nu(\bar{z}) > 0$ then $\bar{z} \in {}^{++}\mathrm{coPt}(Z\mathbf{P}^*)$. As in Case 2.2.1, it follows that $\mu(\mathbf{P}^*) = \nu(\mathbf{P}) = -\infty$. Since $\mathbf{P}$ is non-vacuous, it has a point $\bar{x}_0$: $\mathbf{1} \to \mathbf{P}$ satisfying $x_0 \geq 0$ and $Ax_0 \leq b$. Now, by 3.7, $Z\mathbf{P}^* = (\mathbf{P}Z)^*$ so $\bar{z} \in {}^{++}\mathrm{coPt}(\mathbf{P}Z)^*$. If $\mathbf{P}Z$ is reflexive, then by 2.8, $\bar{z}$ corresponds to a point $\bar{x} \in {}^{++}\mathrm{Pt}(\mathbf{P}Z)$ which satisfies $x \geq 0$ and $Ax \leq 0$. As in 2.2.1 define $\bar{x}_n = \bar{x}_0 + n\bar{x}$. Then

$$Ax_n = Ax_0 + nAx \leq b + 0 \leq b$$

and $x_n \geq 0$ so $\bar{x}_n \in \mathrm{Pt}(\mathbf{P})$. Furthermore

$$\mu(\bar{x}_n) = \mu(\bar{x}_0) + n\mu(\bar{x})$$

which approaches $+\infty$ as n increases since $\mu(\bar{x}) > 0$. Hence $\mu(\mathbf{P}) = +\infty$. Thus $\mathbf{P}$ has no duality gap in this case since $\nu(\mathbf{P}) = -\infty$ and $\mu(\mathbf{P}) = +\infty$.

If $\mathbf{P}Z$ is not reflexive then, by 2.5, $\bar{z}$ corresponds to a point

$$\bar{x} \in {}^{++}\mathrm{Pt}(\mathbf{P}Z)^{**} = {}^{++}\mathrm{Pt}(\mathbf{P}^{**}Z).$$

Regard $\bar{x}_0$ as a point of $\mathbf{P}^{**}$ via the inclusion $\mathbf{P} \hookrightarrow \mathbf{P}^{**}$ and then the same argument as before shows that $\mu(\mathbf{P}^{**}) = +\infty$. Thus it can happen that $\nu(\mathbf{P}) = -\infty$ and $\mu(\mathbf{P})$ is finite so that $\mathbf{P}$ has a no-copoint infinite duality gap. However, $\mu(\mathbf{P}^*) = \nu(\mathbf{P}) = -\infty$ and $\nu(\mathbf{P}^*) = \mu(\mathbf{P}^{**}) = +\infty$ so $\mathbf{P}^*$ has no duality gap.

(ii) If $\mathbf{P}$ is dual regular then, by 3.8, $(\mathbf{P} \boxplus \mathbf{P}^*)R$ satisfies (FII$^{\mathrm{d}}$). The analysis proceeds as before except reflexivity appears in a different place.

Case 1$^{\mathrm{d}}$. $\mathrm{coPt}((\mathbf{P} \boxplus \mathbf{P}^*)R) \neq \varnothing$. But

$$\begin{aligned}\mathrm{coPt}((\mathbf{P} \boxplus \mathbf{P}^*)R) &\simeq {}^{+}\mathrm{coPt}(\mathbf{P} \boxplus \mathbf{P}^*) \\ &\simeq {}^{+}(\mathrm{coPt}(\mathbf{P}) \times \mathrm{coPt}(\mathbf{P}^*)).\end{aligned}$$

If $\mathbf{P}$ is reflexive we can write $\mathrm{coPt}(\mathbf{P}^*) \simeq \mathrm{Pt}(\mathbf{P})$ and conclude that $\mathbf{P}$ has

points and copoints and that $\mu(\mathbf{P}) + \nu(\mathbf{P}) = 0$ so $\mathbf{P}$ has no duality gap. However, if $\mathbf{P}$ is not reflexive, we can only write $\mathrm{coPt}(\mathbf{P}^*) \simeq \mathrm{Pt}(\mathbf{P}^{**})$ and $\mathrm{coPt}(\mathbf{P}) = \mathrm{Pt}(\mathbf{P}^*)$ so that there is a copoint $\bar{w}$ of $\mathbf{P}$ (or equivalently a point $\bar{w}'$ of $\mathbf{P}^*$) and a point $\bar{x}$ of $\mathbf{P}^{**}$ with

$$\nu(\bar{w}) + \mu(\bar{x}) = \mu(\bar{w}') + \nu(\bar{x}) \geq 0.$$

Hence

$$\nu(\mathbf{P}) + \mu(\mathbf{P}^{**}) = \mu(\mathbf{P}^*) + \mu(\mathbf{P}^{**}) = 0.$$

Thus $\nu(\mathbf{P})$ is finite. If $\mathbf{P}$ has no points then there is a no-point infinite duality gap (this seems unlikely), while if $\mathbf{P}$ has points then the gap is only finite. $\mathbf{P}^*$ as before has no duality gap.

Case 2^{d}. ${}^{++}\mathrm{Pt}((\mathbf{P} \boxplus \mathbf{P}^*)R) \neq \varnothing$. As before

$${}^{++}\mathrm{Pt}((\mathbf{P} \boxplus \mathbf{P}^*)R) = {}^{++}\left(\bigcup_{\lambda \geq 0} \lambda\text{-}\mathrm{Pt}(\mathbf{P}) \times \lambda\text{-}\mathrm{Pt}(\mathbf{P}^*)\right).$$

If $\lambda > 0$ occurs one gets a contradiction, so $\lambda = 0$ is the only possibility and

$${}^{++}\mathrm{Pt}((\mathbf{P} \boxplus \mathbf{P}^*)R) \simeq {}^{++}(\mathrm{Pt}(\mathbf{P}Z) \times \mathrm{Pt}(\mathbf{P}^*Z)) \neq \varnothing.$$

Thus there are points $\bar{x} \in \mathrm{Pt}(\mathbf{P}Z)$ and $\bar{w} \in \mathrm{Pt}(\mathbf{P}^*Z)$ with $\mu(\bar{x}) + \mu(\bar{w}) > 0$.

Case $2.2.1^{\mathrm{d}}$. If $\mu(\bar{x}) > 0$ then $\bar{x} \in {}^{++}\mathrm{Pt}(\mathbf{P}Z)$ so $\mathrm{coPt}(\mathbf{P}Z) \simeq \mathrm{coPt}(\mathbf{P}) = \varnothing$ and $\nu(\mathbf{P}) = -\infty$. Since $\mathbf{P}$ is non-vacuous, it has a point $\bar{x}_0 \in \mathrm{Pt}(\mathbf{P})$. As before, one concludes that $\mu(\mathbf{P}) = +\infty$.

Case $2.2.2^{\mathrm{d}}$. If $\mu(\bar{w}) > 0$ then $\bar{w} \in {}^{++}\mathrm{Pt}(\mathbf{P}^*Z)$ so $\mathrm{coPt}(\mathbf{P}^*Z) = \varnothing$. Now $\mathbf{P}^*Z = (Z\mathbf{P})^*$ by 3.7 so $\mathrm{coPt}((Z\mathbf{P})^*) = \varnothing$. Since duality gives a map

$$\mathrm{Pt}(Z\mathbf{P}) \to \mathrm{coPt}((Z\mathbf{P})^*)$$

it follows that $\mathrm{Pt}(Z\mathbf{P}) = \mathrm{Pt}(\mathbf{P}) = \varnothing$. Hence $\mu(\mathbf{P}) = -\infty$. Since $\mathbf{P}$ is non-vacuous, it has a copoint or, equivalently, $\mathbf{P}^*$ has a point $\bar{w}_0$. One uses $\bar{w}$ and $\bar{w}_0$ as before to show that $\mu(\mathbf{P}^*) = +\infty$.

5.5. *Remarks.* It would be interesting to have more elementary conditions that imply reflexivity and regularity.

(i) As long as one sticks to the weak-$*$-topology then reflexity is automatic. Otherwise one can use any functorial topology bigger (more open sets) than the weak-$*$-topology such that the usual map $j\colon V \to V^{**}$ is continuous and such that $R^* \simeq R$. But then reflexivity could fail and we do not know precise conditions for it to hold. (If $\mathrm{Pt}(\mathbf{P}^{**}) = \varnothing$ then $\mathbf{P}$ is reflexive.)

(ii) The conditions given in 4.9 specialize to the following: if V_2 is locally convex and

$$[(A) \times (-A^*), \langle -c, b \rangle](V_1^+ \times V_2^{*+}) + (V_2^+ \times V_1^{*+} \times R^+)$$

is a closed subset of $V_2 \times V_1^* \times R$, then **P** is regular. This is not very enlightening and we hope to find more transparent and easily verified conditions later.

(iii) Although it may not at first look like it, the details of part (i) of this proof are very similar to the details of the proof in Franklin [4]. What seems remarkable is that Franklin's proof goes through in the language and generality considered here. Part (ii) is a straightforward dualization of part (i), but was totally unexpected until I realized that every construction and every step in the proof could be dualized.

6. Special Cases of Duality

By changing the preorders of V_1 or V_2 we can derive two other duality theorems from 5.4. First we need some notation.

6.1. Definition. A preordered vector space $(V, \mathbf{c})$ is called *discrete* if $x \leq y$ iff $x = y$. It is called *indiscrete* if $x \leq y$ holds for all x and y.

6.2. Proposition. *Let V and W be preordered vector spaces.*

(i) *V is discrete iff $V^+ = \{0\}$.*

(ii) *V is indiscrete iff $V^+ = V$.*

(iii) *If V is discrete or W is indiscrete then every linear map $f: V \to W$ is positive.*

(iv) *If V is discrete, or W is indiscrete then the preordered vector space $L(V, W)$ of linear maps from V to W is indiscrete.*

(v) *If V is indiscrete, or W is discrete and V^+ is generating then $L(V, W)$ is discrete.*

(vi) *If V is discrete (resp., indiscrete) then V^* is indiscrete (resp., discrete).*

Proof. Everything is obvious except perhaps the first part of (v). Suppose V is indiscrete. Then $f \leq g$ iff for all $x \in V^+ = V$, $f(x) \leq g(x)$. But then one also has $-f(x) = f(-x) \leq g(-x) = -g(x)$ so $f(x) \geq g(x)$. Hence $f = g$.

6.3. DEFINITION. Let

$$\mathbf{P}\colon R \overset{c}{\leftarrow} V_1 \overset{A}{\rightarrow} V_2 \overset{b}{\leftarrow} R$$

be a linear program. Then $\mathbf{P}_i$ denote the same program as $\mathbf{P}$ except that the preorder on V_1 is replaced by the indiscrete preorder. Similarly $\mathbf{P}_d$ is the same program as $\mathbf{P}$ except that the preorder on V_2 is replaced by the discrete preorder.

6.4. PROPOSITION.

(i) $\mathrm{Pt}(\mathbf{P}_i) = \mathrm{Pt}_a(\mathbf{P})$, $\mathrm{coPt}(\mathbf{P}_i) = \mathrm{cos}(\mathbf{P})$.

(ii) $\mathrm{Pt}(\mathbf{P}_d) = \mathrm{sPt}(\mathbf{P})$, $\mathrm{coPt}(\mathbf{P}_d) = \mathrm{coPt}_a(\mathbf{P})$.

Proof. (i) If V_1 is indiscrete then every $\bar{x}\colon \mathbf{1} \to \mathbf{P}_i$ is composable since every $x\colon R \to V_1$ is positive. On the other hand $\bar{w}\colon \mathbf{P} \to \mathbf{1}^*$ satisfies $c \le wA$ iff $c = wA$ by the first part of 6.2 (v).

(ii) If V_2 is discrete then a point $\bar{x}\colon \mathbf{1} \to \mathbf{P}_d$ satisfies $Ax \le b$ iff $Ax = b$ (since R^+ is generating). On the other hand every $\bar{w}\colon \mathbf{P}_d \to \mathbf{1}^*$ is cocomposable since $w_2\colon V_2 \to R$ is positive.

6.5. COROLLARY.

(i) $\mu(\mathbf{P}_i) = \mu_a(\mathbf{P})$, $\nu(\mathbf{P}_i) = \nu_s(\mathbf{P})$.

(ii) $\mu(\mathbf{P}_d) = \mu_s(\mathbf{P})$, $\nu(\mathbf{P}_d) = \nu_a(\mathbf{P})$.

6.6. PROPOSITION. *The following relations hold*:

(i) $(Z\mathbf{P})_d = Z(\mathbf{P}_d)$, $(Z\mathbf{P})_i = Z(\mathbf{P}_i)$;

(ii) $(\mathbf{P}Z)_d = (\mathbf{P}_d)Z$, $(\mathbf{P}Z)_i = (\mathbf{P}_i)Z$;

(iii) $(\mathbf{P} \boxplus \mathbf{P}')_d = \mathbf{P}_d \boxplus \mathbf{P}'_d$, $(\mathbf{P} \boxplus \mathbf{P}')_i = \mathbf{P}_i \boxplus \mathbf{P}'_i$;

(iv) $(S\mathbf{P})_d = S(\mathbf{P}_d)$, $(\mathbf{P}S)_i = (\mathbf{P}_i)S$;

(v) $(\mathbf{P}R)_d = (\mathbf{P}_d)R$, $(R\mathbf{P})_i = R(\mathbf{P}_i)$;

(vi) $(\mathbf{P}_d)^* = (\mathbf{P}^*)_i$, $(\mathbf{P}_i)^* = (\mathbf{P}^*)_d$.

Proof. These are simple calculations. Note that (iv) and (v) hold only as indicated. The opposite directions, which might be of interest to us, are not valid.

6.7. DEFINITION (cf. 2.7). A linear program $\mathbf{P}$ is called *strict* (resp., *arbitrary*) reflexive if $\mathbf{P} \to \mathbf{P}^{**}$ induces an isomorphism over R

$$\mathrm{sPt}(\mathbf{P}) \xrightarrow{\sim} \mathrm{sPt}(\mathbf{P}^{**})$$

$$(\text{resp., } \mathrm{Pt}_a(\mathbf{P}) \xrightarrow{\sim} \mathrm{Pt}_a(\mathbf{P}^{**}).$$

6.8. PROPOSITION. *If* $\mathbf{P}$ *is strict (resp., arbitrary) reflexive then there is an isomorphism over* R

$$\operatorname{cosPt}(\mathbf{P}^*) \simeq \operatorname{sPt}(\mathbf{P}) \qquad (\text{resp., } \operatorname{coPt}_a(\mathbf{P}^*) \simeq \operatorname{Pt}_a(\mathbf{P})).$$

Proof. Immediate from 2.5.

6.9. PROPOSITION. $\mathbf{P}$ *is arbitrary (resp., strict) reflexive if and only if* $\mathbf{P}_i$ *(resp.,* $\mathbf{P}_d$*) is reflexive.*

Proof. Immediate from 6.4 and 2.5.

6.10 DEFINITION (cf. 5.2). Let $\mathbf{P}$ be a linear program.

(i) $\mathbf{P}$ has a *finite arbitrary–strict* (resp., *strict–arbitrary*) *duality gap* if $\mu_a(\mathbf{P})$ and $\nu_s(\mathbf{P})$ (resp., $\mu_s(\mathbf{P})$ and $\nu_a(\mathbf{P})$) are both finite but $\mu_a(\mathbf{P}) + \nu_s(\mathbf{P}) \neq 0$ (resp., $\mu_s(\mathbf{P}) + \nu_a(\mathbf{P}) \neq 0$).

(ii) *Infinite arbitrary–strict* (resp., *strict–arbitrary*) *duality gaps* are defined similarly.

(iii) We abbreviate arbitrary–strict by "a–s" and strict–arbitrary by "s–a".

6.11. PROPOSITION. *Let* $\mathbf{P}$ *be a linear program such that* $\mathbf{P}_i$ *is non-vacuous.*

(i) *Suppose* $\mathbf{P}_i$ *is regular. Then*

(a) *if* $\mathbf{P}Z$ *is arbitrary reflexive then* $\mathbf{P}$ *does not have an* a–s *duality gap.*

(b) *If* $\mathbf{P}Z$ *is not arbitrary reflexive, then* $\mathbf{P}$ *either does not have an* a–s *duality gap or has a no-strict-copoint infinite* a–s *duality gap.*

(ii) *Suppose* $\mathbf{P}_i$ *is dual regular. Then*

(a) *if* $\mathbf{P}$ *is arbitrary reflexive, then* $\mathbf{P}$ *does not have an* a–s *duality gap.*

(b) *If* $\mathbf{P}$ *is not arbitrary reflexive then* $\mathbf{P}$ *either does not have an* a–s *duality gap or has an* a–s *duality gap which can be either finite or no-arbitrary-point infinite.*

(iii) *If* $\mathbf{P}_i$ *is either regular or dual regular, then* $\mathbf{P}^*$ *does not have an* a–s *duality gap.*

(iv) *The above statements remain valid if* $\mathbf{P}_i$ *is replaced by* $\mathbf{P}_d$, "a–s" *by* "s–a", *and arbitrary and strict are interchanged.*

Proof. If $\mathbf{P}Z$ is arbitrary reflexive then, by 6.9 and 6.6, $(\mathbf{P}Z)_i = (\mathbf{P}_i)Z$ is reflexive. Apply 5.4 to $\mathbf{P}_i$ and use 6.5 to translate the results back to properties of $\mathbf{P}$.

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