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# VARIETIES OF COMPLEX ALGEBRAS

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*Dedicated to Krister Segerberg on his fiftieth birthday*

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## 1. Introduction

It was once commonplace to call a subset of a group a *complex* (e.g. Zassenhaus [52]: the usage originated with Frobenius). There is a natural algebra of complexes, given by the operation

$$X_0 \cdot X_1 = \{x_0 \cdot x_1 : x_0 \in X_0 \ \& \ x_1 \in X_1\}$$
$$= \text{image of } X_0 \times X_1 \text{ under the group operation,} \quad (1)$$

on subsets $X_0$, $X_1$ of a group $(X, \cdot)$. More generally, an $n+1$-placed relation $R$ on a set $X$ induces the $n$-ary operation $f_R : \mathrm{Sb}(X)^n \to \mathrm{Sb}(X)$ on the set $\mathrm{Sb}(X)$ of

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subsets of $X$ that has

$$f_R(X_0, \ldots, X_{n-1}) = R\text{-image of } X_0 \times \cdots \times X_{n-1}$$
$$= \{y: \exists x_0 \cdots \exists x_{n-1}\,(R(x_0, \ldots, x_{n-1}, y)\ \&\ x_i \in X_i\ \forall i < n)\}. \qquad (2)$$

(1) is then the special case of (2) in which $R$ is the graph $\{(x_0, x_1, y): y = x_0 \cdot x_1\}$ of the group operation.

Thus any relation structure $\mathfrak{S} = (X, \mathfrak{R})$, comprising a collection $\mathfrak{R}$ of finitary relations on a set $X$, gives rise to the algebra

$$\mathfrak{S}^+ = (\mathrm{Sb}(X), \{f_R: R \in \mathfrak{R}\}),$$

which we view as an algebra of *operators* $f_R$ on the *Boolean algebra* $\mathrm{Sb}(X)$. Any subalgebra of $\mathfrak{S}^+$ will be called an *algebra of complexes* or *complex algebra.*

The point of departure for this paper is the observation that there are many significant examples of *equational classes,* or *varieties,* of algebras whose members are representable by complex algebras, with the corresponding class of relational structures having a natural description in first-order logic. To explain this, let $K$ be a class of relational structures, and $K^+$ the closure under isomorphism of $\{\mathfrak{S}^+: \mathfrak{S} \in K\}$. Then the class $SK^+$ of subalgebras of members of $K^+$ consists of those algebras that are isomorphic to complex algebras determined by $K$. Our observation is that $SK^+$ is often a variety, and that this is frequently associated with $K$ being *elementary,* i.e. the class of all models of some set of first-order sentences. Examples include the varieties of closure algebras, relation algebras, cylindric algebras, modal algebras, and numerous subvarieties of these. Our aim is to study the general situation here, focusing on these related occurrences of definability: elementary $K$ determining equational $SK^+$.

An equational class of Boolean algebras with operators that is representable in the form $SK^+$ will be called a *complex variety.* Our concern is with their general structure. Not all varieties are complex—indeed there are some that contain no powerset algebras $\mathfrak{S}^+$ at all. On the other hand, there are systematic methods for generating complex varieties. For instance, if $K$ is closed under ultraproducts (e.g. if $K$ is elementary), then the variety $\mathrm{HSP}K^+$ generated by $K$ is complex, i.e. is equal to $SM^+$ for some class $M$ (cf. Theorem 3.6.7). We ask when it is possible to have $M$ equal to $K$ itself, and obtain sufficient conditions (3.6.4) in terms of closure of $K$ under operations that are *dual* to the classical algebraic concepts of homomorphic images, subalgebras, and direct products. The duality is a category-theoretic one between the standard notion of homomorphism of algebras and a notion of *bounded morphism* of relational structures that is stronger than the one usually considered in model theory. To study this idea we use an extension of Stone's Boolean representation theory to the case of operators on Boolean algebras. Since the essential properties of the operators we are interested in do not involve complementation, their theory can be naturally developed in the setting of distributive lattices. Thus in Section 2 the order-

theoretic approach to duality introduced by Priestley [41] is invoked to develop a full representation of homomorphisms of distributive lattices with operators by continuous bounded morphisms of certain topological ordered relational structures.

Many of the results about complex varieties presented in Section 3 arose originally from the study of algebraic and set-theoretic (Kripke) semantics for propositional modal logics. A modal algebra is a single unary operator on a Boolean algebra, while a Kripke model is a particularly simple kind of relational structure: a single binary relation. The notion of bounded morphism we develop generalises that of a special type of truth-preserving function ('p-morphism') between Kripke models that has been widely used in modal logic. One of the main objects of this paper is to explain the extent to which the 'modal case', as we shall call it, can be seen as the simplest illustration of a general theory that forms a chapter of universal algebra. We point out wherever appropriate the way in which a property of a variety interprets a natural question about logical systems.

The remainder of this introduction surveys some examples that illustrate the remarks made above, and motivate some of the general questions to be addressed in what follows.

### *Closure algebras*

A closure algebra is a Boolean algebra carrying a *closure operator,* i.e. a unary operator $c$ satisfying $c0 = 0$, $c(x \sqcup y) = cx \sqcup cy$, $x \sqsubseteq cx = ccx$. The operation of topological closure makes the powerset of any topological space into a closure algebra. The theory of these algebras was extensively developed by Tarski [48].

Jónsson and Tarski [35] showed that the variety $V_{\mathrm{cl}}$ of closure algebras is $SK^{+}_{\mathrm{qo}}$, where $K_{\mathrm{qo}}$ is the elementary class of *quasi-orders* $(X, R)$ (i.e. $R$ is reflexive and transitive). $V_{\mathrm{cl}}$ can also be characterised as $SK^{+}_{\mathrm{po}}$ (cf. remarks following Corollary 3.4.6), where $K_{\mathrm{po}}$ is the elementary class of partial-orders (anti-symmetric quasi-orders), and as the variety generated by $K^{+}_{\mathrm{fqo}}$, where $K_{\mathrm{fqo}}$ is the class of finite quasi-orders (of course $V_{\mathrm{cl}}$ is not $SK^{+}_{\mathrm{fqo}}$, as the latter has only finite members).

There is no closure-algebraic equation that distinguishes $K_{\mathrm{qo}}$ and $K_{\mathrm{po}}$, but there is one that corresponds to anti-symmetry for *finite* structures. The class of finite partial-orders generates the proper subvariety of $V_{\mathrm{cl}}$ consisting of those closure algebras satisfying $x \sqsubseteq c(x - c(cx - x))$. This variety is not complex, i.e. is not of the form $SK^{+}$ for any $K \subseteq K_{\mathrm{qo}}$ (cf. Section 3.7 for an indication of how to prove this).

### *Relation algebras*

The set of all binary relations $R$ on a set $X$ is a Boolean set algebra on which there are natural operations of relational composition $R;S$ and inversion $R^{-1}$, and a distinguished relation $1' = \{(x, x): x \in X\}$. By abstraction this leads to the

variety RA of *relation algebras*, an equationally defined class of Boolean algebras with operations ;, $^{-1}$, $1'$ (Jónsson and Tarski [36], p. 128]). The class RRA of *representable* relation algebras comprises those members of RA embeddable in a product of algebras of binary relations as above. The Jónsson–Tarski theory represents RA as $SK^+$, where $K$ is an elementary class of structures $(X, R, S, U)$ with $R \subseteq X^3$, $S \subseteq X^2$, and $U \subseteq X$ (cf. Maddux [39, Theorem 3.5] for the first-order conditions defining $K$). RRA itself is a variety (Tarski [49, Theorem 2.4]), and was shown by Jónsson and Tarski [36] to be $SM^+$, where $M$ is the class of generalized Brandt groupoids—again an elementary class.

*Cylindric algebras*

The variety $Df_\alpha$ of *diagonal-free cylindric algebras of dimension* $\alpha$ is the class of Boolean algebras with a collection $\{c_\beta : \beta < \alpha\}$ of closure operators satisfying

$$c_\beta(x \sqcap c_\beta y) = c_\beta x \sqcap c_\beta y, \qquad c_\beta c_\gamma x = c_\gamma c_\beta x.$$

$Df_\alpha$ is equal to $SK^+_\alpha$, where $K_\alpha$ is the elementary class of structures $(X, \{R_\beta : \beta < \alpha\})$ having the $R_\beta$'s as a family of pairwise commuting equivalence relations on $X$ (Henkin, Monk and Tarski [28, Remark 2.7.44]).

An $\alpha$-dimensional diagonal-free cylindric *set* algebra is a $Df_\alpha$-subalgebra of the algebra of subsets of the 'Cartesian space' $X^\alpha$ of all $\alpha$-length sequences of elements of a set $X$, with $c_\beta(Y)$, for $Y \subseteq X^\alpha$, being the cylinder generated by moving $Y$ parallel to the $\beta$-axis. A member of $Df_\alpha$ is *geometrically representable* if it is isomorphic to a direct product of such set algebras. The class of geometrically representable $Df_\alpha$'s proves to be a variety [28, p. 402], and is identifiable as $SM_\alpha$, where $M_\alpha$ is a subclass of $K_\alpha$ defined by a set of first-order conditions on the $R_\beta$'s [28, p. 459].

Cylindric algebras are to first-order logic as Boolean algebras are to propositional logic: the laws satisfied by the $c_\beta$'s reflect the logical properties of existential quantifiers. Cylindric algebras proper also have a collection $\{d_{\beta\gamma} : \beta, \gamma < \alpha\}$ of 'diagonal' elements, which are algebraic counterparts to equations $v_\beta = v_\gamma$ between individual variables, and which in the case of cylindric set algebras are given by the sets $\{x \in X^\alpha : x_\beta = x_\gamma\}$. The variety of cylindric algebras with diagonal elements, and its subvariety of geometrically representable algebras, are also each of the form $SK^+$ for some elementary class $K$ [28, pp. 458–459].

The question arises as to which elementary classes generate varieties of complex algebras, i.e.

*What conditions on an elementary class $K$ ensure that $SK^+$ is a variety?*

(Note that $SK^+$ will not always be a variety: for example, it will not be closed under direct products if all members of $K$ are finite.) This question will be answered in Section 3.6 in terms of closure of $K$ under 'bounded' morphisms,

'inner' substructures, and disjoint unions. Section 4 presents preservation theorems that characterise syntactically the first-order theories whose models are closed under these constructions.

### *Modal algebras and modal logic*

In the language of propositional modal logic, formulae are constructed from a set $\Pi$ of propositional variables by using the Boolean connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, and an additional unary connective $\Diamond$ ('possibly'). The dual connective $\Box$ ('necessarily') is defined as $\neg\Diamond\neg$.

Algebraic semantics for this language is provided by algebras $\mathfrak{A} = (\mathfrak{B}, f)$, consisting of a unary function $f$ on a Boolean algebra $\mathfrak{B}$. Each modal formula $\phi$ acts as an algebraic term for $\mathfrak{A}$, with propositional variables serving as individual variables ranging over the elements of $\mathfrak{B}$, and $\Diamond$ being interpreted as $f$. Thus if $\phi$ has $n$ propositional variables, it induces an $n$-ary polynomial function $p^{\phi}$ on $\mathfrak{A}$. $\phi$ is *valid in the algebra* $\mathfrak{A}$, denoted $\mathfrak{A} \vDash \phi$, if $p^{\phi}$ is constantly equal to the greatest element 1 of $\mathfrak{B}$, i.e. if $\mathfrak{A}$ satisfies the equation '$\phi = 1$' when $\phi$ is viewed as a term.

The set-theoretic semantics attributed to Kripke is based on structures $\mathfrak{S} = (X, R)$ (usually called *frames*), with $R \subseteq X^2$. A *model on* $\mathfrak{S}$ is a pair $\mathfrak{M} = (\mathfrak{S}, V)$, with $V: \Pi \rightarrow \mathrm{Sb}(X)$ being a function assigning to each propositional variable $\pi$ a subset $V(\pi)$ of $X$. $V(\pi)$ is thought of as the set of points in $\mathfrak{M}$ at which $\pi$ is 'true'. The satisfaction relation $\mathfrak{M} \vDash_x \phi$—formula $\phi$ is true at $x$ in $\mathfrak{M}$—is defined inductively on the formation of $\phi$, the crucial clause being

$$\mathfrak{M} \vDash_x \Diamond\phi \quad \text{iff} \quad \exists y\,(R(x, y) \,\&\, \mathfrak{M} \vDash_y \phi), \tag{3}$$

and hence

$$\mathfrak{M} \vDash_x \Box\phi \quad \text{iff} \quad \forall y\,(R(x, y) \rightarrow \mathfrak{M} \vDash_y \phi).$$

Sentence $\phi$ is *valid in* $\mathfrak{S}$, $\mathfrak{S} \vDash \phi$, if it is true at every point in every model on $\mathfrak{S}$.

The intuitive basis of this modelling is that $X$ is a set of 'possible worlds', and $R(x, y)$ means that world $y$ is a 'conceivable alternative' to world $x$. Clause (3) then formalises the idea (due to Leibnitz) that a sentence is possibly true if it is true in some possible world. Formally, we could just as well have written $R(y, x)$ for the relation just defined, and presented (3) instead as

$$\mathfrak{M} \vDash_x \Diamond\phi \quad \text{iff} \quad \exists y\,(R(y, x) \,\&\, \mathfrak{M} \vDash_y \phi). \tag{4}$$

But then, writing $\mathfrak{M}(\phi)$ for the 'truth-set' $\{x: \mathfrak{M} \vDash_x \phi\}$, clause (4) becomes

$$\mathfrak{M}(\Diamond\phi) = f_R(\mathfrak{M}(\phi)), \tag{5}$$

where $f_R$ is the operation defined by $R$ according to clause (2) above.

We will confine our discussion to the variety $V_{\mathrm{ma}}$ of so-called *normal* modal algebras: those satisfying $f0 = 0$ and $f(x \sqcup y) = fx \sqcup fy$. For any frame $\mathfrak{S}$, the complex algebra $\mathfrak{S}^+$ is in $V_{\mathrm{ma}}$, and with the aid of (5) it can be seen that $\mathfrak{S}$ and

$\mathfrak{S}^+$ validate the same formulae, according to the two notions of validity just outlined (cf. Lemmon [38], Goldblatt [20]).

Thus a formula that is valid in all normal modal algebras is valid in all frames. The converse is also true, due to the result of Jónsson and Tarski [35] that $V_{\mathrm{ma}} = SK_{\mathrm{fr}}^+$, with $K_{\mathrm{fr}}$ the class of all frames. That each member of $V_{\mathrm{ma}}$ is embeddable in some $\mathfrak{S}^+$ follows from their extension of Stone's representation of Boolean algebras to cover $n$-ary operators on such algebras that preserve 0 and $\sqcup$ in each variable. The case $n = 1$ was rediscovered and applied to modal logic by Dana Scott, as presented in Lemmon [38, Part II].

A *normal modal logic* is a set $\Lambda$ of formula that contains all tautologies and all instances of the schema

$$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi),$$

and is closed under modus ponens, substitution for propositional variables, and the rule of *necessitation*, i.e.

$$\phi \in \Lambda \quad \text{implies} \quad \Box\phi \in \Lambda.$$

The intersection of any collection of normal logics is a normal logic, and so for any set $\Gamma$ of formulae there is a *smallest* normal logic $\Lambda_\Gamma$ containing $\Gamma$.

The lattice of normal modal logics is anti-isomorphic to the lattice of subvarieties of $V_{\mathrm{ma}}$. To see this, observe first that if $\mathfrak{A}$ is a normal modal algebra, then $\Lambda_{\mathfrak{A}} = \{\phi: \mathfrak{A} \models \phi\}$ is a normal logic. Hence if $C \subseteq V_{\mathrm{ma}}$, the set $\Lambda_C$ of formulae valid in all members of $C$ is a normal logic, namely $\Lambda_C = \bigcap \{\Lambda_{\mathfrak{A}}: \mathfrak{A} \in C\}$. $\Lambda_C$ is the logic *determined by* $C$. On the other hand, the class $V_\Gamma$ of normal algebras that validate a set $\Gamma$ of formulae is a variety, defined by the equations '$\phi = 1$' for all $\phi \in \Gamma$. Moreover, $\Gamma$ and the normal logic $\Lambda_\Gamma$ it generates define the same variety, i.e. $V_\Gamma = V_{\Lambda_\Gamma}$. For, if $\mathfrak{A} \models \Gamma$, then $\Gamma \subseteq \Lambda_{\mathfrak{A}}$, and so $\Lambda_{\mathfrak{A}}$ includes the smallest logic containing $\Gamma$, i.e. $\mathfrak{A} \models \Lambda_\Gamma$.

Since any modal-algebraic equation is equivalent to one of the form '$\phi = 1$' for some formula $\phi$, it follows that each variety of normal algebras is of the form $V_\Lambda$ for some normal logic, so that the correspondence $\Lambda \mapsto V_\Lambda$ is a $\subseteq$-reversing mapping of the lattice of normal logics *onto* the lattice of subvarieties of $V_{\mathrm{ma}}$. That the correspondence is an anti-isomorphism follows from the fact that $\Lambda = \Lambda_{V_\Lambda}$, i.e. $V_\Lambda$ determines $\Lambda$. The non-trivial part of the proof of this involves showing that if $\phi \notin \Lambda$, then there is an algebra in $V_\Lambda$ that invalidates $\phi$. For this one uses the familiar *Lindenbaum algebra* $\mathfrak{A}_\Lambda$, whose elements are the equivalence classes of formulae under the relation '$(\phi \leftrightarrow \psi) \in \Lambda$'. $\mathfrak{A}_\Lambda$ determines $\Lambda$ (i.e. validates precisely the members of $\Lambda$) and is in fact the free $V_\Lambda$-algebra generated by the equivalence classes of the propositional variables.

Observe that $V_{\Lambda_C}$ is the smallest equational class containing $C$, for if $C \subseteq V_\Gamma$, then $\Gamma \subseteq \Lambda_C$ and so $V_{\Lambda_C} \subseteq V_\Gamma$. Thus $V_{\Lambda_C} = \mathrm{HSP}C$. Hence $C = V_{\Lambda_C}$ iff $C$ is a variety. It follows that

*$C$ is of the form $V_\Lambda$ for some normal modal logic $\Lambda$ iff it is closed under homomorphic images, subalgebras, and direct products.*

This discussion suggests a corresponding series of questions about logics and classes of frames instead of algebras. A frame $\mathfrak{S}$ determines the logic $\Lambda_{\mathfrak{S}} = \{\phi: \mathfrak{S} \models \phi\}$, while a class $K$ of frames determines the logic $\Lambda_K = \{\phi: K \models \phi\} = \bigcap\{\Lambda_{\mathfrak{S}}: \mathfrak{S} \in K\}$. On the other hand, we define $K_\Gamma = \{\mathfrak{S}: \mathfrak{S} \models \Gamma\}$ to be the class of all frames that validate $\Gamma$, and observe that $K_\Gamma = K_{\Lambda_\Gamma}$ (since in general $\mathfrak{S} \models \Gamma$ iff $\mathfrak{S} \models \Lambda_\Gamma$). A class of frames is *modal-axiomatic* if it is of the form $K_\Gamma$, or equivalently of the form $K_\Lambda$ for some normal logic $\Lambda$. Observe that the smallest modal-axiomatic class containing a given class $K$ must be $K_{\Lambda_K}$, for if $K \subseteq K_\Gamma$, then $\Gamma \subseteq \Lambda_K$ and so $K_{\Lambda_K} \subseteq K_\Gamma$. Thus $K$ is modal-axomatic iff $K = K_{\Lambda_K}$. By analogy with the case of varieties, it is natural to ask

*When is a class $K$ of frames equal to $K_\Lambda$ for some normal modal logic $\Lambda$?*

It turns out (cf. Theorem 3.7.6) that for *elementary* $K$ there is an answer, similar to the algebraic case, in terms of closure conditions for $K$ involving bounded morphisms, inner substructures, disjoint unions, and a notion of *canonical extension* which derives from the Stone representation theory (the canonical extension of a structure $\mathfrak{S}$ is a new structure whose elements are the ultrafilters on $\mathfrak{S}$). We establish this characterisation for elementary classes of arbitrary similarity type.

The property of being modal-axiomatic can be re-expressed in algebraic terms as follows. Let $V_K = \mathbf{HSP}K^+$ be the variety generated by $K^+$. Then $V_K = V_{\Lambda_K}$, the variety defined by the logic $\Lambda_K$ determined by $K$. In asking whether $K$ is modal-axiomatic we are asking whether all frames that validate $\Lambda_K$ are in $K$. Thus

*$K$ is a modal-axiomatic iff $K^+$ is the class of all powerset algebras $\mathfrak{S}^+$ in the variety that it generates.*

Turning now to a different question, we say that a normal logic $\Lambda$ is *complete* if it is determined by some class of frames, i.e. if $\Lambda = \Lambda_K$ for some $K$. This holds iff $\Lambda = \Lambda_{K_\Lambda}$, i.e. iff $\Lambda$ is determined by the class of all $\Lambda$-frames, but whereas in the algebraic case we always have $\Lambda = \Lambda_{V_\Lambda}$ (since $\Lambda$ is determined by the variety $V_\Lambda$), there are normal logics that are *incomplete*, and so are not equal to $\Lambda_K$ for any class $K$. Examples of such modal logics were first constructed by Thomason [51] and Fine [16], following an earlier construction of an incomplete tense logic by Thomason [50].

The algebraic version of completeness goes as follows. If $V$ is a variety of normal modal algebras, we define $K_V = \{\mathfrak{S}: \mathfrak{S}^+ \in V\}$ to be the class of all structures whose powerset complex algebras belong to $V$. In the case that $V$ is $V_\Lambda$, $K_V$ is just the class $K_\Lambda$ of all $\Lambda$-frames. Now completeness means that $\Lambda$ is the logic $\Lambda_{K_\Lambda}$ determined by $K_\Lambda$, or equivalently that $\Lambda$ and $\Lambda_{K_\Lambda}$ define the same subvariety of $V_{\mathrm{ma}}$. But the logic determined by $K_\Lambda$ is the same as the logic determined algebraically by $K_\Lambda^+$, and this logic defines the variety generated by $K_\Lambda^+$ (since in general $V_{\Lambda_C} = \mathbf{HSP}C$). Therefore

*$\Lambda$ is complete iff $K_\Lambda^+$ generates $V_\Lambda$.*

Thus it is appropriate to define a general variety $V$ to be *complete* if it is generated by its class $K_V^+$ of powerset complex algebras. Clearly every complex variety is complete, for if $V = SK^+$, then $K \subseteq K_V$ and so $V = SK_V^+$. An incomplete variety of relation algebras appears in Kramer and Maddux [37]. In the modal case, incompleteness is a measure of the failure of the set-theoretic semantics to differentiate and characterise various logics. The strongest result in this direction is given by Blok [11], who shows that for any variety $V$ of normal modal algebras satisfying $x \sqsubseteq fx$, there are uncountably many varieties $W$ such that $K_V = K_W$, so that $V$ and $W$ are indistinguishable by powerset algebras.

The initial success enjoyed by Kripke semantics was no doubt due to the use of models that were more 'concrete' and tractable than modal algebras, and the fact that so many well-known logics were shown to be determined by elementary classes of frames. Take for example the logic S4, the smallest normal logic containing the schemata

$$\phi \to \Diamond\phi, \tag{6}$$

$$\Diamond\Diamond\phi \to \Diamond\phi. \tag{7}$$

S4 is determined algebraically by the variety $V_{\mathrm{cl}}$ of closure algebras: its Lindenbaum algebra is a closure algebra. Set-theoretically it is determined by the class $K_{\mathrm{qo}}$ of quasi-orders. This could have been deduced from the Jónsson–Tarski result that $V_{\mathrm{cl}} = SK_{\mathrm{qo}}^+$, but, intriguingly, their paper did not initiate the extensive model-theoretic study of modal logic that followed the work of Kripke a decade later (cf. also Dummett and Lemmon [14] for constructions of algebraic S4-models based on ordered sets).

The schema (6), corresponding to the algebraic condition $x \sqsubseteq fx$, is valid in $(X, R)^+$ iff $R$ is reflexive, while (7), corresponding to $ffx \sqsubseteq fx$, is valid iff $R$ is transitive [25, Theorem 3.13]. Hence $K_{\mathrm{qo}}$ is the class of all S4-frames.

By no means all modal axioms have first-order characterisations. A simple example is

$$\Box\Diamond\phi \to \Diamond\Box\phi. \tag{8}$$

The class of frames validating (8) is not elementary (Goldblatt [19], van Benthem [4]). Moreover, if $V_8$ is the variety of normal algebras validating (8), there is no *elementary* class $K$ such that $K^+$ generates $V_8$ (Goldblatt [20, §17]). On the other hand, $V_8$ is complete, and indeed is generated by its finite powerset complex algebras (Fine [18]). It is not known whether $V_8$ is a complex variety.

A similar example concerns the so-called *Gödel–Löb* logic, the smallest normal logic containing

$$\Box(\Box\phi \to \phi) \to \Box\phi. \tag{G}$$

This logic characterises the interpretation of $\Box$ as "it is provable in Peano arithmetic that" (cf. Boolos [12]). The frames validating (G) in the sense of (4) are precisely those in which $R$ is transitive and well-founded—clearly a non-

elementary class. The modal algebras validating (G) are studied under the name of *diagonalisable algebras* (cf. Smorynski [45] for a survey and bibliography). A simple compactness argument shows that the variety $V_d$ of diagonalisable algebras is not generated by $K^+$ for any elementary $K$ (cf. Section 3.7). Moreover this variety is not complex: there exist diagonalisable algebras that cannot be embedded into the algebra of subsets of any transitive well-founded frame (Theorem 3.7.1). However, $V_d$ is complete, and is generated by its finite powerset complex algebras. This follows from the result of Segerberg [44] that the Gödel–Löb logic is determined by its finite frames.

The properties of the last two examples may be contrasted with those of the variety $V_9$ of normal algebras validating

$$\Diamond\Box\phi \to (\Diamond\Box(\phi \wedge \psi) \vee \Diamond\Box(\phi \wedge \neg\psi)). \tag{9}$$

The class $K_{V_9}$ of all those frames that validate (9) is not elementary, but $V_9$ *is* a complex variety, and *does* have an elementary generating class. These results are due to Fine [17], who shows that $V_9 = SK_9^+$, where $K_9$ is the class of all frames satisfying

$$\forall x\, \forall y\, (R(y, x) \to \exists z\, (R(z, x) \wedge \forall u\, \forall v\, (R(u, z) \wedge R(v, z) \to u = v \wedge R(v, y)))).$$

Now, as already mentioned, we will prove in Section 3.6 that for any elementary class $K$, the variety $V_K = \mathrm{HSP}K^+$ it generates is complex. The example $K = K_9$ shows that the clas $\{\mathfrak{S}: \mathfrak{S}^+ \in V_K\}$ of all frames associated with $V_K$ need not itself be elementary.

The question arises as to when a set of modal sentences has an elementary class of frames. Algebraically we are asking

*If $V$ is a variety, when is $K_V = \{\mathfrak{S}: \mathfrak{S}^+ \in V\}$ an elementary class?*

The answer is surprisingly simple: precisely when $K_V$ is closed under ultrapowers. A proof of this is given in Section 3.8 that applies to varieties of Boolean algebras with operators of arbitrary signature.

Our motivational discussion of the modal case may only have been of general significance to those unfamiliar with the subject. The reader is recommended to return to this Introduction from time to time when reading the more technical sections to follow.

*Notation*

We use vectorial notation for finite sequences. Thus if $x$ is a sequence of length $n$, its $i$-th 1-st term is denoted $x_i$, so that $x = (x_0, \ldots, x_{n-1})$. A sequence of length 1 may be identified with its only term, and concatenation is indicated by a comma, so that $(x, y)$ is the sequence $(x_0, \ldots, x_{n-1}, y)$ etc. Operations and relations for sequences are defined from the corresponding operations and

relations for terms in the usual 'direct product' fashion. Thus $x \leqslant X$ means $\forall i < n\ (x_i \leqslant X_i)$, and $\varphi(x)$ denotes $(\varphi(x_0), \ldots, \varphi(x_{n-1}))$, etc.

## 2. Duality

The topological duality for Boolean algebras of Stone [46] has been generalised to distributive lattices in two ways. The approach of Stone himself [44] leads to a representation in terms of spaces that are called *spectral* by Hochster [31] (a spectral space is a compact $T_0$ space in which the compact open sets form a base and are closed under finite intersections, and in which every non-empty irreducible closed subset is the closure of a point). Priestley [41] introduced an alternative approach, using partially ordered spaces, which has been held to be advantageous for applications in providing a representation that is more 'concrete' and 'geometric', being based on a Hausdorff topology related to the partial ordering (cf. Priestley [42] for a discussion of merits).

A spectral-space type duality for operators on distributive lattices is given by Hansoul [29]. In this section we develop the alternative order-theoretic approach to such operators.

### 2.1. *Lattices, structures, and spaces*

Let $\mathfrak{D} = (D, \sqcup, \sqcap, 0, 1)$ be a bounded distributive lattice. A function $f: D^n \to D$ with $n > 0$ is a *join-hemimorphism* if it preserves, in each variable, the join of any finite (possibly empty) subset of $D$, i.e. if the equations

$$f(a_0, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_{n-1}) = 0,$$

$$f(a_0, \ldots, a_{i-1}, x \sqcup y, a_{i+1}, \ldots, a_{n-1})$$
$$= f(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{n-1}) \sqcup f(a_0, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_{n-1}),$$

hold for any $i < n$. Note that any such $f$ is *monotonic* in each variable, i.e.

$$x \sqsubseteq y \quad \text{implies} \quad f(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{n-1}) \sqsubseteq$$
$$f(a_0, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_{n-1}).$$

Dually, $g: D^m \to D$ $(m > 0)$ is a *meet-hemimorphism* if it preserves 1 and $\sqcap$ in each variable, i.e. preserves meets of all finite sets. Meet-hemimorphisms are also monotonic.

For $n = 0$ or $m = 0$, any nullary operation (constant) will be regarded as both a join- and a meet-hemimorphism.

A *distributive lattice with operators* is a bounded distributive lattice carrying a collection of join- and/or meet-hemimorphisms. For the purposes of this exposition an arbitrary pair $(n, m)$ of natural numbers will be fixed, and we study algebras $\mathfrak{A} = (\mathfrak{D}, f, g)$ with $f$ a single $n$-ary join-hemimorphism, and $g$ a single

$m$-ary meet-hemimorphism, on the distributive lattice $\mathfrak{D}$. This will typify the general case.

The name 'hemimorphism' was coined by Halmos [27] for join-preserving functions of Boolean algebras, for the reason that 'roughly speaking, a hemimorphism preserves half the structure of a Boolean algebra'.

**Ordered Structures.** Let $\mathfrak{S} = (X, \leqslant)$ be a partially ordered set. A subset $Y$ of $X$ is called a *cone* if it is closed *upwards* under the partial-ordering, i.e. if

$$y \in Y \quad \text{and} \quad y \leqslant z \text{ implies } z \in Y.$$

The set of cones is denoted $\mathrm{co}(\mathfrak{S})$, and is a complete sublattice of the lattice $(\mathrm{Sb}(X), \cup, \cap)$ of all subsets of $X$.

A relation $R \subseteq X^{n+1}$ is *increasing* if for all $x \in X^n$ and all $y, z \in X$,

$$R(x, y) \quad \text{and} \quad y \leqslant z \text{ implies } R(x, z),$$

i.e. if the set $\{y: R(x, y)\}$ is a cone. A cone itself is thus an *increasing set* (one-placed relation). Reversing the order, a relation $Q \subseteq X^{m+1}$ is *decreasing* if

$$Q(x, y) \quad \text{and} \quad z \leqslant y \text{ implies } Q(x, z).$$

An *ordered relational structure* is a partially ordered set that carries a family of increasing and/or decreasing relations. For this study we deal with structures $\mathfrak{S} = (X, \leqslant, R, Q)$, where $R \subseteq X^{n+1}$ is increasing, and $Q \subseteq X^{m+1}$ is decreasing. $R$ and $Q$ determine hemimorphisms on $\mathrm{co}(\mathfrak{S})$. To define these, it is convenient to introduce the concept of the *sum* $\bigoplus Y$ of a sequence $Y \in \mathrm{Sb}(X)^m$ of subsets of $X$ as the set

$$\{x \in X^m: \text{for some } i < m,\ x_i \in Y_i\}.$$

This is dual to the usual notion of the product $\bigotimes Y$ as the set

$$\{x \in X^m: \text{for all } i < m,\ x_i \in Y_i\}$$

(indeed $-(\bigoplus Y) = \bigotimes(-Y_0, \ldots, -Y_{m-1})$).

The operations $\exists_R: \mathrm{co}(\mathfrak{S})^n \to \mathrm{co}(\mathfrak{S})$ and $\forall_Q: \mathrm{co}(\mathfrak{S})^m \to \mathrm{co}(\mathfrak{S})$ are defined by

$$\exists_R(Y) = \{y \in X: \exists x \in X^n\ (R(x, y) \text{ and } x \in \bigotimes Y)\},$$

$$\forall_Q(Y) = \{y \in X: \forall x \in X^m\ (Q(x, y) \text{ implies } x \in \bigoplus Y)\}.$$

$\exists_R(Y)$ will be called the *existential image* of $Y$, and $\forall_Q(Y)$ the *universal image*. Observe that $\exists_R$ is the operator $f_R$ of (1) of Section 1. Modal logicians are accustomed, in the case $n = 1$, to seeing the places reversed in the definition of $\exists_R(Y)$ as $\{y: \exists x\ (R(y, x)\ \&\ x \in Y\}$. However, as explained in the Introduction, we are following the conventions that lead from the group-theoretic origin to Jónsson and Tarski's general definition of an algebra of complexes.

If, for $y \in X$, the *inverse image* sets $R^{-1}(y) \subseteq X^n$ and $Q^{-1}(y) \subseteq X^m$ are given by

$$R^{-1}(y) = \{x: R(x, y)\}, \qquad Q^{-1}(y) = \{x: Q(x, y)\},$$

then $\exists_R$ and $\forall_Q$ can be defined more succinctly by

$$\exists_R(Y) = \{y: R^{-1}(y) \cap \bigotimes Y \neq \emptyset\},$$
$$\forall_Q(Y) = \{y: Q^{-1}(y) \subseteq \bigoplus Y\}.$$

(Note that if $Y$ is the empty sequence, then $\bigoplus Y = \emptyset$ and $\bigotimes Y = \{\emptyset\}$, so that for the case $n = m = 0$ we get $\exists_R(\emptyset) = R$ and $\forall_Q(\emptyset) = -Q$.)

Since $y \leqslant z$ implies $R^{-1}(y) \subseteq R^{-1}(z)$ etc., it follows readily that $\exists_R$ is a join-hemimorphism, and $\forall_Q$ a meet-hemimorphism, on $\mathrm{co}(\mathfrak{S})$. Hence $\mathfrak{S}^+ = (\mathrm{co}(\mathfrak{S}), \cup, \cap, \emptyset, X, \exists_R, \forall_Q)$ is a distributive lattice with operators.

**Relational Priestley Spaces.** An *ordered topological space* is a triple $\mathfrak{S} = (X, \leqslant, \mathcal{T})$, where $\mathcal{T}$ is a topology on the partially-ordered set $(X, \leqslant)$. The set of *clopen* (closed and open) cones of such a space will be denoted $\mathrm{cl}(\mathfrak{S})$. $\mathrm{cl}(\mathfrak{S})$ is a sublattice of $(\mathrm{co}(\mathfrak{S}), \cup, \cap)$.

An ordered space is *totally order-separated* if, whenever $x \not\leqslant y$, there exists a clopen cone $U \in \mathrm{cl}(\mathfrak{S})$ with $x \in U$ and $y \notin U$. This implies the Hausdorff separation property (Priestley [41] uses 'disconnected' in place of 'separated' here: the present terminology follows that of Johnstone [33]). A *Priestley space* is a totally order-separated space that is also *compact*.

**Lemma 2.1.1.** *In a Priestley space, the clopen cones and their complements form a sub-base.*

**Proof.** Let $S = \mathrm{cl}(\mathfrak{S}) \cup \{-Y: Y \in \mathrm{cl}(\mathfrak{S})\}$. Suppose $Z$ is $\mathcal{T}$-open, and $x \in Z$. Then for each $y \in -Z$, by total order-separation there is a member of $\mathrm{cl}(\mathfrak{S})$ containing one of $x$ and $y$ but not the other. Hence there exists $W_y \in S$ with $y \in W_y$ and $x \notin W_y$. Thus $\{W_y: y \notin Z\}$ is an open cover of the complement $-Z$, which is closed and hence compact. Therefore $-Z \subseteq W_{y_1} \cup \cdots \cup W_{y_p}$, for some $y_i$, with $x \in -W_{y_1} \cap \cdots \cap -W_{y_p} \subseteq Z$. So $Z$ is the union of finite intersections of members of $S$. □

The set $\mathrm{cl}(\mathfrak{S})$ is a base for a topology on $X$ which will be called the *upper* topology. Correspondingly, the topology with $\{-Y: Y \in \mathrm{cl}(\mathfrak{S})\}$ as base is the *lower* topology (this usage of 'upper' and 'lower' differs from that of Priestley [41]). Lemma 2.1.1 implies that the topology of a Priestley space is the join of these two *weak topologies*.

If $\leqslant$ is the *discrete* ordering, i.e. $x \leqslant y$ iff $x = y$, then all subsets of $X$ are cones, and the upper and lower topologies coincide with the topology of the Priestley space: a discretely ordered Priestley space is simply a *Boolean* (or *Stone*) space, i.e. a compact Hausdorff space in which the clopen sets form a base.

An *ordered relational space* is an ordered topological space carrying a collection of increasing and/or decreasing finitary relations. We take $\mathfrak{S} = (X, \leqslant, \mathcal{T}, R, Q)$ here as the typical case. Such a structure is a *relational Priestley space* if

(1) $(X, \leqslant, \mathcal{T})$ is a Priestley space;

(2) *inverse images of points are closed* in the appropriate *weak* topology, i.e. if

$R$ is increasing, then $R^{-1}(y)$ is closed in the product topology on $X^n$ of the upper topology, while if $Q$ is decreasing then $Q^{-1}(y)$ is closed in the product on $X^m$ of the lower topology;

(3) *existential and universal images of sequences of clopen cones are clopen, i.e.*

$$\exists_R(Y), \forall_Q(Y) \in \mathrm{cl}(\mathfrak{S}) \quad \text{if } Y \in \mathrm{cl}(\mathfrak{S})^n \text{ (resp. } \mathrm{cl}(\mathfrak{S})^m).$$

If $\mathfrak{S}$ is an ordered relational space satisfying (3), then $\mathrm{cl}(\mathfrak{S})$ is closed under the operations $\exists_R$ and $\forall_Q$. The algebra

$$\mathfrak{S}^+ = (\mathrm{cl}(\mathfrak{S}), \cup, \cap, \emptyset, X, \exists_R, \forall_Q)$$

is a distributive lattice with operators (and a subalgebra of $\mathrm{co}(\mathfrak{S})$), and will be called the *dual algebra* of $\mathfrak{S}$.

An ordered relational structure $\mathfrak{S} = (X, \leqslant, R, Q)$ will be regarded as an ordered relational space under the *discrete topology* $\mathcal{T} = \mathrm{Sb}(X)$ (making 'structure' synonymous with 'discrete space'). All subsets are then clopen, so $\mathrm{cl}(\mathfrak{S}) = \mathrm{co}(\mathfrak{S})$. Hence our previous use of $\mathfrak{S}^+$ for ordered relational structures is consistent with that now for ordered relational spaces: in all cases $\mathfrak{S}^+$ is the lattice of clopen cones.

Observe further that any relational structure $\mathfrak{S} = (X, R, Q)$ can be ordered by the *discrete ordering* $x \leqslant y$ iff $x = y$, in which case the addition of the discrete topology produces a space that is totally order-separated and satisfies (2) and (3) above, but will be compact iff $X$ is finite. When both $\leqslant$ and $\mathcal{T}$ are discrete we have $\mathrm{cl}(\mathfrak{S}) = \mathrm{co}(\mathfrak{S}) = \mathrm{Sb}(X)$, and the dual algebra $\mathfrak{S}^+ = (\mathrm{Sb}(X), \ldots)$ is a *Boolean* algebra with operators, which we will call the *full complex algebra* of $\mathfrak{S}$ (cf. Section 3).

### *2.2. The dual space of a lattice*

If $\mathfrak{D}$ is a bounded distributive lattice, recall that a *filter* of $\mathfrak{D}$ is a non-empty increasing set $F \subseteq D$ that is closed under the meet operation $\sqcap$. A filter $F$ is *proper* if $0 \notin F$ (equivalently, if $F \neq D$), and *prime* if it is proper and satisfies

$$x \sqcup y \in F \quad \text{implies} \quad x \in F \text{ or } y \in F.$$

(A set $F \subseteq D$ is a prime filter iff its characteristic function $F \to \{0, 1\}$ is a bounded-lattice homomorphism.)

Let $X_{\mathfrak{D}}$ be the set of prime filters of $\mathfrak{D}$, partially ordered by set inclusion. For $x \in D$, let $\mathfrak{D}_x = \{F \in X_{\mathfrak{D}} : x \in F\}$. Then $\mathfrak{D}_1 = X_{\mathfrak{D}}$, $\mathfrak{D}_0 = \emptyset$, $\mathfrak{D}_{x \sqcup y} = \mathfrak{D}_x \cup \mathfrak{D}_y$, and $\mathfrak{D}_{x \sqcap y} = \mathfrak{D}_x \cap \mathfrak{D}_y$, so that the function $\sigma_{\mathfrak{D}} : D \to \mathrm{co}(X_{\mathfrak{D}})$ defined by $\sigma_{\mathfrak{D}}(x) = \mathfrak{D}_x$ is a bounded-lattice homomorphism into the lattice of cones of $(X_{\mathfrak{D}}, \subseteq)$.

To prove that $\sigma_{\mathfrak{D}}$ is injective requires the classical Birkhoff–Stone result on the existence of prime filters, which we employ in the following form. A subset $G$ of $D$ is defined to be *separated* from $H \subseteq D$ if $\bigsqcap I \notin \bigsqcup J$ for all *finite* sets $I$ and $J$ with $I \subseteq G$ and $J \subseteq H$.

**Prime Filter Theorem.** *If $G$ is separated from $H$, then $G$ is contained in a prime filter of $\mathbb{D}$ that is disjoint from $H$.* □

Now if $x \not\leq y$ in $\mathbb{D}$, this theorem implies that there exists $F \in X_{\mathbb{D}}$ with $x \in F$ and $y \notin F$, whence $F \in \mathbb{D}_x - \mathbb{D}_y$, and so $\mathbb{D}_x \neq \mathbb{D}_y$. Hence $\sigma_{\mathbb{D}}$ is injective.

A topology $\mathcal{T}_{\mathbb{D}}$ is defined on $X_{\mathbb{D}}$ by declaring the collection

$$\{\mathbb{D}_x, -\mathbb{D}_x : x \in D\}$$

to be a subbase. This ensures that each cone $\mathbb{D}_x$ is clopen, so that $\sigma_{\mathbb{D}}$ maps $D$ into the distributive lattice $\mathrm{cl}(X_{\mathbb{D}})$.

If $F \not\subseteq G$ in $X_{\mathbb{D}}$, there exists an $x \in F - G$, so that $F \in \mathbb{D}_x$ and $G \notin \mathbb{D}_x$. This shows $(X_{\mathbb{D}}, \subseteq, \mathcal{T}_{\mathbb{D}})$ to be totally order-separated.

Under the correspondence between prime filters and homomorphisms $D \to \{0, 1\}$, $X_{\mathbb{D}}$ corresponds to a closed subspace of the product space $\{0, 1\}^X$ (with the discrete topology on $\{0, 1\}$). Compactness of $\mathcal{T}_{\mathbb{D}}$ then follows by Tychonoff's Theorem. A more direct argument goes as follows. It suffices to show that every subbasic open cover of $X_{\mathbb{D}}$ has a finite subbcover (Alexander Subbase Theorem), so let $\Xi$ be a subbasic cover, and put

$$G = \{x : -\mathbb{D}_x \in \Xi\}, \qquad H = \{x : \mathbb{D}_x \in \Xi\}.$$

If $\Xi$ had no finite subbcover, it would follow that for any finite $I \subseteq G$ and $J \subseteq H$, $\bigcap \{\mathbb{D}_x : x \in I\} \not\subseteq \bigcup \{\mathbb{D}_x : x \in J\}$, and so $\bigsqcap I \not\sqsubseteq \bigsqcup J$. But then the Prime Filter Theorem would imply the existence of an $F \in X_{\mathbb{D}}$ with $G \subseteq F$ and $F \cap H = \emptyset$, making $F \notin \bigcup \Xi$, contrary to $\Xi$ being a cover.

Compactness can now be used to show that $\sigma_{\mathbb{D}}$ maps *onto* $\mathrm{cl}(X_{\mathbb{D}})$ (Priestley [41, p. 187]). Let $Y$ be a clopen cone, and take $G \notin Y$. Then for each $F \in Y$, $F \not\subseteq G$ as $Y$ is increasing, and so there is some $x_F$ with $F \in \sigma_{\mathbb{D}}(x_F)$ and $G \notin \sigma_{\mathbb{D}}(x_F)$. Then $\{\sigma_{\mathbb{D}}(x_F) : F \in Y\}$ is an open cover of $Y$, which is closed and hence compact. Thus $\{\sigma_{\mathbb{D}}(x_F) : F \in Y_0\}$ covers $Y$ for some finite $Y_0 \subseteq Y$. Putting $x_G = \bigsqcup Y_0$ gives $Y \subseteq \sigma_{\mathbb{D}}(x_G)$ and $G \notin \sigma_{\mathbb{D}}(x_G)$. Thus $-Y = \bigcup \{-\sigma_{\mathbb{D}}(x_G) : G \in -Y\}$. But $-Y$ is also closed, so $-Y = \bigcup \{-\sigma_{\mathbb{D}}(x_G) : G \in Y_1\}$ for some finite $Y_1 \subseteq -Y$. Hence

$$Y = \bigcap \{\sigma_{\mathbb{D}}(x_G) : G \in Y_1\} = \sigma_{\mathbb{D}}(\bigsqcap Y_1).$$

We have now reconstructed the Priestley representation of a bounded distributive lattice $\mathbb{D}$ as being isomorphic to the lattice of clopen cones of its *dual space* $(X_{\mathbb{D}}, \subseteq, \mathcal{T}_{\mathbb{D}})$. Our main aim is to extend this representation to hemimorphisms on $\mathbb{D}$.

Let $\mathfrak{A} = (\mathbb{D}, f, g)$ with $f : D^n \to D$ a join-hemimorphism, and $g : D^m \to D$ a meet-hemimorphism. Define $R_f \subseteq X_{\mathbb{D}}^{n+1}$ and $Q_g \subseteq X_{\mathbb{D}}^{m+1}$ by

$$\begin{aligned} R_f(G, F) \quad &\text{iff} \quad f(\otimes G) \subseteq F \\ &\text{iff} \quad (\forall i < n\ (x_i \in G_i)) \text{ implies } f(x) \in F, \quad \text{and} \\ Q_g(G, F) \quad &\text{iff} \quad g^{-1}(F) \subseteq \oplus G \\ &\text{iff} \quad g(x) \in F \text{ implies } \exists i < m\ (x_i \in G_i). \end{aligned}$$

In the case $n = m = 0$, these become

$$R_f(F) \text{ iff } f \in F, \text{ and } Q_g(F) \text{ iff } g \notin F,$$

where the nullary operations $f$ and $g$ are identified with members of $D$.

In general, $R_f$ is increasing and $Q_g$ is decreasing on $(X_{\mathfrak{D}}, \subseteq)$, so that the structure

$$\mathfrak{A}_+ = (X_{\mathfrak{D}}, \subseteq, \mathcal{T}_{\mathfrak{D}}, R_f, Q_g)$$

is a totally order-separated compact ordered relational space, called the *dual space* of $\mathfrak{A}$. In order to show that $\mathfrak{A}_+$ is a relational Priestley space, whose dual algebra $(\mathfrak{A}_+)^+$ is isomorphic to $\mathfrak{A}$ under $\sigma_{\mathfrak{D}}$, it is necessary to know that $\sigma_{\mathfrak{D}}$ preserves the two hemimorphisms, i.e. that

(1) $\sigma_{\mathfrak{D}}(f(x)) = \exists_{R_f}(\sigma_{\mathfrak{D}}(x))$, for $x \in D^n$, and

(2) $\sigma_{\mathfrak{D}}(g(x)) = \forall_{Q_g}(\sigma_{\mathfrak{D}}(x))$, for $x \in D^m$.

Since $\mathrm{cl}(X_{\mathfrak{D}})$ is the image of $D$ under $\sigma_{\mathfrak{D}}$, these equations immediately imply that images of clopen cones are clopen, i.e. $\exists_{R_f}$ and $\forall_{Q_g}$ are operations on $\mathrm{cl}(X_{\mathfrak{D}})$. By definition of $\sigma_{\mathfrak{D}}$, (1) and (2) become

(3) $\mathfrak{D}_{f(x)} = \exists_{R_f}(\mathfrak{D}_{x_0}, \ldots, \mathfrak{D}_{x_{n-1}})$, and

(4) $\mathfrak{D}_{g(x)} = \forall_{Q_g}(\mathfrak{D}_{x_0}, \ldots, \mathfrak{D}_{x_{m-1}})$.

In view of the definitions of existential and universal images it is thus enough, in order to establish (1) and (2), to prove the following.

**Theorem 2.2.1.** (1) *For any $x \in D^n$ and $F \in X_{\mathfrak{D}}$,*

$$f(x) \in F \text{ iff } \exists G \in X^n_{\mathfrak{D}}\,(R_f(G, F) \text{ and } x \in \bigotimes G).$$

(2) *For any $x \in D^m$ and $F \in X_{\mathfrak{D}}$,*

$$g(x) \in F \text{ iff } \forall G \in X^m_{\mathfrak{D}}\,(Q_g(G, F) \text{ implies } x \in \bigoplus G).$$

**Proof.** In the case $n = m = 0$, the theorem requires

$$f \in F \text{ iff } R_f(F), \text{ and } g \in F \text{ iff not } Q_g(F),$$

which, as noted above, hold by definition of $R_f$ and $Q_g$. Hence we can assume $n, m \geq 1$.

(1): If $R_f(G, F)$ and $x \in \bigotimes G$, then $f(\bigotimes G) \subseteq F$, so $f(x) \in F$. Conversely, suppose $f(x) \in F$. The desired sequence $G$ is constructed inductively in such a way that for each $i < n$, the following two clauses hold:

(i) $x_i \in G_i$,

(ii) if $y_k \in G_k$ for all $k \leq i$, then $f(y_0, \ldots, y_i, x_{i+1}, \ldots, x_{n-1}) \in F$.

Then when $i = n - 1$, clause (ii) immediately yields $R_f(G, F)$ by definition of $R_f$. Together with (i), this will complete the proof of (1).

So, fix a $j \leq n - 1$, and suppose inductively that for each $i < j$, $G_i$ has been

defined to satisfy (i) and (ii). Then $G_j$ will be defined such that (i) and (ii) hold with $j$ in place of $i$ (the case $j=0$ is included here, so the following argument covers the construction of $G_0$). Let

$$H_j = \{z: \forall i < j\, \exists y_i \in G_i\, (f(y_0, \ldots, y_{j-1}, z, x_{j+1}, \ldots, x_{n-1}) \notin F)\}.$$

This definition is intended in the sense that

$$H_0 = \{z: f(z, x_1, \ldots, x_{n-1}) \notin F\},$$

and similar observations apply to the case of $j=0$ in what follows.

$H_j$ is closed under joins. For, if $z_0, z_1 \in H_j$, then for each $r \in \{0, 1\}$ there exists $y_i^r \in G_i$ for all $i<j$ such that

$$f(y_0^r, \ldots, y_{j-1}^r, z_r, x_{j+1}, \ldots, x_{n-1}) \notin F.$$

Hence

$$f(y_0, \ldots, y_{j-1}, z_r, x_{j+1}, \ldots, x_{n-1}) \notin F,$$

where $y_i = y_i^0 \sqcap y_i^1 \in G_i$. this follows because $y_i \sqsubseteq y_i^r$, $f$ is monotonic in each variable, and $F$ is increasing.

Since $F$ is prime, it follows that

$$f(y_0, \ldots, y_{j-1}, z_0, x_{j+1}, \ldots, x_{n-1}) \sqcup f(y_0, \ldots, y_{j-1}, z_1, x_{j+1}, \ldots, x_{n-1}) \notin F,$$

and so as $f$ preserves $\sqcup$ in each variable,

$$f(y_0, \ldots, y_{j-1}, z_0 \sqcup z_1, x_{j+1}, \ldots, x_{n-1}) \notin F.$$

This shows that $z_0 \sqcup z_1 \in H_j$, as desired. Notice also that $0 \in H_j$, because $f$ preserves 0 and so $f(1, \ldots, 1, 0, x_{j+1}, \ldots, x_{n-1}) = 0 \notin F$. Hence $\bigsqcup J \in H_j$ for any finite $J \subseteq H_j$, including the case $J = \emptyset$.

Next it will be shown that $\{x_j\}$ is separated from $H_j$. For, if not, then $\bigsqcap I \sqsubseteq \bigsqcup J$ for some finite $I \subseteq \{x_j\}$ and $J \subseteq H_j$. Since $\bigsqcap I$ is either $x_j$ or 1, according as $I$ is $\{x_j\}$ or $\emptyset$, it follows that $x_j \sqsubseteq \bigsqcup J$. But $\bigsqcup J \in H_j$, so

$$f(y_0, \ldots, y_{j-1}, \bigsqcup J, x_{j+1}, \ldots, x_{n-1}) \notin F$$

for some $y_i \in G_i$. By monotonicity, it follows that

$$f(y_0, \ldots, y_{j-1}, x_j, x_{j+1}, \ldots, x_{n-1}) \notin F.$$

However if $j>0$, this contradicts the inductive hypothesis (ii) on $j-1$, while if $j=0$ it contradicts the assumption that $f(x) \in F$.

Thus $\{x_j\}$ is separated from $H_j$, and so by the Prime Filter Theorem there exists $G_j \in X_{\mathfrak{D}}$ such that $\{x_j\} \subseteq G_j$ and $H_j \cap G_j = \emptyset$. Then (i) $x_j \in G_j$, and (ii) if $y_k \in G_k$ for all $k \leqslant j$, we have $f(y_0, \ldots, y_j, x_{j+1}, \ldots, x_{n-1}) \in F$, or else $y_j \in H_j \cap G_j$. Hence clauses (i) and (ii) hold for $j$, as desired.

(2): If $g(x) \in F$, then whenever $Q_g(G, F)$, $g^{-1}(F) \subseteq \bigoplus G$ and so $x \in \bigoplus G$ as desired. Conversely, if $g(x) \notin F$, a sequence $G \in X_{\mathfrak{D}}^m$ is to be constructed such that $Q_g(G, F)$ and $x \notin \bigoplus G$, i.e. $x_i \notin G_i$ for all $i<m$. The inductive clauses for $G_i$ are

this time

(iii) $x_i \notin G_i$,

(iv) if $y_k \notin G_k$ for all $k \leq i$, then $g(y_0, \ldots, y_i, x_{i+1}, \ldots, x_{m-1}) \notin F$.

The proof is then completed by putting $i = m - 1$.

Fixing $j \leq m - 1$ and supposing that $G_i$ satisfying (iii) and (iv) has been defined for all $i < j$, put

$$E_j = \{z: \forall i < j \; \exists y_i \notin G_i \, (g(y_0, \ldots, y_{j-1}, z, x_{j+1}, \ldots, x_{m-1}) \in F)\}.$$

Then dualising the argument used for $H_j$ in (1), the meet-hemimorphism properties of $g$ may be used to show that $E_j$ is closed under finite meets, including $\bigsqcap \emptyset$. From this it follows that $E_j$ is separated from $\{x_j\}$. For if not, then $\bigsqcap I \sqsubseteq x_j$ for some finite $I \subseteq E_j$. Then $\bigsqcap I \in E_j$, giving

$$g(y_0, \ldots, y_{j-1}, \bigsqcap I, x_{j+1}, \ldots, x_{m-1}) \in F,$$

and hence

$$g(y_0, \ldots, y_{j-1}, x_j, x_{j+1}, \ldots, x_{m-1}) \in F$$

for some $y_i \notin G_i$. But this contradicts the inductive hypothesis on $j - 1$ when $j > 0$, and contradicts the fact that $g(x) \notin F$ when $j = 0$.

Hence there exists $G_j \in X_{\mathfrak{D}}$ with $E_j \subseteq G_j$ and $x_j \notin G_j$. Then if $y_k \notin G_k$ for all $k \leq j$, we have $y_j \notin G_j$, whence $y_j \notin E_j$, and so $g(y_0, \ldots, y_j, x_{j+1}, \ldots, x_{m-1}) \notin F$.

Thus (2) is proven by dualising the proof of (1). A 'slicker' (but not simpler) version of this duality could be developed by applying the proof of (1) to the dual lattice $\mathfrak{D}^{op}$ of $\mathfrak{D}$. $\mathfrak{D}^{op}$ has the same elements as $\mathfrak{D}$, but the reverse partial ordering, i.e. $x \sqsubseteq y$ in $\mathfrak{D}^{op}$ iff $y \sqsubseteq x$ in $\mathfrak{D}$. Joins and meets in $\mathfrak{D}^{op}$ are meets and joins in $\mathfrak{D}$, respectively. $F$ is a prime filter in $\mathfrak{D}$ iff its set complement $-F$ is a prime filter in $\mathfrak{D}^{op}$. The meet-hemimorphism $g$ becomes a join-hemimorphism on $\mathfrak{D}^{op}$. Thus if $g(x) \notin F$, $g(x) \in (-F) \in X_{\mathfrak{D}^{op}}$. Applying the proof of (1) in $\mathfrak{D}^{op}$, there exists $G' \in X^m_{\mathfrak{D}^{op}}$ with $x_i \in G'_i$ for all $i < m$ and $R_g(G', -F)$, where $R_g$ is the increasing relation on $X_{\mathfrak{D}^{op}}$ defined by $g$ as a join-hemimorphism. Putting $G_i = -G'_i$ gives $G \in X^m_{\mathfrak{D}}$, $Q_g(G, F)$, and $x_i \notin G_i$ for all $i < m$ as desired. □

**Theorem 2.2.2.** *In $\mathfrak{A}_+$, inverse images of points are closed in the appropriate weak topology.*

**Proof.** Let $F \in X_{\mathfrak{D}}$.

(1): To show $R_f^{-1}(F)$ is closed in the product of the upper topology on $X^n_{\mathfrak{D}}$, suppose $G \in X^n_{\mathfrak{D}}$ has $G \notin R_f^{-1}(F)$. Then not $R_f(G, F)$, so for some $x \in \mathfrak{D}^n$, $x_i \in G_i$ for all $i < n$, but $f(x) \notin F$. Then $G_i \in \mathfrak{D}_{x_i}$, so $N = \bigotimes (\mathfrak{D}_{x_0}, \ldots, \mathfrak{D}_{x_{n-1}})$ is an open neighbourhood of $G$ in the product of the upper topology, since each $\mathfrak{D}_{x_i}$ is a clopen cone in $(X_{\mathfrak{D}}, \subseteq)$. Moreover, if $Z \in N$, then $x_i \in Z_i$ for all $i < n$, so that not $R_f(Z, F)$ since $f(x) \notin F$, and hence $Z \notin R_f^{-1}(F)$. Thus we have shown that any $G$ not in $R_f^{-1}(F)$ has an open neighbourhood $N$ disjoint from $R_f^{-1}(F)$, i.e. $R_f^{-1}(F)$ is closed.

(2): To show $Q_g^{-1}(F)$ is closed in the product of the lower topology on $X_{\mathfrak{D}}^m$, suppose that $G \in X_{\mathfrak{D}}^m - Q_g^{-1}(F)$. Then there exists $x \in D^m$ with $g(x) \in F$ but $x_i \notin G_i$ for all $i < m$. Proceeding dually to (1) to put $N = \bigotimes((-\mathfrak{D}_{x_0}), \ldots, (-\mathfrak{D}_{x_{m-1}}))$ gives $N$ as an open neighbourhood of $G$ in the product of the lower topology that is disjoint from $Q_g^{-1}(F)$. □

Theorem 2.2.2 establishes condition (3) in the definition of a relational Priestley space. Since 2.2.1 showed that $\sigma_{\mathfrak{D}}$ preserves the hemimorphisms, we have now finished the demonstration of

**Theorem 2.2.3.** *If $\mathfrak{A} = (\mathfrak{D}, \ldots)$ is a distributive lattice with operators, then the dual space $\mathfrak{A}_+$ is a relational Priestley space, and*

$$\sigma_{\mathfrak{A}} = \sigma_{\mathfrak{D}} : \mathfrak{A} \to (\mathfrak{A}_+)^+$$

*is an isomorphism.* □

The other side of this coin is that any relational Priestley space $\mathfrak{S} = (X, \leqslant, \mathcal{T}, R, Q)$ is 'equivalent' to the dual space of its dual algebra. Given such an $\mathfrak{S}$, for each $x \in X$ let $F_x = \{Y \in \mathrm{cl}(\mathfrak{S}) : x \in Y\}$. Then $F_x$ is a prime filter of the lattice of clopen cones of $\mathfrak{S}$, so $F_x$ is a point in $(\mathfrak{S}^+)_+$. Put $\tau_{\mathfrak{S}}(x) = F_x$.

**Theorem 2.2.4.** *$\tau_{\mathfrak{S}} : \mathfrak{S} \to (\mathfrak{S}^+)_+$ is a homeomorphism of topological spaces, and an isomorphism of ordered relational structures.*

**Proof.** To show $\tau_{\mathfrak{S}}$ surjective, let $F \in X_{\mathrm{cl}(\mathfrak{S})}$ be any point in $(\mathfrak{S}^+)_+$, so that $F$ is a prime filter of $\mathrm{cl}(\mathfrak{S})$. Suppose, in order to derive a contradiction, that $F \neq F_x$ for all $x \in X$. Then for each $x$ there exists a $Y_x \in \mathrm{cl}(\mathfrak{S})$ such that either

(i) $Y_x \in F_x - F$, i.e. $x \in Y_x$ and $Y_x \notin F$; or else

(ii) $Y_x \in F - F_x$, i.e. $x \notin Y_x$ and $Y_x \in F$.

Let

$$\Xi = \{Y_x : \text{(i) holds}\} \cup \{-Y_x : \text{(ii) holds}\}.$$

Then since each $Y_x$ is clopen, $\Xi$ is an open cover of $X$. But $\mathfrak{S}$ is compact, so $\Xi$ reduces to a finite subcover of the form

$$\{Y_{x_1}, \ldots, Y_{x_p}, -Y_{y_1}, \ldots, -Y_{y_r}\},$$

where the $x_i$'s fall under case (i), and the $y_j$'s under case (ii). Then

$$\bigcap \{Y_{y_j} : 1 \leqslant j \leqslant r\} \subseteq \bigcup \{Y_{x_i} : 1 \leqslant i \leqslant p\}.$$

But each $Y_{y_j}$ is in $F$, and $F$ is a filter, so $\bigcup \{Y_{x_i} : 1 \leqslant i \leqslant p\} \in F$. Since $F$ is prime, it follows that $Y_{x_i} \in F$ for some $1 \leqslant i \leqslant p$, which contradicts case (i). Hence $F = F_x = \tau_{\mathfrak{S}}(x)$ for some $x$.

Now if $x \leqslant y$, then since each member of $\mathrm{cl}(\mathfrak{S})$ is increasing, $x \in Y \in \mathrm{cl}(\mathfrak{S})$ only if $y \in Y$, so $F_x \subseteq F_y$. Conversely, if $x \not\leqslant y$ then by total order-separation there

exists a $Y \in \mathrm{cl}(\mathfrak{S})$ with $x \in Y$ and $y \notin Y$. Hence

$$x \leqslant y \quad \text{iff} \quad F_x \subseteq F_y.$$

Consequently, antisymmetry of $\leqslant$ implies that $F_x = F_y$ only if $x = y$. At this point we have established that $\tau_{\mathfrak{S}}$ is an order isomorphism between $(X, \leqslant)$ and $(X_{\mathrm{cl}(\mathfrak{S})}, \subseteq)$.

To show that $\tau_{\mathfrak{S}}$ is a homeomorphism, let $U$ be a subbasic open set in $(\mathfrak{S}^+)_+$. Then for some $Y \in \mathrm{cl}(\mathfrak{S})$, $U$ is either the set $\{F \in X_{\mathrm{cl}(\mathfrak{S})}: Y \in F\}$ or its complement. In the first case $\tau_{\mathfrak{S}}^{-1}(U) = \{x \in X: Y \in F_x\} = Y$, while in the second $\tau_{\mathfrak{S}}^{-1}(U) = \{x: Y \notin F_x\} = -Y$. Since $Y$ is clopen, in either case $\tau_{\mathfrak{S}}^{-1}(U)$ is open. Hence the bijection $\tau_{\mathfrak{S}}$ is continuous. To see that it has a continuous inverse, recall that the clopen cones and their complements form a subbase for the topology on $\mathfrak{S}$ (Lemma 2.1.1). But if $Y \in \mathrm{cl}(\mathfrak{S})$, then $\tau_{\mathfrak{S}}(Y) = \{F_x: x \in Y\} = \{F_x: Y \in F_x\}$, which is a subbasic open in $\mathcal{T}_{\mathrm{cl}(\mathfrak{S})}$, since every point in $X_{\mathrm{cl}(\mathfrak{S})}$ is of the form $F_x$. Similarly, $\tau_{\mathfrak{S}}(-Y)$ is the $\mathcal{T}_{\mathrm{cl}(\mathfrak{S})}$-open set $\{F_x: Y \notin F_x\}$.

Finally, it has to be shown that the relations $R$ and $Q$ are invariant under $\tau_{\mathfrak{S}}$, i.e. if $x \in X^{n+1}$, then

$$R(x) \quad \text{iff} \quad R_{\exists_R}(F_{x_0}, \ldots, F_{x_n}),$$

and similarly for $Q$.

First, suppose $R(x)$. Then for any $Y \in \mathrm{cl}(\mathfrak{S})^n$, if $Y \in \bigotimes(F_{x_0}, \ldots, F_{x_{n-1}})$, then $(x_0, \ldots, x_{n-1}) \in \bigotimes(Y)$, so $x_n \in \exists_R(Y)$, by definition of $\exists_R$, hence $\exists_R(Y) \in F_{x_n}$. This shows $R_{\exists_R}(F_{x_0}, \ldots, F_{x_n})$.

For the converse, suppose that not $R(x)$. Then $(x_0, \ldots, x_{n-1}) \notin R^{-1}(x_n)$. But $R^{-1}(x_n)$ is closed in the product of the upper topology on $X^n$, so there is some basic open neighbourhood $N$ of $(x_0, \ldots, x_{n-1})$ in this topology that is disjoint from $R^{-1}(x_n)$. By definition of the product topology, $N = \bigotimes(Y)$ for some $Y \in \mathrm{cl}(\mathfrak{S})^n$. Then $R^{-1}(x_n) \cap \bigotimes(Y) = \emptyset$, so $x_n \notin \exists_R(Y)$ and hence $\exists_R(Y) \notin F_{x_n}$. But $(x_0, \ldots, x_{n-1}) \in N$, so $Y \in \bigotimes(F_{x_0}, \ldots, F_{x_{n-1}})$. This shows that not $R_{\exists_R}(F_{x_0}, \ldots, F_{x_n})$.

It remains to show that if $x \in X^{m+1}$, then

$$Q(x) \quad \text{iff} \quad Q_{\forall_Q}(F_{x_0}, \ldots, F_{x_{m-1}}).$$

First, suppose that $Q(x)$. Then for any $Y \in \mathrm{cl}(\mathfrak{S})^m$, if $\forall_Q(Y) \in F_{x_m}$ then $x_m \in \forall_Q(Y)$, so it follows that $(x_0, \ldots, x_{m-1}) \in \bigoplus(Y)$, whence $Y \in \bigoplus(F_{x_0}, \ldots, F_{x_{m-1}})$. This shows $Q_{\forall_Q}(F_{x_0}, \ldots, F_{x_{m-1}})$. Conversely, if not $Q(x)$, then there is some basic open neighbourhood $N$ of $(x_0, \ldots, x_{m-1})$ in the product of the lower topology on $X^m$ that is disjoint from $Q^{-1}(x_m)$. Then for some $Y \in \mathrm{cl}(\mathfrak{S})^m$, $N = \bigotimes(-Y_0, \ldots, -Y_{m-1}) = -(\bigoplus Y)$. Hence $Q^{-1}(x_m) \subseteq \bigoplus(Y)$, implying that $x_m \in \forall_Q(Y)$, i.e. $\forall_Q(Y) \in F_{x_m}$. But since $(x_0, \ldots, x_{m-1}) \notin \bigoplus(Y)$, $Y \notin \bigoplus(F_{x_0}, \ldots, F_{x_{m-1}})$. Hence not $Q_{\forall_Q}(F_{x_0}, \ldots, F_{x_{m-1}})$. □

Observe that if the distributive lattice $\mathfrak{D}$ is a Boolean algebra, then the

ordering on $X_{\mathfrak{A}}$ is discrete, as all prime ideals are maximal, and the representation developed here becomes a topological refinement of the Jónsson–Tarski theory for Boolean algebras with operators. The upper and lower topologies on $X_{\mathfrak{A}}$ both coincide with $\mathcal{T}_{\mathfrak{A}}$ in this case, and the requirement that $R^{-1}(y)$ be closed in $X^n$ for all $y$ become equivalent to the condition that $R$ be a closed subset of $X^{n+1}$, with the corresponding statement holding for $Q$.

### 2.3. *Bounded morphisms*

For a fixed pair $(n, m)$, let **DLO** be the category whose objects are the distributive lattices with operators $\mathfrak{A} = (\mathfrak{D}, f, g)$ as described in Section 2.1, with homomorphisms as arrows. We now develop a notion of arrow between ordered relational spaces that makes the class of relational Priestley spaces into a category **RSP** that is dual to **DLO**.

Let $\mathfrak{S}_i = (X_i, \leqslant, R_i, Q_i)$, for $i = 0, 1$, be ordered relational structures, with $R_i$ increasing $n+1$-ary and $Q_i$ decreasing $m+1$-ary, as in Sections 2.1 and 2.2. A map $\varphi: X_1 \to X_2$ is a *morphism* if it preserves the three relations of $\mathfrak{S}_1$ in passing to $\mathfrak{S}_2$, i.e.

(1) $x \leqslant y$ implies $\varphi(x) \leqslant \varphi(y)$;

(2) $R_1(x)$ implies $R_2(\varphi(x))$, for all $x \in X_1^{n+1}$;

(3) $Q_1(x)$ implies $Q_2(\varphi(x))$ for all $x \in X_1^{m+1}$

(recall that $\varphi(x) = (\varphi(x_0), \ldots, \varphi(x_{n-1}))$ etc.). A function satisfying (1) is *monotone*.

A morphism is *bounded* if for all $z \in X_1$ it satisfies

(4) $R_2(y, \varphi(z))$ implies $\exists x \in X_1^n\,(R_1(x, z)$ and $y \leqslant \varphi(x))$, for $y \in X_2^n$;

(5) $Q_2(y, \varphi(z))$ implies $\exists x \in X_1^m\,(Q_1(x, z)$ and $\varphi(x) \leqslant y)$, for $y \in X_2^m$

(recall $y \leqslant \varphi(x)$ means that $y_i \leqslant \varphi(x_i)$ for all $i < n$ etc.)

Note that when $n = 0$, the condition (4) becomes

$$R_2(\varphi(z)) \quad \text{implies} \quad R_1(z),$$

and similarly for the $Q_i$ when $m = 0$. In the case that $n = 1$ and the orderings $\leqslant$ are discrete, the condition becomes

$$R_2(y, \varphi(z)) \quad \text{implies} \quad \exists x \in X_1\,(R_1(x, z) \text{ and } y = \varphi(x)),$$

(and similarly for the $Q_i$), which the modal logician will recognise as the key condition in the definition of *p-morphism* (Segerberg [44]). The name 'bounded' is suggested by the use of bounded quantifiers, both in (4) and (5), and in the syntactic characterisation of the class of first-order sentences that are preserved by these morphisms (cf. Section 4.2).

A bounded morphism $\varphi: \mathfrak{S}_1 \to \mathfrak{S}_2$ induces the map $\varphi^+: \mathfrak{S}_2^+ \to \mathfrak{S}_1^+$, in the reverse direction, where $\varphi^+(Y) = \varphi^{-1}(Y) = \{x \in X_1: \varphi(x) \in Y\}$, for each $Y \in \mathrm{co}(\mathfrak{S}_2)$.

**Theorem 2.3.1.** (1) *$\varphi^+$ is a homomorphism of algebras.*
(2) *If $\varphi$ is surjective, $\varphi^+$ is injective.*
(3) *If $\varphi$ is injective, $\varphi^+$ is surjective.*

**Proof.** To begin with, we have to check that $\varphi^{-1}$ takes cones to cones, in order that it map $\mathfrak{S}_2^+$ into $\mathfrak{S}_1^+$. But if $x \in \varphi^{-1}(Y)$ and $x \leqslant y$, then $\varphi(x) \in Y$, and $\varphi(x) \leqslant \varphi(y)$ since $\varphi$ is monotone (condition (1) in the definition of 'bounded morphism'), giving $\varphi(y) \in Y$, and hence $y \in \varphi^{-1}(Y)$, if $Y$ is a cone.

That $\varphi^+$ preserves the bounded-lattice structure is standard set theory, as are parts (2) and (3) of the theorem. The essence then is to show preservation of the hemimorphisms.

First, to show that

$$\varphi^{-1}(\exists_{R_2}(Y)) = \exists_{R_1}(\varphi^{-1}(Y)),$$

for any $Y \in \mathrm{co}(\mathfrak{S}_2)^n$, let $z \in \varphi^{-1}(\exists_{R_2}(Y))$. Then $\varphi(z) \in \exists_{R_2}(Y)$, so $R_2(y, \varphi(z))$ for some $y \in \bigotimes(Y)$. Hence by clause (4) of the definition of bounded morphisms, there exists $x \in X_1^n$ with $R_1(x, z)$ and $y \leqslant \varphi(x)$. But then for each $i < n$ we have $\varphi(x_i) \in Y_i$, since $y_i \in Y_i$, $y_i \leqslant \varphi(x_i)$, and $Y_i$ is a cone. Thus $x \in \bigotimes(\varphi^{-1}(Y))$, making $z \in \exists_{R_1}(\varphi^{-1}(Y))$. Conversely, if there exists $x \in \bigotimes(\varphi^{-1}(Y))$ such that $R_1(x, z)$, then $R_2(\varphi(x), \varphi(z))$, since $\varphi$ is a morphism, and $\varphi(x) \in \bigotimes(Y)$, so $\varphi(z) \in \exists_{R_2}(Y)$ and thus $z \in \varphi^{-1}(\exists_{R_2}(Y))$, as desired.

Finally, to show that

$$\varphi^{-1}(\forall_{Q_2}(Y)) = \forall_{Q_1}(\varphi^{-1}(Y)),$$

let $z \in \varphi^{-1}(\forall_{Q_2}(Y))$, with $Y \in \mathrm{co}(\mathfrak{S}_2)^m$. Then if $Q_1(x, z)$, we have $Q_2(\varphi(x), \varphi(z))$ since $\varphi$ is a morphism, so $\varphi(x) \in \bigoplus(Y)$ since $\varphi(z) \in \forall_{Q_2}(Y)$, hence $\varphi(x_i) \in Y_i$ for some $i < m$, giving $x_i \in \varphi^{-1}(Y_i)$ and therefore $x \in \bigoplus \varphi^{-1}(Y)$. This shows $z \in \forall_{Q_1}(\varphi^{-1}(Y))$. Conversely, if the latter holds, then if $Q_2(y, \varphi(z))$ we have some $x$ with $\varphi(x) \leqslant y$ and $Q_1(x, z)$ since $\varphi$ is bounded, so $x \in \bigoplus(\varphi^{-1}(Y))$ as $z \in \forall_{Q_1}(\varphi^{-1}(Y))$, giving some $i$ such that $\varphi(x_i) \in Y_i$, then $y_i \in Y_i$ as $Y_i$ is a cone and $\varphi(x_i) \leqslant y_i$, making $y \in \bigoplus Y$. This shows that $\varphi(z) \in \forall_{Q_2}(Y)$. □

If the $\mathfrak{S}_i$ are ordered relational *spaces*, then if $\varphi^+$ is to map $\mathfrak{S}_2^+$ into $\mathfrak{S}_1^+$ it must take clopen cones in $\mathfrak{S}_2$ to clopen cones in $\mathfrak{S}_1$. A sufficient condition for this is that $\varphi$ be *continuous*, since the continuous inverse image of a clopen set is clopen. If the $\mathfrak{S}_i$ are relational Priestley spaces, this condition is also necessary, since the clopen cones and their complements form a subbase for the topology in each space (Lemma 2.1.1) and if $\varphi^+$ preserves clopen cones it will also preserve their complements (using $-\varphi^{-1}(Y) = \varphi^{-1}(-Y)$). But continuity follows if the inverse images of all members of a subbase are open.

Thus if **RPS** is the category of relational Priestley spaces with *continuous bounded morphisms* as arrows, putting $\Phi(\mathfrak{S}) = \mathfrak{S}^+$ and $\Phi(\varphi) = \varphi^+$ defines a functor $\Phi$: **RPS** → **DLO** that is *contravariant*, since in general $(\Phi_2 \circ \Phi_1)^+ = \Phi_1^+ \circ \Phi_2^+$.

A functor in the opposite direction is given as follows. Let $\mathfrak{A}_i = (\mathfrak{D}_i, f_i, g_i)$, for $i = 0, 1$ be DLO-objects with dual spaces $\mathfrak{A}_{i+} = (X_{\mathfrak{D}_i}, \subseteq, \mathcal{T}_{\mathfrak{D}_i}, R_{f_i}, Q_{g_i})$ (Section 2.2). Let $\theta: \mathfrak{A}_1 \to \mathfrak{A}_2$ be a homomorphism of algebras. For each prime filter $F$ of $\mathfrak{D}_2$, let

$$\theta_+(F) = \theta^{-1}(F) = \{x \in D_1: \theta(x) \in F\}.$$

**Theorem 2.3.2.** (1) *$\theta_+$ is a continuous bounded morphism $\mathfrak{A}_{2+} \to \mathfrak{A}_{1+}$.*
(2) *If $\theta$ is surjective, $\theta_+$ is injective.*
(3) *If $\theta$ is injective, $\theta_+$ is surjective.*

**Proof.** (1): The definition of $\theta_+$ is standard in the representation theory of distributive lattices (cf. e.g. Balbes and Dwinger [1, §IV.2]). That $\theta^{-1}(F)$ is a prime ideal in $\mathfrak{D}_1$ follows because $\theta$ preserves $\sqcup$, $\sqcap$, 0 and 1, and $F$ is a prime ideal. Hence $\theta_+: X_{\mathfrak{D}_2} \to X_{\mathfrak{D}_1}$ (cf. Section 2.2 for notation).

Now the subbasic sets in $\mathfrak{A}_{i+}$ are of the form $(\mathfrak{D}_i)_x = \{F \in X_{\mathfrak{D}_i}: x \in F\}$ and $-(\mathfrak{D}_i)_x$. But for $x \in D_1$,

$$(\theta_+)^{-1}((\mathfrak{D}_1)_x) = \{F \in X_{D_2}: \theta_+(F) \in (\mathfrak{D}_1)_x\} = \{F: \theta(x) \in F\} = (\mathfrak{D}_2)_{\theta(x)},$$

and, similarly,

$$(\theta_+)^{-1}(-(\mathfrak{D}_1)_x) = \{F \in X_{D_2}: \theta(x) \notin F\} = -(\mathfrak{D}_2)_{\theta(x)},$$

which shows that $\theta_+$ is continuous.

Now we have to show that $\theta_+$ is a bounded morphism. Monotonicity is clear, since $F \subseteq G$ implies $\theta^{-1}(F) \subseteq \theta^{-1}(G)$. For the clauses involving the $R_{f_i}$, note first that when $n = 0$ this reduces to showing that $R_{f_2}(F)$ *iff* $R_{f_1}(\theta_+(F))$. But in that case, $f_1$ and $f_2$ are distinguished elements of their respective algebras, with $\theta(f_1) = f_2$, while $R_{f_2}(F)$ iff $f_2 \in F$, and $R_{f_1}(\theta_+(F))$ iff $f_1 \in \theta_+(F)$ iff $\theta(f_1) \in F$. A similar analysis shows that $Q_{g_2}(F)$ iff $Q_{g_1}(\theta_+(F))$ when $m = 0$. Hence we may assume $n, m > 0$ in the rest of the proof.

Suppose that $R_{f_2}(F)$, where $F \in X^{n+1}_{\mathfrak{D}_2}$. Then $x \in \bigotimes(\theta_+(F_0), \ldots, \theta_+(F_{n-1}))$ implies $\theta(x) \in \bigotimes(F_0, \ldots, F_{n-1})$, hence $f_2(\theta(x)) \in F_n$, and so, since $\theta(f_1(x)) = f_2(\theta(x))$, $f_1(x) \in \theta_+(F_n)$, showing that $R_{f_1}(\theta_+(F))$. Hence we have that $R_{f_2}(F)$ implies $R_{f_1}(\theta_+(F))$.

Analogously, if $Q_{g_2}(F)$, we show $g_1(x) \in \theta_+(F_m)$ implies $x \in \bigoplus(\theta_+(F_0), \ldots, \theta_+(F_{m-1}))$, using $\theta(g_1(x)) = g_2(\theta(x))$, and hence that $Q_{g_1}(\theta_+(F))$.

At this point we have demonstrated that $\theta_+$ is a morphism. The difficult part is to prove its boundedness. So, suppose that $R_{f_1}(G, \theta_+(F))$, where $G \in X^n_{\mathfrak{D}_1}$ and $F \in X_{\mathfrak{D}_2}$. Then it must be shown that there is a sequence $F \in X^n_{\mathfrak{D}_2}$ of prime filters of $\mathfrak{D}_2$ such that $R_{f_2}(F, F)$, and $G \leqslant \theta_+(F)$, i.e. $G_i \subseteq \theta^{-1}(F_i)$ for all $i < n$. The construction, which is similar in strategy to the proof of Theorem 2.2.1(1), is

carried out inductively in such a way that for each $i<n$,

(i) $\theta(G_i)\subseteq F_i$, and

(ii) if $y_k\in F_k$ for $k\leqslant i$, and $x_p\in G_p$ for $i<p<n$, then

$$f_2(y_0,\ldots,y_i,\theta(x_{i+1}),\ldots,\theta(x_{n-1}))\in F.$$

Clause (i) immediately gives the desired $G_i\subseteq\theta^{-1}(F_i)$, while putting $i=n-1$ in (ii) establishes $R_{f_2}(F,F)$.

To make the construction, fix a $j\leqslant n-1$ and suppose inductively that $F_i$ has been defined to satisfy (i) and (ii) for all $i<j$. Define $H_j\subseteq D_2$ by putting $z\in H_j$ iff

there exists $y_i\in F_i$ for $i<j$ and $x_p\in G_p$ for $j<p<n$ such that

$$f_2(y_0,\ldots,y_{j-1},z,\theta(x_{j+1}),\ldots,\theta(x_{n-1}))\notin F.$$

Then $H_j$ is closed under joins. For if $z_0,z_1\in H_j$, then for each $r\in\{0,1\}$ there exist $y_i^r\in F_i$ for $i<j$ and $x_p^r\in G_p$ for $j<p<n$, such that

$$f_2(y_0^r,\ldots,y_{j-1}^r,z_r,\theta(x_{j+1}^r),\ldots,\theta(x_{n-1}^r))\notin F,$$

But then

$$f_2(y_0,\ldots,y_{j-1},z_r,\theta(x_{j+1}),\ldots,\theta(x_{n-1}))\notin F,$$

where $y_i=y_i^0\sqcap y_i^1\in F_i$, and $x_p=x_p^0\sqcap x_p^1$. this follows because $F$ is increasing, $f_2$ is monotonic in each variable, $y_i\sqsubseteq y_i^r$, and $\theta(x_p)\sqsubseteq\theta(x_p^r)$. Hence as $F$ is prime and $f_2$ preserves joins,

$$f_2(y_0,\ldots,y_{j-1},z_0\sqcup z_1,\theta(x_{j+1}),\ldots,\theta(x_{n-1}))\notin F.$$

This shows that $z_0\sqcup z_1\in H_j$ as desired. Notice also that $0\in H_j$, because $f_2$ preserves 0, and so $f_2(1,\ldots,1,0,\theta(1),\ldots,\theta(1))=0\notin F$. Hence $\bigsqcup J\in H_j$ for any finite $J\subseteq H_j$, including $J=\emptyset$.

Next it will be shown that $\theta(G_j)$ is separated from $H_j$. For if not, then $\bigsqcap I\sqsubseteq\bigsqcup J$ for some finite $I\subseteq\theta(G_j)$ and $J\subseteq H_j$. Then $\bigsqcup J\in H_j$, so

$$f_2(y_0,\ldots,y_{j-1},\bigsqcup J,\theta(x_{j+1}),\ldots,\theta(x_{n-1}))\notin F$$

for some $y_i\in F_i$ for $i<j$ and $x_p\in G_p$ for $j<p<n$. But $\bigsqcap I\in\theta(G_j)$, since $G_j$ is closed under finite meets, including the empty meet 1, and these are preserved by $\theta$. Hence $\bigsqcap I=\theta(x_j)$ for some $x_j\in G_j$. Monotonicity of $\theta$ then gives

$$f_2(y_0,\ldots,y_{j-1},\theta(x_j),\theta(x_{j+1}),\ldots,\theta(x_{n-1}))\notin F.$$

But if $j>0$ this contradicts the inductive hypothesis (ii) on $j-1$. On the other hand, if $j=0$ then

$$\theta(f_1(x_0,\ldots,x_{n-1}))=f_2(\theta(x_0),\ldots,\theta(x_{n-1}))\notin F,$$

so $f_1(x_0,\ldots,x_{n-1})\notin\theta_+(F)$. Since $x_i\in G_i$ for all $i<n$, this contradicts $R_{f_1}(G,\theta_+(F))$.

Thus $\theta(G_j)$ is separated from $H_j$. Hence by the Prime Filter Theorem, there

exists an $F_j \in X_{\mathfrak{D}_2}$ with $\theta(G_j) \subseteq F_j$, so that clause (i) holds for $j$, and $H_j \cap F_j = \emptyset$. Then if $y_k \in F_k$ for $k \leq j$ and $x_p \in G_p$ for $j < p < n$, we have

$$f_2(y_0, \ldots, y_j, \theta(x_{j+1}), \ldots, \theta(x_{n-1})) \in F,$$

or else $y_j \in H_j \cap F_j$. Thus (ii) holds for $j$, and this completes the proof of the boundedness of $\theta$ with respect to the $R_{f_i}$.

For the case of the $Q_{g_i}$, suppose that $Q_{g_1}(G, \theta_+(F))$, with $G \in X^m_{\mathfrak{D}_1}$ and $F \in X_{\mathfrak{D}_2}$. We must construct $F \in X^m_{\mathfrak{D}_2}$ such that $Q_{g_2}(F, F)$ and $\theta_+(F) \leq G$, i.e. $\theta^{-1}(F_i) \subseteq G_i$ for all $i < m$. The construction is dual to the above arguments, with the inductive clauses for the $F_i$ now being

(iii) $F_i \cap \theta(-G_i) = \emptyset$, and

(iv) if $y_k \notin F_k$ and $k \leq i$, and $x_p \notin G_p$ for $i < p < m$, then

$$g_2(y_0, \ldots, y_i, \theta(x_{i+1}), \ldots, \theta(x_{m-1})) \notin F.$$

Then (iii) implies that if $\theta(z) \in F_i$ then $z \notin -G_i$, so that $\theta^{-1}(F_i) \subseteq G_i$ as desired, while (iv) implies $Q_{g_2}(F, F)$ when $i = m - 1$.

Fixing $j \leq m - 1$ and supposing $F_i$ has been defined to satisfy (iii) and (iv) for all $i < j$, define $E_j \subseteq D_2$ by putting $z \in E_j$ iff

there exists $y_i \notin F_i$ for $i < j$ and $x_p \notin G_p$ for $j < p < m$ such that

$$g_2(y_0, \ldots, y_{j-1}, z, \theta(x_{j+1}), \ldots, \theta(x_{m-1})) \in F.$$

Dualising the arguments used for $H_j$ above, it can be shown that $E_j$ is closed under finite meets, and then that $E_j$ is separated from $\theta(-G_j)$. Hence there is a prime filter $F_j$ of $\mathfrak{D}_2$ such that $E_j \subseteq F_j$ and $F_j \cap \theta(-G_j) = \emptyset$. From this it follows that (iii) and (iv) hold for $j$. The details are left to the reader.

This completes the proof of part (1) of the Theorem. Parts (2) and (3) are standard, (2) being elementary, and (3) requiring the Prime Filter Theorem to show that if $G \in X_{\mathfrak{D}_1}$ then there exists a prime filter $F$ of $\mathfrak{D}_2$ that extends $\theta(G) = \{\theta(x) : x \in G\}$ and is disjoint from $\theta(-G) = \{\theta(x) : x \notin G\}$, so that $G = \theta^{-1}(F)$. But if $\theta$ is injective, then $\theta(G)$ is separated from $\theta(-G)$, as the reader may confirm. □

We now see that putting $\Theta(\mathfrak{A}) = \mathfrak{A}_+$ and $\Theta(\theta) = \theta_+$ defines a contravariant functor $\Theta$: **DLO** → **RPS**.

**Theorem 2.3.3.** *The composites $\Phi \circ \Theta$ and $\Theta \circ \Phi$ are naturally isomorphic to the identity functors on* **DLO** *and* **RPS**, *respectively, and so the two categories are dually equivalent.*

**Proof.** It suffices (cf. Herrlich and Strecker [30], §14]) to show that the isomorphisms

$$\sigma_{\mathfrak{A}} : \mathfrak{A} \to (\mathfrak{A}_+)^+ = \Phi(\Theta(\mathfrak{A})), \qquad \tau_{\mathfrak{S}} : \mathfrak{S} \to (\mathfrak{S}^+)_+ = \Theta(\Phi(\mathfrak{S}))$$

of Theorems 2.2.3 and 2.2.4 are the components of natural transformations, i.e. that the diagrams

$$\begin{array}{ccc} \mathfrak{A}_1 & \xrightarrow{\theta} & \mathfrak{A}_2 \\ \downarrow{\scriptstyle \sigma_{\mathfrak{A}_1}} & & \downarrow{\scriptstyle \sigma_{\mathfrak{A}_2}} \\ (\mathfrak{A}_{1+})^+ & \xrightarrow{(\theta_+)^+} & (\mathfrak{A}_{2+})^+ \end{array} \qquad \begin{array}{ccc} \mathfrak{S}_1 & \xrightarrow{\varphi} & \mathfrak{S}_2 \\ \downarrow{\scriptstyle \tau_{\mathfrak{S}_1}} & & \downarrow{\scriptstyle \tau_{\mathfrak{S}_2}} \\ (\mathfrak{S}_1^+)_+ & \xrightarrow{(\varphi^+)_+} & (\mathfrak{S}_2^+)_+ \end{array}$$

commute for any **DLO**-arrow $\theta$ and **RPS**-arrow $\varphi$ as shown. But for the left diagram we have

$$\begin{aligned}(\theta_+)^+ \circ \sigma_{\mathfrak{A}_1}(x) &= (\theta_+)^+(\{F \in X_{\mathfrak{A}_1} : x \in F\}) \\ &= \{G \in X_{\mathfrak{A}_2} : x \in \theta_+(G)\} = \{G \in X_{\mathfrak{A}_2} : \theta(x) \in G\} = \sigma_{\mathfrak{A}_2} \circ \theta(x),\end{aligned}$$

and for the other,

$$\begin{aligned}(\varphi^+)_+ \circ \tau_{\mathfrak{S}_1}(x) &= (\varphi^+)_+(\{Y \in \mathfrak{S}_1^+ : x \in Y\}) \\ &= \{Z \in \mathfrak{S}_2^+ : x \in \varphi^+(Z)\} = \{Z \in \mathfrak{S}_2^+ : \varphi(x) \in Z\} = \tau_{\mathfrak{S}_2} \circ \varphi(x). \qquad \square\end{aligned}$$

### 2.4. *Heyting algebras*

A *Heyting algebra* is a lattice $\mathfrak{D}$ with a least element and a binary operation $\Rightarrow$ of *relative pseudo-complementation* satisfying

$$x \sqcap z \sqsubseteq y \quad \text{iff} \quad x \sqsubseteq z \Rightarrow y.$$

All Heyting algebras are distributive, and have a greatest element (equal to $z \Rightarrow z$ for any $z$). These algebras provide the basic algebraic models of propositional intuitionistic logic (cf. Rasiowa and Sikorski [43], Goldblatt [22]).

On the lattice of cones of a partially ordered set $(X, \leqslant)$, a relative pseudo-complementation is given by

$$Y \Rightarrow Z = \{x \in X : x \leqslant y \in Y \text{ implies } y \in Z\}.$$

In the lattice $\mathrm{cl}(X_{\mathfrak{D}})$ of clopen cones of the dual space $(X_{\mathfrak{D}}, \subseteq, \mathcal{T}_{\mathfrak{D}})$ of $\mathfrak{D}$, it turns out that

$$\mathfrak{D}_{z \Rightarrow y} = \mathfrak{D}_z \Rightarrow \mathfrak{D}_y,$$

so that $\sigma_{\mathfrak{D}} : \mathfrak{D} \to \mathrm{cl}(X_{\mathfrak{D}})$ becomes an isomorphism of Heyting algebras. The inclusion of $\mathfrak{D}_{z \Rightarrow y}$ in $\mathfrak{D}_z \Rightarrow \mathfrak{D}_y$ follows readily from the fact that $(z \Rightarrow y) \sqcap z \sqsubseteq y$. For the converse, if $F \in \mathfrak{D}_z \Rightarrow \mathfrak{D}_y$, then every prime filter extending $F \cup \{z\}$ must belong to $\mathfrak{D}_y$ and so have $y$ as a member. Hence $F \cup \{z\}$ is not separated from $\{y\}$, so for some $x \in F$, $x \sqcap z \sqsubseteq y$, giving $x \sqsubseteq y \Rightarrow z$ and so $(y \Rightarrow z) \in F$.

In order for a monotonic $\varphi : (X_1, \leqslant) \to (X_2, \leqslant)$ to induce a homomorphism $\mathrm{co}(X_2) \to \mathrm{co}(X_1)$ that preserves relative pseudo-complements, an additional constraint is needed to make

$$\varphi^{-1}(Y \Rightarrow Z) = \varphi^{-1}(Y) \Rightarrow \varphi^{-1}(Z).$$

A suitable condition is

$$\varphi(z) \leqslant y \quad \text{implies} \quad \exists x\,(z \leqslant x \text{ and } \varphi(x) = y).$$

In combination with monotonicity, this is equivalent to

$$\varphi(z) \leqslant y \quad \text{iff} \quad \exists x\,(z \leqslant x \text{ and } \varphi(x) = y),$$

i.e. to

$$\{y\colon \varphi(z) \leqslant y\} = \{\varphi(x)\colon z \leqslant x\},$$

which states that the cone generated in $X_2$ by $\varphi(z)$ is the $\varphi$-image of the cone $\{x\colon z \leqslant x\}$ generated in $X_1$ by $z$.

The dual map $\theta_+\colon X_{\mathfrak{D}_2} \to X_{\mathfrak{D}_1}$ of a Heyting algebra homomorphism $\theta\colon \mathfrak{D}_1 \to \mathfrak{D}_2$ satisfies this new condition on $\leqslant$: if $\theta^{-1}(F) \subseteq G$, where $F \in X_{\mathfrak{D}_2}$ and $G \in X_{\mathfrak{D}_1}$, then $F \cup \theta(G)$ is separated from $\theta(-G)$. For if not, then since $F$ and $\theta(G)$ are closed under meets, and $\theta(-G)$ is closed under joins, there would exist $x \in F$, $z \in G$, and $y \notin G$ such that $x \sqcap \theta(z) \sqsubseteq \theta(y)$. But then $x \sqsubseteq (\theta(z) \Rightarrow \theta(y)) = \theta(z \Rightarrow y)$, so $\theta(z \Rightarrow y) \in F$, making $(z \Rightarrow y) \in \theta^{-1}(F) \subseteq G$. Since $z \in G$, this would lead to the contradiction $y \in G$.

Any $H \in X_{\mathfrak{D}_2}$ that extends $F \cup \theta(G)$ and is disjoint from $\theta(-G)$ then has $F \subseteq H$ and $\theta^{-1}(H) = G$, as desired.

A significant type of meet-hemimorphism $j\colon \mathfrak{D} \to \mathfrak{D}$ on a Heyting algebra is one satisfying $x \sqsubseteq j(x) = j(j(x))$. This arises in topos theory, in the context of an axiomatic development of 'sheaves over a Grothendieck topology' (cf. Johnstone [32, §3.1], Goldblatt [22, §14.2]). A representation of such operators in terms of ordered relational structures appears in Goldblatt [21].

## 3. Varieties of complex algebras

Fix a natural number $n$. This section will discuss *structures* $\mathfrak{S} = (X, R)$ having $R \subseteq X^{n+1}$, and *algebras* $\mathfrak{A} = (\mathfrak{B}, f)$ having $f$ an $n$-ary join-hemimorphism on the *Boolean* algebra $\mathfrak{B}$. This will typify the situation for general relational structures and their associated algebras of complexes.

In the presence of Boolean complementation it is unnecessary to treat both types of hemimorphism on a par: putting

$$g(x) = -f(-x)$$

defines $g$ as a meet-hemimorphism, while any meet-hemimorphism $g$ arises in this way from the join-hemimorphism $f(x) = -g(-x)$. The associated relations $R_f$ and $Q_g$ on $X_{\mathfrak{B}}$ are identical.

### 3.1. *Canonical structures*

We make the following definitions.

(1) The *canonical structure* of $\mathfrak{A} = (\mathfrak{B}, f)$ is the structure

$$\mathfrak{Cs}\,\mathfrak{A} = (X_{\mathfrak{B}}, R_f).$$

(2) The *canonical embedding algebra* of $\mathfrak{A}$ is the algebra

$$\mathfrak{Em}\,\mathfrak{A} = (\mathfrak{Cs}\,\mathfrak{A})^+ = (\mathrm{Sb}(X_{\mathfrak{B}}), \exists_{R_f}).$$

(3) The *canonical extension* of a structure $\mathfrak{S} = (X, R)$ is the structure

$$\mathfrak{Ce}\,\mathfrak{S} = \mathfrak{Cs}\,\mathfrak{S}^+ = (X_{\mathrm{Sb}(X)}, R_{\exists_R}).$$

Recall that $X_{\mathfrak{B}}$ is the set of prime filters of $\mathfrak{B}$, and that these are the same as maximal filters in a Boolean algebra, so that $\subseteq$ is discrete on $X_{\mathfrak{B}}$. Note especially that $\mathfrak{Cs}\,\mathfrak{A}$ is not the same thing as the dual space $\mathfrak{A}_+$, since the definition of the latter (Section 2.3) includes the generally non-discrete topology $\mathcal{T}_{\mathfrak{B}}$. In particular, whereas $(\mathfrak{A}_+)^+$ is isomorphic to $\mathfrak{A}$ (Theorem 2.2.3), it is in general a proper subalgebra of $(\mathfrak{Cs}\,\mathfrak{A})^+ = \mathfrak{Em}\,\mathfrak{A}$, which is in turn the full complex algebra of all subsets of $X_{\mathfrak{B}}$. Thus the homomorphism $\sigma_{\mathfrak{A}}(x) = \{F \in X_{\mathfrak{B}} : x \in F\}$ generally embeds $\mathfrak{A}$ into, and not onto, $\mathfrak{Em}\,\mathfrak{A}$. The question as to what properties are preserved in passing from $\mathfrak{A}$ to $\mathfrak{Em}\,\mathfrak{A}$ is a signficant one algebraically, and model-theoretically in the modal case (cf. Section 3.5), and is the source of some of the theory of this section.

A key relationship between the concepts just introduced is given by the equations

$$(\mathfrak{Ce}\,\mathfrak{S})^+ = (\mathfrak{Cs}\,\mathfrak{S}^+)^+ = \mathfrak{Em}\,\mathfrak{S}^+.$$

The name 'canonical embedding algebra' and notation $\mathfrak{Em}\,\mathfrak{A}$ comes from Henkin, Monk, and Tarski [28, p. 432], where $\sigma_{\mathfrak{A}}$ is called the 'canonical embedding function' (actually, Henkin et al. work with maximal ideals rather than filters, but the two formalisms are interchangeable). Thus 'canonical structure' seems appropriate for $\mathfrak{Cs}\,\mathfrak{A}$, and this usage fits neatly with the common practice in modal logic of referring to the *canonical frame* of a normal logic $\Lambda$ as being the structure $\mathfrak{S}_\Lambda = (X_\Lambda, R_\Lambda)$, where $X_\Lambda$ is the set of maximally $\Lambda$-consistent sets of formulae, and $R_\Lambda(\Gamma, \Delta)$ iff $\{\Diamond\phi : \phi \in \Delta\} \subseteq \Gamma$. Maximally $\Lambda$-consistent sets correspond bijectively to maximal filters in the Lindenbaum algebra $\mathfrak{A}_\Lambda$ of $\Lambda$ described in Section 1. In fact

> *The canonical frame of $\Lambda$ isomorphic to the canonical structure $\mathfrak{Cs}\,\mathfrak{A}_\Lambda$ of the Lindenbaum algebra for $\Lambda$.*

The role of canonical frames and their relation to free algebras will be discussed further in Section 3.5.

The elements of the canonical extension $\mathfrak{Ce}\,\mathfrak{S}$ of a structure $\mathfrak{S}$ are the maximal filters of $\mathrm{Sb}(X)$, i.e. the *ultrafilters* on the set $X$. Hence in the modal case, $\mathfrak{Ce}\,\mathfrak{S}$ is called the *ultrafilter extension* of a Kripke frame $\mathfrak{S}$ by van Benthem [7, 9]. It is the same as the *completion* of a frame in Goldblatt [20, Definition 20.2].

*3.2. Inner substructures and epimorphic images*

In the present context of discrete structures, a bounded morphism $\mathfrak{S}_1 \to \mathfrak{S}_2$ between structures $\mathfrak{S}_i = (X_i, R_i)$ is a function $\varphi: X_1 \to X_2$ that satisfies

$$R_1(x, y) \text{ implies } R_2(\varphi(x), \varphi(y)), \text{ and}$$

$$R_2(y, \varphi(z)) \text{ implies } \exists x \in X_1^n (R_1(x, z) \text{ and } y = \varphi(x)).$$

This is equivalent to requiring that

$$R_2(y, \varphi(z)) \text{ iff } \exists x \in X_1^n (R_1(x, z) \text{ and } y = \varphi(x)),$$

i.e. that for all $z \in X_1$,

$$R_2^{-1}(\varphi(z)) = \varphi(R_1^{-1}(z)),$$

which states that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{R_1^{-1}} & \mathrm{Sb}(X_1^n) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ X_2 & \xrightarrow{R_2^{-1}} & \mathrm{Sb}(X_2^n) \end{array}$$

commutes, where the right-hand arrow is the natural lifting $Y \mapsto \{\varphi(x): x \in Y\}$ of $\varphi$ to $\mathrm{Sb}(X_1^n)$.

**Theorem 3.2.1.** *If $\varphi$ is an injective bounded morphism, then for all $z \in X_1^{n+1}$,*

$$R_2(\varphi(z)) \text{ implies } R_1(z).$$

*Hence a bijective bounded morphism is an isomorphism of relational structures.*

**Proof.** If $R_2(\varphi(z))$, then there exists $x \in X_1^n$ with $R_1(x, z_n)$ and $\varphi(x) = (\varphi(z_0), \ldots, \varphi(z_{n-1}))$. By injectivity, $x_i = z_i$ for all $i < n$. □

A structure $\mathfrak{S}_1$ is an *inner substructure* of $\mathfrak{S}_2$, and $\mathfrak{S}_2$ is an *outer extension* of $\mathfrak{S}_1$, if $X_1 \subseteq X_2$ and the inclusion function $X_1 \hookrightarrow X_2$ is a bounded morphism from $\mathfrak{S}_1$ to $\mathfrak{S}_2$. We write $\mathfrak{S}_1 \Subset \mathfrak{S}_2$ when this holds.

**Lemma 3.2.2.** *If $X_1 \subseteq X_2$, then $\mathfrak{S}_1 \Subset \mathfrak{S}_2$ if, and only if,*

(1) $R_1 = R_2 \cap X_1^{n+1}$; *and*

(2) *for all $z \in X_1$ and $y \in X_2^n$,*

$$R_2(y, z) \text{ implies } y \in X_1^n.$$

**Proof.** (1) holds iff

$$R_1^{-1}(z) = R_2^{-1}(z) \cap X_1^n, \quad \text{for all } z \in X_1,$$

while (2) states that $R_2^{-1}(z) \subseteq X_1^n$. Together these hold iff

$$R_1^{-1}(z) = R_2^{-1}(z), \quad \text{for all } z \in X_1,$$

which is precisely the condition for the commutativity of the above diagram when $\varphi$ is the inclusion.

For a more 'element-wise' proof, observe that the inclusion is a bounded morphism iff

$$R_2(y, z) \quad \text{iff} \quad \exists x \in X_1^n (R_1(x, z) \text{ and } y = x). \qquad \square$$

**Corollary 3.2.3.** (1) *The image of a bounded morphism $\varphi: \mathfrak{S}_1 \to \mathfrak{S}_2$ is an inner substructure of $\mathfrak{S}_2$ which is isomorphic to $\mathfrak{S}_1$ when $\varphi$ is injective.*

(2) *If a function $\varphi: \mathfrak{S}_1 \to \mathfrak{S}_2$ makes $\mathfrak{S}_1$ isomorphic to an inner substructure of $\mathfrak{S}_2$, then $\varphi$ is a bounded morphism.*

**Proof.** (1) Let $\operatorname{Im} \varphi = (\varphi(X_1), R_2')$, where $R_2' = R_2 \cap \varphi(X_1)^{n+1}$ is the restriction of $R_2$ to $\varphi(X_1)$. If $\varphi(z) \in \varphi(X_1)$, then by boundedness of $\varphi$, $R_2^{-1}(\varphi(z)) \subseteq (\varphi(X_1))^n$, so $\operatorname{Im} \varphi \in \mathfrak{S}_2$ by Lemma 3.2.2. Moreover, if $\varphi$ is injective, from Theorem 3.2.1 it follows that

$$R_1(x) \quad \text{iff} \quad R_2(\varphi(x)) \quad \text{iff} \quad R_2'(\varphi(x)).$$

(2) is left to the reader. $\square$

Observe that clause (1) of Lemma 3.2.2 is the usual condition for $\mathfrak{S}_1$ to be a *substructure* of $\mathfrak{S}_2$. In modal logic, inner substructures are known as *subframes*. The present nomenclature was introduced, for binary relations, in Feferman [15]. Outer extensions of models of set theory are known as *end extensions*.

The standard symbols $\rightarrowtail$ and $\twoheadrightarrow$ will be used to denote functions that are injective and surjective respectively. We write $\mathfrak{S}_1 \rightarrowtail \mathfrak{S}_2$, or $\mathfrak{S}_1 \twoheadrightarrow \mathfrak{S}_2$, to indicate that there exists an injective, or surjective, bounded morphism from $\mathfrak{S}_1$ to $\mathfrak{S}_2$. Similarly, the notations $\mathfrak{A}_1 \rightarrowtail \mathfrak{A}_2$ and $\mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$ indicate the existence of injective and surjective homomorphisms between algebras. A surjective bounded morphism will be called a *bounded epimorphism*, and if $\mathfrak{S}_1 \twoheadrightarrow \mathfrak{S}_2$, then $\mathfrak{S}_2$ is a *bounded epimorphic image* of $\mathfrak{S}_1$.

If $\theta: \mathfrak{A}_1 \to \mathfrak{A}_2$ is a homomorphism of algebras, then the function $\theta_+: X_{\mathfrak{B}_2} \to X_{\mathfrak{B}_1}$ defined in Section 2.3 is a bounded morphism from $\mathfrak{Cs}\,\mathfrak{A}_2$ to $\mathfrak{Cs}\,\mathfrak{A}_1$. The duality presented in Theorems 2.3.1 and 2.3.2 can be summarized as follows:

**Theorem 3.2.4.** (1) *A bounded morphism $\varphi: \mathfrak{S}_1 \to \mathfrak{S}_2$ induces a homomorphism $\varphi^+: \mathfrak{S}_2^+ \to \mathfrak{S}_1^+$, such that if $\varphi: \mathfrak{S}_1 \rightarrowtail \mathfrak{S}_2$ then $\varphi^+: \mathfrak{S}_2^+ \twoheadrightarrow \mathfrak{S}_1^+$, and if $\varphi: \mathfrak{S}_1 \twoheadrightarrow \mathfrak{S}_2$ then $\varphi^+: \mathfrak{S}_2^+ \rightarrowtail \mathfrak{S}_1^+$.*

(2) *A homomorphism $\theta: \mathfrak{A}_1 \to \mathfrak{A}_2$ induces a bounded morphism $\theta_+: \mathfrak{Cs}\,\mathfrak{A}_2 \to \mathfrak{Cs}\,\mathfrak{A}_1$ such that if $\theta: \mathfrak{A}_1 \rightarrowtail \mathfrak{A}_2$ then $\theta_+: \mathfrak{Cs}\,\mathfrak{A}_2 \twoheadrightarrow \mathfrak{Cs}\,\mathfrak{A}_1$, and if $\theta: \mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$ then $\theta_+: \mathfrak{Cs}\,\mathfrak{A}_2 \rightarrowtail \mathfrak{Cs}\,\mathfrak{A}_1$.* $\square$

**Corollary 3.2.5.** (1) *If $\mathfrak{S}_1$ is (isomorphic to) an inner substructure of $\mathfrak{S}_2$, then $\mathfrak{S}_1^+$ is a homomorphic image of $\mathfrak{S}_2^+$.*

(2) *If $\mathfrak{S}_2$ is a bounded epimorphic image of $\mathfrak{S}_1$, then $\mathfrak{S}_2^+$ is isomorphic to a subalgebra of $\mathfrak{S}_1^+$.*

(3) *If $\mathfrak{A}_1$ is (isomorphic to) a subalgebra of $\mathfrak{A}_2$, then $\mathfrak{Cs}\,\mathfrak{A}_1$ is a bounded epimorphic image of $\mathfrak{Cs}\,\mathfrak{A}_2$.*

(4) *If $\mathfrak{A}_2$ is a homomorphic image of $\mathfrak{A}_1$, then $\mathfrak{Cs}\,\mathfrak{A}_2$ is isomorphic to an inner substructure of $\mathfrak{Cs}\,\mathfrak{A}_1$.*

(5) *If $\mathfrak{S}_1 \rightarrowtail \mathfrak{S}_2$ or $\mathfrak{S}_1 \twoheadrightarrow \mathfrak{S}_2$, respectively, then $\mathfrak{Ce}\,\mathfrak{S}_1 \rightarrowtail \mathfrak{Ce}\,\mathfrak{S}_2$ or $\mathfrak{Ce}\,\mathfrak{S}_1 \twoheadrightarrow \mathfrak{Ce}\,\mathfrak{S}_2$, respectively.*

(6) *If $\mathfrak{A}_1 \rightarrowtail \mathfrak{A}_2$ or $\mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$, respectively, then $\mathfrak{Em}\,\mathfrak{A}_1 \rightarrowtail \mathfrak{Em}\,\mathfrak{A}_2$ or $\mathfrak{Em}\,\mathfrak{A}_1 \twoheadrightarrow \mathfrak{Em}\,\mathfrak{A}_2$, respectively.*

**Proof.** (1)–(4) follow directly from 3.2.4 with the aid of 3.2.3. For (5), if $\mathfrak{S}_1 \rightarrowtail \mathfrak{S}_2$, then $\mathfrak{S}_2^+ \twoheadrightarrow \mathfrak{S}_1^+$ by (1), and then $\mathfrak{Cs}\,\mathfrak{S}_1^+ \rightarrowtail \mathfrak{Cs}\,\mathfrak{S}_2^+$ by (4), so use $\mathfrak{Ce}\,\mathfrak{S}_i = \mathfrak{Cs}\,\mathfrak{S}_i^+$ etc. Similarly, the proof of (6) uses (1)–(4) and $\mathfrak{Em}\,\mathfrak{A}_i = (\mathfrak{Cs}\,\mathfrak{A}_i)^+$. □

### *3.3. Point-generation and subdirect irreducibility*

The material in this subsection is of conceptual and algebraic importance, but is not strictly essential to what follows. It does illuminate a construction appearing in Theorem 3.6.7, but could be skipped over at a first reading. It concerns the notion of an inner substructure generated by a point.

If $z$ is a point in a structure $\mathfrak{S}$, let $\mathfrak{S}_z = (X_z, R_z)$, where $X_z$ is the intersection of all inner substructures of $\mathfrak{S}$ that contain $z$, and $R_z$ is the restriction of $R$ to $X_z$. Then $\mathfrak{S}_z \in \mathfrak{S}$, and so $\mathfrak{S}_z$ is the smallest inner substructure of $\mathfrak{S}$ containing $z$.

In modal logic, $\mathfrak{S}_z$ is known as the *subframe of $\mathfrak{S}$ generated by $z$*, and is an extensively used construction. Its importance derives from the fact that in any model on a frame $\mathfrak{S}$, truth-values of formulae at $z$ depend only on truth-values at points in $\mathfrak{S}_z$.

Let $\theta_z: \mathfrak{S}^+ \twoheadrightarrow \mathfrak{S}_z^+$ be the homomorphism induced by the inclusion $\mathfrak{S}_z \hookrightarrow \mathfrak{S}$ as a bounded morphism. Then $\theta_z(Y) = Y \cap X_z$, so $\theta_z(Y) \neq \emptyset$ if $z \in Y$. This shows that the product map $\langle \theta_z : z \in X \rangle$ is an injection of $\mathrm{Sb}(X)$ into the product of the algebras $\mathfrak{S}_z^+$, and, since the $\theta_z$'s are surjective, gives a subdirect-product representaion

$$\mathfrak{S}^+ \rightarrowtail \prod_{z \in X} \mathfrak{S}_z^+$$

of $\mathfrak{S}^+$ in terms of the complex algebras of point-generated substructures. In fact this is a subdirect representation by *subdirectly irreducible* algebras, as we will now see.

If $R$ is a binary relation, there is a particularly simple description of $X_z$, for

which it is helpful to use infix notation, writing $yRz$ for $R(y, z)$ etc. We have

$$X_z = \{y \in X : yR^*z\},$$

where $R^*$ is the reflexive transitive closure of $R$. In other words,

(1) $y \in X_z$ iff there is a sequence $(y_0, \ldots, y_p)$ in $X$ (for some $p \geqslant 0$) such that

$$y = y_0 R y_1 R \cdots R y_p = z.$$

In the general case there is a similar, though more intricate, description. If $R$ is $n+1$-ary, then for each $i < n$ a binary relation $R^i$ on $X$ is given by

$$yR^iz \quad \text{iff} \quad \exists x \in X^n\,(R(x, z) \text{ and } y = x_i).$$

Then putting

$$[R] = R^0 \cup \cdots \cup R^{n-1},$$

we have that $X_z = \{y : y[R]^*z\}$, i.e. that

(2) $y \in X_z$ iff there is a sequence $(y_0, \ldots, y_p)$ in $X$ (for some $p \geqslant 0$), and a sequence $(k_0, \ldots, k_{p-1})$ of numbers $k_i < n$ such that

$$y = y_0 R^{k_0} y_1 R^{k_1} \cdots R^{k_{p-1}} y_p = z.$$

For, any $y$ satisfying the right-hand side of this biconditional is in $X_z$, while the set of all such $y$ is closed under $R^{-1}$ (i.e. under $(R^i)^{-1}$ for all $i$), and so is an inner substructure of $\mathfrak{S}$.

The set $X_z$ can also be characterised by means of the hemimorphism $\exists_R$. Since

$$yRz \quad \text{iff} \quad z \in \exists_R(\{y\})$$

when $R$ is binary, it follows in that case from (1) above that

$$y \in X_z \quad \text{iff} \quad \exists p \geqslant 0\,(z \in \underbrace{\exists_R(\exists_R(\cdots(\exists_R}_{p \text{ times}}(\{y\})\cdots).$$

To generalise this, define for each $i < n$ a unary operator $\exists^i_R$ on $\mathfrak{S}^+$ by

$$\exists^i_R(Y) = \exists_R(X, \ldots, X, Y, X, \ldots, X),$$

with the $Y$ on the right of the equation being the $i+1$-st coordinate. $\exists^i_R$ is a join-preserving polynomial function on $\mathfrak{S}^+$, characterising $R^i$ in the sense that

$$yR^iz \quad \text{iff} \quad z \in \exists^i_R(\{y\}).$$

For a given $p$-sequence $k$ of numbers less than $n$, we then put

$$\exists^k_R = \exists^{k_0}_R \circ \cdots \circ \exists^{k_p}_R,$$

and have, from (2) above, that

(3) $y \in X_z$ iff there is a $p$-sequence $k$, for some $p \geqslant 0$, such that

$$z \in \exists^k_R(\{y\}).$$

**Theorem 3.3.1.** *$\mathfrak{S}^+_z$ is subdirectly irreducible.*

**Proof.** Let $\sim_z$ be a congruence on $\mathfrak{S}^+$ that is maximal as a separating congruence for $\{z\}$ and $\emptyset$. Then the quotient algebra $\mathfrak{S}^+/\sim_z$ is subdirectly irreducible on universal-algebraic grounds (Henkin et al. [28, Corollary 0.3.53]).

Let $\sim_{\theta_z}$ be the kernel congruence of $\theta_z: \mathfrak{S}^+ \twoheadrightarrow \mathfrak{S}_z^+$, i.e.

$$Y \sim_{\theta_z} Z \quad \text{iff} \quad \theta_z(Y) = \theta_z(Z) \quad \text{iff} \quad Y \cap X_z = Z \cap X_z.$$

Then $\mathfrak{S}_z^+$ is isomorphic to $\mathfrak{S}^+/\sim_{\theta_z}$, and so it will suffice to show that $\sim_{\theta_z}$ and $\sim_z$ are identical.

Now since $\{z\} \cap X_z = \{z\} \neq \emptyset$, $\sim_{\theta_z}$ separates $\{z\}$ and $\emptyset$, so $\sim_{\theta_z}$ is contained in $\sim_z$ by definition of the latter. For the converse, suppose that $Y \sim_z Z$. If $Y \not\sim_{\theta_z} Z$, then $Y \cap X_z \neq Z \cap X_z$, so there exists some $y \in X_z$ with $y \in Y + Z$, where $+$ denotes symmetric difference. By (3) above, there is a sequence $k$ such that $z \in \exists_R^k(\{y\}) \subseteq \exists_R^k(Y + Z)$. But $(Y + Z) \sim_z \emptyset$, since $Y \sim_z Z$, so as $\exists_R^k$ is a polynomial on $\mathfrak{S}^+$, $\exists_R^k(Y + Z) \sim_z \exists_R^k(\emptyset) = \emptyset$. Hence $\{z\} \cap \exists_R^k(Y + Z) \sim_z \{z\} \cap \emptyset$, i.e. $\{z\} \sim_z \emptyset$, which contradicts the definition of $\sim_z$. Therefore $Y \sim_{\theta_z} Z$ as desired. □

### 3.4. *Complex varieties*

Recall the following operators on a class $W$ of algebras.

- $\mathbf{H}W$: the class of homomorphic images of members of $W$.
- $\mathbf{S}W$: the class of subalgebras of members of $W$.
- $\mathbf{P}W$: the class of algebras isomorphic to direct products of members of $W$.

$W$ is a *variety* if $\mathbf{H}W \subseteq W$, $\mathbf{S}W \subseteq W$, and $\mathbf{P}W \subseteq W$. In general, $\mathbf{HSP}W$ is a variety, and is the smallest variety containing $W$. Thus a variety $V$ is said to be *generated* by $W$ if $V = \mathbf{HSP}W$. The famous characterisation of Birkhoff states that $W$ is a variety iff there is a set of equations such that $W$ consists precisely of those algebras that satisfy all these equations.

Under the duality described in Section 3.2 (cf. Corollary 3.2.5), inner substructures correspond to homomorphic images of algebras, and bounded epimorphic images of structures correspond to subalgebras. The operation on structures that corresponds to direct products of algebras is the formation of *disjoint unions*. If $\{\mathfrak{S}_i : i \in I\}$ is a collection of structures $\mathfrak{S}_i = (X_i, R_i)$, then the disjoint union $\sum_I \mathfrak{S}_i$ is the structure $(\sum_I X_i, R)$, with

$$\sum_I X_i = \{(x, i) : i \in I \text{ and } x \in X_i\} = \bigcup_I (X_i \times \{i\}),$$

and

$$R = \{((x_0, i), \ldots, (x_n, i)) : i \in I \text{ and } R_i(x_0, \ldots, x_n)\}.$$

**Lemma 3.4.1.** *$(\sum_I \mathfrak{S}_i)^+$ is isomorphic to $\prod_I \mathfrak{S}_i^+$.*

**Proof.** For each $j \in I$, the function $\varphi_j : x \mapsto (x, j)$ is a bounded morphism $\mathfrak{S}_j \rightarrowtail \sum_I \mathfrak{S}_i$, which thereby induces the surjective homomorphism $\varphi_j^* : (\sum_I \mathfrak{S}_i)^+ \twoheadrightarrow$

$\mathfrak{S}_j^+$. This then determines a homomorphism $\theta: (\Sigma_I \mathfrak{S}_i)^+ \twoheadrightarrow \Pi_I \mathfrak{S}_i^+$, viz. $\theta(Y)(j) = \varphi_j^+(Y) = \varphi_j^{-1}(Y)$, which is surjective because if $Z_j \in \mathfrak{S}_j^+$ for all $j \in I$,

$$\langle Z_j : j \in I \rangle = \theta\Big(\bigcup_I \varphi_j(Z_j)\Big).$$

But if $\theta(Y) = \theta(Z)$, then $(x, j) \in Y$ implies $x \in \varphi_j^{-1}(Y) = \varphi_j^{-1}(Z)$, and hence $(x, j) \in Z$. Interchanging $Y$ and $Z$ shows $Y = Z$. Thus $\theta$ is injective, and altogether is an isomorphism. □

Now if $K$ is a class of structures, let

$$K^+ = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{S}^+ \text{ for some } \mathfrak{S} \in K\}$$

be the closure under isomorphism of the class of full complex algebras of members of $K$. If I denotes closure under isomorphism, then

$$SK^+ = SI\{\mathfrak{S}^+ : \mathfrak{S} \in K\} = IS\{\mathfrak{S}^+ : \mathfrak{S} \in K\},$$

so the members of $SK^+$ are precisely those algebras isomorphic to complex algebras based on members of $K$. As discussed in the Introduction, there are many significant examples of varieties that have the form $SK^+$ for some $K$, and these we call *complex varieties*. In general, $SK^+$ is contained in any variety that contains $K^+$, and in particular, $SK^+ \subseteq HSPK^+$. But if $K$ is closed under disjoint unions, then Lemma 3.4.1 implies that $PK^+ = K^+$. Thus

**Theorem 3.4.2.** *If $K$ is closed under disjoint unions, then $SK^+$ is a variety if, and only if, $HSK^+ = SK^+ =$ the smallest variety containing $K^+$.* □

Thus to determine when $SK^+$ is a variety, we seek conditions under which it is closed under homomorphic images. For this pupose the notions of the canonical extension $\mathfrak{Ce}\,\mathfrak{S} = (X_{Sb(X)}, R_{\exists_R})$ of a structure $\mathfrak{S} = (X, R)$, and canonical structure $\mathfrak{Cs}\,\mathfrak{A} = (X_{\mathfrak{B}}, R_f)$ of an algebra $\mathfrak{A} = (\mathfrak{B}, f)$, will be used, together with the duality expressed in Corollary 3.2.5.

**Lemma 3.4.3.** *Let $K$ be closed under canonical extensions.*

(1) *If $K$ is closed under bounded epimorphic images, then*

$$\mathfrak{A} \in SK^+ \quad \textit{implies} \quad \mathfrak{Cs}\,\mathfrak{A} \in K.$$

(2) *If $K$ is closed under bounded epimorphic images and inner substructures, then*

$$\mathfrak{A} \in HSK^+ \quad \textit{implies} \quad \mathfrak{Cs}\,\mathfrak{A} \in K.$$

**Proof.** (1) If $\mathfrak{A} \in SK^+$ then there is some $\mathfrak{S} \in K$ with $\mathfrak{A} \rightarrowtail \mathfrak{S}^+$. Hence $\mathfrak{Cs}\,\mathfrak{S}^+ \twoheadrightarrow \mathfrak{Cs}\,\mathfrak{A}$ (Corollary 3.2.5). Since $\mathfrak{Cs}\,\mathfrak{S}^+ = \mathfrak{Ce}\,\mathfrak{S}$, closure of $K$ under canonical extensions and bounded epimorphic images then gives $\mathfrak{Cs}\,\mathfrak{A} \in K$.

(2) If $\mathfrak{A} \in \mathbf{HSK}^+$, then $\mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}$ for some $\mathfrak{A}_1 \in \mathbf{SK}^+$. But then $\mathfrak{Cs}\,\mathfrak{A} \rightarrowtail \mathfrak{Cs}\,\mathfrak{A}_1$ (Corollary 3.2.5), so $\mathfrak{Cs}\,\mathfrak{A}$ is isomorphic to an inner substructure of $\mathfrak{Cs}\,\mathfrak{A}_1$ (Corollary 3.2.3). But $\mathfrak{Cs}\,\mathfrak{A}_1 \in K$ by part (1), and $K$ is closed under inner substructures and isomorphic images (as a special type of bounded epimorphic image), so $\mathfrak{Cs}\,\mathfrak{A} \in K$. □

**Theorem 3.4.4.** *Let $K$ be a class of structures closed under canonical extensions. Then $\mathbf{SK}^+$ is a variety if $K$ is closed under bounded epimorphic images, inner substructures, and disjoint unions.*

**Proof.** by Theorem 3.4.2, it is enough to show $\mathbf{HSK}^+ \subseteq \mathbf{SK}^+$. But if $\mathfrak{A} \in \mathbf{HSK}^+$, then by 3.4.3(2), $\mathfrak{Cs}\,\mathfrak{A} \in K$, so that $(\mathfrak{Cs}\,\mathfrak{A})^+ \in K^+$, i.e. the canonical embedding algebra $\mathfrak{Em}\,\mathfrak{A} = (\mathfrak{Cs}\,\mathfrak{A})^+$ of $\mathfrak{A}$ is in $K^+$. Since $\mathfrak{A} \rightarrowtail \mathfrak{Em}\,\mathfrak{A}$ (Section 3.1), $\mathfrak{A} \in \mathbf{SK}^+$. □

The closure conditions on $K$ of Theorem 3.4.4 are not necessary for $\mathbf{SK}^+$ to be a variety. For instance, if $K$ is the class of *infinite* quasi-orders, then $\mathbf{SK}^+$ is the variety $V_{\mathrm{cl}}$ of closure algebras, but $K$ is not closed under bounded epimorphic images or inner substructures (even though it is closed under canonical extensions—cf. Jónsson and Tarski [35, Section 3], and also the remarks following Corollary 3.4.6 below). To obtain necessary conditions, we must consider the role of the class $K_{\mathbf{S}}$ of relational structures defined by

$$K_{\mathbf{S}} = \{\mathfrak{S} : \mathfrak{S}^+ \in \mathbf{SK}^+\}.$$

**Theorem 3.4.5.** (1) *$K_{\mathbf{S}}$ is the largest class $M$ of structures satisfying $\mathbf{SM}^+ = \mathbf{SK}^+$.*

(2) *$K_{\mathbf{S}}$ is closed under bounded epimorphic images.*

(3) *If $K$ is closed under disjoint unions, then so is $K_{\mathbf{S}}$.*

(4) *$\mathbf{SK}^+$ is closed under direct products iff $K_{\mathbf{S}}$ is closed under disjoint unions.*

(5) *If $K$ is closed under canonical extensions, then so is $K_{\mathbf{S}}$.*

(6) *If $\mathbf{SK}^+$ is closed under homomorphic images, then $K_{\mathbf{S}}$ is closed under inner substructures.*

(7) *If $\mathbf{SK}^+$ is a variety, then $K_{\mathbf{S}}$ is closed under inner substructures and disjoint unions.*

(8) *If $K_{\mathbf{S}}$ is closed under canonical extensions, bounded epimorphic images, and inner substructures, then $\mathbf{SK}^+$ is closed under homomorphic images.*

(9) *If $K$ is closed under canonical extensions, bounded epimorphic images, and inner substructures, then $K_{\mathbf{S}}$ is closed under inner substructures.*

(10) *If $K$ is closed under canonical extensions, bounded epimorphic images, inner substructures, and disjoint unions, then so is $K_{\mathbf{S}}$.*

**Proof.** (1) Since $K \subseteq K_{\mathbf{S}}$, $\mathbf{SK}^+ \subseteq \mathbf{SK}_{\mathbf{S}}^+$. But if $\mathfrak{A} \in \mathbf{SK}_{\mathbf{S}}^+$, then $\mathfrak{A} \rightarrowtail \mathfrak{S}^+$ for some $\mathfrak{S}$ such that $\mathfrak{S} \in K_{\mathbf{S}}$, and hence $\mathfrak{S}^+ \in \mathbf{SK}^+$. But then $\mathfrak{A} \in \mathbf{SK}^+$. Thus $\mathbf{SK}_{\mathbf{S}}^+ = \mathbf{SK}^+$.

Moreover, if $SM^+ = SK^+$, then $\mathfrak{S} \in M$ implies $\mathfrak{S}^+ \in SK^+$ and so $\mathfrak{S} \in K_S$, showing $M \subseteq K_S$.

(2) If $\mathfrak{S} \in K_S$ and $\mathfrak{S} \twoheadrightarrow \mathfrak{S}_1$, then $\mathfrak{S}_1^+ \rightarrowtail \mathfrak{S}^+$ and $\mathfrak{S}^+ \in SK^+$. Closure of $SK^+$ under subalgebras then gives $\mathfrak{S}_1 \in K_S$.

(3) Suppose $\{\mathfrak{S}_i : i \in I\} \subseteq K_S$. Then for each $i \in I$ there exists $\mathfrak{T}_i \in K$ with $\mathfrak{S}_i^+ \rightarrowtail \mathfrak{T}_i^+$. Then

$$\left(\sum_I \mathfrak{S}_i\right)^+ \cong \prod_I \mathfrak{S}_i^+ \rightarrowtail \prod_I \mathfrak{T}_i^+ \cong \left(\sum_I \mathfrak{T}_i\right)^+.$$

Hence if $(\Sigma_I \mathfrak{T}_i) \in K$, $(\Sigma_I \mathfrak{S}_i)^+ \in SK^+$, so $(\Sigma_I \mathfrak{S}_i) \in K_S$.

(4) Suppose $SK^+$ is closed under direct products. Then if $\{\mathfrak{S}_i : i \in I\} \subseteq K_S$, $(\Sigma_I \mathfrak{S}_i)^+ \cong \Pi_I \mathfrak{S}_i^+ \in SK^+$, so $(\Sigma_I \mathfrak{S}_i) \in K_S$. Conversely, suppose $K_S$ is closed under disjoint unions. Then $PK_S^+ = K_S^+$ by Lemma 3.4.1. But in general $PSW \subseteq SPW$, so $PSK^+ = PSK_S^+ \subseteq SPK_S^+ = SK_S^+ = SK^+$.

(5) Let $\mathfrak{S} \in K_S$, so that $\mathfrak{S}^+ \rightarrowtail \mathfrak{T}^+$ for some $\mathfrak{T} \in K$. Then $\mathfrak{Cs}\,\mathfrak{T}^+ \twoheadrightarrow \mathfrak{Cs}\,\mathfrak{S}^+$, i.e. $\mathfrak{Ce}\,\mathfrak{T} \twoheadrightarrow \mathfrak{Ce}\,\mathfrak{S}$, and so $(\mathfrak{Ce}\,\mathfrak{S})^+ \rightarrowtail (\mathfrak{Ce}\,\mathfrak{T})^+$. But if $K$ is closed under canonical extensions then $(\mathfrak{Ce}\,\mathfrak{T})^+ \in K^+$, so $(\mathfrak{Ce}\,\mathfrak{S})^+ \in SK^+$, making $\mathfrak{Ce}\,\mathfrak{S} \in K_S$.

(6) Suppose $\mathfrak{S} \in \mathfrak{T} \in K_S$. Then $\mathfrak{S} \rightarrowtail \mathfrak{T}$, so $\mathfrak{T}^+ \twoheadrightarrow \mathfrak{S}^+$. But $\mathfrak{T}^+ \in SK^+$, so if $SK^+$ is closed under homomorphic images, it contains $\mathfrak{S}^+$, giving $\mathfrak{S} \in K_S$.

(7) By (4) and (6).

(8) Let $\mathfrak{A} \in HSK^+ = HSK_S^+$. Then applying Lemma 3.4.3(2) to $K_S$, under the hypothesised closure conditions on $K_S$, gives $\mathfrak{Cs}\,\mathfrak{A} \in K_S$. But $\mathfrak{A} \rightarrowtail (\mathfrak{Cs}\,\mathfrak{A})^+ = \mathfrak{Em}\,\mathfrak{A}$, so $\mathfrak{A} \in SK_S^+ = SK^+$.

(9) If $\mathfrak{S} \in \mathfrak{T} \in K_S$, then $\mathfrak{T}^+ \twoheadrightarrow \mathfrak{S}^+$ and $\mathfrak{T}^+ \in SK^+$, so $\mathfrak{S}^+ \in HSK^+$. Then by Lemma 3.4.3(2), $\mathfrak{Cs}\,\mathfrak{S}^+ \in K$. But $\mathfrak{S}^+ \rightarrowtail \mathfrak{Em}\,\mathfrak{S}^+ = (\mathfrak{Cs}\,\mathfrak{S}^+)^+$, so $\mathfrak{S}^+ \in SK^+$, i.e. $\mathfrak{S} \in K_S$.

(10) By (2), (3), (5), and (9). □

**Corollary 3.4.6.** *If $K_S$ is closed under canonical extensions, then $SK^+$ is a variety iff $K_S$ is closed under inner substructures and disjoint unions.*

**Proof.** Assume $K_S$ closed under canonical extensions. Then since $K_S$ is always closed under bounded epimorphic images (3.4.5(2)), closure under inner substructures and disjoint unions yields, by Theorem 3.4.4 that $SK_S^+$ is a variety. But $SK_S^+ = SK^+$.

The converse is part (7) of 3.4.5. □

Corollary 3.4.6 uses a weakening of the basic hypothesis of Theorem 3.4.4 (cf. 3.4.5(5)). To show that this is a proper weakening, i.e. that closure of $K_S$ under canonical extensions does not imply the same closure for $K$, consider the variety $V_{cl}$ of closure algebras. If $\mathfrak{A}$ is a closure algebra, then so is $\mathfrak{Em}\,\mathfrak{A} = (\mathfrak{Cs}\,\mathfrak{A})^+$, and if $\mathfrak{S} = (X, R)$ with $R$ binary, then $\mathfrak{S}^+$ is a closure algebra iff $\mathfrak{S}$ is a quasi-order (Jónsson and Tarski [35]). Thus if $\mathfrak{S}$ is a quasi-order, then so is $\mathfrak{Ce}\,\mathfrak{S} = \mathfrak{Cs}\,\mathfrak{S}^+$,

i.e. quasi-orders are preserved by canonical extensions. But for any quasi-order $\mathfrak{S}$ there is a partial-order $\mathfrak{T}$ with $\mathfrak{S}^+ \rightarrowtail \mathfrak{T}^+$ (cf. Grzegorczyk [26], or apply the 'bulldozer' technique of Segerberg [44] to get $\mathfrak{T} \twoheadrightarrow \mathfrak{S}$). These facts show that $V_{cl} = SK^+_{po}$, where $K_{po}$ is the class of partial orders, and that $(K_{po})_S$ is the class of all quasi-orders. Thus $(K_{po})_S$ is closed under canonical extensions, but $K_{po}$ is not itself so closed, as the following example shows.

Let $\mathfrak{A} = (\omega, \geqslant) \in K_{po}$. The hemimorphism $\exists$ on $\mathrm{Sb}(\omega)$ induced by $\geqslant$ has

$$\exists(Y) = \{z : \exists y \geqslant z\ (y \in Y)\},$$

so that if $Y$ is infinite, and hence has no last element, then $\exists(Y) = \omega$. Thus if $F$ and $G$ are two distinct non-principal ultrafilters on $\omega$, then all members of $F$ and $G$ are infinite, so $Y \in F$ implies $\exists(Y) \in G$, and $Y \in G$ implies $\exists(Y) \in F$. This means that $R_\exists(F, G)$ and $R_\exists(G, F)$ in $\mathfrak{Ce}(\omega, \geqslant)$, so the latter is not a partial-order.

We end this subsection by considering the question as to when $K = K_S$. For this we introduce the expression "*$K$ reflects canonical extensions*" to mean that

$$\text{if } \mathfrak{Ce}\,\mathfrak{S} \in K, \quad \text{then } \mathfrak{S} \in K.$$

**Theorem 3.4.7.** (1) *If $K = K_S$, then $K$ is closed under bounded epimorphic images and reflects canonical extensions.*

(2) *If $K$ is closed under canonical extensions, then $K = K_S$ iff $K$ is closed under bounded epimorphic images and reflects canonical extensions.*

**Proof.** (1) Closure of $K_S$ under bounded epimorphic images was shown in 3.4.5(2). But $\mathfrak{S}^+ \rightarrowtail (\mathfrak{Ce}\,\mathfrak{S})^+$ since $(\mathfrak{Ce}\,\mathfrak{S})^+ = (\mathfrak{Cs}\,\mathfrak{S}^+)^+ = \mathfrak{Em}\,\mathfrak{S}^+$, so $\mathfrak{S}^+$ is in $SK^+$ whenever $(\mathfrak{Ce}\,\mathfrak{S})^+$ is, i.e. $\mathfrak{Ce}\,\mathfrak{S} \in K_S$ implies $\mathfrak{S} \in K_S$.

(2) Suppose $K$ is closed under, and reflects, canonical extensions, and is closed under bounded epimorphic images. Let $\mathfrak{S} \in K_S$. Then $\mathfrak{S}^+ \rightarrowtail \mathfrak{T}^+$ for some $\mathfrak{T} \in K$. Hence $\mathfrak{Ce}\,\mathfrak{T} \twoheadrightarrow \mathfrak{Ce}\,\mathfrak{S}$, so the stated closure conditions give $\mathfrak{Ce}\,\mathfrak{T} \in K$, then $\mathfrak{Ce}\,\mathfrak{S} \in K$, and finally $\mathfrak{S} \in K$. This shows that $K_S = K$. □

### *3.5. Canonical varieties*

A class $W$ of algebras will be called *canonical* if it is closed under canonical embedding algebras, i.e.

$$\mathfrak{A} \in W \quad \text{implies} \quad \mathfrak{Em}\,\mathfrak{A} \in W.$$

**Lemma 3.5.1.** *Let $K$ be any class of structures. Then $SK^+$ is canonical iff $K_S$ is closed under canonical extensions.*

**Proof.** If $SK^+$ is canonical, then $\mathfrak{S} \in K_S$ implies $\mathfrak{S}^+ \in SK^+$, so $\mathfrak{Em}\,\mathfrak{S}^+ = (\mathfrak{Ce}\,\mathfrak{S})^+ \in SK^+$, and hence $\mathfrak{Ce}\,\mathfrak{S} \in K_S$.

Conversely, suppose $K_S$ is closed under canonical extensions. Let $\mathfrak{A} \in SK^+$.

Then $\mathfrak{A} \rightarrowtail \mathfrak{S}^+$ for some $\mathfrak{S} \in K \subseteq K_S$. Hence $\mathfrak{Em}\,\mathfrak{A} \rightarrowtail \mathfrak{Em}\,\mathfrak{S}^+ = (\mathfrak{Ce}\,\mathfrak{S})^+$. But $\mathfrak{Ce}\,\mathfrak{S} \in K_S$, by hypothesis, so this gives $\mathfrak{Em}\,\mathfrak{A} \in SK_S^+ = SK^+$ as desired. □

**Corollary 3.5.2.** *If $K$ is closed under canonical extensions, then $SK^+$ is canonical.*

**Proof.** By Theorem 3.4.5(5). □

Note that, by the discussion following 3.4.6, the class of closure algebras is a canonical variety representable in the form $SK^+$ with $K$ not closed under canonical extensions.

It is readily seen that

*Any canonical variety $V$ is a complex variety.*

For, if $K_{cs} = \{\mathfrak{Cs}\,\mathfrak{A}: \mathfrak{A} \in V\}$, then since $(\mathfrak{Cs}\,\mathfrak{A})^+ = \mathfrak{Em}\,\mathfrak{A}$, if $V$ is canonical we get $K_{cs}^+ \subseteq V$, and hence $V = SK_{cs}^+$ since $\mathfrak{A} \rightarrowtail \mathfrak{Em}\,\mathfrak{A}$. It seems prima facie possible that there are non-canonical complex varieties, but no example appears to be known.

The results of the previous subsection can be strengthened as follows.

**Theorem 3.5.3.** *For any class of structures $K$:*

(1) *If $K$ is closed under canonical extensions, bounded epimorphic images, inner substructures, and disjoint unions, then $SK^+$ is a canonical variety.*

(2) *$SK^+$ is a canonical variety iff $K_S$ is closed under canonical extensions, inner substructures, and disjoint unions.*

**Proof.** (1) By Theorem 3.4.4 and Corollary 3.5.2.

(2) From left to right by Theorem 3.4.5(7) and Lemma 3.5.1. The converse is by Corollary 3.4.6 and 3.5.1. □

The next result will enable us to explain the relationship between the notions of canonical variety and *canonical modal logic*.

**Theorem 3.5.4.** *A variety $V$ is canonical if, and only if, it contains the canonical embedding algebras of all its infinitely generated free algebras.*

**Proof.** If $\mathfrak{A} \in V$, then there is an infinitely generated free $\mathfrak{A}_1$ in $V$ with $\mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}$. Then $\mathfrak{Em}\,\mathfrak{A}_1 \twoheadrightarrow \mathfrak{Em}\,\mathfrak{A}$ (3.2.5(6)), so if $V$ contains $\mathfrak{Em}\,\mathfrak{A}_1$, it will contain $\mathfrak{Em}\,\mathfrak{A}$ by closure under homomorphic images. □

Now in Section 3.1 it was explained how a modal logic $\Lambda$ has a canonical frame $\mathfrak{S}_\Lambda$ that can be identified with the canonical structure $\mathfrak{Cs}\,\mathfrak{A}_\Lambda$ of the Lindenbaum algebra $\mathfrak{A}_\Lambda$ of $\Lambda$. There is a model on $\mathfrak{S}_\Lambda$ that falsifies all non-$\Lambda$-theorems, so in order to show that $\Lambda$ is determined by some subclass $K$ of its frames, it is enough to prove that $\mathfrak{S}_\Lambda$ is in $K$. Numerous completeness theorems in modal logic use the canonical frame in this way (cf., e.g., Segerberg [44]).

Thus the question as to whether the frame $\mathfrak{Cs}\,\mathfrak{A}_\Lambda$ validates the logic $\Lambda$ is of particular importance. This is equivalent to asking whether the algebra $(\mathfrak{Cs}\,\mathfrak{A}_\Lambda)^+$, i.e. the canonical embedding algebra $\mathfrak{Em}\,\mathfrak{A}_\Lambda$ of the Lindenbaum algebra, is in the variety $V_\Lambda$ of all algebras validating $\Lambda$. Now $\mathfrak{A}_\Lambda$ is itself freely generated by the (equivalence classes of) propositional variables, and these are usually taken to form a denumerable set. However, for any infinite cardinal $\kappa$ we can consider a version of the logic $\Lambda$ based on a language with $\kappa$-many variables, and construct a canonical frame $\mathfrak{S}^\kappa_\Lambda$ identifiable with $\mathfrak{Cs}\,\mathfrak{A}^\kappa_\Lambda$, where $\mathfrak{A}^\kappa_\Lambda$ is the free (Lindenbaum) algebra in $V_\Lambda$ on $\kappa$ generators. $\Lambda$ is called *canonical* if it is validated by all such canonical frames (Fine [17]). This is equivalent to having the canonical embedding algebras $(\mathfrak{Cs}\,\mathfrak{A}^\kappa_\Lambda)^+$ of all infinitely generated free $V_\Lambda$-algebras be in $V_\Lambda$. Thus according to Theorem 3.5.4, a logic $\Lambda$ is validated by all its canonical frames precisely when its variety $V_\Lambda$ of algebras is canonical in the sense introduced here.

The deepest result about canonical varieties will appear in the next section (Theorem 3.6.7), and for that we need the following result. If $K$ is any class of structures, we write $V_K$ for the variety generated by $K^+$, i.e. $V_K = \mathbf{HSP}K^+$.

**Theorem 3.5.5.** *$V_K$ is canonical if, and only if, it contains the canonical embedding algebra of any of its full complex algebras $\mathfrak{S}^+$.*

**Proof.** Suppose that $\mathfrak{S}^+ \in V_K$ implies $\mathfrak{Em}\,\mathfrak{S}^+ \in V_K$. Let $\mathfrak{A}$ be any member of $V_K$, with the intention of showing that $\mathfrak{Em}\,\mathfrak{A} \in V_K$. Now by definition of $V_K$, there exists an algebra $\mathfrak{A}^*$ and a subfamily $\{\mathfrak{S}_i : i \in I\}$ of $K$ such that

$$\mathfrak{A} \leftarrow \mathfrak{A}^* \rightarrowtail \prod_I \mathfrak{S}_i^+ \cong \left(\sum_I \mathfrak{S}_i\right)^+.$$

Putting $\mathfrak{S} = \sum_I \mathfrak{S}_i$, we then get (3.2.5(6)),

$$\mathfrak{Em}\,\mathfrak{A} \leftarrow \mathfrak{Em}\,\mathfrak{A}^* \rightarrowtail \mathfrak{Em}\,\mathfrak{S}^+.$$

But $\mathfrak{S}^+$ is in $V_K$, by closure under products and isomorphism, so closure under homomorphic images and subalgebras, together with the hypothesis on $\mathfrak{S}$, implies $\mathfrak{Em}\,\mathfrak{A} \in V_K$. □

### *3.6. The elementary case*

If $K$ is an elementary class in the sense of first-order logic, i.e. the class of all models of some set of first order sentences, then the hypothesis in Theorem 3.4.4 that $K$ is closed under canonical extensions can be dispensed with! The reason for this is that $\mathfrak{Ce}\,\mathfrak{S}$ is a bounded epimorphic image of an *elementary extension* of $\mathfrak{S}$. The idea for the proof of this is due to Fine [17], who showed that a 2-saturated model of the language of propositional modal logic has a p-morphism onto the canonical model of its modal theory. We adapt that construction here to general relational structures, for which $\omega$-saturation is needed.

Let $L$ be a first-order language, and $\mathfrak{S} = (X, \ldots)$ a realisation of $L$. A collection $\Gamma$ of $L$-formulae is *satisfiable* in $\mathfrak{S}$ if there is a valuation $y = (y_0, y_1, \ldots)$ of the $L$-variables such that $y$ satisfies every member of $\Gamma$. $\Gamma$ is *finitely satisfiable* in $\mathfrak{S}$ if each finite subset of $\Gamma$ is satisfiable in $\mathfrak{S}$. If $x \in X^m$, for some $m$, let $L_x$ be the associated language $L \cup \{\bar{x}_0, \ldots, \bar{x}_{m-1}\}$, where each $\bar{x}_j$ is an individual constant. $\mathfrak{A}$ is $\omega$-saturated if for any $m < \omega$, and any $x \in X^m$, each set of $L_x$-formulae that is finitely satisfiable in the structure $(\mathfrak{A}, x_0, \ldots, x_{m-1})$ is itself satisfiable in this structure.

Now let $\mathfrak{A} = (X, R)$ with $R \subseteq X^{n+1}$. Let $L$ be the first-order language with an $n+1$-placed relation symbol $\bar{R}$, and a one-placed relation symbol $\bar{Y}$ for each subset $Y$ of $X$. Let $\mathfrak{A}_1 = (X_1, R_1, \{Y^* : Y \subseteq X\})$ be an $\omega$-saturated elementary extension of $\mathfrak{A}_L = (X, R, \{Y : Y \subseteq X\})$. Such an extension exists by the theory of saturated models, and indeed $\mathfrak{A}_1$ may be realised as an *ultrapower* of $\mathfrak{A}_L$, so that $(X_1, R_1)$ is an ultrapower of $\mathfrak{A}$ (cf. Chang and Keisler [13], Theorems 6.1.4 and 6.1.8]).

For each $x \in X_1$, let $F_x = \{Y \subseteq X : x \in Y^*\}$. Now the sentences

$$\forall v\, \bar{X}(v), \qquad \neg\exists v\, \bar{\emptyset}(v),$$
$$\forall v\, (\overline{Y \cap Z}(v) \leftrightarrow \bar{Y}(v) \wedge \bar{Z}(v)),$$
$$\forall v\, (\overline{Y \cup Z}(v) \rightarrow \bar{Y}(v) \vee \bar{Z}(v))$$

are all true in $\mathfrak{A}_L$, for any $Y, Z \in \mathrm{Sb}(X)$, and hence are true in the elementary extension $\mathfrak{A}_1$. Thus

$$X \in F_x, \qquad \emptyset \notin F_x,$$
$$Y \cap Z \in F_x \quad \text{iff} \quad Y \in F_x \text{ and } Z \in F_x,$$
$$Y \cup Z \in F_x \quad \text{implies} \quad Y \in F_x \text{ or } Z \in F_x.$$

Hence $F_x$ is a prime filter of $\mathrm{Sb}(X)$, and so putting $\varphi(x) = F_x$ defines a function $\varphi : X_1 \rightarrow X_{\mathrm{Sb}(X)}$. It will now be shown that $\varphi$ is a bounded morphism from $\mathfrak{T} = (X_1, R_1)$ onto $\mathfrak{Ce}\,\mathfrak{S} = (X_{\mathrm{Sb}(X)}, R_{\exists_R})$.

If $G$ is any member of $X_{\mathrm{Sb}(X)}$, put

$$\Gamma = \{\bar{Y}(v) : Y \in G\} \cup \{\neg\bar{Y}(v) : Y \notin G\}.$$

Then $\Gamma$ is finitely satisfiable in $\mathfrak{A}_L$. For, if $I$ and $J$ are finite subsets of $\mathrm{Sb}(X)$ with $I \subseteq G$ and $J \subseteq -G$, then since $G$ is a prime filter, $\bigcap I \nsubseteq \bigcup J$, so there exists $y \in \bigcap I$ with $y \notin \bigcup J$. Hence $\Gamma$ is finitely satisfiable in $\mathfrak{A}_1$. By $\omega$-saturation, it follows that there is some $x \in X_1$ that satisfies $\Gamma$ in $\mathfrak{A}_1$, so that $F_x = G$. This shows that $\varphi$ maps $X_1$ *onto* $X_{\mathrm{Sb}(X)}$.

Next, to show that $\varphi$ is a morphism, suppose that $R_1(x)$, where $x \in X_1^{n+1}$. Let $Y_i \in F_{x_i}$ for all $i < n$. If $Y = \exists_R(Y_0, \ldots, Y_{n-1})$, then the sentence

$$\forall v_0 \cdots \forall v_n\, (\bar{Y}_0(v_0) \wedge \cdots \wedge \bar{Y}_{n-1}(v_{n-1}) \wedge \bar{R}(v_0, \ldots, v_n) \rightarrow \bar{Y}(v_n))$$

is true in $\mathfrak{A}_L$, and hence true in $\mathfrak{A}_1$. Since $x_i \in Y_i^*$ for all $i < n$, this implies that

$x_n \in Y^* = (\exists_R(Y_0, \ldots, Y_{n-1}))^*$, and so $\exists_R(Y_0, \ldots, Y_{n-1}) \in F_{x_n}$. This shows that $R_{\exists_R}(F_{x_0}, \ldots, F_{x_n})$, i.e. $R_{\exists_R}(\varphi(x))$ as desired.

Finally, we show that $\varphi$ is bounded. Suppose that $R_{\exists_R}(G, \varphi(z))$, where $z \in X_1$ and $G \in X^n_{\mathrm{Sb}(X)}$. Then we must find an $x \in X_1^n$ with $R_1(x, z)$ and $\varphi(x) = G$. Let $\Gamma$ be the following set of formulae in the variables $v_0, \ldots, v_{n-1}$:

$$\Gamma = \{\bar{R}(v_0, \ldots, v_{n-1}, \bar{z})\} \cup \{\bar{Y}(v_0): Y \in G_0\} \cup \cdots \cup \{\bar{Y}(v_{n-1}): Y \in G_{n-1}\}.$$

Now if $x$ satisfies $\Gamma$ in $\mathfrak{A}_1$ when the variable $v_i$ is interpreted as $x_i$, then $R_1(x, z)$, and for each $i < n$, $x_i \in Y^*$ for all $Y \in G_i$, so that $G_i \subseteq F_{x_i} = \varphi(x_i)$. Since $G_i$ is a maximal filter of $\mathrm{Sb}(X)$, this gives $G_i = \varphi(x_i)$ as desired (note that if $\mathfrak{S}$ were an ordered structure, and $G_i$ a prime filter of the distributive lattice $\mathfrak{S}^+$, this would show that $\varphi$ was a bounded morphism of the type appropriate for ordered structures, as discussed in Section 2.3).

Thus it remains to show that $\Gamma$ is satisfiable in $\mathfrak{A}_1$. By the $\omega$-saturation of $\mathfrak{A}_1$, and the closure of each $G_i$ under finite intersections, it is enough to show that if $Y_i \in G_i$ for $i < n$, then

$$\Gamma_0 = \{\bar{R}(v_0, \ldots, v_{n-1}, \bar{z}),\ \bar{Y}_0(v_0), \ldots, \bar{Y}_{n-1}(v_{n-1})\}$$

is satisfiable in $\mathfrak{A}_1$. But given such $Y_i$, since $R_{\exists_R}(G, \varphi(z))$, it follows that $\exists_R(Y_0, \ldots, Y_{n-1})$ belongs to $\varphi(z)$, i.e. $z \in Y^*$, where $Y = \exists_R(Y_0, \ldots, Y_{n-1})$. But for this $Y$, the sentence

$$\forall v\,(\bar{Y}(v) \to \exists v_0 \cdots \exists v_{n-1}(\bar{Y}_0(v_0) \wedge \cdots \wedge \bar{Y}_{n-1}(v_{n-1}) \wedge \bar{R}(v_0, \ldots, v_{n-1}, v)))$$

is true in $\mathfrak{A}_L$, hence true in $\mathfrak{A}_1$. Hence there exists $x \in X_1^n$ such that $x_i \in Y_i^*$ for all $i < n$, and $R_1(x, z)$. But then $x$ satisfies $\Gamma_0$ in $\mathfrak{A}_1$ as required.

This completes the proof of

**Theorem 3.6.1.** *For any relational structure $\mathfrak{S}$ there is an ultrapower $\mathfrak{T}$ of $\mathfrak{S}$ having a bounded epimorphism $\mathfrak{T} \twoheadrightarrow \mathfrak{C} \in \mathfrak{S}$.* □

Note that the set $\Gamma$ used in proving $\varphi$ bounded had only a single constant $\bar{z}$ extending $L$. However the reader should not be misled thereby into thinking that the hypothesis of $\omega$-saturation is stronger than necessary. In dealing with $n+1$-ary relations it was necessary for $\Gamma$ to have $n$ free variables. Since a general relational structure may have relations of arbitrarily large finite arity, the fully general version of 3.6.1 requires consideration of sets of formulae with an arbitrarily large finite set of free variables. But the definition and theory of 'saturation' is usually carried out in terms of satisfaction of sets $\Gamma$ with only one free variable. The essential point is that *for $\omega$-saturated models*, that more restricted definition is equivalent to the one used here (cf. Monk [40, Proposition 28.2], or Barwise [2, Lemma IV.5.2]).

**Theorem 3.6.2.** (1) *If $K$ is a class of structures closed under ultrapowers and bounded epimorphic images, then $K$ is closed under canonical extensions.*

(2) *If $K$ is closed under ultrapowers, then $K_S$ is closed under canonical extensions.*

**Proof.** (1) Immediate from 3.6.1.

(2) Let $\mathfrak{S} \in K_S$, so that $\mathfrak{S}^+ \rightarrowtail \mathfrak{S}_1^+$ for some $\mathfrak{S}_1 \in K$. Let $\mathfrak{T}$ be an ultrapower of $\mathfrak{S}_1$ for which $\mathfrak{T} \twoheadrightarrow \mathfrak{Ce}\,\mathfrak{S}_1$. But $\mathfrak{Cs}\,\mathfrak{S}_1^+ \twoheadrightarrow \mathfrak{Cs}\,\mathfrak{S}^+$, i.e. $\mathfrak{Ce}\,\mathfrak{S}_1 \twoheadrightarrow \mathfrak{Ce}\,\mathfrak{S}$, yielding $\mathfrak{T} \twoheadrightarrow \mathfrak{Ce}\,\mathfrak{S}$ by composition. Since $K$ is closed under ultrapowers, $\mathfrak{T} \in K \subseteq K_S$. But $K_S$ is closed under bounded epimorphic images (3.4.5(2)), so this gives $\mathfrak{Ce}\,\mathfrak{S} \in K_S$. □

**Corollary 3.6.3.** *If $K$ is closed under ultrapowers, then $SK^+$ is canonical.*

**Proof.** $SK^+$ is canonical iff $K_S$ is closed under canonical extensions (Lemma 3.5.1). □

The main results of Sections 3.4 and 3.5 can now be adapted as follows.

**Theorem 3.6.4.** *Let $K$ be closed under ultrapowers.*

(1) *If $K$ is closed under bounded epimorphic images, inner substructures and disjoint unions, then $SK^+$ is a canonical (and hence complex) variety.*

(2) *$SK^+$ is a canonical variety iff $K_S$ is closed under inner substructures and disjoint unions.*

(3) *$K = K_S$ iff $K$ is closed under bounded epimorphic images and reflects canonical extensions.*

**Proof.** By 3.4.7, 3.5.3, and 3.6.2. □

Instead of closure under ultrapowers in 3.6.4, one could hypothesise closure under ($\omega$-saturated) elementary extensions, or under elementary equivalence, any of which is satisfied if $K$ is an elementary class.

A refinement of Fine's saturated-model construction was developed by van Benthem [8], who combined it with a model-theoretic compactness argument to prove that any modal logic which is determined by some elementary class of frames must be validated by its canonical frame. We now consider an algebraic version of this result, to the effect that if $K$ is elementary, then $K^+$ generates a canonical variety. Our proof uses the duality of Section 3.2, and two new ingredients: Jónsson's Theorem, and ultraproducts.

First, some terminology. If $W$ is a class of algebras, we write $V(W)$ for the smallest variety HSP$W$ containing $W$, and Up$W$ for the class of all *ultraproducts* of members of $W$. Likewise, Up$K$ is the closure under ultraproducts of a class $K$ of structures (familiarity with ultraproducts is assumed). Also, a class of algebras

is called *congruence-distributive* if the lattice of congruences of each member of the class is distributive.

**Jónsson's Theorem.** *If $V(W)$ is a congruence-distributive variety, and $\mathfrak{A}$ is a subdirectly irreducible algebra in $V(W)$, then $\mathfrak{A} \in \mathrm{HSUp}W$.*

**Proof.** Cf. Jónsson [34]. □

This fact is relevant to the present context because Boolean algebras with operators are congruence-distributive. Indeed, all lattices are congruence-distributive (Jónsson [34]), so if $\mathfrak{A}$ is any type of algebra having the form of a lattice $\mathfrak{L}$ with additional operations, then the congruence lattice of $\mathfrak{A}$ is a sublattice of the distributive congruence lattice of $\mathfrak{L}$, and hence is distributive itself.

Next we need to know the relationship between an ultraproduct of complex algebras, and the complex algebra of the ultraproduct of the associated structures.

**Lemma 3.6.5.** *If $\{\mathfrak{S}_i : i \in I\}$ is a family of structures, and $U$ is an ultrafilter on $I$, then*

$$\prod_I \mathfrak{S}_i^+/U \rightarrowtail \left(\prod_I \mathfrak{S}_i/U\right)^+.$$

**Proof.** $\prod_I \mathfrak{S}_i^+/U$ is the ultraproduct of the algebras $\mathfrak{S}_i^+$ with respect to $U$, i.e. the quotient of $\prod_I \mathfrak{S}_i^+$ by the congruence $\sim_U$, where

$$\psi \sim_U \chi \quad \text{iff} \quad \{i : \psi(i) = \chi(i)\} \in U.$$

The elements of $\prod_I \mathfrak{S}_i^+/U$, are denoted $\psi/U$, where $\psi \in \prod_I \mathfrak{S}_i^+$. Similarly, the elements of the ultraproduct $\prod_I \mathfrak{S}_i/U$ of the $\mathfrak{S}_i$ are denoted $g/U$, where $g \in \prod_I \mathfrak{S}_i$.

A function $\theta : \prod_I \mathrm{Sb}(X_i)/U \rightarrow \mathrm{Sb}(\prod_I X_i/U)$ that proves the lemma is given by

$$g/U \in \theta(\psi/U) \quad \text{iff} \quad \{i : g(i) \in \psi(i)\} \in U.$$

It is left to the reader to verify that $\theta$ is a well-defined injective homomorphism from $\prod_I \mathfrak{S}_i^+/U$ to $(\prod_I \mathfrak{S}_i/U)^+$ (cf. Goldblatt [20, Theorems 7.5 and 7.7]). □

Note that $(\prod_I \mathfrak{S}_i/U)^+$ is of higher cardinality than $(\prod_I \mathfrak{S}_i/U)$, which will itself in many cases be of the same cardinality as $(\prod_I \mathfrak{S}_i^+/U)$ (Bell and Slomson [3, Ch. 6, §3]). Hence the injection $\theta$ in the proof just given will not in general be an isomorphism.

**Corollary 3.6.6.** *Let $K$ be closed under ultraproducts. Then $\mathrm{HSUp}K^+ \subseteq \mathrm{HS}K^+$, and so if $\mathfrak{A}$ is a subdirectly irreducible algebra in the variety $\mathrm{HSP}K^+$ generated by $K^+$, it follows that $\mathfrak{A} \in \mathrm{HS}K^+$.*

**Proof.** Lemma 3.6.5 implies that $\mathrm{Up}K^+ \subseteq \mathrm{S}(\mathrm{Up}K)^+$. Thus if $\mathrm{Up}K \subseteq K$, we get

$$\mathrm{HSUp}K^+ \subseteq \mathrm{HSS}(\mathrm{Up}K)^+ \subseteq \mathrm{HS}K^+.$$

The rest of the proof then follows from Jónsson's Theorem. □

We are now ready for the result mentioned above.

**Theorem 3.6.7.** *If $K$ is closed under ultraproducts (e.g. if $K$ is elementary), then $V_K = \mathrm{HSP}K^+$ is a canonical, and hence complex, variety.*

**Proof.** By Theorem 3.5.5, it is enough to prove that $V_K$ is closed under canonical embedding algebras of the form $\mathfrak{Em}\,\mathfrak{S}^+$. So, let $\mathfrak{S} = (X, R)$ be a structure having $\mathfrak{S}^+ \in V_K$. We prove $\mathfrak{Em}\,\mathfrak{S}^+ \in V_K$.

For each $x \in X$, let $\sim_x$ be a maximal congruence of $\mathfrak{S}^+$ separating $\{x\}$ and $\emptyset$. The resulting quotient algebra $\mathfrak{A}_x = \mathfrak{S}^+/\sim_x$ is subdirectly irreducible (Henkin et al. [28, Corollary 0.3.53]). The natural homomorphism $\pi_x\colon \mathfrak{S}^+ \twoheadrightarrow \mathfrak{A}_x$ guarantees that $\mathfrak{A}_x \in V_K$, by closure under homomorphic images. Note that $\pi_x(\{x\}) \neq 0$ by definition of $\sim_x$. Let

$$\pi\colon \mathfrak{S}^+ \to \prod_X \mathfrak{A}_x$$

be the product map $\pi(Y) = \langle \pi_x(Y)\colon x \in X\rangle$.

**Lemma 1.** *For any ultrafilter $U \in \mathfrak{Ce}\,\mathfrak{S}$ there exists $F \in \mathfrak{Cs}\,(\prod_X \mathfrak{A}_x/U)$ such that $\theta^{-1}(F) = U$, where*

$$\theta\colon \mathfrak{S}^+ \to \prod_X \mathfrak{A}_x/U$$

*is the composite of $\pi$ with the natural map*

$$\prod_X \mathfrak{A}_x \to \prod_X \mathfrak{A}_x/U,$$

*i.e. $\theta(Y) = \langle \pi_x(Y)\colon x \in X\rangle/U$.*

**Proof.** The set $\theta(U) = \{\theta(Y)\colon Y \in U\}$ is closed under finite meets, as $U$ is so closed and the homomorphism $\theta$ preserves meets. Also, if $Y \in \mathfrak{S}^+$, then $x \in Y$ implies $0 \neq \pi_x(\{x\}) \leqslant \pi_x(Y)$, and so

$$Y \subseteq \{x\colon \pi_x(Y) \neq 0\}.$$

Thus if $Y \in U$, then $\{x\colon \pi_x(Y) \neq 0\} \in U$, implying that $\theta(Y) \neq 0$. Hence $\theta(U)$ has the finite-meet-property (i.e. is separated from $\{0\}$), and so there is a maximal filter $F$ of $\prod_X \mathfrak{A}_x/U$ with $\theta(U) \subseteq F$. But then $U \subseteq \theta^{-1}(F) \in \mathfrak{Ce}\,\mathfrak{S}$, and so $U = \theta^{-1}(F)$, as desired.

(A more constructive proof of this lemma is discussed at the end of the proof of the main theorem below.) □

**Lemma 2.** *For each $U \in \mathfrak{Ce}\,\mathfrak{S}$ there exists an inner substructure $\mathfrak{T}_U$ of $\mathfrak{Ce}\,\mathfrak{S}$ with $U \in \mathfrak{T}_U$ and $\mathfrak{T}_U^+ \in V_K$.*

**Proof.** Let $\mathfrak{T}_U = \operatorname{Im} \theta_+$, the image of the bounded morphism

$$\theta_+ : \mathfrak{Cs}\left(\prod_X \mathfrak{A}_x/U\right) \to \mathfrak{Cs}\,\mathfrak{S}^+ = \mathfrak{Ce}\,\mathfrak{S}$$

induced by the homomorphism $\theta$. Then $\mathfrak{T}_U \in \mathfrak{Ce}\,\mathfrak{S}$, by Corollary 3.2.3. Lemma 1 above states that $U = \theta_+(F)$ for some $F$, so $U \in \mathfrak{T}_U$. Since

$$\mathfrak{Cs}\left(\prod_X \mathfrak{A}_x/U\right) \twoheadrightarrow \mathfrak{T}_U,$$

we get

$$\mathfrak{T}_U^+ \rightarrowtail \left(\mathfrak{Cs}\left(\prod_X \mathfrak{A}_x/U\right)\right)^+ = \mathfrak{Em}\left(\prod_X \mathfrak{A}_x/U\right). \tag{1}$$

Now by Corollary 3.6.6, for each $x \in X$, since $\mathfrak{A}_x$ is subdirectly irreducible there exists an algebra $\mathfrak{A}_x^*$ and a structure $\mathfrak{T}_x \in K$ such that

$$\mathfrak{A}_x \leftarrow \mathfrak{A}_x^* \rightarrowtail \mathfrak{T}_x^+.$$

Forming ultraproducts with respect to $U$ then gives

$$\prod_X \mathfrak{A}_x/U \leftarrow \prod_X \mathfrak{A}_x^*/U \rightarrowtail \prod_X \mathfrak{T}_x^+/U. \tag{2}$$

Let $\mathfrak{T} = \prod_X \mathfrak{T}_x/U$. Then $\prod_X \mathfrak{T}_x^+/U \rightarrowtail \mathfrak{T}^+$, by Lemma 3.6.5, so combining this with (2), applying duality (3.2.5), and adding (1), gives

$$\mathfrak{T}_U^+ \rightarrowtail \mathfrak{Em}\left(\prod_X \mathfrak{A}_x/U\right) \leftarrow \mathfrak{Em}\left(\prod_X \mathfrak{A}_x^*/U\right) \rightarrowtail \mathfrak{Em}\,\mathfrak{T}^+ = (\mathfrak{Ce}\,\mathfrak{T})^+. \tag{3}$$

Now $\mathfrak{T} \in K$, by closure under ultraproducts, and so by Theorem 3.6.1 there is an ultrapower $\mathfrak{T}_\#$ of $\mathfrak{T}$ with a bounded epimorphism $\mathfrak{T}_\# \twoheadrightarrow \mathfrak{Ce}\,\mathfrak{T}$. Hence

$$(\mathfrak{Ce}\,\mathfrak{T})^+ \rightarrowtail \mathfrak{T}_\#^+. \tag{4}$$

But, again, $\mathfrak{T}_\# \in K$, so $\mathfrak{T}_\#^+ \in K^+ \subseteq V_K$. Since $V_K$ is closed under subalgebras and homomorphic images, (4) and then (3) yield $\mathfrak{T}_U^+ \in V_K$. □

Now to complete the proof that $\mathfrak{Em}\,\mathfrak{S}^+ \in V_K$, observe that by Lemma 2, the bounded morphism

$$\sum_{U \in \mathfrak{Ce}\,\mathfrak{S}} \mathfrak{T}_U \to \mathfrak{Ce}\,\mathfrak{S},$$

determined by the inclusions $\mathfrak{T}_U \hookrightarrow \mathfrak{Ce}\,\mathfrak{S}$, is surjective, and so induces

$$(\mathfrak{Ce}\,\mathfrak{S})^+ \rightarrowtail \left(\sum_{U \in \mathfrak{Ce}\,\mathfrak{S}} \mathfrak{T}_U\right)^+ \cong \prod_{U \in \mathfrak{Ce}\,\mathfrak{S}} \mathfrak{T}_U^+.$$

But each $\mathfrak{T}_U^+$ is in the variety $V_K$, and so the closure properties of the latter ensure that it contains $(\mathfrak{Ce}\,\mathfrak{S})^+$, i.e. $\mathfrak{Em}\,\mathfrak{S}^+$.

This completes the proof of Theorem 3.6.7. □

In proving Lemma 1 above, it is possible to define a suitable maximal filter $F$ explicitly, rather than appealing to the Prime Filter Theorem. For each $x \in X$, the non-zero element $\pi_x(\{x\})$ is an atom in $\mathfrak{A}_x$, and so $a = (\pi_x(\{x\}): x \in X)/U$ is an atom in $\prod_X \mathfrak{A}_x/U$, by Łoś's Theorem. $F$ can be taken as the principal ultrafilter of $\prod_X \mathfrak{A}_x/U$ generated by $a$ (this observation is due to C. Bailey).

In fact, by the analysis in Section 3.3, $\mathfrak{A}_x$ is isomorphic to the complex algebra $\mathfrak{S}_x^+$, where $\mathfrak{S}_x = (X_x, R_x)$ is the inner substructure of $\mathfrak{S}$ generated by $x$. The associated homomorphism $\mathfrak{S}^+ \twoheadrightarrow \mathfrak{S}_x^+$ acts by $Y \mapsto Y \cap X_x$, and so leaves $\{x\}$ fixed. The ultrafilter $F$ on $\prod_X \mathfrak{S}_x^+/U$ generated by the atom $(\{x\}: x \in X)/U$ has

$$\psi/U \in F \quad \text{iff} \quad \{x: x \in \psi(x)\} \in U.$$

This yields a more informative picture of what is going on in Lemma 1. However, its use depends on the rather complicated demonstration in Theorem 3.3.1 that $\mathfrak{S}_x^+$ is subdirectly irreducible, whereas the argument given follows from more general considerations.

### *3.7. Complete varieties*

A variety $V$ is *complete* if it is generated by full complex algebras, i.e. if there is some class $K$ of structures such that

$$V = V_K = \mathrm{HSP}K^+.$$

Now if

$$K_V = \{\mathfrak{S}: \mathfrak{S}^+ \in V\}$$

is the class of *all* structures whose full complex algebras are in $V$, then whenever $K^+ \subseteq V$, we have $K^+ \subseteq K_V^+ \subseteq V$, so that if $V = V_K$ then $V = V_{K_V}$. Thus

*$V$ is complete iff it is generated by $K_V$, i.e. iff $V = V_{K_V}$.*

(The question as to when $K = K_{V_K}$ will be taken up later in this section.)

It is clear that if $V = \mathrm{S}K^+$, then $V = \mathrm{HSP}K^+$, and so

*Every complex variety is complete.*

This was pointed out in the Introduction, where the logical significance of completeness of the variety $V_\Lambda$ of algebras for a modal logic $\Lambda$ was explained.

**Theorem 3.7.1.** *There exist complete varieties that are not complex.*

**Proof.** The class $V_d$ of diagonalisable algebras is such a variety. A diagonalisable algebra $\mathfrak{A} = (\mathfrak{B}, \exists)$ consists of a unary join-hemimorphism $\exists$ on a Boolean algebra $\mathfrak{B}$ satisfying

$$\exists x \sqsubseteq \exists(x - \exists x). \tag{1}$$

Any such algebra satisfies $\exists\exists x \sqsubseteq \exists x$, corresponding to transitivity of $R$.

A structure $\mathfrak{S} = (X, R)$ has $\mathfrak{S}^+ \in V_d$ if, and only if, $R$ is transitive, and well-founded, the latter meaning that there are no *infinitely descending sequences*

$$\cdots Rx_{n+1}Rx_nR \cdots Rx_0.$$

Indeed, if there is such a sequence, then putting $Y = \{x_n : n < \omega\}$ gives $Y \subseteq \exists_R(Y)$, since $x_{n+1}Rx_n$, so $\exists_R(Y - \exists_R(Y)) = \exists_R(\emptyset) = \emptyset$, while $\exists_R(Y) \neq \emptyset$, showing that (1) is not fulfilled by $\mathfrak{S}^+$.

It is known that $V_d$ is a complete variety. In fact, as mentioned in the Introduction, it is generated by its *finite* full complex algebras. To show that it is not complex, we must exhibit a diagonalisable algebra $\mathfrak{C}$ such that for any structure $\mathfrak{T}$, if $\mathfrak{T}^+$ is diagonalisable then there is no injective homomorphism $\theta: \mathfrak{C} \rightarrowtail \mathfrak{T}^+$. $\mathfrak{C}$ will be obtained as a (proper) subalgebra of $\mathfrak{S}^+$, for some $\mathfrak{S} = (X, R)$, in such a way that there is a sequence $X_0, \ldots, X_n, \ldots$ of members of $\mathfrak{C}$ satisfying

$$\exists_R(X_0) \neq \emptyset, \quad \text{and} \tag{2}$$

$$X_n \subseteq \exists_R(X_{n+1}), \quad \text{for all } n < \omega. \tag{3}$$

To see that this suffices, let $\mathfrak{T} = (Y, S)$ have $\mathfrak{T}^+$ diagonalisable, and suppose that there is an injective homomorphism $\theta: \mathfrak{C} \rightarrowtail \mathfrak{T}^+$. Putting $Y_n = \theta(X_n)$, (2) and (3), and the preservation properties of $\theta$, then imply

$$\exists_S(Y_0) \neq \emptyset, \quad \text{and} \tag{4}$$

$$Y_n \subseteq \exists_S(Y_{n+1}), \quad \text{for all } n < \omega. \tag{5}$$

But from this, an infinitely descending sequence $\cdots Sy_{n+1}Sy_nS \cdots Sy_0$ can be produced, contradicting the hypothesis that $\mathfrak{T}^+ \in V_d$. For, by (4), there exists $x$ such that $y_0Sx$ for some $y_0 \in Y_0$. Then supposing inductively that $y_n \in Y_n$ has been defined, (5) immediately gives a $y_{n+1} \in Y_{n+1}$ such that $y_{n+1}Sy_n$.

Having explained the strategy of the proof, it remains to construct $\mathfrak{C}$ as described above. To begin with, let $\mathfrak{A} = (\mathfrak{B}, \exists)$ be a free diagonal algebra on denumerably many generators $x_0, \ldots, x_n, \ldots$. Consider the elements

$$y_0 = \exists x_0,$$

$$y_{n+1} = (x_n \Rightarrow \exists x_{n+1}) \sqcap \forall(x_n \Rightarrow \exists x_{n+1}), \quad \text{for all } n < \omega,$$

where $\forall$ is the meet-hemimorphism dual to $\exists$, i.e. $\forall x = -\exists -x$, and $\Rightarrow$ is the Boolean relative pseudo-complement $x \Rightarrow y = -x \sqcup y$.

**Lemma.** *$\{y_n : n < \omega\}$ has the finite-meet-property: every finite subset has non-zero meet.*

**Proof.** To show that $y_0 \sqcap \cdots \sqcap y_n \neq 0$ for any given $n$, consider the structure $\mathfrak{S}_n = (X, T)$, where $X = \{-1, 0, \ldots, n\}$, and $iRj$ iff $i > j$. Then $R$ is transitive and well-founded, so $\mathfrak{S}_n^+$ is diagonalisable. The hemimorphisms induced by $R$

have

$$\exists_R(Y) = \{i: \exists j \supset i\ (j \in Y)\}, \quad \text{and} \quad \forall_R(Y) = \{i: \forall j > i\ (j \in Y)\}.$$

In particular $i \in \exists_R(\{i+1\})$, so putting $X_i = \{i\}$ for $0 \leqslant i < n$ gives $X_i \subseteq \exists_R(X_{i+1})$. Hence putting

$$Y_{i+1} = (X_i \Rightarrow \exists_R(X_{i+1})) \cap \forall_R(X_i \Rightarrow \exists_R(X_{i+1})),$$

for $0 \leqslant i < n$ gives

$$Y_{i+1} = X \cap \forall_R(X) = X.$$

But $-1 \in \exists_R(X_0)$, so putting $Y_0 = \exists_R(X_0)$, we have

$$Y_0 \cap \cdots \cap Y_n \neq \emptyset. \tag{†}$$

Now since $\mathfrak{A}$ is free in $V_{\mathrm{d}}$, there is a homomorphism $\theta: \mathfrak{A} \to \mathfrak{S}_n^+$ such that $\theta(x_i) = X_i$ for $i \leqslant n$. Then $\theta(y_i) = Y_i$, so (†) implies $y_0 \sqcap \cdots \sqcap y_n \neq 0$. □

In view of the Lemma, there is a maximal filter $F$ of $\mathfrak{A}$ that contains $\{y_n: n < \omega\}$. Let $\mathfrak{Cs}\,\mathfrak{A} = (X_{\mathfrak{B}}, R_\exists)$ be the canonical structure of $\mathfrak{A}$ (Section 3.1), and $\mathfrak{S} = (X, R)$ the inner substructure of $\mathfrak{Cs}\,\mathfrak{A}$ generated by $F$, i.e. the smallest inner substructure containing $F$ (Section 3.3). Let $\sigma: \mathfrak{A} \rightarrowtail (\mathfrak{Cs}\,\mathfrak{A})^+$ be the canonical embedding function

$$\sigma(x) = \{G \in X_{\mathfrak{B}}: x \in G\},$$

and $\theta: (\mathfrak{Cs}\,\mathfrak{A})^+ \twoheadrightarrow \mathfrak{S}^+$ the homomorphism induced by the inclusion bounded morphism $\mathfrak{S} \hookrightarrow \mathfrak{Cs}\,\mathfrak{A}$, so that $\theta(Z) = Z \cap X$. Finally, let $\mathfrak{C}$ be the image of the homomorphism $\theta \circ \sigma$, and

$$X_n = \theta \circ \sigma(x_n) = \{G \in X: x_n \in G\}.$$

Then $\mathfrak{C}$ is a subalgebra of $\mathfrak{S}^+$ that is diagonalisable, since $\mathfrak{A} \twoheadrightarrow \mathfrak{C}$ and $\mathfrak{A} \in V_{\mathrm{d}}$. Moreover, $\mathfrak{C}$ contains the $X_n$'s for $n < \omega$, and so it remains to show that the clauses (2) and (3) above are satisfied.

Now for any $n$,

$$\exists_R(X_n) = \exists_R(\theta \circ \sigma(x_n)) = \theta \circ \sigma(\exists x_n) = \{G \in X: \exists x_n \in G\}. \tag{6}$$

Thus as $\exists x_0 = y_0 \in F$, we have $F \in \exists_R(X_0)$, giving $\exists_R(X_0) \neq \emptyset$ and proving clause (2).

For clause (3), we use that fact that $R_\exists$ is transitive (since $\mathfrak{A}$ satisfies $\exists\exists x \sqsubseteq \exists x$), to conclude that

$$X = \{F\} \cup \{G \in X_{\mathfrak{B}}: R_\exists(G, F)\}. \tag{7}$$

Then to show that $X_n \subseteq \exists_R(X_{n+1})$, let $G \in X_n$. There are two cases. If $G = F$, then $y_{n+1} \in G$ by construction, and so $(x_n \Rightarrow \exists x_{n+1}) \in G$. But $x_n \in G$, so $\exists x_{n+1} \in G$, giving $G \in \exists_R(X_{n+1})$ by (6). Alternatively, by (7), we have $R_\exists(G, F)$. But

$\forall(x_n \Rightarrow \exists x_{n+1}) \in F$, since $y_{n+1} \in F$, so again $(x_n \Rightarrow \exists x_{n+1}) \in G$, by definition of $R_\exists$ and $\forall$, leading to the same conclusion that $G \in \exists_R(X_{n+1})$.

This completes the proof of Theorem 3.7.1. □

Since $V_d$ is not complex, it cannot be generated by any elementary class of structures. For if it were, by Theorem 3.6.7 it would be canonical, and hence complex. However a simpler proof of this non-generation can be obtained by a compactness argument, as follows.

For each $n$, the equation

$$\underbrace{\exists \cdots \exists}_{n \text{ times}} 1 = 0$$

fails in any complex algebra $\mathfrak{S}^+$ that has an $R$-chain $x_n R \cdots R x_0$ of length $n$, e.g. in the diagonalisable algebra $\mathfrak{S}_n$ used in the lemma in Theorem 3.7.1. Thus if $V_d = V_K$, then since the above equation fails in $V_d$ it must fail in some member of $K^+$. Hence for any $n$, there is a structure in $K$ that contains an $R$-chain of length $n$. But then if $K$ were elementary, by the Compactness Theorem, or an ultraproduct construction, there would be an $\mathfrak{S} \in K$ that contained an infinitely descending sequence $\cdots R x_{n+1} R x_n R \cdots R x_0$, which is impossible since $\mathfrak{S}^+ \in V_d$.

In the Introduction it was mentioned that the class of finite partial orders generates a proper subvariety of the variety of closure algebras that is not complex. A proof of the latter fact can be obtained by a modification of the argument of Theorem 3.7.1, in which the $y_n$'s are modified to the ele- ·s

$$y_0 = x_0, \quad \text{and}$$

$$y_{n+1} = \forall(x_n \Rightarrow \exists(-x_0 \sqcap \cdots \sqcap -x_n \sqcap x_{n+1})).$$

We turn now to the study of general properties of complete varieties, focusing on the role of the class $K_V = \{\mathfrak{S}: \mathfrak{S}^+ \in V\}$.

**Theorem 3.7.2.** *Let $V$ be any variety.*

(1) $(K_V)_S = K_V$.

(2) *$K_V$ is closed under bounded epimorphic images, inner substructures, and disjoint unions, and reflects canonical extensions.*

(3) *The variety generated by $K_V$ is $HSK_V^+$.*

(4) *If $W$ is a canonical subvariety of $V$, then $W \subseteq SK_V^+$.*

**Proof.** (1) If $\mathfrak{T} \in (K_V)_S$, then $\mathfrak{T}^+ \in SK_V^+ \subseteq V$, giving $\mathfrak{T} \in K_V$.

(2) Closure under bounded epimorphic images follows from (1) by the argument of Theorem 3.4.5(2). Or, observe that if $\mathfrak{S} \in K_V$ and $\mathfrak{S} \twoheadrightarrow \mathfrak{T}$, then $\mathfrak{T}^+ \rightarrowtail \mathfrak{S}^+ \in V$, so $\mathfrak{T}^+ \in V$, since $V$ is closed under subalgebras. For inner substructures, if $\mathfrak{T} \Subset \mathfrak{S} \in K_V$, then $\mathfrak{S}^+ \twoheadrightarrow \mathfrak{T}^+$, giving $\mathfrak{T}^+ \in V$ since $V$ is closed under homomorphic images. Closure under disjoint unions follows from $(\Sigma_I \mathfrak{S}_i)^+ \cong \Pi_I \mathfrak{S}_i^+$ and closure of $V$ under direct products. Finally, for reflection

of canonical extensions, if $\mathfrak{Ce}\,\mathfrak{S} \in K_V$, then as $\mathfrak{S}^+ \rightarrowtail (\mathfrak{Ce}\,\mathfrak{S})^+ \in \mathbf{V}$, we get $\mathfrak{S}^+ \in V$ as desired.

(3) Since $K_V$ is closed under disjoint unions, $PK_V^+ = K_V^+$.

(4) If $W$ is a canonical subvariety of $V$, then $\mathfrak{A} \in W$ implies $\mathfrak{Em}\,\mathfrak{A} = (\mathfrak{Cs}\,\mathfrak{A})^+ \in V$, and so $\mathfrak{Cs}\,\mathfrak{A} \in K_V$. But $\mathfrak{A} \rightarrowtail \mathfrak{Em}\,\mathfrak{A}$, and so $\mathfrak{A} \in SK_V^+$. □

**Corollary 3.7.3.** *For any variety $V$, the following are equivalent.*

(1) *$K_V$ is closed under canonical extensions.*

(2) *$SK_V^+$ is a canonical variety.*

(3) *$SK_V^+$ is the largest canonical subvariety of $V$.*

**Proof.** As $K_V$ is closed under inner substructures and disjoint unions, and $K_V = (K_V)_S$, the equivalence of (1) and (2) is given by Theorem 3.5.3(2). The equivalence of (2) and (3) is by 3.7.2(4) □

Now Theorem 3.5.5 stated in effect that a complete variety $V$ is canonical iff $K_V$ is closed under canonical extensions. This can be reformulated in the light of the last theorems as

**Theorem 3.7.4.** *A variety $V$ is canonical if, and only if, it is complete and $K_V$ is closed under canonical extensions.*

**Proof.** If $V$ is canonical, then it is complex and hence complete, while closure of $K_V$ under canonical extensions means that $\mathfrak{S}^+ \in V$ implies $(\mathfrak{Ce}\,\mathfrak{S})^+ \in V$, a special case of canonicity.

Conversely, if $V$ is complete, then $V = HSK_V^+$ by Theorem 3.7.2(3). But closure of $K_V$ under canonical extensions implies that $SK_V^+$ is a canonical variety, by 3.7.3, hence is closed under homomorphic images, so that altogether $V$ is the canonical variety $SK_V^+$. □

**Corollary 3.7.5.** *If $K_V$ is closed under ultrapowers, then the following are equivalent.*

(1) *$V$ is complete.*

(2) *$V$ is complex.*

(3) *$V$ is canonical.*

**Proof.** If $K_V$ is closed under ultrapowers, then since it is always closed under bounded epimorphic images (3.7.2(2)), it is closed under canonical extensions (3.6.2(1)). □

Recalling the discussion in Section 3.5 of the meaning of canonicity for varieties of modal algebras, it is apparent that Corollary 3.7.5 encompasses the result of Fine [17] that a modal logic whose class of frames is closed under elementary equivalence is complete iff it is canonical.

To conclude this subsection, we take up the question mentioned earlier as to when a class $K$ of structures has $K = K_{V_K}$, i.e. when does $K^+$ contain all full powerset algebras from the variety $V_K$ that it generates. We saw in the Introduction that for modal logic, this is equivalent to asking when a class $K$ of Kripke frames is modal-axiomatic—the class of all frames validating some set of modal formulae.

**Theorem 3.7.6.** (1) *If $K = K_{V_K}$, then $K$ is closed under bounded epimorphic images, inner substructures, and disjoint unions, and reflects canonical extensions.*

(2) *Let $K$ be closed under canonical extensions, or under ultrapowers. Then if $K$ is closed under bounded epimorphic images, inner substructures and disjoint unions, and reflects canonical extensions, it follows that $K = K_{V_K}$.*

**Proof.** (1) By Theorem 3.7.2(2).

(2) Assume that $K$ has the stated closure properties, and hence in the ultrapower case that $K$ is also closed under canonical extensions. By Theorem 3.4.4, $SK^+$ is a variety, and so must be $V_K$ itself. Hence if $\mathfrak{S} \in K_{V_K}$, then $\mathfrak{S}^+ \in V_K$, and so $\mathfrak{S}^+ \in SK^+$. Thus by Lemma 3.4.3(1), $\mathfrak{Cs}\,\mathfrak{S}^+ = \mathfrak{Ce}\,\mathfrak{S} \in K$. Reflection of canonical extensions then gives $\mathfrak{S} \in K$. This shows that $K_{V_K} \subseteq K$. The converse inclusion holds in general. □

Now since $V_K$ is complete, by definition, Theorem 3.7.4 implies that $V_K$ is canonical iff $K_{V_K}$ is closed under canonical extensions. Hence when $K = K_{V_K}$, $V_K$ is canonical iff $K$ is closed under canonical extensions. The overall situation is as follows.

**Theorem 3.7.7.** *For any $K$, the following are equivalent.*

(1) *$K$ is closed under canonical extensions, bounded epimorphic images, inner substructures, and disjoint unions, and reflects canonical extensions.*

(2) *$K = K_{V_K}$ and $V_K$ is canonical.*

(3) *$K = K_V$ for some canonical variety $V$.*

**Proof.** If (1) holds, then $K = K_{V_K}$ by 3.7.6(2). But then $K_{V_K}$ is closed under canonical extensions, and so as $V_K$ is complete, it is canonical by Theorem 3.7.4. Hence (1) implies (2).

That (2) implies (3) is immediate. Finally then, suppose (3) holds. Then $K$ is closed under canonical extensions by 3.7.4, and the other closure conditions of (1) are given by 3.7.2(2). □

The modal logic version of Theorem 3.7.6 first appeared in Goldblatt and Thomason [23], as a characterisation of modal-axiomatic classes of Kripke frames. Theorem 3.7.7 for $K$ a class of Kripke frames is in van Benthem [7].

### *3.8. First-order definability*

A number of the foregoing results have been based on the assumption that $K$ is closed under ultrapowers. In the case of a class of the form $K_V$, it turns out that this is both necessary and sufficient for $K_V$ to be elementary, as will now be shown.

In addition to the notion of an *elementary class* as being the class of all models of a set of first-order sentences, we will use the concepts of a *basic elementary class*: the class of all models of a single first-order sentence; and an *elementarily closed* class: one such that if it contains a structure $\mathfrak{S}$, then it contains any structure that is elementarily equivalent to (i.e. satisfies the same first-order sentences as) $\mathfrak{S}$. These three kinds of class have the following algebraic characterisations (where $-K$ denotes the class of relational structures of the similarity type of $K$ that are not in $K$):

(E1) $K$ is elementary iff $K$ is closed under isomorphism and ultraproducts and $-K$ is closed under ultrapowers.

(E2) $K$ is a basic elementary class iff both $K$ and $-K$ are closed under isomorphism and ultraproducts.

(E3) $K$ is elementarily closed iff both $K$ and $-K$ are closed under isomorphism and ultrapowers. (Cf. Bell and Slomson [3, Ch. 7].)

It is clear that in general $K$ is closed under isomorphism iff $-K$ is. Also, if $V$ is closed under isomorphism, then so is $K_V$, since $\mathfrak{S}^+ \cong \mathfrak{T}^+$ whenever $\mathfrak{S} \cong \mathfrak{T}$.

Recall from Lemma 3.6.5 the relationship

(E4) $$\prod_I \mathfrak{S}_i^+/U \rightarrowtail \left(\prod_I \mathfrak{S}_i/U\right)^+.$$

between an ultraproduct of complex algebras, and the complex algebra of the ultraproduct of the associated structures. In the special case of an ultrapower (when $\mathfrak{S}_i = \mathfrak{S}$ for all $i \in I$), this takes the form

$$(\mathfrak{S}^+)^I/U \rightarrowtail (\mathfrak{S}^I/U)^+.$$

But any algebra is isomorphically embeddable into any of its ultrapowers, so that $\mathfrak{S}^+ \rightarrowtail (\mathfrak{S}^+)^I/U$. Hence by composition,

(E5) $$\mathfrak{S}^+ \rightarrowtail (\mathfrak{S}^I/U)^+.$$

A variety $V$ is *finitely based* if there is a finite set of equations such that $V$ is the class of all algebras that satisfy these equations. By a standard argument, this holds iff the class $-V$ of all similar algebras not in $V$ is closed under ultraproducts.

**Theorem 3.8.1.** *For any variety $V$,*

(1) *$-K_V$ is closed under ultrapowers; and*

(2) *if $V$ is finitely based, then $-K_V$ is closed under ultraproducts.*

**Proof.** (1) Suppose $\mathfrak{S} \notin K_V$. Then $\mathfrak{S}^+ \notin V$, so by (E5) and closure of $V$ under subalgebras and isomorphism, $(\mathfrak{S}^I/U)^+ \notin V$, giving $(\mathfrak{S}^I/U) \notin K_V$.

(2) Suppose $\mathfrak{S}_i \notin K_V$ for all $i \in I$. Then if $\prod_I \mathfrak{S}_i/U \in K_V$, (E4) would give $\prod_I \mathfrak{S}_i^+/U \in V$. But $\mathfrak{S}_i^+ \notin V$ for all $i \in I$, and $-V$ is closed under ultraproducts. □

**Corollary 3.8.2.** (1) *$K_V$ is elementary iff it is closed under ultraproducts.*

(2) *If $V$ is finitely based, $K_V$ is a basic elementary class iff it is closed under ultraproducts.*

**Proof.** As noted earlier, $K_V$ and $-K_V$ are closed under isomorphism. The results now follow by 3.8.1(1) and (E1), and 3.8(2) and (E2), respectively. □

Corollary 3.8.2 implies that if $V$ is finitely based, then $K_V$ is elementary iff it is basic elementary (Goldblatt [19]). However this can be strengthened by means of the following result.

**Theorem 3.8.3.** *If a class $K$ of structures is closed under isomorphism, inner substructures, ultrapowers, and disjoint unions, then $K$ is closed under ultraproducts.*

**Proof.** Suppose $\{\mathfrak{S}_i : i \in I\} \subseteq K$, and $U$ is an ultrafilter on $I$. Define

$$\varphi : \prod_I X_i \to \left(\sum_I X_i\right)^I$$

by $\varphi(f)(i) = (f(i), i)$. Thus $\varphi$ identifies a member $f$ of the product of the $X_i$'s with a function from $I$ into the disjoint union of the $X_i$'s. Then we find that $f \sim_U g$ iff $\varphi(f) \sim_U \varphi(g)$, so putting $\varphi^*(f/u) = \varphi(f)/U$ makes $\varphi^*$ a well-defined injection of $\prod_I X_i/U$ into $(\sum_I X_i)^I/U$. Moreover, $\varphi^*$ can be shown to be a bounded morphism making the ultraproduct $\prod_I \mathfrak{S}_i/U$ isomorphic to an inner substructure of the ultrapower $(\sum_I \mathfrak{S}_i)^I/U$ of the disjoint union $\sum_I \mathfrak{S}_i$ of the $\mathfrak{S}_i$'s. Hence $\prod_I \mathfrak{S}_i/U \in K$. □

**Theorem 3.8.4.** *For any variety $V$, the following are equivalent.*

(1) *$K_V$ is elementary.*

(2) *$K_V$ is elementarily closed.*

(3) *$K_V$ is closed under ultrapowers.*

(4) *$K_V$ is closed under ultraproducts.*

**Proof.** By (E1) and (E3), (1) implies (2), and (2) implies (3). Since $K_V$ is closed under isomorphism, inner substructures, and disjoint unions (3.7.2(2)), (3) implies (4) by 3.8.3. But (4) implies (1) by 3.8.2. □

**Theorem 3.8.5.** *The following are equivalent.*
(1) $-K_V$ *is elementary.*
(2) $K_V$ *is basic elementary.*

**Proof.** That (2) implies (1) is clear. But if $-K_V$ is elementary, then by (E1), $-K_V$ is closed under ultraproducts and $K_V$ is closed under ultrapowers. Then $K_V$ is closed under ultraproducts by 3.8.4, and hence is basic elementary by E2. □

If $V$ is not finitely based, it is possible for $K_V$ to be elementary but not basic-elementary. An example is the variety $W$ of modal algebras $(\mathfrak{B}, f)$ satisfying

$$fx_0 \sqcap \cdots \sqcap fx_{n-1} \sqsubseteq f(fx_0 \sqcap \cdots \sqcap fx_{n-1})$$

for all $n > 0$. A full complex algebra $\mathfrak{S}^+$ satisfies this inequality iff it satisfies the first-order condition

$$(\lambda_n) \qquad \left(\bigwedge_{i<n} R(y_i, z)\right) \to \exists w \left(R(w, z) \wedge \left(\bigwedge_{i<n} R(y_i, w)\right)\right).$$

Thus $K_W$ is the elementary class of all structures satisfying $\{\lambda_n : n > 0\}$. Now if $\mathfrak{S}_n = (\{0, \ldots, n\}, R)$, where $R(y, z)$ iff $y \neq z$, then $\mathfrak{S}_n$ does not satisfy $\lambda_n$, so $\mathfrak{S}_n \notin K_W$. But any non-trivial ultraproduct of the $\mathfrak{S}_n$'s is infinite, and so satisfies every $\lambda_n$. Thus $-K_W$ is not closed under ultraproducts.

**Theorem 3.8.6.** *If $V$ is a finitely based variety, the following are equivalent.*
(1) $K_V$ *is basic elementary.*
(2) $-K_V$ *is elementary.*
(3) $K_V$ *is elementary.*
(4) $K_V$ *is elementarily closed.*
(5) $K_V$ *is closed under ultrapowers.*
(6) $K_V$ *is closed under ultraproducts.*

**Proof.** (1) and (2) are equivalent in general by 3.8.5, as are (3)–(6) by 3.8.4. But (1) and (6) are equivalent by 3.8.2(2). □

The fact that the class of frames of a modal logic is elementary iff it is elementarily closed was first proved by van Benthem, using a model-theoretic compactness argument. Theorem 3.8.3, and its use in proving 3.8.4–3.8.6, are due to the present author.

## 4. Preservation theorems

The purpose of this section is to give syntactic characterisations of the first-order sentences whose truth is preserved by bounded epimorphic images, inner substructures, and disjoint unions.

### 4.1. *Structures and languages*

For some fixed $n$, we deal with relational structures that carry an $n+1$-placed relation, and possibly constants and other relations as well. Such structures will be denoted $\mathfrak{S} = (X_{\mathfrak{S}}, R_{\mathfrak{S}}, \ldots)$, $\mathfrak{T} = (X_{\mathfrak{T}}, R_{\mathfrak{T}}, \ldots)$ etc. The notation $f: \mathfrak{S} \rightsquigarrow \mathfrak{T}$ will mean that $f$ is a *partial* function from $\mathfrak{S}$ to $\mathfrak{T}$, i.e. $\mathrm{dom} f \subseteq X_{\mathfrak{S}}$ and $\mathrm{im} f \subseteq X_{\mathfrak{T}}$, where $\mathrm{dom} f$ is the domain of $f$, and $\mathrm{im} f = \{f(x): x \in \mathrm{dom} f\}$ is the image of $f$.

The letter $L$ denotes a first-order language having an $n+1$-placed relation symbol $\bar{R}$, and possibly other relation symbols and individual constants as well, but no function symbols. Put

$$\mathrm{Fma}_L = \{\phi: \phi \text{ is an } L\text{-formula}\},$$
$$\mathrm{Sent}_L = \{\sigma: \sigma \text{ is an } L\text{-sentence}\},$$
$$\mathrm{Th}_L \mathfrak{S} = \{\sigma \in \mathrm{Sent}_L: \sigma \text{ is true in } \mathfrak{S}\}$$

(the letter $\sigma$ will always be reserved for sentences).

If $L_C = L \cup C$, where $C$ is a set of individual constants, then an $L_C$-structure is of the form $(\mathfrak{S}, Y)$, where $\mathfrak{S}$ is an $L$-structure, and $Y \subseteq X_{\mathfrak{S}}$ is the set of $\mathfrak{S}$-elements named by the member of $C$. Often this $L_C$-structure will be displayed as $(\mathfrak{S}, y \in Y)$ to emphasise the elements (viz. $y$) interpreting the constants. Even more briefly, if $f: \mathfrak{S} \rightsquigarrow \mathfrak{T}$, then $(\mathfrak{T}, \mathrm{im} f)$ will on occasion be denoted $(\mathfrak{T}, f(x))$ to indicate that the constant interpreted as $x \in \mathrm{dom} f$ in $\mathfrak{S}$ is to be interpreted as $f(x)$ in $\mathfrak{T}$.

For a given structure $\mathfrak{T}$, $L_{\mathfrak{T}}$ denote the language of $\mathfrak{T}$.

*Syntactic categories*

Let $S = \{\bot, \top, \alpha, \neg\alpha, \wedge, \vee, \forall_0, \exists_0, \forall, \exists\}$. Then for any language $L$, each symbol in $S$ is associated with a closure property of sets $\Delta \subseteq \mathrm{Fma}_L$, as follows.

$\bot$: $\bot \in \Delta$.

$\top$: $\top \in \Delta$.

$\alpha$: $\{\phi \in \mathrm{Fma}_L: \phi \text{ is atomic}\} \subseteq \Delta$.

$\neg\alpha$: $\{\neg\phi \in \mathrm{Fma}_L: \phi \text{ is atomic}\} \subseteq \Delta$.

$\wedge$: $\phi, \psi \in \Delta$ implies $\phi \wedge \psi \in \Delta$.

$\vee$: $\phi, \psi \in \Delta$ implies $\phi \vee \psi \in \Delta$.

$\forall_0$: $\phi \in \Delta$ implies $\forall v_0 \cdots \forall v_{n-1} (\bar{R}(v_0, \ldots, v_{n-1}, u) \rightarrow \phi)$ belongs to $\Delta$, whenever $u$ is an $L$-term, i.e. variable or constant, distinct from $v_0, \ldots, v_{n-1}$.

$\exists_0$: $\phi \in \Delta$ implies $\exists v_0 \cdots \exists v_{n-1} (\bar{R}(v_0, \ldots, v_{n-1}, u) \wedge \phi)$ belongs to $\Delta$, if $v_0, \ldots, v_{n-1}$ are distinct from $u$.

$\forall$: $\phi \in \Delta$ implies $\forall v\, \phi \in \Delta$, for any variable $v$.

$\exists$: $\phi \in \Delta$ implies $\exists v\, \phi \in \Delta$, for any variable $v$.

Formulae constructed by the closure conditions corresponding to $\forall_0$ and $\exists_0$ are said to be obtained by *bounded quantification*.

For any language $L$, and any subset $T$ of $S$, let $T_L^{\mathrm{F}}$ denote the smallest subset of $\mathrm{Fma}_L$ that has the closure properties associated with each member of $T$. Put

$$T_L = T_L^{\mathrm{F}} \cap \mathrm{Sent}_L.$$

Then for structures $\mathfrak{S}$ and $\mathfrak{T}$ whose language includes $L$, define

$$\mathfrak{S}\, T_L\, \mathfrak{T} \quad \text{iff} \quad \mathrm{Th}_L\mathfrak{S} \cap T_L \subseteq \mathrm{Th}_L\mathfrak{T}$$
$$\text{iff} \quad \text{every } T_L\text{-sentence true in } \mathfrak{S} \text{ is true in } \mathfrak{T}.$$

If $L = L_{\mathfrak{S}}$, then $\mathfrak{S}\, T_L\, \mathfrak{T}$ will be abbreviated to $\mathfrak{S}\, T\, \mathfrak{T}$.

**Lemma 4.1.1.** *If* $\neg\alpha \in T$, *and* $(\mathfrak{S}, x \in \mathrm{dom} f)\, T\, (\mathfrak{T}, f(x))$, *then* $f: \mathfrak{S} \rightsquigarrow \mathfrak{T}$ *is injective.*

**Proof.** If $y \neq z$ in $\mathfrak{S}$, then the sentence $\neg(\bar{y} = \bar{z})$, where $\bar{y}$ and $\bar{z}$ are constants naming $y$ and $z$, is true in $(\mathfrak{S}, x \in \mathrm{dom} f)$ and belongs to $T_L$, where $L$ is the language of $(\mathfrak{S}, x \in \mathrm{dom} f)$. Hence this sentence is true in $(\mathfrak{T}, f(x))$, making $f(y) \neq f(z)$. □

Different choices of $T$ generate different sets of $L$-formulae. The particular $T$'s we shall deal with, and the names of the associated types of formulae (i.e. members of $T_L^{\mathrm{F}}$), are as follows.

| | |
|---|---|
| $\mathrm{EA} = \{\bot, \top, \alpha, \wedge, \vee, \forall_0, \exists_0\}$ | essentially atomic, |
| $\mathrm{EB} = \mathrm{EA} \cup \{\neg\alpha\}$ | essentially basic, |
| $\mathrm{EP} = \mathrm{EA} \cup \{\forall, \exists\}$ | essentially positive, |
| $\mathrm{EU} = \mathrm{EB} \cup \{\forall\}$ | essentially universal, |
| $\mathrm{EPU} = \mathrm{EA} \cup \{\forall\}$ | essentially positive and universal, |
| $\mathrm{EE} = \mathrm{EB} \cup \{\exists\}$ | essentially existential, |
| $\mathrm{EPE} = \mathrm{EA} \cup \{\exists\}$ | essentially positive and existential |

($\exists_0$ is of course redundant in EP, EE, and EPE).

The symbol $\bot$ denotes the constant *false* sentence (and $\top$ the constant true sentence), and the negation $\neg\phi$ of a formula is to be identified with $(\phi \rightarrow \bot)$. Hence bounded universal quantification admits limited forms of negation of atomic formulae. For instance,

$$\forall v\, (\neg \bar{R}(v, u))$$

is an essentially atomic formula.

If $\mathfrak{S}$ and $\mathfrak{T}$ are $L$-structures, then a map $f: X_{\mathfrak{S}} \rightarrow X_{\mathfrak{T}}$ is a *bounded morphism of L-structures* if $f$ is a bounded morphism from $(X_{\mathfrak{S}}, R_{\mathfrak{S}})$ to $(X_{\mathfrak{T}}, R_{\mathfrak{T}})$, and furthermore $f$ preserves $L$-constants, i.e. if $c \in L$ is a constant that denotes $x$ in $\mathfrak{S}$ and $y$ in $\mathfrak{T}$, then $f(x) = y$. If $X_{\mathfrak{S}} \subseteq X_{\mathfrak{T}}$, then $\mathfrak{S}$ is an *inner L-substructure of* $\mathfrak{T}$,

and $\mathfrak{T}$ an *outer L-extension of* $\mathfrak{S}$, if the inclusion of $X_{\mathfrak{S}}$ in $X_{\mathfrak{T}}$ is a bounded morphism of $L$-structures.

**Lemma 4.1.2.** *Let $f: \mathfrak{S} \to \mathfrak{T}$ be a bounded morphism of L-structures. If $x = (x_0, \ldots)$ is a sequence in $X_{\mathfrak{S}}$, let $f(x) = (f(x_0), \ldots)$. Suppose that $\phi$ is an L-formula such that for any sequence $x$ in $X_{\mathfrak{S}}$,*

(*) *if $x$ satisfies $\phi$ in $\mathfrak{S}$, then $f(x)$ satisfies $\phi$ in $\mathfrak{T}$.*

*Then (*) holds for all $x$ when $\phi$ is replaced by $\psi$, where $\psi$ is either*

(1) $\forall v_0 \cdots \forall v_{n-1} (\bar{R}(v_0, \ldots, v_{n-1}, u) \to \phi)$, *or*

(2) $\exists v_0 \cdots \exists v_{n-1} (\bar{R}(v_0, \ldots, v_{n-1}, u) \wedge \phi)$,

*provided that the variables $v_i$ are distinct from the term $u$.*

**Proof.** Consider case (1), and suppose $x$ satisfies $\psi$ in $\mathfrak{S}$. Let $b$ be the value of $u$ in $\mathfrak{T}$ under the valuation $f(x)$. If $u$ is the variable $v_j$, put $a = x_j$, while if $u$ is a constant, let $a$ be the interpretation of $u$ in $\mathfrak{S}$. In either case we have $f(a) = b$, and $a$ is the value of $u$ in $\mathfrak{S}$ under $x$.

Now let $y$ be any sequence in $\mathfrak{T}$ differing from $f(x)$ at most in its assignment to $v_0, \ldots, v_{n-1}$. Suppose $y$ satisfies $\bar{R}(v_0, \ldots, v_{n-1}, u)$. Then as the $v_i$'s are distinct from $u$, $y$ and $f(x)$ agree on $u$, so $R_{\mathfrak{T}}(y_0, \ldots, y_{n-1}, f(a))$. Hence there exists $z_0, \ldots, z_{n-1} \in X_{\mathfrak{S}}$ such that $R_{\mathfrak{S}}(z_0, \ldots, z_{n-1}, a)$ and $f(z_i) = y_i$ for $i < n$. Let $z = (z_0, \ldots, z_{n-1}, x_n, x_{n+1}, \ldots)$. Then $z$ satisfies $\bar{R}(v_0, \ldots, v_{n-1}, u)$ in $\mathfrak{S}$ and differs from $x$ only on $v_0, \ldots, v_{n-1}$. Since $x$ satisfies $\psi$, it follows that $z$ satisfies $\phi$ in $\mathfrak{S}$, and so by hypothesis $f(z)$ satisfies $\phi$ in $\mathfrak{T}$. But $f(z) = y$. This shows that $f(x)$ satisfies $\psi$ in $\mathfrak{T}$.

The case of (2) is similar. □

If the function $f$ in this lemma is a bounded epimorphism, then every sequence in $\mathfrak{T}$ is of the form $f(x)$, so that if $f$ preserves validity of a formula $\phi$, then it will preserve validity of any formula constructed from $\phi$ by bounded quantification. Combined with arguments familiar from standard model theory, this may be used to confirm that truth of any essentially positive sentence is preserved by bounded epimorphisms. Similar arguments show that essentially universal sentences are preserved by inner substructures, and essentially positive and universal sentences are preserved by both bounded epimorphisms and inner substructures. The essentially atomic sentences are preserved by these two constructions, and also by outer extensions (hence are *invariant* with respect to the latter). Outer extensions in general preserve all essentially existential sentences.

For the case of disjoint unions, define

$$\forall_1 T_L = \{\forall v\, \phi : \phi \in T_L^{\mathrm{F}}\} \cap \mathrm{Sent}_L.$$

Thus $\forall_1 T_L$ is the class of all *sentences* formed by a single universal quantification of a member of $T_L^{\mathrm{F}}$.

**Theorem 4.1.3.** *Any member of* $\forall_1 \mathrm{EE}_L$ *is preserved by disjoint unions.*

**Proof.** Let $\{\mathfrak{S}_i : i \in I\}$ be a collection of $L$-structures. For each $j \in I$, the function $f_j(x) = (x, j)$ is a bounded morphism making $\mathfrak{S}_j$ isomorphic to an inner substructure of the disjoint union $\sum_I \mathfrak{S}_i$. Then $f_j$ preserves satisfaction of essentially existential formulae, i.e. if $\phi \in \mathrm{EE}_L$, and $y$ is a sequence in $\mathfrak{S}_j$ satisfying $\phi$, then $f_j(y)$ satisfies $\phi$ in $\sum_I \mathfrak{S}_i$. Now suppose that such a $\phi$ has at most the variable $v$ free, and that $\forall v\, \phi$ is true in $\mathfrak{S}_i$ for all $i \in I$. If $z$ is any element of $\sum_I \mathfrak{S}_i$, then $z = (x, j)$ for some $j \in I$ and $x$ in $\mathfrak{S}_j$. Then $\phi$ is satisfied in $\mathfrak{S}_j$ when $v$ is assigned $x$, since $\forall v\, \phi$ is true in $\mathfrak{S}_j$. But then $\phi$ is satisfied in $\sum_I \mathfrak{S}_i$ when $v$ is assigned $f_j(x) = z$. Since this holds for all $z$ in $\sum_I \mathfrak{S}_i$, $\forall v\, \phi$ is true in the latter. □

**Corollary 4.1.4.** *Any sentence in* $\forall_1 \mathrm{EA}_L$ *is preserved by bounded epimorphic images, inner substructures, and disjoint unions of L-structures.* □

The converse of 4.1.4, up to logical equivalence, will be proved in Theorem 4.3.4.

### *4.2. Preservation by bounded morphisms and inner substructures*

If $\mathfrak{S} = (X_{\mathfrak{S}}, R_{\mathfrak{S}}, \ldots)$ is a structure, then for each $Y \subseteq X_{\mathfrak{S}}$, the subset $R_{\mathfrak{S}}[Y]$ of $X_{\mathfrak{S}}$ is defined by

$$y \in R_{\mathfrak{S}}[Y] \quad \text{iff} \quad \text{for some } z \in Y, \text{ and some } x \in X_{\mathfrak{S}}^n \text{ such that } R_{\mathfrak{S}}(x, z), \text{ there exists } i < n \text{ with } y = x_i.$$

In terms of the notation introduced in Section 3.3, we have

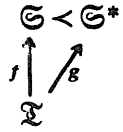
$$y \in R_{\mathfrak{S}}[Y] \quad \text{iff} \quad yR^i z \quad \text{for some } z \in Y \text{ and some } i < n$$
$$\text{iff} \quad Y \cap \exists^i_{R_{\mathfrak{S}}}(\{y\}) \neq \emptyset \quad \text{for some } i < n.$$

The standard notation $\mathfrak{S} < \mathfrak{T}$ will be used to signify that $\mathfrak{T}$ is an elementary extension of $\mathfrak{S}$.

**Theorem 4.2.1.** *Suppose* $\{\bot, \vee, \forall_0\} \subseteq T$. *If* $\mathfrak{S}$ *and* $\mathfrak{T}$ *are structures with* $L_{\mathfrak{S}} \subseteq L_{\mathfrak{T}}$, *and* $f : \mathfrak{S} \rightsquigarrow \mathfrak{T}$ *satisfies*

$$(\mathfrak{S}, x \in \mathrm{dom}\, f) T(\mathfrak{T}, f(x)),$$

*then there exists an* $L_{\mathfrak{S}}$*-structure* $\mathfrak{S}^*$ *such that* $\mathfrak{S} < \mathfrak{S}^*$, *and a function* $g : \mathfrak{T} \rightsquigarrow \mathfrak{S}^*$



*such that*

(1) $\mathrm{dom}\, g = R_{\mathfrak{T}}[\mathrm{im}\, f]$;

(2) $(\mathfrak{S}^*, x \in \mathrm{dom}\, f, g(y))T(\mathfrak{T}, f(x), y \in \mathrm{dom}\, g)$;

(3) $R_{\mathfrak{T}}(y, f(x))$ *implies* $R_{\mathfrak{S}^*}(g(y), x)$; *and*

(4) $\mathrm{im}\, g \subseteq R_{\mathfrak{S}^*}[\mathrm{dom}\, f]$.

*Moreover, if* $\forall \in T$, (1) *can be strengthened to*

(5) $\mathrm{dom}\, g = X_{\mathfrak{T}}$, *i.e. g is total.*

**Proof.** Let $Y = R_{\mathfrak{T}}[\mathrm{im}\, f]$, and take new disjoint sets of constants $C = \{\bar{x}: x \in X_{\mathfrak{S}}\}$ and $D = \{\bar{y}: y \in Y\}$. Put $C_f = \{\bar{x} \in C: x \in \mathrm{dom}\, f\}$, and define $L_0 = L_{\mathfrak{S}} \cup C$, $L_1 = L_{\mathfrak{S}} \cup C_f$, and $L_2 = L_1 \cup D$. Then $(\mathfrak{S}, X_{\mathfrak{S}})$ is an $L_0$-structure, $(\mathfrak{S}, x \in \mathrm{dom}\, f)$ is an $L_1$-structure, $(\mathfrak{T}, f(x))$ is an expansion of an $L_1$-structure, and $(\mathfrak{T}, f(x), y \in Y)$ is an expansion of an $L_2$-structure. Let

$$\Lambda = \{\sigma \in \mathrm{Sent}_{L_0}: (\mathfrak{S}, X_{\mathfrak{S}}) \vDash \sigma\},$$

$$\Delta = \{\neg\sigma: \sigma \in T_{L_2} \text{ and } (\mathfrak{T}, f(x), y \in Y) \nvDash \sigma\},$$

$$\Gamma = \{\bar{R}(\bar{y}_0, \ldots, \bar{y}_{n-1}, \bar{x}): x \in \mathrm{dom}\, f \text{ and } R_{\mathfrak{T}}(y_0, \ldots, y_{n-1}, f(x))\}.$$

Then $\Lambda \cup \Delta \cup \Gamma$ has a model. To prove this it is enough, by the Compactness Theorem, to prove that if $\Delta_0$ and $\Gamma_0$ are finite subsets of $\Delta$ and $\Gamma$, respectively, then $\Delta_0 \cup \Gamma_0$ has a model that is an expansion of $(\mathfrak{S}, X_{\mathfrak{S}})$, and hence is also a model of $\Lambda$.

At this point, the fact that $\bot \in T$ is used to observe that $\neg\bot \in \Delta$, hence $\Delta \neq \emptyset$, and so it may be assumed that $\Delta_0 \neq \emptyset$. This is vital to the argument to follow, which is, on the other hand, compatible with the possibility that $\Gamma = \emptyset$.

Also, for readability it will be assumed that $n = 3$, the construction that ensues being entirely typical of the general case.

Suppose then that

$$\Delta_0 = \{\neg\sigma_1, \ldots, \neg\sigma_m\},$$

$$\Gamma_0 = \{\bar{R}(\bar{y}_{11}, \bar{y}_{12}, \bar{x}_1), \ldots, \bar{R}(\bar{y}_{p1}, \bar{y}_{p2}, \bar{x}_p)\}$$

for some $m$ and $p$. If there are members of $D$ occurring in $\Delta_0 - \Gamma_0$, let these be denoted $\bar{y}_{p+1}, \ldots, \bar{y}_k$. Then for $p < j \leq k$, by definition of $Y$ there exist $y_{j1}, y_{j2} \in X_{\mathfrak{T}}$ and $x_j \in \mathrm{dom}\, f$ such that $R_{\mathfrak{T}}(y_{j1}, y_{j2}, f(x_j))$, and $y_j$ is one of $y_{j1}$ and $y_{j2}$.

Replace each constant $\bar{y}_{ir}$, where $1 \leq i \leq k$ and $r = 1, 2$, by a new variable $v_{ir}$ not occurring in $\Delta_0 \cup \Gamma_0$. Let $\sigma_q'$ be the result of making these replacements uniformly in $\sigma_q$. Then the sentence

$$\exists v_{11}\exists v_{12}(\bar{R}(v_{11}, v_{12}, \bar{x}_1) \wedge (\cdots \wedge (\exists v_{k1}\exists v_{k2}(\bar{R}(v_{k1}, v_{k2}, \bar{x}_k) \wedge \neg\sigma_1' \wedge \cdots \wedge \neg\sigma_m') \cdots),$$

all of whose constants $\bar{x}$ have $x \in \mathrm{dom}\, f$, is true in $(\mathfrak{T}, f(x))$. But this sentence is

equivalent to $\neg\sigma$, where $\sigma$ is

$$\forall v_{11}\forall v_{12}(\bar{R}(v_{11}, v_{12}, \bar{x}_1)\to(\cdots\to(\forall v_{k1}\forall v_{k2}(\bar{R}(v_{k1}, v_{k2}, \bar{x}_k)\to\sigma'_1\vee\cdots\vee\sigma'_m)\cdots).$$

Now $\sigma\in T_{L_1}$, since $\{\vee, \forall_0\}\subseteq T$, and so by hypothesis, as $\sigma$ is false in $(\mathfrak{T}, f(x))$, it is also false in the $L_1$-reduct $(\mathfrak{S}, x\in\mathrm{dom}f)$ of $(\mathfrak{S}, x\in X_{\mathfrak{S}})$. Hence $\sigma$ is false in the latter, so that there exists a sequence $(d_{11}, d_{12}, \ldots, d_{k1}, d_{k2})$ of $\mathfrak{S}$-elements that satisfy in $(\mathfrak{S}, x\in X_{\mathfrak{S}})$ the formula

$$\bar{R}(v_{11}, v_{12}, \bar{x}_1)\wedge\cdots\wedge\bar{R}(v_{k1}, v_{k2}, \bar{x}_k)\wedge\neg\sigma'_1\wedge\cdots\wedge\neg\sigma'_m.$$

Hence $\Delta_0\cup\Gamma_0$ is true in $(\mathfrak{S}, x\in X_{\mathfrak{S}}, d_{ir})$, as desired.

Thus $\Lambda\cup\Delta\cup\Gamma$ has a model. From the definition of $\Lambda$, this model may be taken to have the form $(\mathfrak{S}^*, x\in X_{\mathfrak{S}}, \{y^*: y\in Y\})$, with $\mathfrak{S}<\mathfrak{S}^*$, and $y^*$ the $\mathfrak{S}^*$-element named by $\bar{y}$. Hence putting $g(y)=y^*$ defines $g$ as a function satisfying (1). Then $\Delta$ establishes that (2) holds, and $\Gamma$ yields (3).

For (4), observe that if $g(z)\in\mathrm{im}\,g$, then by (1), $z\in R_{\mathfrak{T}}[\mathrm{im}f]$, and so there exist $x\in\mathrm{dom}f$, $y\in X^n_{\mathfrak{T}}$, and $i<n$ such that $z=y_i$ and $R_{\mathfrak{T}}(y, f(x))$. Hence by (3), $R_{\mathfrak{S}^*}(g(y), x)$, giving $g(z)=g(y_i)\in R_{\mathfrak{S}^*}[\mathrm{dom}f]$.

Finally, to prove (5) when $\forall\in T$, the above argument is reconstructed, starting instead with $Y=X_{\mathfrak{T}}$, and so $D=\{\bar{y}: y\in X_{\mathfrak{T}}\}$. Then the unbounded universal quantifier $\forall$ can be applied directly to the variables, $v_{ir}$ for $p<j\leqslant k$ that replace members of $D$ from $\Delta_0-\Gamma_0$, to form $\sigma$ as the universal closure of

$$\forall v_{11}\forall v_{12}(\bar{R}(v_{11}, v_{12}, \bar{x}_1)\to(\cdots\to(\forall v_{p1}\forall v_{p2}(\bar{R}(v_{p1}, v_{p2}, \bar{x}_p)\to\sigma'_1\vee\cdots\vee\sigma'_m)\cdots).$$

The above argument then goes through to define $g(y)=y^*$ for all $y\in X_{\mathfrak{T}}$. □

**Theorem 4.2.2.** *Suppose* $\{\top, \alpha, \wedge, \exists_0\}\subseteq T$. *Then given structures* $\mathfrak{S}$, $\mathfrak{S}^*$, *and* $\mathfrak{T}$, *with* $L_{\mathfrak{S}}=L_{\mathfrak{S}^*}\subseteq L_{\mathfrak{T}}$, *and a diagram*

$$\begin{array}{ccc}\mathfrak{S} & < & \mathfrak{S}^*\\ {\scriptstyle f}\downarrow & \nearrow_{g} & \\ \mathfrak{T} & & \end{array}$$

*with the functions* $f: \mathfrak{S}\rightsquigarrow\mathfrak{T}$ *and* $g: \mathfrak{T}\rightsquigarrow\mathfrak{S}^*$ *satisfying conditions* (1)–(4) *of the conclusion of Theorem* 4.2.1, *there exists an* $L_{\mathfrak{T}}$*-structure* $\mathfrak{T}^*$ *with* $\mathfrak{T}<\mathfrak{T}^*$ *and a function* $f^*: \mathfrak{S}^*\rightsquigarrow\mathfrak{T}^*$

$$\begin{array}{ccc}\mathfrak{S} & < & \mathfrak{S}^*\\ {\scriptstyle f}\downarrow & \nearrow_{g} & \downarrow{\scriptstyle f^*}\\ \mathfrak{T} & < & \mathfrak{T}^*\end{array}$$

*such that*

(1) $\operatorname{im} g \subseteq \operatorname{dom} f^* = \operatorname{dom} f \cup R_{\mathfrak{S}^*}[\operatorname{dom} f]$;

(2) $(\mathfrak{S}^*, x \in \operatorname{dom} f, g(y), z \in \operatorname{dom} f^*) T (\mathfrak{T}^*, f(x), y \in \operatorname{dom} g, f^*(z))$;

(3) $R_{\mathfrak{S}^*}(x)$ *implies* $R_{\mathfrak{T}^*}(f^*(x))$, *if* $x \in (\operatorname{dom} f^*)^{n+1}$;

(4) $f \subseteq f^*$;

(5) $f^*(g(y)) = y$, *if* $y \in \operatorname{dom} g$; *and*

(6) *if* $\neg\alpha \in T$, *then* $f^*$ *is injective.*

*Moreover, if* $\exists \in T$, *then* (1) *can be strengthened to*

(7) $\operatorname{dom} f^* = X_{\mathfrak{S}^*}$, *i.e.* $f^*$ *is total.*

**Proof.** Let $Z = \operatorname{dom} f \cup R_{\mathfrak{S}^*}[\operatorname{dom} f]$, so that $\operatorname{im} g \subseteq Z$ by 4.2.1(4). Take new mutually disjoint sets of constants $C = \{\bar{x}: x \in \operatorname{dom} f\}$, $D = \{\bar{y}: y \in X_{\mathfrak{T}}\}$, and $E = \{\bar{z}: z \in Z\}$. Note that elements $x$ of $\operatorname{dom} f$ are associated with two distinct constants, viz. $\bar{x} \in C$ and $\bar{x} \in E$. Put $D_g = \{\bar{y} \in D: y \in \operatorname{dom} g\}$, $L_0 = L_{\mathfrak{T}} \cup C \cup D$, $L_1 = L_{\mathfrak{T}} \cup C \cup D_g$, and $L_2 = L_1 \cup E$. Let

$$\Lambda = \{\sigma \in \operatorname{Sent}_{L_0}: (\mathfrak{T}, f(x), X_{\mathfrak{T}}) \models \sigma\},$$
$$\Delta = \{\sigma \in T_{L_2}: (\mathfrak{S}^*, x \in \operatorname{dom} f, g(y), z \in Z) \models \sigma\}.$$

To show that $\Lambda \cup \Delta$ has a model, it suffices to show that any finite $\Delta_0 \subseteq \Delta$ has a model that is an expansion of $(\mathfrak{T}, f(x), X_{\mathfrak{T}})$ and hence is a model of $\Lambda$. Since $\top \in T$, $\top \in \Delta$, and so $\Delta_0$ can be assumed non-empty. As in Theorem 4.2.1, we present the case $n = 3$.

Let $\Delta_0 = \{\sigma_1, \ldots, \sigma_m\}$, and let $\bar{z}_1, \ldots, \bar{z}_p$ be all the members of $E$ that occur in $\Delta_0$. If there are any $z_i$ in $R_{\mathfrak{S}^*}[\operatorname{dom} f]$ but not in $\operatorname{dom} f$, let the indexing be such that for some $j \leqslant p$, $\bar{z}_1, \ldots, \bar{z}_j$ are all the $z_i$ in $R_{\mathfrak{S}^*}[\operatorname{dom} f] - \operatorname{dom} f$. Then if $1 \leqslant i \leqslant j$, there exist $x_i \in \operatorname{dom} f$ and $z_{i1}, z_{i2}$ in $\mathfrak{S}^*$ such that $R_{\mathfrak{S}^*}(z_{i1}, z_{i2}, x_i)$ and $z_i$ is one of $z_{i1}$ and $z_{i2}$. Replace $\bar{z}_{i1}$ and $\bar{z}_{i2}$ by new variables $v_{i1}$, $v_{i2}$ that do not occur in $\Delta_0$ (note that $z_{i1}$ and $z_{i2}$ are both in $Z$). For the remaining $z_i$, i.e. for $j < i \leqslant p$, we have $z_i \in \operatorname{dom} f$. Replace $\bar{z}_i$ by $\bar{z}_i \in C$.

Let $\sigma_q'$ be the result of making all these replacements in $\sigma_q$, so that $\sigma_q'$ contains no members of $E$. Then if $\sigma$ is the sentence

$$\exists v_{11} \exists v_{12} (\bar{R}(v_{11}, v_{12}, \bar{x}_1) \wedge (\cdots \wedge (\exists v_{j1} \exists v_{j2} (\bar{R}(v_{j1}, v_{j2}, \bar{x}_j) \wedge \sigma_1' \wedge \cdots \wedge \sigma_m') \cdots),$$

it follows that $\sigma$ is true in $(\mathfrak{S}^*, x \in \operatorname{dom} f, g(y))$. But $\sigma \in T_{L_1}$, since $\{\wedge, \exists_0\} \subseteq T$, and so by the assumed condition (2) of Theorem 4.2.1, $\sigma$ is true in $(\mathfrak{T}, f(x), y \in \operatorname{dom} g)$, and hence true in $(\mathfrak{T}, f(x), y \in X_{\mathfrak{T}})$. Thus there exists a sequence $(d_{11}, d_{12}, \ldots, d_{j1}, d_{j2})$ that satisfies $\sigma_1' \wedge \cdots \wedge \sigma_m'$ in $(\mathfrak{T}, f(x), X_{\mathfrak{T}})$ and has $R_{\mathfrak{T}}(d_{i1}, d_{i2}, f(x_i))$ for $1 \leqslant i \leqslant j$. Then $(\mathfrak{T}, f(x), X_{\mathfrak{T}}, d_{ir})$ is a model of $\Delta_0$, in which $\bar{z}_i$ is interpreted as $f(z_i)$ when $z_i \in \operatorname{dom} f$.

Thus by Compactness, $\Lambda \cup \Delta$ has a model, which, in view of $\Lambda$, may be taken to have the form $(\mathfrak{T}^*, f(x), y \in X_{\mathfrak{T}}, \{z^*: z \in Z\})$, with $\mathfrak{T}^*$ an elementary extension of $\mathfrak{T}$. Putting $f^*(z) = z^*$ defines $f^*$ as a function satisfying conclusion (1) of

this Theorem. $\Delta$ then establishes that (2) holds. To prove (3), observe that if $R_{\mathfrak{S}^\bullet}(z)$, with $z \in Z^{n+1}$, then since $\alpha \in T$, the sentence $\bar{R}(\bar{z}_0, \ldots, \bar{z}_n)$ belongs to $\Delta$, and so is true in $(\mathfrak{T}^*, f^*(z))$, making $R_{\mathfrak{T}^\bullet}(f^*(z))$.

For (4), we have $\mathrm{dom} f \subseteq \mathrm{dom} f^*$ by definition of $f^*$, and if $x \in \mathrm{dom} f$, then the two constants $\bar{x} \in C$ and $\bar{x} \in E$ both name $x$, so that the equation $(\bar{x} = \bar{x})$ belongs to $\Delta$, and hence is true in $(\mathfrak{T}, f(x), f^*(z))$, giving $f(x) = f^*(x)$.

For (5), if $y \in \mathrm{dom}\, g$ then in $(\mathfrak{S}^*, g(y), z \in Z)$, $g(y)$ is named by $\bar{y} \in D_g$ and $\overline{g(y)} \in E$ (since $g(y) \in Z$ by (1)). Hence $(\bar{y} = \overline{g(y)})$ belongs to $\Delta$ and is true in $(\mathfrak{T}^*, X_{\sim}, f^*(z))$, showing that $y = f^*(g(y))$.

Now (6) follows by (2) and Lemma 4.1.1, and so its remains to prove (7) when $\exists \in T$. In this case, the proof is reconstructed, starting instead with $Z = X_{\mathfrak{S}^\bullet}$, and so $E = \{\bar{z} : z \in X_{\mathfrak{S}^\bullet}\}$. The analysis of the $z_i$ in the formation of $\sigma$ is abandoned, and we simply replace $\bar{z}_i$ by a new variable $v_i$ for $1 \leqslant i \leqslant p$, and take $\sigma$ as

$$\exists v_1 \cdots \exists v_p (\sigma'_1 \wedge \cdots \wedge \sigma'_m).$$

The argument then proceeds, to obtain the definition $f^*(z) = z^*$ for all $z \in X_{\mathfrak{S}^\bullet}$. □

**Theorem 4.2.3.** *Suppose* $\mathrm{EA} = \{\bot, \top, \alpha, \wedge, \vee, \forall_0, \exists_0\}$ *is a subset of* $T$. *If* $\mathfrak{S}$ *and* $\mathfrak{T}$ *are structures with* $L_{\mathfrak{S}} \subseteq L_{\mathfrak{T}}$, *and* $f: \mathfrak{S} \rightsquigarrow \mathfrak{T}$ *satisfies*

$$(\mathfrak{S}, x \in \mathrm{dom} f) T(\mathfrak{T}, f(x)),$$

*then there exists an* $L_{\mathfrak{S}}$*-structure* $\mathfrak{S}^*$ *with* $\mathfrak{S} < \mathfrak{S}^*$, *and a function* $f^*: \mathfrak{S}^* \rightsquigarrow \mathfrak{T}^*$

$$\begin{array}{ccc} \mathfrak{S} & < & \mathfrak{S}^* \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathfrak{T} & < & \mathfrak{T}^* \end{array}$$

*such that*

(1) $f \subseteq f^*$;

(2) $\mathrm{dom} f^*$ *is an inner substructure of* $\mathfrak{S}^*$, *and* $f^*$ *is a bounded morphism on* $\mathrm{dom} f^*$;

(3) *if* $\forall \in T$, *then* $f^*$ *is surjective, i.e.* $\mathrm{im} f^* = X_{\mathfrak{T}^\bullet}$;

(4) *if* $\exists \in T$, *then* $f^*$ *is total, i.e.* $\mathrm{dom} f^* = X_{\mathfrak{S}^\bullet}$; *and*

(5) *if* $\neg\alpha \in T$, *then* $f^*$ *is injective.*

**Proof.** Put $\mathfrak{S}_0 = \mathfrak{S}$, $\mathfrak{T}_0 = \mathfrak{T}$, and $f_0 = f$. Then by iterating 4.2.1 and 4.2.2 alternately denumerably many times we obtain a diagram

$$\begin{array}{ccccccccccc} \mathfrak{S}_0 & < & \mathfrak{S}_1 & < & \cdots & < & \mathfrak{S}_m & < & \cdots \\ {\scriptstyle f_0}\downarrow & \nearrow{\scriptstyle g_0} & \downarrow{\scriptstyle f_1} & \nearrow & \cdots & \nearrow & \downarrow{\scriptstyle f_m} & \nearrow{\scriptstyle g_m} & \\ \mathfrak{T}_0 & < & \mathfrak{T}_1 & < & \cdots & < & \mathfrak{T}_m & < & \cdots \end{array}$$

which for all $m \geqslant 0$ has $\mathfrak{S}_m < \mathfrak{S}_{m+1}$, $\mathfrak{T}_m < \mathfrak{T}_{m+1}$, and

(i) $\operatorname{dom} g_m = R_{\mathfrak{T}_m}[\operatorname{im} f_m]$;

(ii) $\operatorname{im} g_m \subseteq \operatorname{dom} f_{m+1} = \operatorname{dom} f_m \cup R_{\mathfrak{S}_{m+1}}[\operatorname{dom} f_m]$;

(iii) $R_{\mathfrak{S}_{m+1}}(x)$ implies $R_{\mathfrak{T}_{m+1}}(f_{m+1}(x))$, if $x \in (\operatorname{dom} f_{m+1})^{n+1}$;

(iv) $R_{\mathfrak{T}_m}(y, f_m(x))$ implies $R_{\mathfrak{S}_{m+1}}(g_m(y), x)$, if $y \in (\operatorname{dom} g_m)^n$;

(v) $f_m \subseteq f_{m+1}$; and

(vi) $f_{m+1}(g_m(y)) = y$.

Let $\mathfrak{S}^* = \bigcup_m \mathfrak{S}_m$ and $\mathfrak{T}^* = \bigcup_m \mathfrak{T}_m$, so that $\mathfrak{S} < \mathfrak{S}^*$ and $\mathfrak{T} < \mathfrak{T}^*$. Put $f^* = \bigcup_m f_m$, so that by (v), $f^*$ is a partial function from $\mathfrak{S}^*$ to $\mathfrak{T}^*$ that extends $f_0$. Hence (1) holds.

To prove that $\operatorname{dom} f^*$ is an inner substructure of $\mathfrak{S}^*$, suppose that $z \in \operatorname{dom} f^*$ and $R_{\mathfrak{S}^*}(y, z)$, where $y \in X^n_{\mathfrak{S}^*}$. Then for some $m$, $z \in \operatorname{dom} f_m$ and $R_{\mathfrak{S}_{m+1}}(y, z)$. But then (ii) implies that $y_i \in \operatorname{dom} f_{m+1} \subseteq \operatorname{dom} f^*$ for all $i < n$. Hence $\operatorname{dom} f^* \in \mathfrak{S}^*$. Moreover, for such $z$ and $y$ we also have by (iii) that $R_{\mathfrak{T}_{m+1}}(f_{m+1}(y, z))$, and so $R_{\mathfrak{T}^*}(f^*(y, z))$. This shows $f^*$ to be a morphism.

To prove $f^*$ is bounded, let $z \in \operatorname{dom} f^*$ and $R_{\mathfrak{T}^*}(y, f^*(z))$, where $y \in X^n_{\mathfrak{T}^*}$. Then for some $m$, $R_{\mathfrak{T}_m}(y, f^m(z))$. Hence by (i), $y \in (\operatorname{dom} g_m)^n$, so with $x = g_m(y)$ we get $R_{\mathfrak{S}_{m+1}}(x, z)$ from (iv), whence $R_{\mathfrak{S}^*}(x, z)$, and $f^*(x) = f_{m+1}(g_m(y)) = y$ by (vi). This establishes (2).

(3): If $\forall \in T$, then by 4.2.1(5) each $g_m$ can be taken to have domain $X_{\mathfrak{T}_m}$. Then if $y \in X_{\mathfrak{T}^*}$, $y \in X_{\mathfrak{T}_m}$ for some $m$. and so by (ii) and (iii), $y = f_{m+1}(g_m(y)) = f^*(g_m(y))$. Hence $f^*$ is surjective.

(4): If $\exists \in T$, then by 4.2.2(7) we can assume $\operatorname{dom} f_{m+1} = X_{\mathfrak{S}_{m+1}}$ for all $m$, so $\operatorname{dom} f^* = X_{\mathfrak{S}^*}$.

(5): If $\neg\alpha \in T$, then each $f_{m+1}$ is injective by 4.2.2(6), and hence so if $f^*$. □

The following is a variant of a well known model-theoretic principle (Chang and Keisler [13, Lemma 3.2.1]).

**Lemma 4.2.4.** *Suppose that* $\{\bot, \vee\} \subseteq T$, *and* $\Gamma$ *is a set of L-sentences satisfying, for all L-structures* $\mathfrak{S}$ *and* $\mathfrak{T}$,

$$\mathfrak{S} \models \Gamma \quad \text{and} \quad \mathfrak{S} \equiv_L \mathfrak{T} \text{ implies } \mathfrak{T} \models \Gamma.$$

*Then*:

(1) *there exists a set* $\Delta \subseteq T_L$ *that is logically equivalent to* $\Gamma$ (*i.e. has exactly the same models as* $\Gamma$); *and*

(2) *if* $\Gamma = \{\sigma\}$ *and also* $\{\top, \wedge\} \subseteq T$, *then* $\sigma$ *is logically equivalent to some sentence belonging to* $T_L$.

**Proof.** (1) Let $\Delta = \{\sigma \in T_L : \Gamma \models \sigma\}$. Then $\Gamma \models \Delta$. Suppose $\mathfrak{T} \models \Delta$, and let $\Lambda = \{\neg\sigma : \sigma \in T_L \ \& \ \mathfrak{T} \not\models \sigma\}$. Then $\neg\bot \in \Lambda$, as $\bot \in T$, so $\Lambda \neq \emptyset$. If $\neg\sigma_1, \ldots, \neg\sigma_m \in \Lambda$, then $\mathfrak{T} \not\models \sigma_1 \vee \cdots \vee \sigma_m$. But $(\sigma_1 \vee \cdots \vee \sigma_m) \in T_L$, as $\vee \in T$, so as $\mathfrak{T} \models \Delta$, $(\sigma_1 \vee \cdots \vee \sigma_m) \notin \Delta$. Hence $\Gamma \not\models \sigma_1 \vee \cdots \vee \sigma_m$, so $\Gamma \cup \{\neg\sigma_1, \ldots, \neg\sigma_m\}$ has a model.

By Compactness, it follows that there exists a model $\mathfrak{S}$ of $\Gamma \cup \Delta$. Then $\mathfrak{S} \models \Gamma$, and $\mathfrak{S} \equiv_L \mathfrak{T}$ since $\mathfrak{S} \models \Delta$, so by hypothesis, $\mathfrak{T} \models \Gamma$ as desired.

(2) If $\top \in T$, then $\top \in \Delta$, where $\Delta$ is as in the proof of (1), so $\Delta \neq \emptyset$. Then if $\Gamma = \{\sigma\}$, by Compactness $\{\sigma\}$ is logically equivalent to some non-empty subset $\{\sigma_1, \ldots, \sigma_m\}$ of $\Delta$. Then since $\wedge \in T$, $\sigma$ is logically equivalent to $\sigma_1 \wedge \cdots \wedge \sigma_m \in T_L$. □

The basic groundwork for our first preservation result has now been completed.

**Theorem 4.2.5.** *For any set $\Gamma$ of L-sentences:*

*(1) If $\Gamma$ is preserved by inner substructures, bounded epimorphisms, and outer extensions, then $\Gamma$ is logically equivalent to a set of essentially atomic L-sentences.*

*(2) If $\Gamma$ is preserved by inner substructures and outer extensions, then $\Gamma$ is logically equivalent to a set of essentially basic L-sentences.*

*(3) If $\Gamma$ is preserved by inner substructures and bounded epimorphisms, then $\Gamma$ is logically equivalent to a set of essentially positive and universal L-sentences.*

*(4) If $\Gamma$ is preserved by inner substructures, then $\Gamma$ is logically equivalent to a set of essentially universal L-sentences.*

*(5) If $\Gamma$ is preserved by bounded epimorphisms, then $\Gamma$ is logically equivalent to a set of essentially positive L-sentences.*

*(6) If $\Gamma$ is preserved by outer extensions, then $\Gamma$ is logically equivalent to a set of essentially existential L-sentences.*

*(7) If $\Gamma$ is preserved by bounded epimorphisms and outer extensions, then $\Gamma$ is logically equivalent to a set of essentially positive and existential L-sentences.*

**Proof.** (1) Recall that $\mathrm{EA} = \{\bot, \top, \wedge, \vee, \forall_0, \exists_0\}$. To show that $\Gamma$ is logically equivalent to some $\Delta \subseteq \mathrm{EA}_L$, apply Lemma 4.2.4 with $T = \mathrm{EA}$.

Suppose $\mathfrak{S}\,\mathrm{EA}\,\mathfrak{T}$ and $\mathfrak{S} \models \Gamma$. Let $f: \mathfrak{S} \rightsquigarrow \mathfrak{T}$ be the empty function. Then by Theorem 4.2.3 there exist elementary extensions $\mathfrak{S}^*$ and $\mathfrak{T}^*$ of $\mathfrak{S}$ and $\mathfrak{T}$, respectively, and a bounded morphism $f^*: \mathfrak{S}^* \rightsquigarrow \mathfrak{T}^*$, with $\mathrm{dom} f^*$ an inner substructure of $\mathfrak{S}^*$. Since $\mathfrak{S} \prec \mathfrak{S}^*$, $\mathfrak{S}^* \models \Gamma$. But $\Gamma$ is preserved by inner substructures and bounded epimorphisms, giving $\mathrm{dom} f^* \models \Gamma$ and then $\mathrm{im} f^* \models \Gamma$. Hence as $\mathfrak{T}^*$ is an outer extension of $\mathrm{im} f^*$ (Corollary 3.2.3), $\mathfrak{T}^* \models \Gamma$. Since $\mathfrak{T} \prec \mathfrak{T}^*$, we finally get $\mathfrak{T} \models \Gamma$ as desired.

(Note that if $L$ has no constants, the only essentially atomic $L$-sentences are $\wedge, \vee$-combinations of $\bot$ and $\top$, so all members of $\mathrm{EA}_L$ are either valid or inconsistent.)

(2) Repeat the proof of part (1) with $T = \mathrm{EB} = \mathrm{EA} \cup \{\neg\alpha\}$ instead of EA. Then by 4.2.3(5), the function $f^*$ is injective, and so is an isomorphism between $\mathrm{dom} f^*$ and $\mathrm{im} f^*$. But *all* sentences are preserved by isomorphisms, so the argument goes through without the additional hypothesis of (1).

(3) This time take $T = \mathrm{EPU} = \mathrm{EA} \cup \{\forall\}$. Then by 4.2.3(3), $f^*$ maps $\mathrm{dom} f^*$

onto $\mathfrak{T}^*$, leading directly from $\mathrm{dom} f^* \models \Gamma$ to $\mathfrak{T}^* \models \Gamma$ if $\Gamma$ is preserved by bounded epimorphisms.

(4) When $T = \mathrm{EU} = \mathrm{EA} \cup \{\neg\alpha, \forall\}$, $f^*$ becomes an isomorphism between $\mathrm{dom} f^*$ and $\mathfrak{T}^*$.

(5) When $T = \mathrm{EP} = \mathrm{EA} \cup \{\forall, \exists\}$, $f^*$ becomes total (4.2.3(4)) and surjective (4.2.3(3)), making $\mathfrak{T}^*$ a bounded epimorphic image of $\mathfrak{S}^*$.

(6) When $T = \mathrm{EE} = \mathrm{EA} \cup \{\neg\alpha, \exists\}$, $f^*$ becomes total and injective, making $\mathfrak{S}^*$ isomorphic to an inner substructure of $\mathfrak{T}^*$.

(7) When $T = \mathrm{EPE} = \mathrm{EA} \cup \{\exists\}$, $\mathrm{im} f^*$ is a bounded epimorphic image of $\mathfrak{S}^*$. □

**Corollary 4.2.6.** *In* 4.2.5, *if* $\Gamma = \{\sigma\}$, *then in each of* (1)–(7) *it can be concluded that* $\sigma$ *is logically equivalent to a sentence in the appropriate* $T_L$.

**Proof.** By Theorem 4.2.4(2). □

Results (2) and (3) in the case $n = 2$ were obtained for many-sorted admissible fragments of $L_{\omega_1\omega}$ by Feferman [15], using proof-theoretic methods.

There is one obvious case of $T_L$ that was not considered, viz. $T = \mathrm{EA} \cup \{\neg\alpha, \forall, \exists\}$. In that case $f^*$ becomes total, surjective, and injective, making $\mathfrak{S}^* \equiv \mathfrak{T}^*$. It follows that in this case *every* $L$-sentence is logically equivalent to a member of $T_L$.

In applying Theorem 4.2.3 to the proof of 4.2.5, we used the case $L_{\mathfrak{S}} = L_{\mathfrak{T}}$ $(= L)$ of 4.2.3. The possibility that $L_{\mathfrak{T}}$ may be a proper extension of $L_{\mathfrak{S}}$ will be crucial in the next section (cf. 4.2.2 and the proof of 4.3.3).

### 4.3. *And disjoint unions*

An $L$-sentence preserved by bounded epimorphisms, inner substructures, and disjoint unions is logically equivalent to a member of $\forall_1\mathrm{EA}_L$. This was shown by van Benthem [6] (in the case $n = 2$), building on the constructions of Section 4.2 to produce families of elementary-chain systems of the type that appear in the proof of Theorem 4.2.3. In this section, a new proof of the result will be obtained by invoking the concept of *bounded union*. A structure $\mathfrak{S}$ is the bounded union of a collection $\{\mathfrak{S}_i : i \in I\}$ if

(1) $X_{\mathfrak{S}} = \bigcup \{X_{\mathfrak{S}_i} : i \in I\}$, and

(2) each $\mathfrak{S}_i$ is an inner substructure of $\mathfrak{S}$.

A disjoint union $\sum_I \mathfrak{S}_i$ is the bounded union of the isomorphic copies $\mathfrak{S}_i \times \{i\}$ of the $\mathfrak{S}_i$'s. Hence any set $\Gamma$ of sentences that is preserved by bounded unions will be preserved by disjoint unions.

**Lemma 4.3.1.** *If* $\Gamma$ *is preserved by disjoint unions and bounded epimorphisms, then* $\Gamma$ *is preserved by bounded unions.*

**Proof.** If $\mathfrak{S}$ is the bounded union of $\{\mathfrak{S}_i : i \in I\}$, define $f : \sum_I \mathfrak{S}_i \to \mathfrak{S}$ by $f(x, i) = x$ for all $i \in I$ and $x \in X_{\mathfrak{S}_i}$. Then $f$ is surjective by (1) above, and is a bounded morphism by (2)—the fact that $\mathfrak{S}_i$ is an *inner* substructure of $\mathfrak{S}$ being essential to showing that $f$ is bounded. Hence $\mathfrak{S}$ is a bounded epimorphic image of $\sum_I \mathfrak{S}_i$. □

Thus, given preservation by bounded epimorphisms, the hypothesis of preservation by disjoint unions can be replaced by that of preservation by the more readily constructed bounded unions.

Recall that $\forall_1 T_L = \{\forall v\, \phi \in \mathrm{Sent}_L : \phi \in T_L^{\mathrm{E}}\}$. If $\Gamma \subseteq \mathrm{Sent}_L$, define

$$\forall_1 T_L(\Gamma) = \{\sigma \in \forall_1 T_L : \Gamma \vDash \sigma\}.$$

**Theorem 4.3.2.** *Let $\Gamma$ be a set of $L$-sentences such that either*

(1) *$\Gamma$ is preserved by bounded epimorphisms and inner substructures; or*

(2) *$\Gamma$ is preserved by bounded epimorphisms and $\exists \in T$; or*

(3) *$\Gamma$ is preserved by inner substructures and $\neg\alpha \in T$; or*

(4) *$\exists, \neg\alpha \in T$.*

*Suppose $\mathfrak{T} \vDash \forall_1 T_L(\Gamma)$, where $\mathrm{EA} \subseteq T$ and $L \subseteq L_{\mathfrak{T}}$. Then for any $y \in X_{\mathfrak{T}}$ there exists an elementary extension $\mathfrak{T}^*$ of $\mathfrak{T}$ and an $L$-structure $\mathfrak{S}$ such that*

(5) *$y \in X_{\mathfrak{S}}$;*

(6) *$\mathfrak{S}$ is an inner $L$-substructure of $\mathfrak{T}^*$; i.e. an inner substructure of the $L$-reduct of $\mathfrak{T}^*$; and*

(7) *$\mathfrak{S} \vDash \Gamma$.*

**Proof.** Let $L_y = L \cup \{\bar{y}\}$, where $\bar{y}$ is a new constant. Then $(\mathfrak{T}, y)$ is an expansion of an $L_y$-structure. Put

$$\Delta_y = \Gamma \cup \{\neg\sigma : \sigma \in T_{L_y} \text{ and } (\mathfrak{T}, y) \nvDash \sigma\}.$$

We show that $\Delta_y$ has a model, noting first that $\neg\bot \in \Delta_y$, since $\bot \in T$, so $\Delta_y \neq \emptyset$. Let $\neg\sigma_1, \ldots, \neg\sigma_m \in \Delta_y - \Gamma$. Then if $\Gamma \cup \{\neg\sigma_1, \ldots, \neg\sigma_m\}$ had no model it would follow that $\Gamma \vDash \sigma_1 \vee \cdots \vee \sigma_m$. But $\bar{y}$ does not occur in $\Gamma$, so if $v$ were a variable not occurring in $\sigma_1, \ldots, \sigma_m$, then $\Gamma \vDash \forall v\, \sigma$, where $\sigma$ is the result of replacing $\bar{y}$ by $v$ in $\sigma_1 \vee \cdots \vee \sigma_m$. But then $\forall v\, \sigma \in \forall_1 T_L(\Gamma)$, since $\vee \in T$, so $\mathfrak{T} \vDash \forall v\, \sigma$. Then $y$ would satisfy $\sigma$ in $\mathfrak{T}$, so that $(\mathfrak{T}, y) \vDash \sigma_1 \vee \cdots \vee \sigma_m$, in contradiction with the fact that $(\mathfrak{T}, y) \nvDash \sigma_i$ for $1 \leq i \leq m$.

Thus by Compactness, there exists a model $(\mathfrak{S}^*, x)$ of $\Delta_y$ with $\mathfrak{S}^*$ an $L$-structure, and $x$ the interpretation of $\bar{y}$. Let $f = \{(x, y)\}$. Then in view of $\Delta_y$, $(\mathfrak{S}^*, x)\ T\ (\mathfrak{T}, f, (x))$. Hence by Theorem 4.2.3, there are structures $\mathfrak{S}^{**}$ and $\mathfrak{T}^*$, with $\mathfrak{S} < \mathfrak{S}^{**}$ and $\mathfrak{T} < \mathfrak{T}^*$, and a function $f^*$ extending $f$ that is a bounded morphism from an inner substructure of $\mathfrak{S}^{**}$ to $\mathfrak{T}^*$. Let $\mathfrak{S}$ be the $L$-substructure of $\mathfrak{T}^*$ based on $\mathrm{im} f^*$. Then $y = f^*(x) \in \mathrm{im} f^*$, so (5) holds. Since the image of a bounded morphism is an inner substructure of its codomain, (6) holds. For (7), observe that since $\mathfrak{S}^* < \mathfrak{S}^{**}$, $\mathfrak{S}^{**} \vDash \Gamma$. Hence (1) if $\Gamma$ is preserved by bounded

epimorphisms and inner substructures, we get $\mathrm{dom} f^* \models \Gamma$ as $\mathrm{dom} f^* \in \mathfrak{S}^{**}$, and then $\mathfrak{S} \models \Gamma$ as $\mathfrak{S} = \mathrm{im} f^*$; (2) if $\exists \in T$, then by 4.2.3(4) we can assume $\mathrm{dom} f^* = \mathfrak{S}^{**}$, so we need only preservation by bounded epimorphisms to get $\mathfrak{S} \models \Gamma$; (3) if $\neg\alpha \in T$, then by 4.2.3(5) we can assume $f^*$ is injective making $\mathrm{dom} f^* \cong \mathfrak{S}$, so that $\mathfrak{S} \models \Gamma$ if $\mathrm{dom} f^* \models \Gamma$; and (4) if $\exists, \neg\alpha \in T$, then $\mathfrak{S}^{**} \cong \mathfrak{S}$, whence $\mathfrak{S} \models \Gamma$ immediately. □

**Theorem 4.3.3.** *Let $\Gamma$ be a set of $L$-sentences that is preserved by bounded unions and satisfies one of the hypotheses* (1)–(4) *as in* 4.3.2. *Suppose $\mathfrak{T} \models \forall_1 T_L(\Gamma)$, where $\mathrm{EA} \subseteq T$ and $L \subseteq L_{\mathfrak{T}}$. Then there exists an elementary extension $\mathfrak{T}^*$ of $\mathfrak{T}$ and an $L$-structure $\mathfrak{S}$ such that*

(5) $X_{\mathfrak{T}} \subseteq X_{\mathfrak{S}}$,

(6) $\mathfrak{S}$ *is an inner $L$-substructure of $\mathfrak{T}^*$, and*

(7) $\mathfrak{S} \models \Gamma$.

**Proof.** Let $X_{\mathfrak{T}} = \{y_\lambda : \lambda < \beta\}$, for some ordinal $\beta$. We construct a sequence $\{\mathfrak{S}_\lambda : \lambda < \beta\}$ of $L$-structures, and an elementary chain $\{\mathfrak{T}_\lambda : \lambda < \beta\}$ of structures $\mathfrak{T}_\lambda$ with $\mathfrak{T} < \mathfrak{T}_\lambda$, such that

(a) $y_\lambda \in X_{\mathfrak{S}_\lambda}$,

(b) $\mathfrak{S}_\lambda$ is an inner $L$-substructure of $\mathfrak{T}_\lambda$, and

(c) $\mathfrak{S}_\lambda \models \Gamma$.

Also constructed is a collection $\{\mathfrak{S}_{\kappa\lambda} : \kappa \leq \lambda < \beta\}$ of $L$-structures such that for each $\lambda < \beta$,

(d) $\mathfrak{S}_{\kappa\lambda}$ is an $L$-substructure of $\mathfrak{T}_\lambda$ if $\kappa \leq \lambda$,

(e) $\mathfrak{S}_{\lambda\lambda} = \mathfrak{S}_\lambda$, and

(f) $(\mathfrak{T}_\mu, \mathfrak{S}_{\kappa\mu}) < (\mathfrak{T}_\lambda, \mathfrak{S}_{\kappa\lambda})$ if $\kappa \leq \mu \leq \lambda$.

With regard to (f), a pair $(\mathfrak{T}^\#, \mathfrak{S}^\#)$ with $\mathfrak{S}^\#$ an $L$-substructure of $\mathfrak{T}^\#$ is to be regarded as a single structure for $L_{\mathfrak{T}} \cup \{\pi\}$, where $\pi$ is a unary relational symbol whose extension in $\mathfrak{T}^\#$ is $X_{\mathfrak{S}^\#}$. Then $\mathfrak{S}^\# \models \sigma$ iff $(\mathfrak{T}^\#, \mathfrak{S}^\#) \models \sigma^\pi$, where $\sigma^\pi$ is obtained by relativising the quantifiers in $\sigma$ to $\pi$ in the standard way. In particular, (f) implies that $\mathfrak{T}_\mu < \mathfrak{T}_\lambda$ and $\mathfrak{S}_{\kappa\mu} < \mathfrak{S}_{\kappa\lambda}$.

The above construction is carried but by transfinite induction on $\beta$. At stage $\gamma$, $\mathfrak{T}_\gamma$ and $\mathfrak{S}_\gamma$ are constructed to include $y_\gamma$ in $\mathfrak{S}_\gamma$ and fulfill (a)–(f) with $\gamma$ in place of $\lambda$. $\mathfrak{S}_{\kappa\gamma}$ is the 'enlargement' of previously constructed $\mathfrak{S}_\kappa$ that takes place at stage $\gamma$.

Supposing for now that the construction has been made, put $\mathfrak{T}^* = \bigcup \{\mathfrak{T}_\lambda : \lambda < \beta\}$, so that $\mathfrak{T} < \mathfrak{T}^*$. For each $\kappa < \beta$, put $\mathfrak{S}^*_\kappa = \bigcup \{\mathfrak{S}_{\kappa\lambda} : \kappa \leq \lambda < \beta\}$. Then $\mathfrak{S}^*_\kappa$ is an $L$-substructure of $\mathfrak{T}^*$, and from (e) and (f) we have

$$(\mathfrak{T}_\kappa, \mathfrak{S}_\kappa) = (\mathfrak{T}_\kappa, \mathfrak{S}_{\kappa\kappa}) < (\mathfrak{T}_\lambda, \mathfrak{S}_{\kappa\lambda})$$

whenever $\kappa \leq \lambda < \beta$. Hence

(g) $(\mathfrak{T}_\kappa, \mathfrak{S}_\kappa) < (\mathfrak{T}^*, \mathfrak{S}^*_\kappa)$.

But $\mathfrak{S}_\kappa$ is an inner substructure of $\mathfrak{T}_\kappa$, by (b), so the sentence

$$\forall v_0 \cdots \forall v_n (\pi(v_n) \wedge \bar{R}(v_0, \ldots, v_n) \to \pi(v_0) \wedge \cdots \wedge \pi(v_{n-1}))$$

is true in $(\mathfrak{T}_\kappa, \mathfrak{S}_\kappa)$, hence by (g) is true in $(\mathfrak{T}^*, \mathfrak{S}^*_\kappa)$. This shows that each $\mathfrak{S}^*_\kappa$ is an inner substructure of $\mathfrak{T}^*$. Put $\mathfrak{S} = \bigcup \{\mathfrak{S}^*_\kappa : \kappa < \beta\}$. Then $\mathfrak{S}$ is an $L$-substructure of $\mathfrak{T}^*$. We have:

(5) For each $\lambda < \beta$, $y_\lambda \in X_{\mathfrak{S}_\lambda} = X_{\mathfrak{S}_{\lambda\lambda}} \subseteq X_{\mathfrak{S}^*_\lambda}$, so $y_\lambda \in X_{\mathfrak{S}}$. Hence $X_{\mathfrak{T}} \subseteq X_{\mathfrak{S}}$.

(6) $\mathfrak{S}$ is an inner $L$-substructure of $\mathfrak{T}^*$, since it is the union of inner $L$-substructures of $\mathfrak{S}^*$.

(7) Each $\mathfrak{S}^*_\kappa$ is an inner substructure of $\mathfrak{S}$, since it is a substructure of $\mathfrak{S}$ and an inner substructure of $\mathfrak{T}^*$. Hence $\mathfrak{S}$ is the *bounded* union of $\{\mathfrak{S}^*_\kappa : \kappa < \beta\}$. But $\mathfrak{S}_\kappa < \mathfrak{S}^*_\kappa$, by (g), so $\mathfrak{S}^*_\kappa \models \Gamma$ from (c). Since $\Gamma$ is preserved by bounded unions, this gives $\mathfrak{S} \models \Gamma$ as desired.

Thus to complete the theorem, it remains to show that the above construction can be carried out. Take $\gamma < \beta$, and assume inductively that for each $\lambda < \gamma$, and each $\kappa \leq \lambda$, $\mathfrak{S}_\lambda$, $\mathfrak{T}_\lambda$, and $\mathfrak{S}_{\kappa\lambda}$ have been constructed such that $\mathfrak{T} < \mathfrak{T}_\lambda$ and (a)–(f) hold. If $\gamma = 0$, put $\mathfrak{T}^0_\gamma = \mathfrak{T}^1_\gamma = \mathfrak{T}$. Otherwise, let $\mathfrak{T}^0_\gamma = \bigcup \{\mathfrak{T}_\lambda : \lambda < \gamma\}$, and for $\kappa < \gamma$ put $\mathfrak{S}^0_\kappa = \bigcup \{\mathfrak{S}_{\kappa\lambda} : \kappa \leq \lambda < \gamma\}$. Then $\mathfrak{S}^0_\kappa$ is an $L$-substructure of $\mathfrak{T}^0_\gamma$ by (d). Note also, from (f), that

(h) $(\mathfrak{T}_\mu, \mathfrak{S}_{\kappa\mu}) < (\mathfrak{T}^0_\gamma, \mathfrak{S}^0_\kappa)$ if $\kappa \leq \mu < \gamma$.

Now let

$$\mathfrak{T}^1_\gamma = (\mathfrak{T}^0_\gamma, \{\mathfrak{S}^0_\kappa : \kappa < \gamma\}).$$

Regard $\mathfrak{T}^1_\gamma$ as a single structure (in the manner described above) for a language that adds to $L_{\mathfrak{T}}$ predicates $\pi_\kappa$ to name $X_{\mathfrak{S}^0_\kappa}$ in $\mathfrak{T}^0_\gamma$. Then $\mathfrak{T}^1_\gamma \models \forall_1 T_L(\Gamma)$, since $\mathfrak{T} < \mathfrak{T}^0_\gamma$, and so by Theorem 4.3.2, with $\mathfrak{T}^1_\gamma$ in place of $\mathfrak{T}$ and $y_\gamma$ for $y$, there exists an elementary extension

$$\mathfrak{T}^*_\gamma = (\mathfrak{T}_\gamma, \{\mathfrak{S}_{\kappa\gamma} : \kappa < \gamma\})$$

of $\mathfrak{T}^1_\gamma$, and an $L$-structure $\mathfrak{S}_\gamma$ such that (a) $y_\gamma \in X_{\mathfrak{S}_\gamma}$; (b) $\mathfrak{S}_\gamma$ is an inner $L$-substructure of the $L_{\mathfrak{T}}$-structure $\mathfrak{T}_\gamma$; and (c) $\mathfrak{S}_\gamma \models \Gamma$. Each $\mathfrak{S}_{\kappa\gamma}$ is an $L$-substructure of $\mathfrak{T}_\gamma$, giving (d) for $\lambda = \gamma$. Put $\mathfrak{S}_{\gamma\gamma} = \mathfrak{S}_\gamma$, so that $\mathfrak{S}_{\kappa\lambda}$ is now defined whenever $\kappa \leq \lambda \leq \gamma$, and (e) is fulfilled for $\gamma$. Finally, since $\mathfrak{T}^1_\gamma < \mathfrak{T}^*_\gamma$ we get $(\mathfrak{T}^0_\gamma, \mathfrak{S}^0_\kappa) < (\mathfrak{T}_\gamma, \mathfrak{S}_{\kappa\gamma})$, which by (h) and the inductive hypothesis ensures that (f) holds with $\gamma$ in place of $\lambda$. Hence the inductive construction for $\gamma$ is complete. □

**Theorem 4.3.4.** *Let $\Gamma$ be a set of $L$-sentences.*

(1) *If $\Gamma$ is preserved by bounded epimorphisms, inner substructures, and disjoint unions, then $\Gamma$ is logically equivalent to some $\Delta \subseteq \forall_1 \mathrm{EA}_L$.*

(2) *If $\Gamma$ is preserved by bounded epimorphisms and disjoint unions, then $\Gamma$ is logically equivalent to some $\Delta \subseteq \forall_1 \mathrm{EPE}_L$.*

(3) If $\Gamma$ is preserved by inner substructures, and bounded unions, then $\Gamma$ is logically equivalent to some $\Delta \subseteq \forall_1 EB_L$.

(4) *If $\Gamma$ is preserved by bounded unions, then $\Gamma$ is logically equivalent to some* $\Delta \subseteq \forall_1 EE_L$.

*Moreover, in each of* (1)–(4), *if* $\Gamma = \{\sigma\}$, *then $\sigma$ is logically equivalent to some sentence belonging to the appropriate* $\forall_1 T_L$.

**Proof.** To prove (1), let $\Delta = \forall_1 EA_L(\Gamma)$. Since $\Gamma \vdash \Delta$ by definition, we have to show $\Delta \vdash \Gamma$. Let $\mathfrak{X}$ be any $L$-model of $\forall_1 EA_L(\Gamma)$. Since, by Lemma 4.3.1, $\Gamma$ is also preserved by bounded unions, we can apply Theorem 4.3.3, using alternative (1) of the hypotheses of 4.3.2, and taking $T = EA$. Sequences $\{\mathfrak{S}_m : m < \omega\}$ and $\{\mathfrak{X}_m : m < \omega\}$ of $L$-structures are constructed, with $\mathfrak{X} < \mathfrak{X}_m$, such that for each $m < \omega$,

(a) $X_{\mathfrak{X}_{m-1}} \subseteq X_{\mathfrak{S}_m}$,

(b) $\mathfrak{S}_m$ is an inner substructure of $\mathfrak{X}_m$, and

(c) $\mathfrak{S}_m \vdash \Gamma$.

Also constructed are $L$-structures $\mathfrak{S}_{pm}$ for $p \leq m$ such that for each $m < \omega$,

(d) $\mathfrak{S}_{pm}$ is a substructure of $\mathfrak{X}_m$,

(e) $\mathfrak{S}_{mm} = \mathfrak{S}_m$, and

(f) $(\mathfrak{X}_r, \mathfrak{S}_{pr}) < (\mathfrak{X}_m, \mathfrak{S}_{pm})$ if $p \leq r \leq m$.

The construction follows exactly the same pattern as the construction in the proof of 4.3.3, taking $\beta = \omega$ and using 4.3.3(5) to obtain (a). The details need not be gone over again.

Now put $\mathfrak{X}^* = \bigcup \{\mathfrak{X}_m : m < \omega\}$, so that $\mathfrak{X} < \mathfrak{X}^*$. For each $p < \omega$, let $\mathfrak{S}_p^* = \bigcup \{\mathfrak{S}_{pm} : p \leq m < \omega\}$, and put $\mathfrak{S} = \bigcup \{\mathfrak{S}_p^* : p < \omega\}$. Then again as in the proof of 4.3.3, we get that $(\mathfrak{X}_p, \mathfrak{S}_p) < (\mathfrak{X}^*, \mathfrak{S}^*)$, and $\mathfrak{S}_p^*$ and $\mathfrak{S}$ are inner substructures of $\mathfrak{X}^*$. Hence $\mathfrak{S}_p^* \in \mathfrak{S}$, so that $\mathfrak{S}$ is the bounded union of the $\mathfrak{S}_p^*$'s. Since $\mathfrak{S}_p < \mathfrak{S}_p^*$, $\mathfrak{S}_p^* \vdash \Gamma$ from (c), so that $\mathfrak{S} \vdash \Gamma$ by preservation of $\Gamma$ under bounded unions. But now by (a), $X_{\mathfrak{X}_{p-1}} \subseteq X_{\mathfrak{S}_{pp}} \subseteq X_{\mathfrak{S}_p^*}$, and so $\mathfrak{S} = \mathfrak{X}^*$. Since $\mathfrak{X} < \mathfrak{X}^*$, this finally gives $\mathfrak{X} \vdash \Gamma$, completing the proof that $\Gamma$ is logically equivalent to $\forall_1 EA_L(\Gamma) \subseteq \forall_1 EA_L$.

If, moreover, $\Gamma = \{\sigma\}$, then by Compactness, $\sigma$ is logically equivalent to

$$(\dagger) \qquad \forall v_1 \phi_1 \wedge \cdots \wedge \forall v_m \phi_m$$

for some sentences $\forall v_i \phi_i \in \forall_1 EA_L$. But $(\dagger)$ is logically equivalent to $\forall v\,(\phi_1' \wedge \cdots \wedge \phi_m') \in \forall_1 EA_L$, where $v$ is a new variable, and $\phi_i'$ is the result of replacing $v_i$ by $v$ in $\phi_i$.

For (2), apply the proof of (1), using instead the hypothesis (2) from 4.3.2, and taking $T = EPE = EA \cup \{\exists\}$. For (3) use hypothesis 4.3.2(3) and $T = EB = EA \cup \{\neg\alpha\}$, and finally for (4), use 4.3.2(4) and $T = EE = EA \cup \{\exists, \neg\alpha\}$. □

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