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GLOBAL INTUITIONISTIC ANALYSIS

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Let Ω be a complete Heyting algebra $\mathcal{O}(X)$ of open sets of a complex manifold X and $V^{(\Omega)}$ be the Ω-valued sheaf model. $V^{(\Omega)}$ is a model for the intuitionistic set theory $\mathrm{ZF_I}$ (cf. Grayson [4]), and the set $\mathbb{C}^{\Omega}$ of all complex numbers in $V^{(\Omega)}$ is, viewed externally, the sheaf of germs of continuous functions on X. Let $\mathbb{C}^{\infty}$, $\mathcal{H}$ and $\check{\mathbb{C}}$ be subfields of $\mathbb{C}^{\Omega}$ in $V^{(\Omega)}$ that are the sheaves of germs of C^{∞}-functions, holomorphic functions and constant functions, respectively. C. Rousseau showed in [9] that one can develop an intuitionistic complex analysis on each of $\mathbb{C}^{\infty}$, $\mathcal{H}$, $\check{\mathbb{C}}$ in $V^{(\Omega)}$. Further, she showed that Weierstrass theorems in the complex analysis of several variables can be translated and proved in an intuitionistic complex analysis of one variable. Local theory can be translated into the intuitionistic set theory $\mathrm{ZF_I}$ in this manner. However, most of the mathematically interesting theorems are global. In order to develop an intuitionistic version of global theory, we strengthened $\mathrm{ZF_I}$ in [13] so that global concepts are expressible in it, and obtained the global intuitionistic set theory GIZF. $V^{(\Omega)}$ is a model of GIZF, and a sheaf model $W^{(H)}$ which is isomorphic to the universe can be constructed in GIZF.

In this paper we formulate a global intuitionistic analysis GIA in GIZF, and consider a GIA-version of global analysis; that is, we define in GIA cohomology group with coefficients in the inner model $W^{(H)}$ and translate the deRham theorem and the Dolbeault theorem in GIA; further, we translate a vector bundle as an apartness vector space in GIA, a family of complex structures as a complex manifold in GIA, and an infinitesimal deformation of a family of complex structures as a notion on a complex manifold in GIA.

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1. Preliminaries

In this section we review briefly the intuitionistic set theory in Grayson [4]; the intuitionistic linear algebra in Ruitenberg [10]; and the global intuitionistic set theory in Takeuti and Titani [13].

1.1. *Intuitionistic set theory* ZF_1

We refer to the stronger intuitionistic set theory ZF_I' in Grayson [4] as ZF_I. ZF_I is a first-order intuitionistic theory with relation symbols $\in$ and $=$, and with the following nonlogical axioms.

A1. Equality axioms: $u=u$, $u=v\to v=u$, and $u=v\wedge\varphi(u)\to\varphi(v)$.

A2. Extensionality: $\forall z\,(z\in u\leftrightarrow z\in v)\to u=v$.

A3. Pairing: $\exists x\,\forall z\,(z\in x\leftrightarrow z=u\vee z=v)$.

A4. Union: $\exists x\,\forall z\,(z\in x\leftrightarrow\exists y\in u\,(z\in y))$.

A5. Power set: $\exists x\,\forall z\,(z\in x\leftrightarrow\forall y\in z\,(y\in u))$.

A6. $\in$-induction: $\forall x\,(\forall y\in x\,\varphi(y)\to\varphi(x))\to\forall x\,\varphi(x)$.

A7. Separation: $\exists x\,\forall z\,(z\in x\leftrightarrow z\in u\wedge\varphi(z))$.

A8. Collection: $\forall x\in u\,\exists y\,\varphi(x,y)\to\exists v\,\forall x\in u\,\exists y\in v\,\varphi(x,y)$.

A9. Infinity: $\exists x\,(\exists y\,(y\in x)\wedge\forall y\in x\,\exists z\in x\,(y\in z))$.

In ZF_I, we have a recursion theorem, so that the set ω of natural numbers and the set $\mathbb{Q}$ of rational numbers are defined as usual, and they satisfy the full trichotomy,

$$\forall x,y\,(x<y\vee x=y\vee x>y).$$

Real numbers are defined in ZF_I as pairs of subsets of $\mathbb{Q}$, $\langle L,U\rangle$, such that

(1) $\exists r,s\in\mathbb{Q}\,(r\in L\wedge s\in U)$,

(2) $\forall r\in\mathbb{Q}\,\neg(r\in L\wedge r\in U)$,

(3) $\forall r\in\mathbb{Q}\,(r\in U\leftrightarrow\exists s\in U\,(s<r))$,

(4) $\forall r\in\mathbb{Q}\,(r\in L\leftrightarrow\exists s\in L\,(r<s))$,

(5) $\forall r,s\in\mathbb{Q}\,(s<r\to s\in L\vee r\in U)$.

The set of all real numbers is denoted by $\mathbb{R}$, and the order $<$, $\leqslant$ on $\mathbb{R}$ is given by:

$$\langle L,U\rangle<\langle L',U'\rangle \quad\text{iff}\quad \exists x\in U\cap L',$$
$$\langle L,U\rangle\leqslant\langle L',U'\rangle \quad\text{iff}\quad L\subseteq L'\wedge U'\subseteq U.$$

The arithmetic on $\mathbb{R}$ is induced from the arithmetic on $\mathbb{Q}$ in a natural way (cf. Takeuti and Titani [12]). The set of complex numbers is $\mathbb{C}=\mathbb{R}+i\mathbb{R}$ as usual.

Let Ω be a complete Heyting algebra (cHa) in $\mathrm{ZF_I}$. Then a Ω-valued model $V^{(\Omega)}$ of $\mathrm{ZF_I}$ is constructed in $\mathrm{ZF_I}$ as follows. For each ordinal α the set $V_\alpha^{(\Omega)}$ is defined as the set of all Ω-valued functions u on a subset $\mathscr{D}(u)$ of $V_\beta^{(\Omega)}$ with $\beta \in \alpha$. $V^{(\Omega)}$ is then $\bigcup_\alpha V_\alpha^{(\Omega)}$. The evaluation $[\![\cdot]\!]$ for atomic sentences of $V^{(\Omega)}$ is given by simultaneous recursion:

$$[\![u \in v]\!] = \bigvee_{x \in \mathscr{D}(v)} [\![u = x]\!] \wedge v(x),$$

$$[\![u = v]\!] = \bigwedge_{x \in \mathscr{D}(u)} (u(x) \to [\![x \in v]\!]) \wedge \bigwedge_{x \in \mathscr{D}(u)} (v(x) \to [\![x \in u]\!]).$$

From this, the evaluation for non-atomic sentences is defined in natural way. It is proved in $\mathrm{ZF_I}$ that $V^{(\Omega)}$ is a model of $\mathrm{ZF_I}$. $V^{(\Omega)}$ is called a (*Ω-valued*) *sheaf model*. It is known that the universe V of $\mathrm{ZF_I}$ is embedded in the sheaf model $V^{(\Omega)}$ constructed in V as follows. For each set x in V define a member $\check{x}$ of $V^{(\Omega)}$ by

$$\mathscr{D}(\check{x}) = \{\check{t} : t \in x\}, \quad \text{and} \quad \check{x}(\check{t}) = 1.$$

Then the mapping $x \mapsto \check{x}$ is an embedding of V into $V^{(\Omega)}$ in the following sense. If $\varphi(x_1, \ldots, x_n)$ is a sentence built up from atoms using $\wedge$, $\vee$, $\exists$ and bounded $\forall$, then $\varphi(x_1, \ldots, x_n)$ implies $[\![\varphi(\check{x}_1, \ldots, \check{x}_n)]\!] = 1$. For the sets ω and $\mathbb{Q}$ of natural numbers and rational numbers in V, $\check{\omega}$, $\check{\mathbb{Q}}$ are the sets of natural numbers and rational numbers in $V^{(\Omega)}$. We call a set of the form $\check{x}$ a *check set*.

In the rest of this paper, Ω denotes a cHa, and $V^{(\Omega)}$ the Ω-valued sheaf model.

1.2. Intuitionistic linear algebra

Definition. A relation $\neq$ on a set S is called *apartness*, if for $a, b, c \in S$

(1) $a = b \leftrightarrow \neg(a \neq b)$,

(2) $a \neq b \to a \neq c \vee b \neq c$.

On the set $\mathbb{R}$ of real numbers, the relation $\neq$ such that $u \neq v$ iff $v < v \vee v < u$ is an apartness.

Definition. A ring S (by ring we mean commutative ring) with apartness $\neq$ is called an *apartness ring* if for $a, b, c \in S$

(1) $a \neq b \to a + c \neq b + c$,

(2) $a \neq b,\ c \neq 0 \to ac \neq bc$.

An apartness ring S with 1 is an *apartness field* if

(3) $1 \neq 0$,

(4) $a \neq 0 \to \exists a^{-1}(a^{-1} \neq 0 \wedge aa^{-1} = 1)$.

Lemma 1.2.1 (cf. Heyting [7]). *On an apartness field S,*

(1) $ab \neq 0 \rightarrow a \neq 0 \wedge b \neq 0$,

(2) $a + b \neq c + d \rightarrow a \neq c \vee b \neq d$,

(3) $ab \neq cd \rightarrow a \neq c \vee b \neq d$.

Examples. The fields $\mathbb{R}$ and $\mathbb{C}$ of real numbers and complex numbers are apartness fields.

Definitions. Let F be an apartness ring. S is called an *apartness F-module* if S is and F-module with apartness $\neq$ satisfying the following conditions. For $a, a' \in F$ and $\alpha, \alpha', \beta, \beta' \in S$,

(1) $a\alpha \neq a'\alpha' \rightarrow a \neq a' \vee \alpha \neq \alpha'$,

(2) $\alpha + \beta \neq \alpha' + \beta' \rightarrow \alpha \neq \alpha' \vee \beta \neq \beta'$.

If F is an apartness field, then an apartness F-module is called an *apartness vector space over F.*

Let S be an apartness F-module and $n \in \omega$. A function $\alpha : n \rightarrow S$ is called an *n-sequence in S* and written as $\{\alpha_1, \ldots, \alpha_n\}$. An apartness on the set S^n of n-sequences is given by

$$\alpha \neq \beta \leftrightarrow \exists i \leq n\ (\alpha(i) \neq \beta(i)).$$

For each $\alpha \in S^n$, the sum $\sum_{i \in n} \alpha(i) = \alpha_1 + \cdots + \alpha_n$ is defined and

$$\alpha_1 + \cdots + \alpha_n \neq \beta_1 + \cdots + \beta_n \rightarrow \alpha \neq \beta.$$

Definition. Let T be an apartness vector space over an apartness field F. A finite sequence of vectors $\{e_1, \ldots, e_n\} \in T^n$ is *apartness linearly independent (over F)* if

$$\forall \{a_1, \ldots, a_n\} \in F^n (\exists i\ (a_i \neq 0) \rightarrow a_1 e_1 + \cdots + a_n e_n \neq 0).$$

$\{e_1, \ldots, e_n\} \in T^n$ is called an *apartness basis for T,* if

(1) $\{e_1, \ldots, e_n\}$ is apartness linearly independent,

(2) T is generated by $\{e_1, \ldots, e_n\}$, i.e.

$$\forall \alpha \in T\ \exists \{a_1, \ldots, a_n\} \in F^n (\alpha = a_1 e_1 + \cdots + a_n e_n).$$

The following theorem is proved in Routenberg [10].

Theorem 1.2.2. *Let T be an apartness vector space over F. If* $\{e_1, \ldots, e_m\}$ *and* $\{e'_1, \ldots, e'_n\}$ *are apartness bases for T, then* $m = n$.

Definition. If an apartness vector space T has an apartness basis $\{e_1, \ldots, e_n\}$, then we say T has *dimension n.*

Definition. Let T_1 and T_2 be apartness F-modules. A mapping $h: T_1 \to T_2$ is a *homomorphism* if

$$\forall \alpha, \beta \in T_1 \; \forall a, b \in F \; [h(a\alpha + b\beta) = ah(\alpha) + bh(\beta)].$$

A homomorphism $h: T_1 \to T_2$ is an *embedding* if

$$\forall \alpha, \beta \in T_1 \; [h(\alpha) = h(\beta) \to \alpha = \beta],$$

and h is an *apartness embedding* if

$$\forall \alpha, \beta \in T_1 \; [\alpha \neq \beta \to h(\alpha) \neq h(\beta)].$$

An (apartness) embedding $h: T_1 \to T_2$ is called an (*apartness*) *isomorphism*, if

$$\forall \beta \in T_2 \; \exists \alpha \in T_1 \; (h(\alpha) = \beta).$$

Let F be an apartness field and $n \in \omega$. Then F^n is an n-dimensional apartness vector space over F. If $m, n \in \omega$ and a function $h: F^n \to F^m$ is a homomorphism, then h can be expressed as an $m \times n$ matrix (a_{ij}) over F. The determinant of an $n \times n$ matrix M, $\det M$, is the well known polynomial in n^2 variables.

Theorem 1.2.3. *If F is an apartness field, $h: F^n \to F^n$ is a homomorphism and M is the $n \times n$ matrix of h, then*

(1) $\forall \xi \in F^n (\xi \neq 0 \leftrightarrow h\xi \neq 0)$ *iff* $\det M \neq 0$,

(2) *M is invertible iff* $\det M \neq 0$.

The proof is in Ruitenberg [10].

1.3. Global intuitionistic set theory GIZF

A global intuitionistic logic GI is an intuitionistic modal logic that is expressed in the language of Gentzen's intuitionistic prediate logic in [2] with an additional logical symbol $\Box$. The new logical symbol $\Box$ for modality is called *globalization* and read 'it is globally true that $\cdots$'. A *sequent* of GI is of the form $A_1, \ldots, A_n \Rightarrow B$ or $A_1, \ldots, A_n \Rightarrow$ where $n \geq 0$ and $A_1, \ldots, A_n$, B are formulas. The axiom schemata and inference rules of GI are those of Gentzen's intuitionistic predicate logic LJ plus the following G1–G4.

G1. $\Box A \Rightarrow A$,

G2. $\Box A \to \Box B \Rightarrow \Box(\Box A \to B)$,

G3. $\forall x \, \Box A(x) \Rightarrow \Box \forall x \, A(x)$,

G4. $$\frac{\Box A_1, \ldots, \Box A_n \Rightarrow B}{\Box A_1, \ldots, \Box A_n \Rightarrow \Box B}.$$

Let Ω be a cHa $\mathcal{O}(X)$ of open sets of a nonempty topological space X, and

denote $\exists x\ (x \in a)$ by $a \neq 0$. Then: on Ω,

(1) $a = 0 \leftrightarrow \neg(a \neq 0)$,

(2) $1 \neq 0$,

(3) $a \wedge b \neq 0 \rightarrow a \neq 0 \wedge b \neq 0$,

(4) $A \subset \Omega \wedge \bigvee A \neq 0 \rightarrow \exists a \in A\ (a \neq 0)$,

(5) $(a \neq 0 \rightarrow a \leqq b) \rightarrow a \leqq b$.

A predicate $\neq 0$ on a cHa satisfying (1)–(5) will be called an *apartness from* 0.

Let Ω be a cHa on which apartness from 0, $a \neq 0$, is defined. Then an operation $\Box$ on Ω given by

(1.3.1) $\Box a = \bigvee \{v \in \Omega \mid a = 1\}$

satisfies the following condition corresponding to G1–G4.

$\Box 1$. $\Box a \leqq a$,

$\Box 2$. $\Box a \rightarrow \Box b \leqq \Box(\Box a \rightarrow b)$,

$\Box 3$. $\bigwedge_i \Box a_i \leqq \Box(\bigwedge_i a_i)$,

$\Box 4$. $\Box a_i \wedge \cdots \wedge \Box a_n \leqq b$ implies $\Box a_1 \wedge \cdots \wedge \Box a_n \leqq \Box b$.

An operation $\Box$ on a cHa is called a *globalization* if it satisfies $\Box 1$–$\Box 4$.

Let V be our universe of $\mathrm{ZF_I}$, Ω be a cHa with apartness from 0 and let $\Box$ be the globalization on Ω given by (1.3.1). $V^{(\Omega)}$ is then a model of GI, and if $\varphi(x_1, \ldots, x_n)$ is a sentence in which all quantifiers are bounded, then

$$\varphi(x_1, \ldots, x_n) \quad \text{iff} \quad [\![\varphi(\check{x}_1, \ldots, \check{x}_n)]\!] = 1 \text{ on } V^{(\Omega)},$$

hence V is embedded in $V^{(\Omega)}$ as a submodel, where we identify two members x and y in $V^{(\Omega)}$ if $[\![x = y]\!] = 1$. We introduce a predicate $\mathrm{ck}(x)$ to mean that x is a check set, i.e.

(1.3.2) $[\![\mathrm{ck}(x)]\!] = \bigvee_{z \in V} [\![\Box(x = \check{z})]\!]$

and a function π on $V^{(\Omega)}$ defined by

(1.3.3) $\pi(x) = [\![0 \in x]\!]^{\vee}$.

Then the truth value $[\![\varphi]\!]$ of a sentence φ on $V^{(\Omega)}$ is expressed as $\pi\{x \in 1 : \varphi\}$ $(= [\![\varphi]\!]^{\vee})$ in $V^{(\Omega)}$.

The global intuitionistic set theory GIZF introduced in Takeuti and Titani [13] is an intuitionistic set theory governed by the global intuitionistic logic GI with symbols ck and π, for which the above $V^{(\Omega)}$ with the interpretation (1.3.1), (1.3.2), (1.3.3) is a model. We call the model $V^{(\Omega)}$ a *standard sheaf model of* GIZF.

Nonlogical axioms of GIZF are as follows, where we denote $\Box(x=y)$, $\Box(x\in y)$, $\Box(x\subset y)$ by $x\overset{\sqcup}{=}y$, $x\overset{\sqcup}{\in}y$, $x\overset{\sqcup}{\subset}y$; $\Box(\varphi\to\psi)$ by $\varphi\overset{\sqcup}{\to}\psi$; and $\forall x\,(x\in y\to\varphi(x))$, $\exists x\,(x\in y\wedge\varphi(x))$ by $\forall x\in y\,\varphi(x)$, $\exists x\in y\,\varphi(x)$, respectively.

A1′. *Equality axioms*: $u\overset{\Box}{=}u$, $u=v\overset{\Box}{\to}v=u$, $u=v\wedge v=w\overset{\Box}{\to}u=w$, $u=v\wedge u\in w\overset{\Box}{\to}v\in w$, and $u=v\wedge w\in u\overset{\Box}{\to}w\in v$.

A2′. *Extensionality*: $\forall z\,(z\in u\leftrightarrow z\in v)\overset{\Box}{\to}u=v$.

A3′. *Pairing*: $\exists x\,\forall z\,(z\in x\overset{\Box}{\leftrightarrow}z=u\vee z=v)$

A4′. *Union*: $\exists x\,\forall z\,(z\in x\overset{\Box}{\leftrightarrow}\exists y\in u\,(z\in y))$

A5′. *Power-set*: $\exists x\,\forall z\,(z\in x\overset{\Box}{\leftrightarrow}\forall y\in z\,(y\in u))$

A6′. *ϵ-induction*: $\mathrm{Reg}\,\varphi(x)\wedge\forall x\,(\forall y\in x\,\varphi(y)\to\varphi(x))\overset{\Box}{\to}\forall x\,\varphi(x)$.
where $\mathrm{Reg}\,\varphi(x)$ stands for $\forall x,x'(x=x'\wedge\varphi(x)\to\varphi(x'))$.

A7′. *Separation*: $\forall x\,\exists y\,\forall z\,[z\in y\overset{\Box}{\leftrightarrow}z\in x\wedge\exists z'(z=z'\wedge\varphi(z'))]$.

A8′. *Collection*: $\mathrm{Reg}_x\,\varphi(x,y)\overset{\Box}{\to}\exists v\,[\forall x\in u\,\exists y\,\varphi(x,y)\overset{\Box}{\to}\forall x\in u\,\exists y\overset{\Box}{\in}v\ \varphi(x,y)]$, where
$\mathrm{Reg}_x\,\varphi(x,y)$ stands for $\forall x,x',y\,(x=x'\wedge\varphi(x,y)\to\varphi(x',y))$.

A9′. *Infinity*: $\exists x\,\Box(\exists y\,(y\in x)\wedge\forall y\in x\,\exists z\in x\,(y\in z)]$.

A10′. *Axiom of check sets*: $\mathrm{ck}(u)\overset{\Box}{\leftrightarrow}\Box\forall x\in u\,\exists y\overset{\Box}{\in}u\,(\mathrm{ck}\,(y)\wedge y=x)$.

A11. *Axiom of inner valuation*:

$$\forall x\,(x\overset{\Box}{\subset}1\to\mathrm{ck}(\pi(x))]\wedge\forall x,y\,[x\overset{\Box}{\subset}1\wedge y\overset{\Box}{\subset}1\overset{\Box}{\to}(x\overset{\Box}{=}y\leftrightarrow\pi(x)=\pi(y))].$$

It is obvious that every theorem of $\mathrm{ZF_I}$ is provable in GIZF. Let W be the class of check sets,

$$W=\{x:\exists x'(x=x'\wedge\mathrm{ck}(x'))\}.$$

The quantifiers relativized to the check sets, $\forall x\,(\mathrm{ck}(x)\to\cdots)$ and $\exists x\,(\mathrm{ck}(x)\wedge\cdots)$ are denoted by $\check{\forall}x\cdots$ and $\check{\exists}x\cdots$. If φ is a formula, then the result of replacing all quantifiers $\forall x$, $\exists x$ in φ by $\check{\forall}x$, $\check{\exists}x$ is denoted by φ^W. A^W for axioms A of $\mathrm{ZF_I}$ and $\check{\forall}x\,(x\in W)$ are provable in GIZF. Hence W is a model of $\mathrm{ZF_I}$ interpreting $\forall x$ and $\exists x$ as $\check{\forall}x$ and $\check{\exists}x$, respectively. Each sentence φ of GIZF is equivalent to $0\in\{t\in 1:\varphi\}\overset{\Box}{\subset}1$. We call $\pi\{t\in 1:\varphi\}$ the *inner truth value* of φ. Now let H be the set of inner truth values, i.e.

$$H=\{p:\exists x\,(x\overset{\Box}{\subset}1\wedge p=\pi(x))\}.$$

Then H is a check set and a cHa in W, i.e.,

$$\mathrm{ck}(H)\wedge(H\text{ is a cHa})^W.$$

Thus, we can construct the H-valued sheaf model W^H in W. It is proved in [13] that the sheaf model W^H represents the universe, that is,

Theorem 1.3.4. *Let $[\![\cdot]\!]$ be the evaluation in W^H and $|\varphi|$ be the inner truth value of φ, $\pi\{t \in 1 : \varphi\}$. Then there exists a function F on W^H such that*

(a) $\check{\forall} x_1, \ldots, x_n \in W^H([\![\varphi(x_1, \ldots, x_n)]\!] = |\varphi(F(x_1), \ldots, F(x_n))|)$,

(b) $\forall u\, \check{\exists} x \in W^H(F(x) \overset{\square}{=} u)$.

As a result of Theorem 1.3.4, we have

Corollary 1.3.5. $[\![\varphi(x_1, \ldots, x_n)]\!] = 1$ *in* W^H *iff* $\square\varphi(F(x_1), \ldots, F(x_n))$.

2. Intuitionistic sheaf theory and sheaf model $V^{(\Omega)}$

Here we discuss sheaves and sheaf models in the intuitionistic set theory $\mathrm{ZF_I}$. Algebraic systems such as rings and modules in $V^{(\Omega)}$ are sheaves of the corresponding algebras viewed externally. Thus, in $\mathrm{ZF_I}$ we can define a cohomology group of Ω with coefficients in $V^{(\Omega)}$ as a cohomology group with coefficients in a sheaf.

In what follows, **Definition** and **Theorem** are those in $\mathrm{ZF_I}$ unless otherwise indicated. Those in other set theories will be followed by the name of the set theory, e.g. **Theorem 2.4.1.** (ZFC).

2.1. Sheaves and sheaf representations

The following definition of sheaf is from Fourman and Scott [1].

Definition. A *presheaf over a cHa* Ω is a set A equipped with maps $E : A \to \Omega$ and $\lceil : A \times \Omega \to A$ satisfying:

(1) $a\lceil 0 = b\lceil 0$,

(2) $a\lceil Ea = a$,

(3) $E(a\lceil p) = Ea \wedge p$,

(4) $(a\lceil p)\lceil q = a\lceil p \wedge q$.

for all $a, b \in A$ and $p, q \in \Omega$.

Definition. Members f, g of a presheaf A over Ω are said to be *compatible* if $f\lceil Eg = g\lceil Ef$. A subset F of A is *compatible* if the elements are pairwise compatible.

Definition. A presheaf A over Ω is called a *sheaf over* Ω if for any compatible subset F of A there exists a unique $g \in A$ such that

(1) $f \in F \to g\lceil Ef = f$,

(2) $Eg = \bigvee \{Ef : f \in F\}$.

g is called the *join* of F and denoted by $\vee F$.

If A is a sheaf, then the subset $\{f \in A : Ef = p\}$ is denoted by $\Gamma(p, A)$.

We now consider $\lceil$ on a sheaf model $V^{(\Omega)}$.

Definition. If $u \in V^{(\Omega)}$ and $p \in \Omega$, then $u\lceil p$ is an element of $V^{(\Omega)}$ defined by

$$\mathcal{D}(u\lceil p) = \mathcal{D}(u) \quad \text{and} \quad (u\lceil p)(t) = u(t) \wedge p.$$

Then it is easy to see:

Lemma 2.1.1. *For $u \in V^{(\Omega)}$ and $p \in \Omega$,*

$$p \leqslant [\![u = u\lceil p]\!] \quad \textit{and} \quad [\![t \in u\lceil p]\!] = [\![t \in u]\!] \wedge p.$$

Definition. $u \in V^{(\Omega)}$ is called a *sheaf representation* SR if

(1) for each $x \in \mathcal{D}(u)$, $[\![x = x\lceil [\![x \in u]\!]]\!] = 1$ and $u(x) = [\![x \in u]\!]$, i.e. u is *definite*,
(2) for each $v \in V^{(\Omega)}$ there exists $v' \in \mathcal{D}(u)$ such that $[\![v \in u]\!] \leqslant [\![v = v']\!]$,
(3) if $v \in \mathcal{D}(u)$ and $p \in \Omega$, then $v\lceil p \in \mathcal{D}(u)$.

We will show that each member u of $V^{(\Omega)}$ such that $[\![\forall x \in u \exists t \in x]\!] = 1$ is represented by a sheaf.

Remark. A relation $\sim$ on a set X is an *equivalence relation* if (i) $a \sim b \to a \in X \wedge b \in X$, (ii) $a \in X \to a \sim a$, (iii) $a, b \in X \wedge a \sim b \to b \sim a$, and (iv) $a, b, c \in X \wedge a \sim b \wedge b \sim c \to a \sim c$. Let the equivalence class containing a, $\{b \in X : b \sim a\}$, be denoted by $[a]$, and the quotient set, $\{[a] : a \in G\}$, by $G/\sim$. Then it is obvious that $a \in [b] \leftrightarrow a \sim b$ and $[a] = [b] \leftrightarrow a \sim b$ for $a, b \in X$. The relation $\sim$ on $V^{(\Omega)}$ defined by: $x \sim y \leftrightarrow [\![x = y]\!] = 1$ is an equivalence relation. We denote $V^{(\Omega)}/\sim$ and $[x]$ simply by $V^{(\Omega)}$ and x when this does not cause confusion.

Lemma 2.1.2. *For each member u of $V^{(\Omega)}$ there exists a SR u' in $V^{(\Omega)}$ such that $[\![u = u']\!] = 1$.*

Proof. Define $v \in V^{(\Omega)}$ by:

$$\mathcal{D}(v) = \{x\lceil [\![x \in u]\!] : x \in \mathcal{D}(u)\}, \quad \text{and} \quad v(t) = [\![t \in u]\!] \quad \text{for } t \in \mathcal{D}(v).$$

Then $[\![u = v]\!] = 1$ and v is definite. So we assume that u is definite. Let

$$D = \{x\lceil p : x \in \mathcal{D}(u), p \in \Omega\} \quad \text{and}$$
$$Et = [\![\exists x\ (x \in t) \wedge t \in u]\!] \quad \text{for } t \in D.$$

Then D is a presheaf, and each compatible subset A of D has its join $\vee A$ in $V^{(\Omega)}$, where $\vee A$ is defined by $\mathcal{D}(\vee A) = \bigcup_{a \in A} \mathcal{D}a$ and $(\vee A)(t) = \bigvee_{a \in A} [\![t \in a]\!]$. The desired SR is then defined by

$$\mathcal{D}u' = \{\vee A : A \subseteq D \text{ compatible}\}, \quad \text{and} \quad u'(t) = [\![t \in u]\!].$$

Definition. A set u is said to be *special* if each member has an element, i.e. $\forall x \in u\ \exists t \in x$.

Theorem 2.1.3. *If a member u of $V^{(\Omega)}$ is SR and special in $V^{(\Omega)}$(i.e. $[\![\forall x \in u\ \exists t \in x]\!] = 1$), then $(\mathfrak{D}u, E, \lceil)$ is a sheaf, where $Ex = [\![x \in u]\!]$.*

Proof. $x = x\lceil Ex$ for $x \in \mathfrak{D}u$, since u is definite. For $x \in \mathfrak{D}u$ and $p \in \Omega$, $E(x\lceil p) = Ex \wedge p$ follows from:

$$E(x\lceil p) = [\![x\lceil p \in u \wedge \exists t\ (t \in x\lceil p)]\!] = [\![x\lceil p \in u]\!] \wedge p = Ex \wedge p.$$

If $A \subset \mathfrak{D}u$ is compatible, then $\bigvee_{a \in A} Ea = E(\vee A)$ and $\forall a \in A(a = \vee A\lceil Ea)$. From these properties it follows that $(\mathfrak{D}u, E, \lceil)$ is a sheaf.

Definition. If u is a special set in $V^{(\Omega)}$ and u' is a sheaf representation such that $[\![u = u']\!] = 1$, then the sheaf $(\mathfrak{D}u', E, \lceil)$ determined uniquely by u is called the *sheaf representing* u and denoted by $\tilde{u}$. The function E on $\tilde{u}$ is written as $E_{\tilde{u}}$ if it is necessary to indicate $\tilde{u}$.

Theorem 2.1.4. *Let S_1 and S_2 be special sets in $V^{(\Omega)}$. If f is a function $S_1 \to S_2$ in $V^{(\Omega)}$, then there exists a unique function $g : \tilde{S}_1 \to \tilde{S}_2$ such that for each $a \in \tilde{S}_1$*

(2.1.4.1) $E_{\tilde{S}_2} g(a) = E_{\tilde{S}_1} a = [\![\langle a, g(a)\rangle \in f]\!]$.

Conversely, if $g : \tilde{S}_1 \to \tilde{S}_2$ satisfies $E_{\tilde{S}_2} g(a) = E_{\tilde{S}_1} a$ and $g(a)\lceil p = g(a\lceil p)$ for $a \in \tilde{S}_1$, then there exists a unique function $f : S_1 \to S_2$ in $V^{(\Omega)}$ satisfying (2.1.4.1). *We denote the function $g : \tilde{S}_1 \to \tilde{S}_2$ by $\tilde{f}$.*

Proof. If f is a function $S_1 \to S_2$ in $V^{(\Omega)}$, then

$$[\![a \in S_1]\!] \leqslant [\![\exists!\, c \in S_2\ (\langle a, c\rangle \in f)]\!].$$

Hence $A = \{c \in \tilde{S}_2 : c = c\lceil [\![\langle a, c\rangle \in f]\!] \wedge Ea\}$ is compatible. The function $a \mapsto \vee A$ is the desired $\tilde{f}$. For the second part, define $f \in V^{(\Omega)}$ by

$$\mathfrak{D}f = \{\langle a, g(a)\rangle : a \in \tilde{S}_1\} \quad \text{and} \quad f(\langle a, g(a)\rangle) = E_{\tilde{S}_1} a.$$

Since $(S_1 \times S_2)^{\sim} \cong \tilde{S}_1 \times \tilde{S}_2$, we have:

Corollary 2.1.5. *Let S_1, S_2, S_3 be special sets in $V^{(\Omega)}$. If f is a function $S_1 \times S_2 \to S_3$ in $V^{(\Omega)}$, then there exists a unique function $g : \tilde{S}_1 \times \tilde{S}_2 \to \tilde{S}_3$ such that for $a_1 \in \tilde{S}_1$, $a_2 \in \tilde{S}_2$,*

(2.1.5.1) $E_{\tilde{S}_3} g(a_1, a_2) = E_{\tilde{S}_1} a_1 \wedge E_{\tilde{S}_2} a_2 = [\![\langle \langle a_1, a_2\rangle, g(a_1, a_2)\rangle \in f]\!]$.

Conversely, if $g : \tilde{S}_1 \times \tilde{S}_2 \to \tilde{S}_3$ satisfies

$$E_{\tilde{S}_3} g(a_1, a_2) = E_{\tilde{S}_1} a_1 \wedge E_{\tilde{S}_2} a_2 \quad \text{and} \quad g(a_1, a_2)\lceil p = g(a_1\lceil p, a_2\lceil p)$$

for $\langle a_1, a_2\rangle \in \tilde{S}_1 \times \tilde{S}_2$, *then there exists a unique function* $f: S_1 \times S_2 \to S_3$ *in* $V^{(\Omega)}$ *satisfying* (2.1.5.1). *We denote the function* $g: \tilde{S}_1 \times \tilde{S}_2 \to \tilde{S}_3$ *by* $\tilde{f}$.

Definition. Let $\neq$ be a relation on a sheaf $(\mathcal{S}, E, \lceil)$ over Ω such that $a \neq b \to a\lceil p \neq b\lceil p$ for $a, b \in \mathcal{S}$ and $p \in \Omega$. Let

$$[a \neq b] = \bigvee \{r \leq Ea \wedge Eb : a\lceil r \neq b\lceil r\},$$
$$[a = b] = \bigvee \{r \leq Ea \wedge Eb : a\lceil r = b\lceil r\}.$$

Then $\neq$ is called an *apartness on* $\mathcal{S}$ if

(1) $[a = b] = Ea \wedge Eb \wedge \neg[a \neq b]$,

(2) $Ec \wedge [a \neq b] \leq [a \neq c] \vee [b \neq c]$.

If $\neq$ is a relation on a special set S in $V^{(\Omega)}$, then we define a relation $\tilde{\neq}$ on $\tilde{S}$ by

$$a \tilde{\neq} b \quad \text{iff} \quad Ea \wedge Eb \leq [\![a \neq b]\!].$$

The following lemma is immediate from the definition.

Lemma 2.1.6. *If* $\neq$ *is a relation on a special set* S *in* $V^{(\Omega)}$, *and* $[a = b]$, $[a \tilde{\neq} b]$ *are defined on* $\tilde{S}$ *as above, then*

$$[a = b] = Ea \wedge Eb \wedge [\![a = b]\!] \quad \textit{and} \quad [a \tilde{\neq} b] = Ea \wedge Eb \wedge [\![a \neq b]\!].$$

Hence we have:

Theorem 2.1.7. *Let* S *be a special set with a relation* $\neq$ *in* $V^{(\Omega)}$. *Then* $\neq$ *is an apartness on* S *in* $V^{(\Omega)}$ *iff* $\tilde{\neq}$ *is an apartness on the sheaf* $\tilde{S}$.

Definition. Let $\mathcal{S}$ be a sheaf over Ω and let $+, \cdot$ be operations on $\mathcal{S}$. Then $(\mathcal{S}, +, \cdot)$ is a *sheaf of rings over* Ω if

(a) for each $p \in \Omega$, $\Gamma(p, \mathcal{S}) = \{a \in \mathcal{S} : Ea = p\}$ is a commutative ring,

(b) For $p, q \in \Omega$ with $p \leq q$, the restriction $a \mapsto a\lceil p$ is a homomorphism of $\Gamma(q, \mathcal{S})$ into $\Gamma(p, \mathcal{S})$.

A sheaf of ring $(\mathcal{S}, +, \cdot, \neq)$ with apartness $\neq$ *is a sheaf of apartness rings over* Ω if

(c) $[a \neq b] \wedge Ec \leq [a + c \neq b + c]$,

(d) $[a \neq b] \wedge [c \neq 0] \leq [ac \neq bc]$.

A sheaf of apartness rings $(\mathcal{S}, +, \cdot, \neq, 1)$ with unit element 1, i.e., $\forall a \in \mathcal{S}$ $(a \cdot 1 = a)$, is a *sheaf of apartness fields over* Ω if

(e) $1 \neq 0$,

(f) for each $a \in \mathcal{S}$ with $a \neq 0$, there exists $a^{-1} \in \mathcal{S}$ such that

$$Ea^{-1} = Ea = [a^{-1} \neq 0] \wedge [aa^{-1} = 1].$$

Theorem 2.1.8. *Let* S *be a special in* $V^{(\Omega)}$, $+, \cdot$ *be operations and* $\neq$ *be a relation*

on S. Then:

(1) $(S, +, \cdot)$ is a ring in $V^{(\Omega)}$ *iff* $(\tilde{S}, \tilde{+}, \tilde{\cdot})$ *is a sheaf of rings over* Ω.

(2) $(S, +, \cdot, \neq)$ *is an apartness ring in* $V^{(\Omega)}$ *iff* $(\tilde{S}, \tilde{+}, \tilde{\cdot}, \tilde{\neq})$ *is a sheaf of apartness rings over* Ω.

(3) $(S, +, \cdot, \neq, 1)$ *is an apartness field in* $V^{(\Omega)}$ *iff* $(\tilde{S}, \tilde{+}, \tilde{\cdot}, \tilde{\neq}, 1)$ *is a sheaf of apartness fields.*

Proof. Straightforward by using Theorem 2.1.3, 2.1.5 and 2.1.7.

Definition. Let $\mathscr{F}$ be a sheaf of rings over Ω. Then a sheaf $\mathscr{S}$ over Ω with operations $+ : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ and $\cdot : \mathscr{F} \times \mathscr{S} \to \mathscr{S}$ is called a *sheaf of* $\mathscr{F}$*-modules over* Ω if

(a) $\Gamma(p, \mathscr{S})$ is a $\Gamma(p, \mathscr{F})$-module for each $p \in \Omega$,

(b) if $p, q \in \Omega$ and $p \leqq q$, then the restriction $\alpha \mapsto \alpha \restriction p$ is a homomorphism of $\Gamma(q, \mathscr{S})$ into $\Gamma(p, \mathscr{S})$.

If $\mathscr{F}$ is a sheaf of apartness rings over Ω, then a sheaf $\mathscr{S}$ of $\mathscr{F}$-modules over Ω with apartness $\neq$ satisfying the following (c), (d) is called a *sheaf of apartness* $\mathscr{F}$*-modules.*

(c) $[a\alpha \neq a'\alpha'] \leqq [a \neq a'] \vee [\alpha \neq \alpha']$ for $a, a' \in \mathscr{F}$, $\alpha, \alpha' \in \mathscr{S}$,

(d) $[\alpha + \beta \neq \alpha' + \beta'] \leqq [\alpha \neq \alpha'] \vee [\beta \neq \beta']$ for $\alpha, \alpha', \beta, \beta' \in \mathscr{S}$.

Theorem 2.1.9. *Let F, S be special sets, F be a ring and* $+ : S \times S \to S$, $\cdot : F \times S \to S$ *in* $V^{(\Omega)}$. *Then* $(S, +, \cdot)$ *is a F-module in* $V^{(\Omega)}$ iff $(\tilde{S}, \tilde{+}, \tilde{\cdot})$ *is a sheaf of* $\tilde{F}$*-modules. Further if F is an apartness ring and* $\neq$ *is a relation on S in* $V^{(\Omega)}$, *then* $(S, +, \cdot, \neq)$ *is an apartness F-module in* $V^{(\Omega)}$ *iff* $(\tilde{S}, \tilde{+}, \tilde{\cdot}, \tilde{\neq})$ *is a sheaf of apartness* $\tilde{F}$*-modules.*

Proof. Similar to Theorem 2.1.8.

In what follows we assume that all algebraic systems are special sets.

Definition. Let $\mathscr{F}$ be a sheaf of rings over Ω and $(\mathscr{S}_1, E_1, \restriction_1)$, $(\mathscr{S}_2, E_2, \restriction_2)$ be sheaves of $\mathscr{F}$-modules over Ω. Then $\mathscr{S}_1$ is a *subsheaf of* $\mathscr{S}_2$ if

(a) $\mathscr{S}_1 \subset \mathscr{S}_2$, $E_1 \subset E_2$, $\restriction_1 \subset \restriction_2$, and

(b) $\Gamma(p, \mathscr{S}_1)$ is a submodule of $\Gamma(p, \mathscr{S}_2)$ for each $p \in \Omega$.

Theorem 2.1.10. *Let F be a ring and* S_1, S_2 *be F-modules in* $V^{(\Omega)}$. *Then* S_1 *is a submodule of* S_2 *in* $V^{(\Omega)}$ *iff* $\tilde{S}_1$ *is a subsheaf of* $\tilde{S}_2$.

Proof. If $S_1 \subset S_2$ *in* $V^{(\Omega)}$, then for $a \in \tilde{S}_1$,

$$[\![a \in S_2]\!] \leqq [\![\exists x\, (x \in a)]\!] \leqq [\![a \in S_1]\!] \leqq [\![a \in S_2]\!],$$

hence, $S_1 \subset S_2$ in $V^{(\Omega)}$ iff $\forall a \in \tilde{S}_1$ $([\![a \in S_1]\!] = [\![a \in S_2]\!])$, and thus the theorem follows immediately from Theorem 2.1.9.

Definition. Let $\mathcal{F}$ be a sheaf of rings over Ω and $\mathcal{S}_1$, $\mathcal{S}_2$ be sheaves of $\mathcal{F}$-modules over Ω. A mapping $h: \mathcal{S}_1 \to \mathcal{S}_2$ is called a *sheaf homomorphism* (*isomorphism*) if

(a) $h: \Gamma(p, \mathcal{S}_1) \to \Gamma(p, \mathcal{S}_2)$ is homomorphism (isomorphism) for each $p \in \Omega$, and

(b) $h(\alpha)\lceil p = h(\alpha \lceil p)$ for each $\alpha \in \mathcal{S}_1$ and $p \in \Omega$.

If $\mathcal{S}_1$, $\mathcal{S}_2$ are sheaves of apartness $\mathcal{F}$-modules over Ω, then the sheaf isomorphism $h: \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$\forall \alpha, \beta \in \mathcal{S}_1 \ (\alpha \neq \beta \to h(\alpha) \neq h(\beta))$$

is called an *apartness sheaf isomorphism.*

By using Theorems 2.14, 2.1.7, 2.1.8 we have:

Theorem 2.1.11. *Let F be a ring and S_1, S_2 be F-modules in $V^{(\Omega)}$. Then $h: S_1 \to S_2$ is a homomorphism (isomorphism) in $V^{(\Omega)}$ iff $\tilde{h}: \tilde{S}_1 \to \tilde{S}_2$ is a sheaf homomorphism (isomorphism). Further if S_1, S_2 are apartness F-modules in $V^{(\Omega)}$, then $h: S_1 \to S_2$ is an apartness isomorphism in $V^{(\Omega)}$ iff $\tilde{h}: \tilde{S}_1 \to \tilde{S}_2$ is an apartness sheaf isomorphism.*

In ZFC if Ω is a cHa $\mathcal{O}(X)$ of open sets of a topological space X, then special sets and mappings in $V^{(\Omega)}$ are the sheaves and sheaf mappings in the usual sense. Namely, let u be a special set in $V^{(\Omega)}$. Then let

$$U = \{(x, f) \in X \times \tilde{u} \mid x \in [\![f \in u]\!]\}/\sim$$

where $(x_1, f_1) \sim (x_2, f_2)$ iff $x_1 = x_2 \in [\![f_1 = f_2]\!]$. We denote the equivalence class containing (x, f) by f_x. U is a topological space with the set of $\{f_x \in U : x \in [\![f \in u]\!]\}$ for $f \in \tilde{u}$ as its open base. Then U and the mapping $\pi: f_x \to x$ form a sheaf in the usual sense. If S_1, S_2 are special sets and h is a mapping of S_1 into S_2 in $V^{(\Omega)}$, then the mapping $H: f_x \leftrightarrow (\tilde{h}f)_x$ is a sheaf mapping in the usual sense. In what follows, we identify U with $\tilde{u}$, H with $\tilde{h}$ and the mapping $x \leftrightarrow f_x$ on Ef with f.

2.2. Properties of cHa

In this section we assume that Ω is a cHa with apartness from 0 ($\neq 0$).

Definition. A subset $\{p_\alpha\}_{\alpha \in A}$ of Ω indexed with a set A, i.e. $p: A \to \Omega$, is called a *covering of* Ω if $\bigvee_{\alpha \in A} p_\alpha = 1$. A covering $\{p_\alpha\}_{\alpha \in A}$ is *locally finite* if A satisfies $\forall \alpha, \beta \in A$ $(\alpha = \beta \vee \neg\neg \alpha = \beta)$ and there exists a covering $\{q_\beta\}_{\beta \in B}$ of Ω such that $\{\alpha \in A \mid q_\beta \wedge p_\alpha \neq 0\}$ is finite for each $\beta \in B$, where 'F is finite' means that there exist $n \in \omega$ and a bijection $f: n \to F$. Ω is *paracompact* if every covering $\{p_\alpha\}_{\alpha \in A}$

of Ω has a locally finite refinement $\{q_\beta\}_{\beta\in B}$ with $s:B\to A$, that is, a locally finite covering $\{q_\beta\}_{\beta\in B}$ of Ω and a function $s:B\to A$ such that $q_\beta \leqslant p_{s(\beta)}$ for each $\beta\in B$.

Lemma 2.2.1. *Let S be a module. A be a set such that $\forall\alpha,\beta\in A\ (\alpha=\beta\vee\neg\alpha=\beta)$, and $m, n\in\omega$. If $f:m\to A$ and $g:n\to A$ are one-to-one functions and a function $h:A\to S$ satisfies:*

$$\forall\alpha\in A\ (h(\alpha)=0\vee\exists i<m\ (f(i)=\alpha))$$

$$\text{and}\quad \forall\alpha\in A\ (h(\alpha)=0\vee\exists j<n\ (g(j)=\alpha)),$$

then $\sum_{i<m}h(f(i))=\sum_{j<n}h(g(j))$.

Proof. We proceed by mathematical induction on $\max(m, n)$. By the assumption we have three cases:

Case 1. If $h(f(m-1))=0$ and $h(g(n-1))=0$, then it is obvious from the induction hypothesis.

Case 2. If there exists $i<m$ such that $f(i)=g(n-1)$, then define $f':(m-1)\to A$ by

$$f'(k)=\begin{cases}f(k) & \text{if } k<i,\\ f(k+1) & \text{if } i\leqslant k<m-1,\end{cases}$$

and let $B=A-\{g(n-1)\}$. By induction hypothesis we have

$$\sum_{k<m}h(f(k))=\sum_{k<m-1}h(f'(k))+h(g(n-1))=\sum_{j<n}h(g(j)).$$

Case 3. If there exists $j<n$ such that $g(j)=f(m-1)$, then the proof is similar to Case 2.

Theorem 2.2.2. *Let $\{p_\alpha\}_{\alpha\in A}$ be a locally finite covering of Ω, S be a module in $V^{(\Omega)}$ and $\{s_\alpha\}_{\alpha\in A}\subset V^{(\Omega)}$. If $p\in\Omega$ and for each $\alpha\in A$*

$$p=[\![s_\alpha\in S]\!]\leqslant[\![s_\alpha=0]\!]\vee p_\alpha,$$

then $\sum_{\alpha\in A}s_\alpha$ can be defined as a member of $V^{(\Omega)}$ such that $[\![\sum_{\alpha\in A}s_\alpha\in S]\!]=p$.

Proof. Define $h:\check{A}\to S$ with $h(\check{\alpha})=s_\alpha$ in $V^{(\Omega)}$. Since $\forall\alpha,\beta\in\check{A}\ (\alpha=\beta\vee\neg\alpha=\beta)$ in $V^{(\Omega)}$, it will suffice by Lemma 2.2.1 to show that:

$$p\leqslant[\![\exists n\in\check{\omega}\ \exists f:n\xrightarrow{1-1}\check{A}\ \forall\alpha\in\check{A}\ (h(\alpha)=0\vee\exists i<n\ f(i)=\alpha))]\!].$$

Let $\{q_\beta\}_{\beta\in B}$ be a covering of Ω such that $\{\alpha\in A\mid q_\beta\wedge p_\alpha\neq 0\}$ is finite for each $\beta\in B$. Let $\beta\in B$ and $p_{\alpha_0},\ldots,p_{\alpha_{n-1}}$ be all the elements of $\{p_\alpha\}$ with nonempty

intersection with q_β. We now define $f:\check{n}\to\check{A}$ with $f(\check{\imath})=\check{\alpha}_i$, $i<n$. Then

$$[\![\forall\alpha\in\check{A}\ (h(\alpha)=0\vee\exists i<\check{n}\ (f(i)=\alpha))]\!]=\bigwedge_{\alpha\neq\alpha_0,\dots,\alpha_{n-1}}[\![s_\alpha=0]\!]\geqslant p\wedge q_p.$$

Since $\bigvee_{\beta\in B}q_\beta=1$, the left side is larger than or equal to p.

Definition. A cHa Ω is said to be *normal* if for $p, q\in\Omega$ such that $p\vee q=1$ there exist $r, s\in\Omega$ satisfying $r\wedge s=0$, $p\vee r=1$ and $q\vee s=1$.

2.3. Cohomology with coefficients in $V^{(\Omega)}$

The cohomology of a cHa Ω with coefficients in a sheaf is defined as follows: Let Ω be a paracompact cHa and $\mathcal{S}$ be a sheaf of modules over Ω. Let $\mathcal{U}=\{u_i\}_{i\in I}$ be a locally finite covering of Ω and $q\in\omega$. A *q-cochain σ of $\mathcal{U}$ with values in $\mathcal{S}$* is a function $I^{q+1}\to\mathcal{S}$ which assigns to $(i_0,\dots,i_q)\in I^{q+1}$ a section $\sigma(i_0,\dots,i_q)\in\Gamma(u_{i_0}\wedge\cdots\wedge u_{i_q},\mathcal{S})$ so that $\sigma(i_0,\dots,i_q)$ is an alternating function of $(i_0,\dots,i_q)$. The set $C^q(\mathcal{U},\mathcal{S})$ of all q-cochains of $\mathcal{U}$ with values in $\mathcal{S}$ is a module. A *coboundary operator* δ from $C^q(\mathcal{U},\mathcal{S})$ to $C^{q+1}(\mathcal{U},\mathcal{S})$ is defined as follows: If $\sigma\in C^q(\mathcal{U},\mathcal{S})$, then

$$(\delta\sigma)(i_0,\dots,i_{q+1})=\sum_{j=0}^{q+1}(-1)^j\sigma(i_0,\dots,\hat{\imath}_j,\dots,i_{q+1}),$$

where the notation $\hat{\imath}_j$ means that the index i_j shall be removed. From the definition of δ it follows immediately that $\delta\circ\delta=0$ and δ is a homomorphism. If we introduce

$$Z^q(\mathcal{U},\mathcal{S})=\{\sigma\in C^q(\mathcal{U},\mathcal{S})\mid\delta\sigma=0\},$$

the module of q-cocyles with values in $\mathcal{S}$, then the module of q-coboundaries $\delta C^{q-1}(\mathcal{U},\mathcal{S})=\{\delta\sigma\mid\sigma\in C^{q-1}(\mathcal{U},\mathcal{S})\}$ is a submodule of $Z^q(\mathcal{U},\mathcal{S})$. We can therefore introduce the quotient model

$$H^q(\mathcal{U},\mathcal{S})=Z^q(\mathcal{U},\mathcal{S})/\delta C^{q-1}(\mathcal{U},\mathcal{S})\quad\text{and}\quad H^0(\mathcal{U},\mathcal{S})=Z^0(\mathcal{U},\mathcal{S})$$

which is called the *qth cohomology group of $\mathcal{U}$ with values in $\mathcal{S}$.*

Now let $\mathcal{V}=\{v_j\}_{j\in J}$ be a locally finite refinement of $\mathcal{U}$ with $s:J\to I$, that is, $v_j\leqslant u_{s(j)}$ for every $j\in J$. If $\sigma\in C^p(\mathcal{U},\mathcal{S})$, we can define a cochain $\prod_v^{s,u}\sigma\in C^p(\mathcal{V},\mathcal{S})$ by setting

$$\left(\prod_v^{s,u}\sigma\right)(j_0,\dots,j_q)=\sigma(s(j_0),\dots,s(j_q))\restriction v_{j_0}\wedge\cdots\wedge v_{j_q}.$$

This map $\prod_v^{s,u}$ commutes with δ, so it induces a map $\prod_v^{s,u}:H^q(\mathcal{U},\mathcal{S})\to H^q(\mathcal{V},\mathcal{S})$ which does not depend on the choice of $s:J\to I$. By $\mathcal{V}>\mathcal{U}$ we mean that $\mathcal{V}$ is a refinement of $\mathcal{U}$. $>$ is a partial order on the set of locally finite coverings, and for any two locally finite coverings $\mathcal{U}$, $\mathcal{V}$ there exists a locally finite covering which is

a refinement of both of $\mathcal{U}$ and $\mathcal{V}$. We can therefore introduce the direct limit $H^q(\Omega, \mathcal{S})$ of the modules $H^q(\mathcal{U}, \mathcal{S})$ and call it the *cohomology of Ω with coefficients in $\mathcal{S}$*. If S is a module in $V^{(\Omega)}$, we define $H^q(\Omega, S)$ as $H^q(\Omega, \tilde{S})$.

We can prove the following theorems for a paracompact cHa Ω.

Theorem 2.3.1. *If $\mathcal{S}$ is a sheaf of modules over Ω, then $H^0(\Omega, \mathcal{S}) = \Gamma(1, \mathcal{S})$.*

Theorem 2.3.2. *If $\mathcal{S}$ is a sheaf of modules over Ω and $\mathcal{U}$ is a locally finite covering of Ω, then the canonical mapping $\prod^u : H^1(\mathcal{U}, \mathcal{S}) \to H^1(\Omega, \mathcal{S})$ is injective.*

If $\mathcal{S}$ and $\mathcal{S}'$ are sheaves of modules over Ω, then a sheaf homomorphism $h : \mathcal{S} \to \mathcal{S}'$ induces a homomorphism $h : C^q(\mathcal{U}, \mathcal{S}) \to C^q(\mathcal{U}, \mathcal{S}')$, $h : H^q(\mathcal{U}, \mathcal{S}) \to H^q(\mathcal{U}, \mathcal{S}')$ and $h : H^q(\Omega, \mathcal{S}) \to H^q(\Omega, \mathcal{S}')$ in natural way. For convenience we write simply $H^q(\mathcal{S})$ instead of $H^q(\Omega, \mathcal{S})$ and $H^q(S)$ instead of $H^q(\tilde{S})$ if S is a module in $V^{(\Omega)}$.

2.4. *Exact sequence*

In the set theory ZFC, let $\mathcal{S}, \mathcal{S}', \mathcal{S}''$ be sheaves of modules over a normal paracompact cHa Ω and g, h be sheaf homomorphisms such that

$$0 \to \mathcal{S}' \xrightarrow{g} \mathcal{S} \xrightarrow{h} \mathcal{S}'' \to 0$$

is exact.

Then the following theorem is proved as for Čech cohomology.

Theorem 2.4.1. (ZFC) *There exists a homomorphism $\delta^* : H^q(\mathcal{S}'') \to H^{q+1}(\mathcal{S}')$ for each $q \geq 0$ such that the following sequence is exact:*

$$0 \to H^0(\mathcal{S}') \xrightarrow{g} H^0(\mathcal{S}) \xrightarrow{h} H^0(\mathcal{S}'') \xrightarrow{\delta^*} H^1(\mathcal{S}')$$

$$\to H^1(\mathcal{S}) \xrightarrow{h} H^1(\mathcal{S}'') \xrightarrow{\delta^*} H^2(\mathcal{S}') \to \cdots$$

$$\to H^q(\mathcal{S}) \xrightarrow{h} H^q(\mathcal{S}'') \xrightarrow{\delta^*} H^{q+1}(\mathcal{S}') \xrightarrow{g} H^{q+1}(\mathcal{S}) \to \cdots$$

2.5. *Fine resolution*

Let Ω be a normal paracompact cHa in ZFC.

Definition. (ZFC) A module S in $V^{(\Omega)}$ is said to be *fine* if for every locally finite covering $\{u_i\}_{i \in I}$ of Ω there exists a set $\{h_i\}_{i \in I}$ of homomorphisms $h_i : S \to S$ in $V^{(\Omega)}$

such that

(a) for each $i \in I$ and $\alpha \in \tilde{S}$, $[\![\alpha \in S]\!] \leqq [\![h_i(\alpha) = 0]\!]^{\vee} u_i$,

(b) $[\![\forall \alpha \in S\ (\Sigma_{i \in I} h_i(\alpha) = \alpha)]\!] = 1$.

Remark. In the above (b) $\Sigma_{i \in I} h_i(\alpha)$ is well defined by Theorem 2.2.2.

Theorem 2.5.1. (ZFC) *If a module S in $V^{(\Omega)}$ is fine, then $H^q(S) = 0$ for $q \geqslant 1$.*

Proof. Let $\mathcal{U} = \{u_i\}_{i \in I}$ be a locally finite covering and $\sigma \in Z^q(\mathcal{U}, \tilde{S})$. Then we prove that σ is a coboundary. Since S is fine, there exists a set $\{h_i\}_{i \in I} \subset V^{(\Omega)}$ of homorphisms $h_i : S \to S$ in $V^{(\Omega)}$ satisfying (a), (b). Define $\tau \in C^{q-1}(\mathcal{U}, \tilde{S})$ by

$$\tau(k_1, \ldots, k_q) = \sum_{i \in I} (h_i \sigma)(i, k_1, \ldots, k_q).$$

Then we have $\delta\tau = \sigma$.

Definition. (ZFC) Let $S, A^0, A^1, \ldots$ be modules in $V^{(\Omega)}$. Then a sequence of modules in $V^{(\Omega)}$

$$(R) \qquad 0 \to S \xrightarrow{i} A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \to A^{q-1} \xrightarrow{d} A^q \to \cdots$$

is called a *fine resolution of S*, if the corresponding sequence of sheaves

$$(\tilde{R}) \qquad 0 \to \tilde{S} \xrightarrow{\tilde{i}} \widetilde{A^0} \xrightarrow{\tilde{d}} \widetilde{A^1} \to \cdots \to \widetilde{A^{q-1}} \xrightarrow{\tilde{d}} \widetilde{A^q} \to \cdots$$

is exact and all of $A^0, A^1, \ldots$ are fine.

The following theorem follows from Theorem 2.4.1 and 2.5.1.

Theorem 2.5.2. (ZFC) *If (R) is a fine resolution, then*

$$H^q(S) \cong \frac{H^0(dA^{q-1})}{\tilde{d}H^0(A^{q-1})} \qquad (q \geqslant 1).$$

2.6. Real and complex numbers in $V^{(\Omega)}$

Here we assume that Ω is a cHa $\mathcal{O}(X)$ of open sets of a complex manifold X and construct the standard sheaf model $V^{(\Omega)}$ of GIZF in ZFC. By a complex manifold we mean a paracompact complex analytic manifold. It is known that the set $\mathbb{R}^{\Omega}$ (or $\mathbb{C}^{\Omega}$) of all real (or complex) numbers is a complete apartness field in $V^{(\Omega)}$, where $u \neq v$ iff $u < v \vee v < u$, and is represented by the sheaf $\widetilde{\mathbb{R}^{\Omega}}$ (or $\widetilde{\mathbb{C}^{\Omega}}$) of germs of real (or complex) continuous functions, that is, each $u \in \widetilde{\mathbb{R}^{\Omega}}$ is represented by a continuous function $\underline{u}$ of Eu into $\mathbb{R}$ such that

$$r < \underline{u}(x) \quad \text{iff} \quad x \in [\![\check{r} < u]\!] \quad \text{for } r \in \mathbb{Q}$$

and each $u + iv \in \widetilde{\mathbb{C}^{\Omega}}$ by $\underline{u} + i\underline{v} : Eu \cap Ev \to \mathbb{C}$. We identify $\underline{u}$ with u if it does not cause confusion. It is obvious that for $u, v \in \widetilde{\mathbb{C}^{\Omega}}$

$$x \in [\![u \neq v]\!] \quad \text{iff} \quad u(x) \neq v(x).$$

Let $\mathbb{R}^{\infty}$, $\mathbb{C}^{\infty}$, $\mathscr{H}$ be the subsets of $\mathbb{C}^{\Omega}$ in $V^{(\Omega)}$ such that $\widetilde{\mathbb{R}^{\infty}}$, $\widetilde{\mathbb{C}^{\infty}}$, $\widetilde{\mathscr{H}}$ are sheaves of germs of real valued C^{∞}-functions, complex valued C^{∞}-functions and holomorphic functions, respectively. That is,

$$\mathscr{D}(\mathbb{R}^{\infty}) = \{u \in \widetilde{\mathbb{R}^{\Omega}} \mid u : Eu \to \mathbb{R} \text{ is } C^{\infty}\text{-function}\},$$
$$\mathbb{R}^{\infty}(u) = Eu,$$

and similarly for $\mathbb{C}^{\infty}$ and $\mathscr{H}$. Then it is obvious that in $V^{(\Omega)}$:

(1) $\mathbb{R}^{\infty}$, $\mathbb{C}^{\infty}$, $\mathscr{H}$ are subfields of $\mathbb{C}^{\Omega}$,

(2) $\mathscr{H}$ is complete, that is, every Cauchy sequence convergences in $\mathscr{H}$,

(3) $\mathbb{C}^{\infty} = \mathbb{R}^{\infty} + i\mathbb{R}^{\infty}$.

It is known that for any covering $\mathscr{U}$ of X there exists a partition of unity $\{\varphi_{\beta}\}_{\beta \in B}$ subordinate to $\mathscr{U}$. Hence we have:

Theorem 2.6.1. (ZFC) *For any covering $\mathscr{U}$ of Ω there exists a locally finite refinement $\{v_j\}_{j \in J}$ and $\{a_j\}_{j \in J} \subset \widetilde{\mathbb{R}^{\infty}}$ such that*

(a) *$Ea_j (= [\![a_j \in \mathbb{R}^{\infty}]\!]) = [\![a_j = 0]\!]^{\vee} v_j = 1$ for each $j \in J$.*

(b) $[\![\sum_{j \in J} a_j = 1]\!] = 1$.

Theorem 2.6.2. (ZFC) *$\mathbb{R}^{\infty}$-modules and $\mathbb{C}^{\infty}$-modules in $V^{(\Omega)}$ are fine.*

Proof. Let S be a $\mathbb{R}^{\infty}$-module in $V^{(\Omega)}$ and let $\{v_j\}_{j \in J}$ be a locally finite covering. Let $\{a_j\}_{j \in J}$ be the set $\subseteq \widetilde{\mathbb{R}^{\infty}}$ satisfying (a), (b) in Theorem 2.6.1. Then the mapping $h_j : a \mapsto a_j \alpha$ of S into S satisfies $[\![\alpha \in S]\!] \leqq [\![h_j(\alpha) = 0]\!]^{\vee} v_j$ for each $j \in J$ and $\alpha \in \tilde{S}$, and $[\![\alpha \in S]\!] \leqq [\![\sum_{j \in J} h_j(\alpha) = \alpha]\!]$ for $\alpha \in \tilde{S}$, that is, S in $V^{(\Omega)}$ is fine. Since $\mathbb{R}^{\infty} \subset \mathbb{C}^{\infty}$, a $\mathbb{C}^{\infty}$-module in $V^{(\Omega)}$ is also fine.

Let $\{(u_i, z_i)\}_{i \in I}$ be a system of coordinate neighbourhoods of X, where $z_i : u_i \to \mathbb{C}^n$ for $i \in I$. The notion of vector field on X is translated into the language of $\mathrm{ZF_I}$ on $V^{(\Omega)}$ as follows. Let T be the set of $\check{\mathbb{C}}$-linear derivation on $\mathbb{C}^{\infty}$ in $V^{(\Omega)}$ that is, the set of mappings $D : \mathbb{C}^{\infty} \to \mathbb{C}^{\infty}$ such that

$$\forall a \in \check{\mathbb{C}}\ \forall f, g \in \mathbb{C}^{\infty}\ (D(af + g) = aDf + Dg)$$

and

$$\forall f, g \in \mathbb{C}^{\infty}\ (D(fg) = (Df)g + f(Dg)).$$

Then T is an apartness vector space over the apartness field $\mathbb{C}^{\infty}$ in $V^{(\Omega)}$, where $D_1 \neq D_2$ iff $\exists f \in \mathbb{C}^{\infty}\ (D_1 f \neq D_2 f)$ for $D_1, D_2 \in T$.

If $D \in \tilde{T}$ and $p \in ED = [\![D \in T]\!]$, then the mapping $D_p : f \mapsto (Df)(p)$ for $f \in \widetilde{\mathbb{C}^{\infty}}$ with $p \in Ef = [\![f \in \mathbb{C}^{\infty}]\!]$ is a tangent vector at p, and $\underline{D} : p \mapsto D_p$ is a vector field on $E(D)$.

Further, for each $i \in I$, there exists φ_k, $\psi_k \in \widetilde{\mathbb{C}^\infty}$ such that

(a) $\underline{D} = \sum_{k=1}^{n} \varphi_k \frac{\partial}{\partial z_i^k} + \sum_{k=1}^{n} \varphi_k \frac{\partial}{\partial \bar{z}_i^k}$ on $E(D) \cap u_i$, and

(b) $E(D) \wedge \bigvee_{k \leqq n} [\![\varphi_k \neq 0 \vee \psi_k \neq 0]\!] \leqq [\![D \neq 0]\!]$.

Let $D_1, \ldots, D_{2n}$ be mappings of $\mathbb{C}^\infty[u_i$ into $\mathbb{C}^\infty$ in $V^{(\Omega)}$ such that $\underline{D}_\alpha = \partial/\partial z_i^\alpha$ and $\underline{D}_{\alpha+n} = \partial/\partial \bar{z}_i^\alpha$ for $\alpha = 1, \ldots, n$. Then we have:

Theorem 2.6.3. (ZFC) (1) $u_i \leqq [\![D_1, \ldots, D_{2n}$ *form an apartness basis of* $T]\!]$.

(2) $u_i \leqq [\![[D_\alpha, D_\beta] = 0]\!]$ *for* $\alpha, \beta = 1, \ldots, 2n$, *where* $[D_\alpha, D_\beta] = D_\alpha D_\beta - D_\beta D_\alpha$.

(3) $u_i \leqq [\![\forall f \in \mathbb{C}^\infty\ (\widetilde{D_\alpha f} = D_{n+\alpha} f)]\!]$ *for* $\alpha = 1, \ldots, n$.

(4) $u_i \leqq [\![\mathscr{H} = \{f \in \mathbb{C}^\infty : n < \forall \alpha \leqq 2n\ (D_\alpha f = 0)\}]\!]$.

(5) $u_i \leqq [\![\check{\mathbb{C}} = \{f \in \mathbb{C}^\infty : \forall \alpha \leqq 2n\ (D_\alpha f = 0)\}]\!]$.

(6) *If* $u_i \leqq [\![H_0, \ldots, H_k \in T$ *are apartness linearly independent over* $\mathbb{C}^\infty$, $H_\alpha H_\beta = H_\beta H_\alpha$ *for* $\alpha, \beta \leqq k$ *and* $f \in \mathbb{C}^\infty$, $H_1 f = \cdots = H_k f = 0]\!]$, *then there exists* $h \in \widetilde{\mathbb{C}^\infty}$ *such that* $u_i \leqq Eh \wedge [\![H_0 h = f$ *and* $H_1 h = \cdots = H_k h = 0]\!]$.

Further, for an open set U *of* $\mathscr{H}^k$ *and a holomorphic function* $h = h(\zeta_1, \ldots, \zeta_k)$ *from* U *into* $\mathscr{H}$ *in* $V^{(\Omega)}$, *let*

$$\underline{U} = \{(p, t_1, \ldots, t_k) \in X \times \mathbb{C}^k : p \in [\![(\check{t}_1, \ldots, \check{t}_k) \in U]\!]\},$$

$$\underline{h}(p, t_1, \ldots, t_k) = h(\check{t}_1, \ldots, \check{t}_k)(p) \quad \text{for } (p, t_1, \ldots, t_k) \in \underline{U}.$$

$\underline{U}$ *is an open set of* $X \times \mathbb{C}^k$ *and* $\underline{h}$ *is a holomorphic function* (cf. Rousseau [9]). $D_\alpha h$ *is defined as a holomorphic function on* U *such that* $\underline{D_\alpha h} = (\partial/\partial z_i^\alpha)\underline{h}$. *Then:*

(7) $$u_i \leqq \left[\!\!\left[D_\alpha \frac{\partial}{\partial \zeta_\beta} h = \frac{\partial}{\partial \zeta_\beta} D_\alpha h \right]\!\!\right] \quad \text{for } \alpha = 1, \ldots, n,\ \beta = 1, \ldots, k.$$

(8) *If* $u_i \leqq [\![(f_1, \ldots, f_k)$ *is a holomorphic function from an open set* Δ *of* $\mathscr{H}^k$ *into* U *and* $g = h(f_1, \ldots, f_k)]\!]$, *then*

$$u_i \leqq \left[\!\!\left[D_\alpha g = (D_\alpha h)(f_1, \ldots, f_k) + \sum_{\beta=1}^{k} \left(\frac{\partial}{\partial \zeta_\beta} h\right)(f_1, \ldots, f_k) \cdot D_\alpha f_\beta \text{ on } \Delta \right]\!\!\right].$$

3. Global intuitionistic analysis GIA

The global intuitionistic analysis GIA is a global intuitionistic set theory with constants $\mathbb{C}^\infty$, $\mathscr{H}$, C_0, that is interpreted as the set theory on $V^{(\Omega)}$ given in § 2.6.

3.1. Axioms of GIA

The axions of GIA consist of those of GIZF, i.e. A1′–A9′, A10 and A11 and extra axioms for the new constants. In GIZF the class of check sets $\check{W} = \{x \mid \exists x'(x = x')\}$ is defined and proved to be a model of $\mathrm{ZF_I}$ interpreting $\forall x\ \varphi(x)$

and $\exists x\, \varphi(x)$ as $\check{\forall} x\, \varphi(x)$ and $\check{\exists} x\, \varphi(x)$. It is known that there is an isomorphism F between the universe and a sheaf model W^H constructed in W, that is,

$$\check{\forall} x \in W^H\, \exists!\, y\, F(x, y) \wedge \forall y\, \check{\exists} x \in W^H\, (F(x) \overset{\square}{=} y)$$

$$\wedge\, \check{\forall} x_1, \ldots, x_n \in W^H\, (\llbracket \varphi(x_1, \ldots, x_n) \rrbracket = |\varphi(F(x_1), \ldots, F(x_n))|)$$

where $\llbracket \cdot \rrbracket$ is the evaluation in W^H and $|\varphi|$ is the inner truth value of φ, i.e. $|\varphi| = \pi\{x \in 1 : \varphi\}$.

The extra axioms of GIA are as follows.

Axiom 12. AC^W, that is, the axiom of choice in W. Hence W is a model of the set theory ZFC.

Axiom 13. $\mathbb{C}^\infty$ is a field of complex numbers such that $\mathbb{C}^\infty = \mathbb{R}^\infty + \mathrm{i}\mathbb{R}^\infty$ where $\mathbb{R}^\infty$ is the subfield of all reals in $\mathbb{C}^\infty$, $\mathscr{H}$ is a complete subfield of $\mathbb{C}^\infty$ and C_0 is a subfield of $\mathscr{H}$ such that $C_0 = R_0 + \mathrm{i}R_0$ where $R_0 = C_0 \cap \mathbb{R}^\infty$.

By $T(\mathbb{C}^\infty, C_0)$ or simply T we denote the set of C_0-linear derivations on $\mathbb{C}^\infty$. T is an apartness vector space over $\mathbb{C}^\infty$.

Axiom 14. There exists a natural number $n > 0$ and an apartness basis $\{D^1, \ldots, D^{2n}\}$ for T over $\mathbb{C}^\infty$ such that:

(a) $[D^j, D^k] = 0$ for $j, k = 1, \ldots, 2n$, where $[D^j, D^k] = D^jD^k - D^kD^j$.

(b) $D^{n+j}f = D^j\bar{f}$ for $f \in \mathbb{C}^\infty$ and $j = 1, \ldots, n$, where $\bar{f}$ is the conjugate of f.

(c) $\mathscr{H} = \{f \in \mathbb{C}^\infty : D^jf = 0 \text{ for } j = n+1, \ldots, 2n\}$.

(d) $C_0 = \{f \in \mathbb{C}^\infty : D^jF = 0 \text{ for } j = 1, \ldots, 2n\}$.

Axiom 15. There exists h such that the following statement is globally true: if $H_0, \ldots, H_k \in T$ are apartness linearly independent over $\mathbb{C}^\infty$ and

$$[H_i, H_j] = 0 \quad \text{for } i, j = 0, \ldots, k \quad \text{and} \quad H_1f = \cdots = H_kf = 0,$$

then

$$H_0h = f \quad \text{and} \quad H_1h = \cdots = H_kh = 0.$$

Note. 'φ is globally ture' means $\square\varphi$. By the Corollary 1.3.5, $\exists h\, \square\varphi(h, F(x_1), \ldots, F(x_n)) \leftrightarrow \check{\exists} h\, (\llbracket \varphi(h, x_1, \ldots, x_n) \rrbracket = 1 \text{ in } W^H)$.

Axiom 16. In W, H is a normal and paracompact cHa and if $\{p_\alpha\}_{\alpha \in A}$ is a locally finite covering of H, then there exists $\{g_\alpha\}_{\alpha \in A} \subset W^H$ such that

(a) $\llbracket g_\alpha \in F^{-1}(\mathbb{R}^\infty) \rrbracket = 1$,

(b) $\bigvee_{\alpha \in A} \llbracket g_\alpha \neq 0 \rrbracket = 1$.

(c) $\forall \alpha \in A\, (\llbracket g_\alpha = 0 \rrbracket^\vee p_\alpha = 1)$.

(d) $\llbracket \Sigma_\alpha\, g_\alpha = 1 \rrbracket = 1$.

Thus GIA is GIZF + Axioms 12–16. It is obvious from §2.6 that in ZFC if X is a complex manifold, $\Omega = \mathcal{O}(X)$, $C_0 = \check{\mathbb{C}} \in V^{(\Omega)}$ and $\mathcal{H}$, $\mathbb{C}^\infty \in V^{(\Omega)}$ are represented by the sheaf of germs of holomorphic functions, the sheaf of germs of C^∞-functions on X respectively, then the standard sheaf model $V^{(\Omega)}$ of GIZF is a model of GIA. We call this model a *standard sheaf model of GIA*. In what follows, $V^{(\Omega)}$ denotes a standard sheaf model of GIZF or GIA according as Ω is cHa with globalization $\square$ or $\mathcal{O}(X)$ for a complex analytic manifold X.

3.2. deRham and Dolbeault theorem in GIA

The deRham theorem and Dolbeault theorems can be translated into theorems in GIA and proved in GIA. For this purpose we first consider intuitionistic tensors and exterior algebras with apartness.

3.2.1. Induced apartness

If S is a set with apartness $\neq$, and $\sim$ is an equivalence relation on S, and if each element A of $S/\sim$ contains exactly one element a such that $\varphi(a)$, then the relation $\neq$ on $S/\sim$ given by:

$$A \neq B \quad \text{iff} \quad \exists a \in A\ \exists b \in B\ (\varphi(a) \wedge \varphi(b) \wedge a \neq b)$$

is an apartness. From this fact we have:

Lemma 3.2.1.1. *Let F be an apartness ring and S be an apartness F-module. If S' is a submodule of S and $S'' \subset S$ such that $\forall a \in S\ \exists!\, b \in S''\ (a - b \in S')$, then S/S' is an apartness F-module, where*

$$A \neq B \quad \textit{iff} \quad \exists a \in A \cap S''\ \exists b \in B \cap S''\ (a \neq b).$$

Let T be an apartness vector space over a field F and $\{e_1, \ldots, e_n\}$ be an apartness basis for T. Then the dual space $T^* \subset F^T$ and the dual basis $\{f_1, \ldots, f_n\}$ of $\{e_1, \ldots, e_n\}$ are defined as usual, and apartness $\neq$ on T^* is given by

$$f \neq g \quad \text{iff} \quad \exists \alpha \in T\ (f(\alpha) \neq g(\alpha)).$$

It is easy to check that T^* is an apartness vector space over F and $\{f_1, \ldots, f_n\}$ is an apartness basis for T^*.

3.2.2. Tensors and exterior algebras with apartness

Let T and S be an apartness vector space over an apartness field F such that $n \cdot 1 \neq 0$ for nonzero n, and let $\{e_1, \ldots, e_n\}$, $\{f_1, \ldots, f_m\}$ be apartness bases for T and S respectively. The tensor product $T \otimes S$ is defined as the dual space of the apartness vector space of all bilinear forms on $T \oplus S$. $T \otimes S$ is an apartness vector space over F with apartness basis $\{e_i \otimes f_j : i = 1, \ldots, n; j = 1, \ldots, m\}$. Let $I_p(T)$ be the subspace of the tensor space $T \otimes \cdots \otimes T$ of type $(p, 0)$ generated by

the elements $t_1 \otimes \cdots \otimes t_p$ such that $t_k = t_{k+1}$ for some $k < p$. The p-fold exterior power $\bigwedge^p T$ is the quotient vector space of $T \otimes \cdots \otimes T$($p$-times) by $I_p(T)$. The equivalence class $[t_1 \otimes \cdots \otimes t_p]$ in $\bigwedge^p T$ is denoted by $t_1 \wedge \cdots \wedge t_p$. Since each element of $\bigwedge^p T$ can be written in the form $\sum_{i_1<\cdots<i_p} a_{i_1\cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$ uniquely, $\bigwedge^p T$ is an apartness vector space with apartness basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid i_1 < \cdots < i_p\}$. The exterior product of $\varphi \in \bigwedge^p T$ and $\psi = \bigwedge^q T$ is also defined as usual, that is, if

$$\varphi = \sum_{i_1<\cdots<i_p} a_{i_1\cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p} \quad \text{and} \quad \psi = \sum_{j_1<\cdots<j_q} b_{j_1\cdots j_q} e_{j_1} \wedge \cdots \wedge e_{j_q}$$

then

$$\varphi \wedge \psi = \sum_{i_1<\cdots<i_p, j_1<\cdots<j_q} a_{i_1\cdots i_p} b_{j_1\cdots j_q} e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}.$$

3.2.3. DeRham Theorem

Let $\{D^1, \ldots, D^{2n}\}$ be an apartness basis of $T = T(\mathbb{C}^\infty, C_0)$ satisfying (a)–(d) in Axiom 14, and let

$$\frac{\partial}{\partial x^{2\alpha-1}} = D^\alpha + D^{n+\alpha}, \qquad \frac{\partial}{\partial x^{2\alpha}} = \frac{1}{\mathrm{i}}(D^\alpha - D^{n+\alpha}),$$

for $\alpha = 1, \ldots, n$. Then $\{\partial/\partial x^1, \ldots, \partial/\partial x^{2n}\}$ is an apartness basis for T. Let A be the dual space of T and $\{dx^1, \ldots, dx^{2n}\}$ be the dual basis of $\{\partial/\partial x^1, \ldots, \partial/\partial x^{2n}\}$. We denote the p-fold exterior power $\bigwedge^p A$ by A^p. Every element of A^p, which is a (differential) p-form, can be written in the form

$$\varphi = \frac{1}{p!} \sum_{\alpha_1,\ldots,\alpha_p} \varphi_{\alpha_1\cdots\alpha_p}\, dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p},$$

where $\varphi_{\alpha_1\cdots\alpha_p} \in \mathbb{C}^\infty$ and $\varphi_{\alpha_1\cdots\alpha_p}$ is an alternating function of $\alpha_1, \ldots, \alpha_p$. A^p is an apartness vector space over $\mathbb{C}^\infty$ which does not depend on the choice of the bases $\{D^1, \ldots, D^{2n}\}$ of T. To see this, let $\{D_j^1, \ldots, D_j^{2n}\}$ and $\{D_k^1, \ldots, D_k^{2n}\}$ be two apartness bases for T satisfying (a)–(d) in Axiom 14, and let $\partial/\partial x_j^\alpha$, dx_j^α and $\partial/\partial x_k^\alpha$, dx_k^α be the corresponding $\partial/\partial x^\alpha$, dx^α ($\alpha = 1, \ldots, 2n$). Each $\partial/\partial x_j^\alpha$ can be written as a linear combination of $\{\partial/\partial x_k^\beta\}_{\beta=1,\ldots,2n}$. We denote the coefficient of $\partial/\partial x_j^\alpha$ by $\partial x_j^\alpha/\partial x_k^\beta$, that is,

$$\frac{\partial}{\partial x_k^\beta} = \sum_{\alpha=1}^{2n} \frac{\partial x_j^\alpha}{\partial x_k^\beta} \frac{\partial}{\partial x_j^\alpha}.$$

Then

$$dx_j^\alpha = \sum_\beta \frac{\partial x_j^\alpha}{\partial x_k^\beta} dx_k^\beta, \quad \text{where } \frac{\partial x_j^\alpha}{\partial x_k^\beta} \in \mathbb{C}^\infty.$$

It follows that there exist $\varphi'_{\beta_1\cdots\beta_p} \in \mathbb{C}^\infty$ such that

$$\frac{1}{p!} \sum_{\alpha_1,\ldots,\alpha_p} \varphi_{\alpha_1\cdots\alpha_p}\, dx_j^{\alpha_1} \wedge \cdots \wedge dx_j^{\alpha_p} = \frac{1}{p!} \sum_{\beta_1,\ldots,\beta_p} \varphi'_{\beta_1\cdots\beta_p}\, dx_k^{\beta_1} \wedge \cdots \wedge dx_k^{\beta_p}.$$

The exterior differentiation operator $d: A^p \to A^{p+1}$ $(p \geqslant 0)$ is defined as usual:

$$\text{if } f \in A^0 \in \mathbb{C}^\infty, \quad \text{then} \quad df = \sum_\alpha \frac{\partial f}{\partial x^\alpha} dx^\alpha,$$

$$\text{if } \varphi = \frac{1}{p!} \sum_{\alpha_1, \ldots, \alpha_p} \varphi_{\alpha_1 \cdots \alpha_p} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p} \in A^p,$$

then

$$d\varphi = \frac{1}{p!} \sum d\varphi_{\alpha_1 \cdots \alpha_p} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p}$$

$$= \frac{1}{p!} \sum_{\alpha, \alpha_1, \ldots, \alpha_p} \frac{\partial \varphi_{\alpha_1 \cdots \alpha_p}}{\partial x^\alpha} dx^\alpha \wedge dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p}.$$

Then it is routine to check that $d: A^p \to A^{p+1}$ is a C_0-homomorphism which does not depend on the choice of $\{D^1, \ldots, D^{2n}\}$, and $dd = 0$.

Since the universe is isomorphic with the inner sheaf model W^H by Theorem 1.3.4, each special set u can be considered as a sheaf in W, that is, there exists a check set $x \in W^H$ such that $F(x) \overset{\square}{=} u$, and x is represented by a sheaf $\tilde{x}$ in W. For convenience we identify x such that $F(x) \overset{\square}{=} u$ with u and denote x by u.

Lemma 3.2.3.1. (GIA) *A^p in W^H is fine in W.*

Proof. In W, if $\{p_\alpha\}_{\alpha \in A}$ is a locally finite covering of H, then there exists $\{g_\alpha\}_{\alpha \in A} \subseteq W^H$ satisfying the conditions (a)–(d) in Axiom 16. The function $h_\alpha: t \mapsto g_\alpha \cdot t$ on A^p satisfies the conditions of fine sheaf.

Lemma 3.2.3.2. (GIA) *There exists h such that the following statement is globally true. If $f \in \mathbb{R}^\infty$, $a \neq \alpha_1, \ldots, \alpha_s$ and $\partial f/\partial x^{\alpha_1} = \cdots = \partial f/\partial x^{\alpha_s} = 0$, then $\partial h/\partial x^\alpha = f$ and $\partial h/\partial x^{\alpha_1} = \cdots \partial h/\partial x^{\alpha_s} = 0$.*

Proof. By Axiom 15.

Lemma 3.2.3.3. (Poincaré's lemma in GIA) *There exists ψ such that the following statement is globally true. If φ is a p-form, $p \geqslant 1$ and $d\varphi = 0$, then $\varphi = d\psi$.*

Proof. This is proved by interpreting the usual Poincaré's lemma into GIA, using Lemma 3.2.3.2.

By Lemma 3.2.3.1 and 3.2.3.3, the following sequence in W^H is a fine resolution of C_0 in W.

$$0 \to C_0 \xrightarrow{i} A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \to A^{p-1} \xrightarrow{d} A^p \to \cdots \xrightarrow{d} A^{2n} \xrightarrow{d} 0.$$

Hence, by Theorem 2.5.2, we have:

Theorem 3.2.3.4. (deRham Theorem in GIA)

$$H^q(C_0) \cong \frac{H^0(dA^{q-1})}{\bar{d}H^0(A^{q-1})} \quad (q \geqslant 1) \quad \textit{in } W.$$

3.2.4. Dolbeault Theorem

Define $\partial/\partial z^\alpha$, $\partial/\partial \bar{z}^\alpha \in T = T(\mathbb{C}^\infty, C_0)$ *by*

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^{2\alpha-1}} - \mathrm{i}\frac{\partial}{\partial x^{2\alpha}}\right), \qquad \frac{\partial}{\partial \bar{z}^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^{2\alpha-1}} + \mathrm{i}\frac{\partial}{\partial x^{2\alpha}}\right).$$

$\{\partial/\partial z^1, \ldots, \partial/\partial z^n, \partial/\partial \bar{z}^1, \ldots, \partial/\partial \bar{z}^n\}$ *is an apartness basis for* T. *Let* $\{dz^1, \ldots, dz^n, d\bar{z}^1, \ldots, d\bar{z}^n\}$ *be its dual basis. That is,*

$$dz^\alpha = dx^{2\alpha-1} + \mathrm{i}\, dx^{2\alpha}, \qquad d\bar{z}^\alpha = dx^{2\alpha-1} - \mathrm{i}\, dx^{2\alpha}.$$

We now consider differential forms generated by dz^α, $d\bar{z}^\alpha$. The r-forms can be written as a linear combination of (differential) (p, q)-form:

$$\varphi^{(p,q)} = \frac{1}{p!\,q!} \sum_{\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q} \varphi_{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}\, dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}$$

with $p + q = r$. For $f \in \mathbb{C}^\infty$,

$$df = \sum_{\lambda=1}^{2n} \frac{\partial f}{\partial x^\lambda} dx^\lambda = \sum_{\alpha=1}^{n} \frac{\partial f}{\partial z^\alpha} dz^\alpha + \sum_{\alpha=1}^{n} \frac{\partial f}{\partial \bar{z}^\alpha} d\bar{z}^\alpha.$$

$$\sum_{\alpha=1}^{n} \frac{\partial f}{\partial z^\alpha} dz^\alpha, \qquad \sum_{\alpha=1}^{n} \frac{\partial f}{\partial \bar{z}^\alpha} d\bar{z}^\alpha$$

are denoted by ∂f, $\bar{\partial} f$. If φ is a (p, q)-form

$$\frac{1}{p!\,q!} \sum \varphi_{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}\, dz^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}$$

then $\partial\varphi$, $\bar{\partial}\varphi$ are defined as follows.

$$\partial\varphi = \frac{1}{p!\,q!} \sum (\partial\varphi_{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}) \wedge dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}$$

$$\bar{\partial}\varphi = \frac{1}{p!\,q!} \sum (\bar{\partial}\varphi_{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}) \wedge dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}.$$

It is easy to see:

Lemma 3.2.4.1. (GIA) $d = \partial + \bar{\partial}$, $\partial\partial = \overline{\partial\partial} = 0$, $\bar{\partial}\partial = -\partial\bar{\partial}$.

Lemma 3.2.4.2. (GIA) *If* φ *is a* (p, q)*-form, then* $\partial\varphi$ *is* $(p+1, q)$*-form,* $\bar{\partial}\varphi$ *is* $(p, q+1)$*-form and* $\overline{\partial\varphi} = \bar{\partial}\bar{\varphi}$.

The following lemma is proved by translating the proof of usual Dolbeault Lemma into GIA and using Axiom 15.

Lemma 3.2.4.3. (Dolbeault lemma in GIA) *There exists ψ such that the following statement is globally true. If φ is a (p, q)-form and $\bar{\partial}\varphi = 0$, then ψ is a $(p, q-1)$-form and $\bar{\partial}\psi = \varphi$.*

A $(p, 0)$-form

$$\varphi = \frac{1}{p!}\sum_{\alpha_\nu} \varphi_{\alpha_1 \cdots \alpha_p}\, dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p}$$

is called *holomorphic p-form* if $\varphi_{\alpha_1 \cdots \alpha_p} \in \mathcal{H}$ for each $\alpha_1, \ldots, \alpha_p$; equivalently if $\bar{\partial}\varphi = 0$. Let $\mathcal{O}^p$ be the module of holomorphic p-forms, and $A^{p,q}$ be the module of (p, q)-forms. Since $A^{p,q}$ in $W^{(H)}$ is fine, by Dolbeault Lemma in GIA,

$$0 \to \mathcal{O}^p \to A^{p,0} \xrightarrow{\bar{\partial}} A^{p,1} \to \cdots \xrightarrow{\bar{\partial}} A^{p,q} \xrightarrow{\bar{\partial}} \cdots \to A^{p,n} \to 0$$

is a fine resolution of $\mathcal{O}^p$, where $\mathcal{O}^0 = \mathcal{H}$. Therefore, by using Theorem 2.5.2, we have:

Theorem 3.2.4.4. (Dolbeault Theorem in GIA)

$$H^q(\mathcal{O}^p) \cong \frac{H^0(\bar{\partial}A^{p,q-1})}{\bar{\partial}H^0(A^{p,q-1})} \quad (q \geqslant 1) \quad \textit{in } W.$$

3.3. Intuitionistic topology

Definitions in the intuitionistic theory of topology in this section are from Grayson [5].

Definition. A set $\mathcal{O}(X)$ of subsets of X is called a *topology on X* if $X \in \mathcal{O}(X)$ and $\mathcal{O}(X)$ is a cHa with respect to intersection $\cap$ and infinite union $\bigcup$, i.e.

(a) $X \in \mathcal{O}(X)$,

(b) $\forall u, v \in \mathcal{O}(X)\ [u \cap v \in \mathcal{O}(X)]$,

(c) $\forall \mathcal{U} \subset \mathcal{O}(X)\ [\bigcup \mathcal{U} \in \mathcal{O}(X)]$.

A special set X (i.e. $\forall x \in X\ \exists t \in x$) with topology $\mathcal{O}(X)$ is called a *topological space*. A topological space X with apartness $\neq$ is called a *separated space* if

(d) $\forall x \in X\ [\{y \in X : y \neq x\} \in \mathcal{O}(X)]$, and

(e) each open set $u \in \mathcal{O}(X)$ is *strongly extensional*, i.e. $\forall x \in u\ \forall y \in X\ (y \in u \vee y \neq x)$.

A topological space X is *Hausdorff* if it is separated and

(f) $\forall x, y \in X\ [x \neq y \to \exists u, v \in \mathcal{O}(X)\ (x \in u \wedge y \in v \wedge (u \cap v = \emptyset))]$.

Immediately from the above definition we have:

Lemma 3.3.1. *In a separated space X,*

$$\forall x, y \in X\,[x \neq y \leftrightarrow \exists u \in \mathcal{O}(X)\ (x \in u \wedge \neg y \in u)].$$

In ZFC, let Ω be a topology $\mathcal{O}(M)$ and X be a separated space in $V^{(\Omega)}$. We define a topological space $\bar{X}$ from X as follows. Let

(3.3.2) $\quad \bar{X} = \{(t, x) \in M \times \check{X} : t \in [\![x \in X]\!]\}/\sim$

where $(t, x) \sim (t', x')$ iff $t = t' \notin [\![x \neq x']\!]$. $\sim$ is an equivalence relation since $\neq$ is an apartness. Note that $\bar{X}$ is a quotient space of the sheaf $\check{X}$ representing X. We denote the equivalence class containing (t, x) by $\langle t, x\rangle$. For each $u \in \widetilde{\mathcal{O}(X)}$, the corresponding subset $\bar{u}$ of $\bar{X}$ is well defined as follows:

$$\bar{u} = \{\langle t, x\rangle \in \bar{X} : t \in [\![x \in u]\!]\}.$$

Set $\mathcal{O}(\bar{X}) = \{\bar{u} \subset \bar{X} : u \in \widetilde{\mathcal{O}(X)}\}$.

Theorem 3.3.3. (ZFC) *If Ω is a topology $\mathcal{O}(M)$ of M and X is a separated space in $V^{(\Omega)}$, then $\bar{X}$ is a topological space with topology $\mathcal{O}(\bar{X})$. In particular, if $V^{(\Omega)}$ is a standard sheaf model of* GIA, *then $\mathbb{R}^{\Omega}$, $\check{\mathbb{R}}$, $\mathbb{R}^{\infty}$, $\mathbb{C}^{\Omega}$, $\check{\mathbb{C}}$, $\mathbb{C}^{\infty}$, $\mathcal{H}$ are Hausdorff spaces in $V^{(\Omega)}$ and*

$$\overline{\mathbb{R}^{\Omega}} = \check{\mathbb{R}} = \overline{\mathbb{R}^{\infty}} \cong M \times R, \qquad \overline{\mathbb{C}^{\Omega}} = \check{\mathbb{C}} = \overline{\mathbb{C}^{\infty}} = \bar{\mathcal{H}} \cong M \times \mathbb{C}.$$

Proof. $\bar{X} \in \mathcal{O}(\bar{X})$ since $X \in \widetilde{\mathcal{O}(X)}$; if $\bar{u}, \bar{v} \in \mathcal{O}(\bar{X})$ then $\bar{u} \cap \bar{v} = \overline{u \cap v} \in \mathcal{O}(\bar{X})$; if $\mathcal{U} \subset \mathcal{O}(\bar{X})$, define $A \in V^{(\Omega)}$ by

$$\mathcal{D}(A) = \{u \in \widetilde{\mathcal{O}(X)} : \bar{u} \in \mathcal{U}\}, \quad \text{and} \quad A(u) = [\![u \in \mathcal{O}(X)]\!],$$

then $\bigcup \mathcal{U} = \overline{\bigcup A} \in \mathcal{O}(\bar{X})$. Therefore, $\mathcal{O}(\bar{X})$ is a topology on $\bar{X}$. Let R be $\mathbb{R}^{(\Omega)}$, $\check{\mathbb{R}}$ or $\mathbb{R}^{\infty}$. $\mathcal{O}(R)$ consisting of unions of open intervals $(a_\lambda b_\lambda) = \{x \in R : a_\lambda < x < b_\lambda\}$, where $a_\lambda, b_\lambda \in \check{\mathbb{Q}}$, is a topology on R, and it is obvious, from the definition of reals and $\neq$, that R is a Hausdorff space. As in §2.6, $x \in \bar{R}$ is identified with a function $\underset{\sim}{x} : Ex \to \mathbb{R}$ such that $r < \underset{\sim}{x}(t)$ iff $t \in [\![\check{r} < x]\!]$ for $r \in \mathbb{Q}$ and $t \in Ex$. Define $\varphi : \bar{R} \to M \times \mathbb{R}$ by $\varphi\langle t, x\rangle = (t, x(t))$. φ is well defined and a homeomorphism $\bar{R} \cong M \times \mathbb{R}$. Similar for complex number fields.

Theorem 3.3.4. (ZFC) *Let Ω be a topology $\mathcal{O}(M)$ of a Hausdorff space M and let X be a separated space in $V^{(\Omega)}$. Then X is a Hausdorff space in $V^{(\Omega)}$ iff $\bar{X}$ is a Hausdorff space.*

Proof. Suppose that X is a Hausdorff space in $V^{(\Omega)}$, and $\langle t, x\rangle$, $\langle t', x'\rangle \in \bar{X}$ and $\langle t, x\rangle \neq \langle t', x'\rangle$. Then $t \neq t'$ or $t \in [\![x \neq x']\!]$. If $t \neq t'$, then there exist $p, q \in \mathcal{O}(M)$ such that $t \in p$, $t' \in q$ and $p \cap q = \emptyset$, hence $\overline{X \lceil p}$ and $\overline{X \lceil q}$ separate $\langle t, x\rangle$ and

$\langle t', x' \rangle$. If $t \in [\![x \neq x']\!]$, then there exist $u, u' \in \widetilde{\mathcal{O}(X)}$ and $r \in \mathcal{O}(M)$ such that

$$t \in r \leqslant [\![u, u' \in \mathcal{O}(X) \wedge x \in u \wedge x' \in u' \wedge u \cap u' = \emptyset]\!],$$

hence $\overline{u \restriction r}$ and $\overline{u' \restriction r}$ separate $\langle t, x \rangle$ and $\langle t', x' \rangle$. Conversely, if $\tilde{X}$ is Hausdorff and $t \in [\![x \neq x']\!]$, then there exist $u, u' \in \widetilde{\mathcal{O}(X)}$ such that $t \in [\![x \in u \wedge x' \in u' \wedge u \cap u' = \emptyset]\!]$.

Definition. Let X and Y be topological spaces. Then a function $f: X \to Y$ is *continuous* if $\forall u \in \mathcal{O}(Y)\ [f^{-1}(u) \in \mathcal{O}(X)]$.

Let $\Omega = \mathcal{O}(M)$ and $V^{(\Omega)}$ is constructed in ZFC. If X, Y are separated spaces and $f: X \to Y$ is a continuous function in $V^{(\Omega)}$, then a function $\tilde{f}: \tilde{X} \to \tilde{Y}$ is well defined by

$$\tilde{f}\langle t, x \rangle = \langle t, \tilde{f}(x) \rangle.$$

For, if $t \in [\![\tilde{f}(x) \neq \tilde{f}(x')]\!]$ then there exists $u \in \widetilde{\mathcal{O}(X)}$ such that

$$t \in [\![\tilde{f}(x) \in u \wedge \neg \tilde{f}(x') \in u]\!] \leqslant [\![\tilde{f}^{-1}(u) \in \mathcal{O}(X) \wedge x \in \tilde{f}^{-1}(u) \wedge \neg x' \in \tilde{f}^{-1}(u)]\!]$$

by Lemma 3.3.1, hence $t \in [\![x \neq x']\!]$. It follows that $(t, x) \sim (t', x')$ implies $(t, \tilde{f}(x)) \sim (t', \tilde{f}(x'))$.

Theorem 3.3.5. (ZFC) *If $\Omega = \mathcal{O}(M)$ and if X, Y are separated spaces and $f: X \to Y$ is a continuous function in $V^{(\Omega)}$, then the function $\tilde{f}: \tilde{X} \to \tilde{Y}$ is continuous; further, if f is an apartness homeomorphism in $V^{(\Omega)}$, i.e. $[\![\forall x, x' \in X\ (x \neq x' \to f(x) \neq f(x')) \wedge \forall y \in Y\ \exists x \in X\ (f(x) = y) \wedge f^{-1}: Y \to X$ continuous$]\!] = 1$, then $\tilde{f}: \tilde{X} \to \tilde{Y}$ is a homeomorphism.*

Proof. We prove only that $\tilde{f}$ is one-to-one, since the other part is obvious. Suppose that $\tilde{f}\langle t, x \rangle = \tilde{f}\langle t', x' \rangle$. Then $t = t' \notin [\![\tilde{f}(x) \neq \tilde{f}(x')]\!]$. Since $[\![x \neq x']\!] \leqslant [\![\tilde{f}(x) \neq \tilde{f}(x')]\!]$, $\langle t, x \rangle = \langle t', x' \rangle$. that is, $\tilde{f}$ is one-to-one.

3.4. Apartness vector space in GIA and vector bundle

Here we show that an apartness vector space over $\mathcal{H}$ or $\mathbb{C}^{\Omega}$ in a standard sheaf model $V^{(\Omega)}$ of GIA is a vector bundle viewed externally.

Definition. (ZFC) Let M be a complex manifold. An *m-dimensional holomorphic (or differentiable) vector bundle over M* is a pair (F, ϖ) of complex analytic (or differentiable) manifold F and a holomorphic (or differentiable) function ϖ from F onto M together with a covering $\{v_j\}_{j \in J}$ of M and a set $\{h_j\}_{j \in J}$ of homeomorphisms $h_j: \varpi^{-1}(v_j) \cong v_j \times \mathbb{C}^m$ satisfying:

(a) $\varpi \circ h_j^{-1}(z, \xi_j) = z$ for $z \in v_j$ and $\xi_j \in \mathbb{C}^m$, and

(b) the transition function $h_j \circ h_k^{-1}: v_j \cap v_k \times \mathbb{C}^m \to v_j \cap v_k \times \mathbb{C}^m$ is determined by a matrix valued holomorphic (or differentiable) function $f_{jk}(z) = (f_{jk\nu}^{\lambda}(z))_{\nu,\lambda=1,\ldots,m}$, i.e., $h_j \circ h_k^{-1}(z, \xi_k) = (z, f_{ik}(z)\xi_k)$.

Definition. (ZFC) Two vector bundles (F, ϖ) and (G, π) are *holomorphically (or differentiably) equivalent* if there exists a biholomorphic (or bidifferentiable) function $h: F \to G$ such that (a) $\pi h = \varpi$ and (b) $h: \varpi^{-1}(z) \to \pi^{-1}(z)$ is linear.

It is known that if (F, ϖ) and (G, π) are holomorphic (or differentiable) vector bundles over M with covering $\{v_j\}_{j\in J}$ and systems of transition functions $\{f_{jk}\}$, $\{g_{jk}\}$ respectively, then (F, ϖ) and (G, π) are holomorphically (or differentiably) equivalent iff there exits a matrix-valued holomorphic (or differentiable) function $h_j(z)$ on v_j for each $j \in J$ such that $\det h_j(z) \neq 0$ on v_j and $g_{jk}(z) = h_j(z) f_{jk}(z) h_k^{-1}(z)$ on $v_j \cap v_k$.

Theorem 3.4.1. (ZFC) *Let M be a complex manifold, and $\Omega = \mathcal{O}(M)$. Then*

(1) *If B is a m-dimensional apartness vector space over $\mathcal{H}$ (or $\mathbb{C}^\infty$) in $V^{(\Omega)}$, then $\bar{B} = (\bar{B}, \varpi)$, where $\varpi\langle z, u\rangle = z$ for $\langle z, u\rangle \in \bar{B}$, is an m-dimensional holomorphic (or differentiable) vector bundle. Conversely, for a given m-dimensional holomorphic (or differentiable) vector bundle F, there exists a m-dimensional apartness vector space B over $\mathcal{H}$ (or $\mathbb{C}^\infty$) in $V^{(\Omega)}$ such that $\bar{B}$ is holomorphically (or differentially) equivalent to F.*

(2) *Let B_1, B_2 be m-dimensional apartness vector spaces over $\mathcal{H}$ (or $\mathbb{C}^\infty$) in $V^{(\Omega)}$. Then $\varphi: B_1 \to B_2$ is an apartness isomorphism over $\mathcal{H}$ (or $\mathbb{C}^\infty$) in $V^{(\Omega)}$ iff $\bar{B}_1, \bar{B}_2$ are holomorphically (or differentiably) equivalent, with $\bar{\varphi}: \langle z, u\rangle \mapsto \langle z, \tilde{\varphi}(u)\rangle$.*

Proof. (1) Suppose that $[\![B$ is an m-dimensional apartness vector space over $\mathcal{H}]\!] = 1$. Then there exists a covering $\{v_j\}_{j\in J}$ of M and $\{e_j^1, \ldots, e_j^m\} \subset V^{(\Omega)}$ for each $j \in J$ such that

$$v_j = [\![\{e_j^1, \ldots, e_j^m\} \text{ is an apartness basis of } B]\!].$$

Hence for $\langle z, u\rangle \in \varpi^{-1}(v_j)$ there exist $\xi_j^1, \ldots, \xi_j^m \in \check{\mathcal{H}}$ such that $z \in [\![u = \xi_j^1 e_j^1 + \cdots + \xi_j^m e_j^m]\!]$. Let

$$\varphi_j\langle z, u\rangle = \langle z, \xi_j^1(z), \ldots, \xi_j^m(z)\rangle.$$

Then $\bar{B}$, ϖ, $\{v_j\}_{j\in J}$ and $\{\varphi_j\}_{j\in J}$ form an m-dimensional holomorphic vector bundle. Conversely, for a given m-dimensional holomorphic vector bundle (F, ϖ) determined by a covering $\{v_j\}_{j\in J}$ and the system of transition functions $\{\varphi_{jk}\}$, there exists $f_{jk} \in V^{(\Omega)}$ such that

$$v_j \cap v_k \leqq [\![f_{jk} = (f_{jkv}^\lambda)_{v,\lambda=1,\ldots,m} \text{ is a } m \times n \text{ matrix over } \mathcal{H}]\!]$$

and $(f_{jkv}^\lambda)_{v,\lambda=1,\ldots,m} = \varphi_{jk}$, for each $j, k \in J$. Define $B \in V^{(\Omega)}$ as $\check{J} \times \mathcal{H}^m/\sim$ where

$$(j, \xi) \sim (k, \zeta) \quad \text{iff} \quad \xi = f_{jk}\zeta \quad \text{for} \quad \xi, \zeta \in \mathcal{H}^m.$$

Then B is an m-dimensional apartness vector space over $\mathcal{H}$ and $\bar{B}$ is equivalent to F. The differentiable case is proved similarly.

(2) follows from the fact that the matrix ρ of an apartness isomorphism $B_1 \to B_2$ relative to given bases is a matrix over $\mathcal{H}$ (or $\mathbb{C}^\infty$) with $\det \rho \neq 0$, and that $v \leqslant [\![\rho = (\rho_\nu^\lambda)_{\nu,\lambda=1,\ldots,m}$ is a matrix over $\mathcal{H}$ (or $\mathbb{C}^\infty$) and $\det \rho \neq 0]\!]$ iff $\rho = (\rho_\nu^\lambda)_{\nu,\lambda=1,\ldots m}$ is a matrix-valued holomorphic (or differentiable) function such that $\det \rho(z) \neq 0$ for $z \in v$.

3.5. *Manifold in* GIA *and family of complex structures*

A manifold in a standard sheaf model of GIA is a family of complex structure, viewed externally.

Definition. (GIA) Let X be a topological space. A subset $\{u_i\}_{i\in I}$ of $\mathcal{O}(X)$ is an *open covering of* X if $\bigcup_{i\in I} u_i = X$, i.e. $\{u_i\}_{i\in I}$ is a covering of cHa $\mathcal{O}(X)$. Similarly, an open covering $\{u_i\}_{i\in I}$ of X is *locally finite* if $\{u_i\}_{i\in I}$ is a locally finite covering of $\mathcal{O}(X)$, and X is *paracompact* if $\mathcal{O}(X)$ is paracompact. (cf. §2.2.).

Theorem 3.5.1. (ZFC) *Let $\Omega = \mathcal{O}(M)$ and let X be a separated space in $V^{(\Omega)}$. Then $\{u_i\}_{i\in I}$ is an open covering of X in $V^{(\Omega)}$ iff $\{\bar{u}_i\}_{i\in I} \subset \mathcal{O}(\bar{X})$ is an open covering of $\bar{X}$.*

Proof. Obvious from the fact that $t \in [\![x \in \bigcup_{i\in I} u_i]\!]$ iff $\langle t, x\rangle \in \bigcup_{i\in I} \bar{u}_i$.

Theorem 3.5.2. (ZFC) *Let M be a complex manifold and $\Omega = \mathcal{O}(M)$, and let X be a separated space in $V^{(\Omega)}$. Then X is paracompact in $V^{(\Omega)}$ iff $\bar{X}$ is paracompact.*

Proof. Suppose that X is paracompact in $V^{(\Omega)}$, and $\{\bar{u}_i\}_{i\in I}$ is a covering of $\bar{X}$. By Theorem 3.5.1, $\{u_i\}_{i\in I}$ is a covering of X in $V^{(\Omega)}$. Hence there exists a locally finite refinement $\{v_j\}_{j\in J}$ of $\{u_i\}_{i\in I}$ in $V^{(\Omega)}$. It follows that there exists a locally finite covering $\{p_\alpha\}_{\alpha\in A}$ of M and $\{v_j^\alpha\}_{j\in J_\alpha}$ such that

$$p_\alpha \leqslant [\![\{v_j^\alpha\}_{j\in J_\alpha} \text{ is a locally finite refinement of } \{u_i\}_{i\in I}]\!],$$

and there exists a locally finite covering $\{p'_\alpha\}_{\alpha\in A}$ of M such that $\overline{p'_\alpha} \subset p_\alpha$ for $\alpha \in A$, where $\overline{p'_\alpha}$ is the closure of p'_α. We now show that $\{\overline{v_j^\alpha \cap p'_\alpha}\}_{j\in J_\alpha, \alpha\in A}$ is a locally finite refinement of $\{\bar{u}\}_{i\in I}$. Let $\langle t, x\rangle \in \bar{X}$. The set of $\alpha \in A$ such that $t \in \overline{p'_\alpha}$ is finite, let it be $\{\alpha_1, \ldots, \alpha_s\}$. We may assume that p_α is connected, hence $p_\alpha \leqslant [\![j \in J_\alpha]\!]$ or $p_\alpha \leqslant \neg[\![j \in J_\alpha]\!]$. For each α_k,

$$p_{\alpha_k} \leqslant [\![\exists w \in \mathcal{O}(X)\ \exists f\ \exists n \in \omega\ [x \in w \wedge f : n \to J_{\alpha_k} \wedge \forall j \in J_{\alpha_k}$$
$$(v_j^{\alpha_k} \cap w \neq \emptyset \to \exists \beta < n\ (f(\beta) = j))]]\!].$$

Hence there exist $w_{\alpha_k} \in \widetilde{\mathcal{O}(X)}$ and $f_{\alpha_k} : n_{\alpha_k} \to \tilde{J}_{\alpha_k}$ such that

$$t \in [\![w_{\alpha_k} \in \mathcal{O}(X) \wedge x \in w_{\alpha_k} \wedge \forall_j \in J_{\alpha_k}\ (v_j^{\alpha_k} \cap w_{\alpha_k} \neq 0 \to \exists \beta < n\ (f(\beta) = j))]\!] = q_{\alpha_k},$$

and there exists $q \in \mathcal{O}(M)$ such that

$$t \in q \subseteq \bigcap_{k=1}^{s} q_{\alpha_k} \quad \text{and} \quad q \cap p'_\alpha = \emptyset \quad \text{for each } \alpha \notin \{\alpha_1, \ldots, \alpha_s\}.$$

$\bigcap_{k=1}^{s} \overline{w_{\alpha_k}\lceil q}$ is a neighbourhood of $\langle t, x\rangle$ with nonempty intersection with only a finite number of $\overline{v_j^\alpha \lceil p'_\alpha}$. Therefore, $\bar{X}$ is paracompact. Conversely, suppose that $\bar{X}$ is paracompact, and $p = [\![\{u_i\}_{i\in I}$ is a covering of $X]\!]$. Then for any $p' \in \mathcal{O}(M)$ such that $\overline{p'} \subset p$, $\{\overline{u_i \lceil p}\}_{i\in(\bar{I})} \cup \{\overline{X \lceil M - p'}\}$ is a covering of $\bar{X}$. Let $\{\bar{v}_j\}_{j\in J} \subset \mathcal{O}(\bar{X})$ be a locally finite refinement of it. Then

$$p' \leqslant [\![\{v_j\}_{j\in J} \text{ is a locally finite refinement of } \{u_i\}_{i\in I}]\!].$$

Definition. A topological space X is *connected* if

$$\forall u, v \in \mathcal{O}(X)\ [u \cup v = X \wedge \exists x \in u \wedge \exists y \in v \rightarrow \exists x \in u \cap v],$$

and X is *arcwise connected* if

$$\forall x, y \in X\ \exists f\ [f : [0, 1] \rightarrow X \text{ continuous} \wedge f(0) = x \wedge f(1) = y],$$

where $[0, 1] = \{t \in \mathbb{R} \mid 0 \leqslant t \leqslant 1\}$.

Theorem 3.5.3. (ZFC) *Let M be a connected topological space, $\Omega = \mathcal{O}(M)$ and let X be a separated space in $V^{(\Omega)}$.*

(1) *If X is connected in $V^{(\Omega)}$, then $\bar{X}$ is connected.*

(2) *If X is arcwise connected in $V^{(\Omega)}$, then $\bar{X}_t = \{\langle t, x\rangle \in \bar{X} \mid x \in \bar{X}\}$ is connected for each $t \in M$.*

(3) *If $\bar{X}_t$ is connected for each $t \in M$, then X is connected in $V^{(\Omega)}$.*

Proof. (1) Assume that $\bar{u}, \bar{v} \in \mathcal{O}(\bar{X})$, $\bar{u} \neq \emptyset$, $\bar{v} \neq \emptyset$ and $\bar{u} \cup \bar{v} = \bar{X}$. Then $[\![u \cup v = X]\!] = M$, and hence $[\![\exists x \in u \vee \exists y \in v]\!] = M$ and $[\![\exists x \in u]\!] \neq \emptyset$, $[\![\exists y \in v]\!] \neq \emptyset$. By the connectedness of M, there exists $t \in [\![\exists x \in u \wedge \exists y \in v]\!]$. It follows from the connectedness of X in $V^{(\Omega)}$ that $t \in [\![\exists x \in u \cap v]\!]$, hence $\bar{u} \cap \bar{v} \neq \emptyset$.

(2) $\mathcal{O}(\bar{X}_t) = \{\bar{u}_t \mid \bar{u} \in \mathcal{O}(\bar{X})\}$, where $\bar{u}_t = \{\langle t, x\rangle \in \bar{X}_t \mid t \in [\![x \in u]\!]\}$, is a topology on $\bar{X}_t$. Suppose that $\bar{u}_t, \bar{v}_t \in \mathcal{O}(\bar{X}_t)$ satisfy $\bar{u}_t \cup \bar{v}_t = \bar{X}_t$, $\bar{u}_t \neq 0$ and $\bar{v}_t \neq 0$. Then there exist $x, y \in \bar{X}$ and $f \in V^{(\Omega)}$ such that

$$t \in [\![x \in u \wedge y \in v \wedge f : [0, 1] \rightarrow X \text{ continuous} \wedge f(0) = x \wedge f(1) = y]\!].$$

From the connectedness of $[0, 1]$ it follows that there exists $r \in [0, 1]$ such that $\langle t, f(\check{r})\rangle \in \bar{u}_t \cap \bar{v}_t$.

(3) For $u, v \in \widetilde{\mathcal{O}(X)}$, if $t \in [\![u \cup v = X \wedge \exists x \in u \wedge \exists y \in v]\!]$, then $\bar{u}_t \cup \bar{v}_t = \bar{X}_t$, $\bar{u}_t \neq \emptyset$ and $\bar{v}_t \neq \emptyset$, hence $t \in [\![\exists x \in u \cap v]\!]$.

Definition. (GIA) Let C be $\mathbb{C}^\infty$ or $\mathcal{H}$. A paracompact Hansdorff space X is called a *complex analytic manifold of dimension n over C* if there exist an open covering

$\{u_j\}_{j\in J}$ of X and a system of local coordinates $h_j: u_j \to U_j \in \mathcal{O}(C^n)$ such that for each $j, k \in J$ the transition function $h_j \circ h_k^{-1}$ is a biholomorphic function of $h_k(u_j \cap u_k)$ into $h_j(u_j \cap u_k)$. X is a *differentiable (or C^∞)-manifold of dimension n over C* if the transition function $h_j \circ h_k^{-1}$ is C^∞-diffeomorphism in the above definition.

Lemma 3.5.4. (ZFC) *Let $\Omega = \mathcal{O}(M)$ where M is a manifold and let C be $\mathbb{C}^\infty$ or $\mathcal{H}$ in $V^{(\Omega)}$. Then*

(1) *$\overline{C^n}$ is homeomorphic to $M \times \mathbb{C}^n$, where $\bar{u} \in \mathcal{O}(\overline{C^n})$ coincides with the open set $\underline{u} = \{(t, \xi) \in M \times \mathbb{C}^n \mid t \in [\![\check{\xi} \in u]\!]\}$ of $M \times \mathbb{C}^n$.*

(2) *Let u be an open set of C^n in $V^{(\Omega)}$. A function $f: u \to C^n$ is holomorphic in $V^{(\Omega)}$ iff $\underline{f}: \underline{u} \to \mathbb{C}^n$ is continuous and holomorphic in the second variable, where $\underline{f}: \underline{u} \to \mathbb{C}^n$ is the function given by $\underline{f}(t, \xi) = \tilde{f}(\check{\xi})(t)$ (cf. §2.6).*

Proof. See Rousseau [9].

Definition. (Kodaira–Spencer [8]) Let M and $\mathcal{V}$ be connected C^∞-manifolds and $\varpi: \mathcal{V} \to M$ be a C^∞-function such that for each $t \in M$, $\mathcal{V}_t = \varpi^{-1}(t)$ is a connected complex analytic manifold. Then $(\mathcal{V}, M, \varpi)$ is called a *differentiable family of complex structures*, if each point of $\mathcal{V}$ has a neighbourhood U satisfying the following conditions: there exists a differentiable homeomorphism h of U into $\varpi(U) \times \mathbb{C}^n$ such that for each point $t \in \varpi(U)$, the restriction h_t of h to $U \cap \mathcal{V}_t$ is biholomorphic on $U \cap \mathcal{V}_t$ into $t \times \mathbb{C}^n$. In the above definition, if $\mathcal{V}$ and M are connected complex analytic manifolds and $\varpi: \mathcal{V} \to M$ is holomorphic, then $(\mathcal{V}, M, \varpi)$ is called a *complex analytic family of complex structures.*

Theorem 3.5.5. (ZFC) *Let M be a connected C^∞- (or complex analytic) manifold of dimension m, $\Omega = \mathcal{O}(M)$ and let X be a separated space in $V^{(\Omega)}$.*

(1) *If X is an arcwise connected complex analytic manifold of dimension n over $\mathbb{C}^\infty$ (or $\mathcal{H}$) in $V^{(\Omega)}$, then $(\bar{X}, M, \varpi)$ with $\varpi\langle t, x\rangle = t$ is a differentiable (or complex analytic) family of complex structures.*

(2) *If $(\bar{X}, M, \varpi)$ is a differentiable family of complex structure with $\varpi\langle t, x\rangle = t$, then X is a connected complex analytic manifold on $\mathbb{C}^\infty$ in $V^{(\Omega)}$.*

Proof. (1) Let $\{p_\alpha\}_{\alpha\in A}$ be a covering of M, $\{(p_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ be a system of coordinate neighbourhood of M and let $p_\alpha \leqslant [\![\{u_j^\alpha\}_{j\in J_\alpha}$ be a locally finite covering of X and $\{(u_i, h_j)\}_{j\in J_\alpha}$ be a system of coordinate neighbourhoods of $X]\!]$. Then $\{\overline{u_j^\alpha \lceil p_\alpha} \mid j \in \bar{J}_\alpha, \alpha \in A\}$ is a covering of $\bar{X}$. It is easy to check that $\overline{u_j^\alpha \lceil p_\alpha}$ and the mappings $\psi_{j\alpha}: \langle t, x\rangle \mapsto (\varphi_\alpha(t), h_j(x)(t))$ of $\overline{u_j^\alpha \lceil p_\alpha}$ into $\mathbb{C}^{m+n}$ for $\alpha \in A$, $j \in J_\alpha$ form a system of coordinate neighbourhoods, and $\bar{X}$ is a manifold and also that the mapping $\varpi: \langle t, x\rangle \mapsto t$ of $\bar{X}$ onto M is differentiable (or holomorphic) and $\bar{X}_t = \varpi^{-1}(t)$ is a complex analytic manifold.

(2) Let $(\bar{X}, M, \varpi)$ be a differentiable family of complex structure. Then there

exists a covering $\{\bar{u}_j\}_{j\in J}$ of $\bar{X}$ and a system $\{\psi_j\}_{j\in J}$ of differentiable homeomorphisms ψ_j of $\bar{u}_j$ into $\varpi(\bar{u}_j)\times\mathbb{C}^n$ such that for each point $t\in\varpi(\bar{u}_j)$, the restriction $(\psi_j)_t$ of ψ_j to $\bar{u}_j\cap\varpi^{-1}(t)$ is biholomorphic on $\bar{u}_j\cap\varpi^{-1}(t)$ into $t\times\mathbb{C}^n$. Define $h_j: u_j\to(\mathbb{C}^\infty)^n$ in $V^{(\Omega)}$ by $\check{h}_j(x)(t)=\psi_j\langle t, x\rangle$. Then h_j is a homeomorphism of u_j into $(\mathbb{C}^\infty)^n$ in $V^{(\Omega)}$, and for $j, k\in J$ with $u_j\cap v_k\neq 0$, $h_jh_k^{-1}$ is biholomorphic in $V^{(\Omega)}$, since $\underline{h_jh_k^{-1}}(t,\xi)=\psi_j\psi_k^{-1}\langle t,\check{\xi}\rangle$ is biholomorphic in the second variable. Therefore X is a complex analytic manifold over $\mathbb{C}^\infty$.

3.6. *Infinitesimal deformation in* GIA

A complex analytic manifold over $\mathscr{H}$ in a standard sheaf model $V^{(\Omega)}$ of GIA is a complex analytic family of complex structures, as shown in §3.5. Here we consider the infinitesimal deformations of complex analytic family of complex structures in GIA. For this purpose, we add the following condition to the Axiom 14. This is satisfied in standard sheaf models of GIA (Theorem 2.6.3).

(e) If $h\equiv h(z_1,\ldots,z_k)$ is a holomorphic function on an open set U of $\mathscr{H}^k$ into $\mathscr{H}$, then for each $j=1,\ldots,n$, the mapping $D^jh:(z_1,\ldots,z_k)\mapsto D^j(h(z_1,\ldots,z_k))$ on $C_0^k\cap U$ is extended to a holomorphic function $D^jh: U\to\mathscr{H}$ and

$$\frac{\partial}{\partial z_\beta}D^jh=D^j\frac{\partial}{\partial z_\beta}h \quad \text{for } \beta=1,\ldots,k.$$

Further if $f(z_1,\ldots,z_k)=(f_1(z_1,\ldots,z_k),\ldots,f_k(z_1,\ldots,z_k))$ is a holomorphic function from an open set Δ of $\mathscr{H}^k$ into U, then

$$D^j(h\circ f)=(D^jh)\circ f+\sum_{\beta=1}^{k}\left(\frac{\partial h}{\partial z_\beta}\circ f\right)(D^jf_\beta).$$

In GIA, let X be a complex analytic manifold over $\mathscr{H}$ of dimension n. Let $\{(U_j,\zeta_j)\}_{j\in J}$ be a system of coordinate neighbourhoods of X with coordinate functions $\zeta_j^1,\ldots,\zeta_j^n$ such that $\zeta_j(U_j)=\{(z_j^1,\ldots,z_j^n)=z_j\in\mathscr{H}^n:|z_j|<1\}$. Then the set Θ of holomorphic vector fields $v=\sum_{\alpha=1}^n v_j^\alpha(\partial/\partial\zeta_j^\alpha)$, where v_j^α's are holomorphic functions on $W\cap U_j$, is a sheaf over X. For $D^\gamma\in T(\mathbb{C}^\infty, C_0)$, $\gamma=1,\ldots,n$, let

$$\theta(j,k)=\sum_{\alpha=1}^{n}D^\gamma(\zeta_j^\alpha\circ\zeta_k^{-1})\frac{\partial}{\partial\zeta_j^\alpha}.$$

$\theta(j,k)\in\Gamma(U_j\cap U_k,\Theta)$ and θ is a 1-cocycle under the axiom (e). The cohomology class of θ is the *infinitesimal deformation in the direction of* D^γ.

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