Geometry of Interaction VI: a blueprint for Transcendental Syntax

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À Corrado, sans qui le λ-calcul
ne serait pas ce qu'il est.
Et donc, la logique non plus.

Abstract

The paper makes use of a finitary version of Geometry of Interaction to fix most pending problems in propositional proof-nets. The methodological guidelines are those of transcendental syntax.

1 Transcendental Syntax

Transcendental syntax starts where semantics fails: foundational issues[1].

1.1 Splendor...

Semantics is very valuable tool: nothing is better than a *good* model-theoretic argument. Typically the independence of Euclide's postulat is explained in the most convincing ways by non-euclidian models: the sphere or the one-sheet hyperboloid. I myself made several crucial uses of semantics: linear logic literally jumped out of *coherent spaces*.

This use of semantics is, unfortunately, accidental. There are dozens of systems of arithmetic, all of them different; but since there is only one decent model for arithmetic — styled standard for this reason —, semantics cannot be used to separate them. This shortage of models, known as *incompleteness*, casts a doubt as to the foundational relevance of semantics.

1.2 ... and misery of semantics

Foundationnaly speaking, the semantic credo is the idea that logic is about « something else », a reality exterior and prior to language. This *realism* can take two aspects: *pleonasmatic* and *structural*.

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1.2.1 The pleonasm

The pleonasm styles the « something else » as *truth* or *meta*. The language is supposed to imitate this preexistent model: when we write $A \Rightarrow A$, we mean « if A is true, then A is true »; justifying $A \Rightarrow A$ by $B \Rightarrow B$ (with $B{:=}$ « A is true ») is dubious: indeed, this is but a metaphysical gimmick condoning our prejudices, a reality imitating the language, not the other way around[2]!

1.2.2 Structuration

Coming from the BHK[3] interpretation, functional (or category-theoretic) interpretations are not pleonasmatic. For instance, $A \Rightarrow A$ is interpreted by the identity morphism. This category-theoretic structure is most precious; the problem is that its is foundationally misleading. A typical example: the BHK interpretation of Π^0_1 formulas is trivial, conveys no information[4] and this cannot be fixed within the category-theoretic universe which forces the identification *of all proofs* in this case. Now to claim that there is nothing inside the proof of Π^0_1 statements like the quadratic reprocity law, Fermat's last theorem, etc. is preposterous. In other terms, the category-theoretic universe is not primitive; by the way, categories are heavily *formatted*[5]; when analysing the format (second « light », *infra*), we clearly step beyond categories.

In the theory of proof-nets, categories have been a nuisance: typically, they compel us to identify the proof of $\vdash A^1_1 \,\&\, A^2_1, \ldots, A^1_n \,\&\, A^2_n, B_1 \otimes \ldots \otimes B_n$ obtained from proofs of A^1_i, B_i and A^2_i, B_i $(i = 1, \ldots, n)$ with the 2^n « slices » $\vdash A^{\sigma(1)}_1, \ldots, A^{\sigma(n)}, B_1 \otimes \ldots \otimes B_n$, neglecting the monstrous difference between the two proofs, $2n$ vs. 2^n.

1.2.3 Subjectivism

The only argument for realism is the fear of *subjectivism*; this is the case with irrelevant subjective distinctions, styled « intensional », e.g., a left-handed vs. right-handed cups. But, subjectivism can take the perverse disguise of *objectivism*, i.e., the denial of the part taken by the subject in the construction of the object: this leads to absurdities like Ptolemaic astronomy which never realised that the « backwards motion » of planets was indeed the movement of the Earth, i.e., of the observer. The Ptolemaic *epicycles* survive in the modern *Kripke models* where cognitive distinctions are handled by means of *ad hoc* parallel universes. And what to say about the « objective » treatment of databases which erases all punctations (`begin,end`) on the basis that they are semantically meaningless, thus making the reckoning of the end-of-file a particular case of the undecidable *halting problem*?

Logic is indeed subjective, since it deals with reasoning: the subject is *a priori* involved in a proof. The question is to shed some *light* on this complex involvement. Once removed the irrelevant bureaucratic details, the subject occurs in three ways: in an *implicit* handling of *answers*, in the choice of a *format* for *questions*, in producing a *reasonable* form of *certainty*.

[2] *Dixit* Voltaire: *On prétend que Dieu a fait l'homme à son image, mais l'homme le lui a bien rendu.*

[3] Brouwer-Heyting-Kolmogoroff, $\sim$ 1930.

[4] The unique BHK proof of $\forall n A$ is the constant fonction $f(n) := \texttt{true}$.

[5] Witness the etymology of « morphism ».

1.3 The three lights

Semantics being discarded, one has to develop a specific methodology; in the absence of any realistic reference, three basic questions corresponding to the fundamental tensions of logic: *implicit* vs. *explicit, formatted* vs. *unformatted, apodictic* vs. *epidictic* will enlighten our way.

1.3.1 First light: what is an answer?

Proofs are always *implicit*, i.e., yield *indirect* answers. Asking logic to be explicit is like asking a bank to milk cows or print newspapers. Logical coherence corresponds to the fact that proofs can — eventually — be « explicited ». This process is known as *cut-elimination*: proofs, as cheques on the possible, can be « cashed ». Various negative theorems, mainly Turing's undecidability of the *halting problem* (1936), ensure that this « explicitation » is extremely hazardous, in particular that the implicit cannot be *a priori* reduced to the explicit.

The first task of transcendental syntax is to find a decent space where implicit and explicit coexist, where a hazardous process of *explicitation* can take place. This space, this process should be syntax-free, i.e., use none of the usual wedding-cake recipes of traditional proof-theory: in other terms, *synthetic*. Usual rewriting systems are not synthetic enough; among them, the best approximation remains pure λ-calculus; various π-calculi are everything but synthetic, since they end into cookbooks. By the way, category-theory which equates the two terms t (implicit) and u (explicit) of a rewriting $t \mapsto u$ — through commutative diagrams of which one side is more commutative than the other — fails in the sole light of « answers ».

1.3.2 Second light: what is a question?

The hazardous character of explicitation means that, without adequate hypotheses, it may fail. Choosing the adequate restrictions amounts at *formatting* logic, i.e., discriminating which questions are *legitimate*. The problem with formats is that they are a *necessary nuisance*: typically, the original, unformated, induction axiom of Peano[6] is inconsistent; but any consistent arithmetic is incomplete. By the way, Richard's paradox (1905) « the smallest number not definable in less than twenty words » shows that *definable* must be formatted, but also that this formatting will be unsatisfactory.

The question of the format is usually evacuated by restricting to such and such system enjoying cut-elimination: inside the usual wedding-cake approach to proof-theory, the formatting is far more hidden than explicitation. For instance, category theory takes it for granted: *morphisms* preserve the form, the format, but which one?

In linear logic, the format occurs in *proof-nets*: some structure claims to be a proof, but we have forgotten in which order the rules have been performed, hence the question of *sequentialisation*. The ancestor of proof-nets is Herbrand's theorem: in a prenex form the existentials should be given as functions $y = t[x_1, \ldots, x_n]$ of the universals. Assuming we forgot the step-by-step construction of t, Herbrand replaces x with $f(y)$ in the case of a formula $\exists y \forall x$; if x actually occurs in t, then we get a cycle (failed unification) $y = t[f(y)]$.

[6] Induction on all « properties ».

The choice of the appropriate questions is thus operated through a *deontic* dialogue whose purpose is not to determine truth, but what makes sense. Just like those judicial discussions « Objection, your Honor! Objection sustained/overruled ». In other terms, refutation has been replaced with *recusal*. This *deontic* standpoint has been central in *Geometry of Interaction* (GoI).

Proof-nets are based on connectedness and acyclicity, which correspond to very old intuitions on logic, completely inaccessible to truth, i.e., to semantics. For instance, it is common to speak of a « vicious circle »; but a vicious circle is neither true nor false, it is just vicious. In proof-nets, a vicious circle becomes a *cycle*, i.e., a *deontic* mistake. There are legitimate objections against *material implication*, on the grounds that the premise should contribute to the conclusion; semantics can't help: $\texttt{false} \Rightarrow \texttt{true} := \texttt{false}$ would equate $A \Rightarrow B$ with $B \Rightarrow A$. In proof-nets, this translates as « a proof should be connected ».

1.3.3 Third light: what conveys certainty?

As we already observed, the ultimate blow against subjectivism is to reckon that logic is subjective. This means that we can actually *write* and *manipulate* proofs and that we *believe* in them. In terms of answers, this means that the syntax-free space should be combinatorial in a certain way. This is is why I eventually decided to replace the von Neumann algebras of GoI [Gir11a] with sorts of graphs based upon *unification*[7].

As to questions, GoI explains logic in an existentialist way, through a notion of *deontic test*, a.k.a. *ordeal*. These ordeals are homegeneous to proofs of the negation. GoI thus relies upon a duality between proofs of A and proofs of $\sim A$, thus vindicating Hegel's *contradictory foundations*[8].

The subjective character of logic forces the set of ordeals for A to be finite. A *gabarit*[9], i.e., a finite set of ordeals, surely conveys absolute, *apodictic*, certainty. However, the second incompleteness theorem says that proofs are not quite apodictic, that there is room for a reasonable — albeit limited — doubt. This relative certainty, I style it *epidictic*. Oustside first order logic, as soon as second order quantifiers (indeed, existential ones) are used, we lose absolute certainty: this is the case for Π^0_1 formulas. We shall see in a forthcoming paper how epidictic certainty is still conveyed by gabarits. The difference with the first order case being that the loss of the *subformula property* casts a *reasonable doubt*[10] as the *relevance* of the gabarits.

1.4 Analytic vs. synthetic, a priori vs. a posteriori

Coming back to the pleonasm, observe that it is hardly more than a translation of the language into another one. And, by the way, this cannot be avoided: language can only be explained by language. What is wrong with the pleonasm is that this other language is *extraterritorial*; an *honest* approach should explain the language by itself, *by its own properties*. To avoid possible circularities in the explanation, we shall look for an approach *synthetic a posteriori*.

[7] A technique invented by Herbrand, decidedly a milestone in logic!

[8] On the base of negation as *refutation*, it stumbles onconsistency, but it works well with negation as *recusal*.

[9] A sort of mould.

[10] Reasonable, but not too serious!

Kant styled mathematics *synthetic a priori*. This means that mathematical definitions take place in a general framework which cannot be put into question. Traditional logic is *analytic a priori*, like a cookbook: it takes the form of list of cases « if this rule is like this, do that ». It works... but don't ask why!

Our main concern will thus be to make logic *synthetic*, i.e., to accommodate our three lights (answers, questions, certainty) independently of any syntactical commitment. This means finding a space in which all normalisations take place (answers); this includes the general notion of typing (questions) and axiomatics (certainty), which should result from a general paradigm in which these notions can be discussed. To sum up, we want logic to be *synthetic a posteriori*.

A posteriori means that language is not primitive, that it is but a convenience, the name for certain remarkable artifacts in our synthetic universe. The fact that we can individualise, give names, formulate rules... will be made possible by various structural results, typically the Church-Rosser property, the mother of all category-theoretic interpretations.

Synthetic a posteriori, this is the only way in which we can speak of the *conditions of possibility* of language, i.e., of transcendental syntax. The whole program of GoI was about those conditions of possibility. Since there have been many fluctuations in the program, it would be preposterous to style these conditions *necessary*; so let us style them *sufficient*. Which does not forbid us from seeking *necessary* conditions; we should only be aware that, necessary or not, we shall never be sure of their necessity.

1.5 This paper

1.5.1 Logical scope

I shall restrict myself to first order propositional linear logic.

Why linear? The expressive powers of classical, intuitionistic or linear logics are, roughly speaking, the same. But linear logic allows for subtler distinctions which are sometimes blurred in the other logics: typically, if we want to understand conjunction from a deontic standpoint, we must distinguish between its *additive* and *multiplicative* versions; one corresponds to &, the other to $\otimes$; the worse is reached when one side uses $\otimes$ while the other uses &.

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

Why propositional? Because predicate calculus, is very delicate to explain; indeed, first order equality remains *terra incognita*.

Why first order? This is because the *epidictic* part of logic involves a quantification over gabarits; something which requires, to start with, some familiarity with gabarits. The real task of this paper is thus the definite version of gabarits. First order is *apodictic*, a world of irrefragable certainty, which takes its source in the *subformula property*.

1.5.2 Proof-nets

The idea is that logic should be completely understood in terms of interaction between graphs, thus expelling any sort of syntactical decorations. In other terms, we shall develop a notion of *proof-net* mainly as a short hand for certain « graphs ». Everything should be contained in the « graphs », in particular, still following our three lights:

Explicitation: should be a matter of « graph contraction », *normalisation*; it should be Church-Rosser: two cuts normalise like a single cut.

Formatting: find the right *gabarits* associated to logical formulas.

Certainty: through adequate *sequentialisation* results.

1.5.3 Technical details

The original proof-net criterion was formulated in terms of *trips*, translated as *permutations*, thus leading to operators of functional analysis. The Danos-Regnier criterion [DR89], formulated in terms of graphs, was considered more manageable, but not quite the right thing. The third light (certainty) tells us that we should stay finitary, thus work in a *unification algebra*, which generalises permutations, i.e., the « trip » approach.

Tethers, i.e., formulas linked to axioms, have been usually rejected from proof-nets for category-theoretic reasons, on the grounds of « irrelevance ». This objection can be overruled by reckoning that irrelevant data are often (e.g., for Π^0_1 formulas) the only way to convey *certainty*. Tethers will thus be accepted or rejected depending whether or not one can normalise them according the general, synthetic, pattern. Tethered axioms being links of arity ≥ 2, we must give up the original criterion — trips — in favour of graphs, the only decent approach to n-ary links. The common generalisation of the unification algebra and graphs yields a sort of « thick graphs »[11]: *galaxies* in which an alternative GoI, closer to the Danos-Regnier version of proof-nets, can be developed.

In order to take care of the second light (questions), it is necessary to find the synthetic version of the correctness condition for proof-nets. Correctness should be stated independently from the proof, which is quite untractable in the case of the contraction rule: we cannot foresee and locate the various premises of this n-ary rule. This problem can be overcome using an idea of de Naurois and Mogbil [dNM11]: the synthetic approach now embodies *virtual switches*.

The paper leaves many questions open. Typically, it hardly goes beyond the definition of proof-nets and their gabarits. So, see this paper as a sort of *blueprint*.

[11] « Graphes épais » to use the terminology of Seiller [Sei12].

2 Stars and galaxies

Here we define the space in which answers dwell.

2.1 Unification

Consider a term language with infinitely many functional symbols of any arity. An equation $t = u$ between terms can be solved by means of *substitutions*, i.e., values for all variables, even those not occurring in t, u: $x_1 = \theta_1, x_2 = \theta_2, \ldots$; if θ is the substitution $x_i = \theta_i$, then $t\theta := t[x_i/\theta_i]$ denotes the result of the replacement of the x_i with the θ_i. We say that t, u are *unifiable* when $t\theta = u\theta$ for some *unifier* θ. The point is that substitutions do compose, hence Herbrand's result:

THEOREM 1
There is a mother of all unifiers for t, u, i.e., a substitution θ_0 such that any unifier θ for t, u can be uniquely written $\theta_0\theta'$.

Due to possible repetitions of variables, the size of this *principal unifier* can grow exponentially; by presenting it as a product of substitutions, one can keep its size linear in the inputs.

2.2 The unification algebra

2.2.1 Flows

A *flow* is an expression $t \leftharpoonup t'$ where t, t' are terms with quite the same variables. These common variables are internal to the flow, in other terms *bound*. In particular, when combining two flows, one must always rename the variables so as to make them distinct. Composition between $t \leftharpoonup t'$ and $u \leftharpoonup u'$ is obtained by *matching* t' and u: matching is the particular case of unification where the terms have no variable in common, what is the case when the variables of t', u have been made distinct. If θ is the principal unifier, we define composition by $(t \leftharpoonup t')(u \leftharpoonup u') := t\theta \leftharpoonup u'\theta$. Composition is thus a partial operation; if we formally add an empty flow 0 to take care of a possible failure of the matching; $(t \leftharpoonup t')(u \leftharpoonup u') := 0$, composition becomes associative, with neutral element $I := x \leftharpoonup x$.

If $\mathcal{T}$ is the set of closed terms and if c is a constant, then $t \leftharpoonup t'$ induces a partial bijection $[t \leftharpoonup t']$ between two subsets of $\mathcal{T}$: $[t \leftharpoonup t']t_0$ is defined when $(t \leftharpoonup t')(t_0 \leftharpoonup c) \neq 0$, in case it writes $[t \leftharpoonup t']t_0 \leftharpoonup c$; the condition « quite the same variables » ensures that $[t \leftharpoonup t']t_0$ is closed and that $[t \leftharpoonup t']$ is injective. Any flow $u \leftharpoonup u$ is idempotent; its associated function is the identity of a subset $[u] \subset \mathcal{T}$. More generally, $[t \leftharpoonup t']$ is a bijection bijection between $[t']$ and $[t]$.

2.2.2 The convolution algebra

One can introduce the *convolution algebra* of the monoid, i.e., the set of finite formal sums $\sum \lambda_i \phi_i$ where the ϕ_i are flows and the λ_i are complex coefficients, the improper flow 0 being identified with the empty sum. This algebra acts on the Hilbert space $\ell^2(\mathcal{T})$ by means of $(t \leftharpoonup t')(\sum_i \lambda_i t_i) := \sum_i \lambda_i [t \leftharpoonup t']t_i$. The involution $(\sum \lambda_i t_i \leftharpoonup t'_i)^* := \sum \bar{\lambda}_i t'_i \leftharpoonup t_i$ is implemented by the usual adjunction. The idempotents $t \leftharpoonup t$ correspond to the projections on the subpaces $\ell^2([t])$ and

$t \leftharpoonup t'$ induces a partial isometry of *source* $\ell^2([t'])$ and *target* $\ell^2([t])$. The early versions of GoI [Gir89, Gir95] did associate to proofs finite sums of flows. These sums were indeed *partial isometries*; $u = \sum t_i \leftharpoonup t'_i$ is partial isometry (i.e., $uu^*u = u$) if the targets t_i are pairwise « disjoint », not unifiable, *idem* for the t'_i. Indeed, the operators of GoI are even *partial symmetries*, i.e., hermitian partial isometries ($u = u^3 = u^*$): typically the identity axioms $t \leftharpoonup t' + t' \leftharpoonup t$ (t, t' disjoint).

2.3 A few examples

The unification algebra enables one to internalise many algebraic constructions.

2.3.1 Matrixes

If I is a finite set of closed terms, the $I \times I$ matrix (λ_{ij}) can be naturally represented by $\sum_{ij} \lambda_{ij}\, i \leftharpoonup j$.

2.3.2 Direct sums

The flows $P := p(x) \leftharpoonup x, Q := q(x) \leftharpoonup x$ induce an isometric embedding of $\ell^2(\mathcal{T}) \oplus \ell^2(\mathcal{T})$ in $\ell^2(\mathcal{T})$: $x \oplus y \mapsto [P]x \oplus [Q]y$. The isometricity comes from $P^*P = Q^*Q = I$, $P^*Q = Q^*P = 0$. The embedding is not surjective: this would require $PP^* + QQ^* = I$, in other terms that every term matches either $p(x)$ or $q(x)$.

P and Q have been heavily used in the early GoI, in particular for multiplicatives — and, modulo tensorisation with I, for contraction. They enable one to change the size of matrices in a flexible way. Usually, the only possibility is to *divide* the size, typically $\mathcal{M}_{mn}(\mathbb{C}) \simeq \mathcal{M}_m(\mathcal{M}_n(\mathbb{C}))$ replaces a $mn \times mn$ matrix with a $m \times m$ matrix whose entries are $n \times n$ matrices, i.e., blocks of size $n \times n$. Thanks to P, Q, one can replace a 3×3 matrix with a 2×2 one (with four « blocks » of sizes $2 \times 2, 2 \times 1, 1 \times 2, 1 \times 1$).

2.3.3 Tensor products

The tensor product of two flows makes use of a binary function « $\cdot$ » and is defined by $(t \leftharpoonup t') \otimes (u \leftharpoonup u') := t \cdot u \leftharpoonup t' \cdot u'$; the variables of the two flows must first be made distinct. This corresponds to an internalisation of the tensor product, which plays an essential role in the handling of exponentials, i.e., of repetition. The flow $T := (x \cdot y) \cdot z \leftharpoonup x \cdot (y \cdot z)$ is such that
$T^*((t \leftharpoonup t') \otimes (u \leftharpoonup u')) \otimes (v \leftharpoonup v')T = (t \leftharpoonup t') \otimes ((u \leftharpoonup u') \otimes (v \leftharpoonup v'))$, thus compensating the want of associativity of the internal tensor.

2.3.4 Crown products

In the same style as T, the flow
$\sigma := x_1 \cdot (x_2 \cdot (\ldots (x_{n-1} \cdot x_n) \ldots)) \leftharpoonup x_{\sigma(1)} \cdot (x_{\sigma(2)} \cdot (\ldots (x_{\sigma(n-1)} \cdot x_{\sigma(n)}) \ldots))$ induces a permutation of the constituants af a n-ary tensor. In [Gir12], crown products have been used to develop a LOGSPACE GoI.

2.4 Stars and galaxies

2.4.1 Stars

A *star* $[\![t_1, \ldots, t_{n+1}]\!]$ consists in $n+1$ terms; these terms — the *rays* of the star — should have *exactly* the same variables and be pairwise disjoint. Stars will be used to represent the links of proof-nets.

Stars generalise the unification algebra; thus, the axiom link $t \leftharpoonup t' + t' \leftharpoonup t$ becomes $[\![t, t']\!]$. The possibility of handling a n-ary link as a single element is of crucial importance in the additive case. However, since our objects are no longer operators, there are some difficulties in defining the analogue of composition. For this we shall use *coloured stars*.

We select pairs of complementary colours, e.g., (`green`, `magenta`) together with the *neutral* colour `black`; a *coloured star* is a star in which each ray has been given a colour: typically, $[\![t, u, v, w]\!]$. Pairwise disjointness is required for the sole rays of the same colour. Coloured stars are not yet another notion, just a shorthand: indeed, consider three unary functions g, m, b and replace $[\![t, u, v, w]\!]$ with $[\![g(t), g(u), b(v), m(w)]\!]$; the translation explains why pairwise disjointness is not required between rays of different colours.

2.4.2 Galaxies

A *galaxy* is a finite set of coloured stars. Cut-free proofs will be represented by black galaxies, whereas the cut-rule will make use of complementary colours.

In order to normalise a galaxy, we first form its *diagrams*. By this I mean any tree (in the topological acception) obtained by attaching $N+1$ stars of the galaxy by means of N *vertices*. By a vertex, I mean a pair $t = u$ of rays of complementary colours. All diagrams can be obtained in the following incremental way, starting with a single star: we select a *free* coloured ray in the diagram, i.e., a coloured ray not yet used in a vertex and form a vertex with any ray of the complementary colour occurring in the galaxy, thus aggregating a new star to the diagram. Since a star may be used several times in a diagram, a galaxy is likely to generate infinitely many diagrams.

The *unification* of a diagram consists in unifying its vertices, so that $t\theta = u\theta$ is an actual equality. Most unifications will fail; we are basically concerned with *correct* diagrams. A correct diagram is a diagram in which unification *has been* successfully *performed*.

2.4.3 Normalisation

In usual GoI, the cut-rule is handled by a partial symmetry σ; the normal form of the proof (u, σ) is given by:

$$(I - \sigma^2)u(I - \sigma u)^{-1}(I - \sigma^2)$$

Here σ corresponds to the swapping of complementary colours: σ exchanges `green` and `magenta` and « kills » `black`. Under reasonable hypotheses (nilpotency), $u(I - \sigma u)^{-1}$ can be written as a finite sum $u + u\sigma u + u\sigma u\sigma u + \ldots$, which corresponds to the plugging of u with itself through complementary colours. The two $I - \sigma^2$ correspond to the restriction to the « black stars ».

Summing up, we obtain the following conditions (*strong normalisation)*, which generalise the nilpotency of σu:

1. There are only finitely many correct diagrams. In other terms, for an appropriate N, all diagrams of size $N+1$ fail; this finite N accounts for *strong* normalisation.

2. Among the correct diagrams, none of them is *closed*, i.e., with a free ray. The simplest closed diagram is made of two unary stars: $\{[\![t]\!], [\![t]\!]\}$ with the vertex $t = t$. This condition is thus reminiscent of consistency.

3. In a correct diagram, free rays of complementary colours are disjoint[12]. The simplest diagram thus excluded consists of a single binary star: $\{[\![t, u]\!]\}$, with t, u not disjoint. This last condition is by far the most important.

The normal form is obtained by collecting the correct diagrams whose free rays are black. And to replace them with their *residual* star, i.e., the star whose rays are its free rays.

The condition of isometricity of GoI translates as: the rays of the same colour of the galaxy are pairwise disjoint. It is easy to show (using condition 3. above) that the normal form of an « isometric galaxy » is still « isometric ».

3 Proof-nets, vehicles and gabarits

3.1 Multiplicative case

3.1.1 Proof-nets

We assume some familiarity with *multiplicative* proof-nets, see, e.g., [Gir11b]. It remains to get rid of syntactical decorations as much as possible so as to describe proof-nets in a purely *locative* way[13]: in order to represent a proof of $\vdash A, B, C$ unary functions p_A, p_B, p_C will be used to distinguish between the various *locations* available in the sequent; I could as well use p_1, p_2, p_3, but this would compel me into a systematic renumbering of the indices.

3.1.2 Vehicles: cut-free case

To each proof π we associate its *vehicle*, i.e., a galaxy $\pi^\bullet$; this galaxy is black in the cut-free case.

Identity axiom: if π is the axiom $\vdash A, \sim A$, then $\pi^\bullet := \{[\![p_A(x), p_{\sim A}(x)]\!]\}$.

$\parr$-rule: if the proof π of $\vdash \Gamma, A \parr B$ has been obtained from a proof ν of $\vdash \Gamma, A, B$, then $\pi^\bullet$ is defined as $\nu^\bullet$ in which p_A and p_B are now defined symbols: $p_A(x) := p_{A\otimes B}(\mathtt{l} \cdot x)$, $p_B(x) := p_{A\otimes B}(\mathtt{r} \cdot x)$. $\mathtt{l}, \mathtt{r}$ are distinct closed terms and $\cdot$ is a binary function chosen once for all.

$\otimes$-rule: if the proof π of $\vdash \Gamma, A \otimes B$ has been obtained from proofs ν of $\vdash \Gamma, A$ and μ of $\vdash B, \Delta$, then $\pi^\bullet := \nu^\bullet \cup \mu^\bullet$, with p_A, p_B *defined* by $p_A(x) := p_{A\otimes B}(\mathtt{l} \cdot x)$, $p_B(x) := p_{A\otimes B}(\mathtt{r} \cdot x)$.

The vehicle is thus a galaxy of axiom-links, seen as stars. The rules $\parr, \otimes$ have been used to relocate these links. For instance, the axiom $[\![p_A(x), p_{\sim A}(x)]\!]$ relocates as $[\![p_{A\parr(\sim A\otimes B)}(\mathtt{l} \cdot x), p_{A\parr(\sim A\otimes B)}(\mathtt{r} \cdot (\mathtt{l} \cdot x))]\!]$.

[12] This is stronger than « non unifiable ».
[13] Only the locations matter: *Locus Solum*.

3.1.3 Vehicles: general case

In presence of cuts, additional coloured functions will be needed. The standard solution consists in using a pair of complementary colours, typically $p_B, p_{\sim B}$ and $p_B, p_{\sim B}$. The interpretation $\pi^\bullet$ now looks as a union $\mathcal{V} \cup \mathcal{C}$: $\mathcal{V}$ (in black and green) is the vehicle proper, $\mathcal{C}$ — its *feedback* — is easily identified as the magenta part of the vehicle.

Cut rule: if the proof π of $\vdash \Gamma$ has been obtained from proofs ν of $\vdash \Gamma, A$ and μ of $\vdash \sim A, \Delta$, then $\pi^\bullet := \nu^\bullet \cup \mu^\bullet \cup \{[\![p_A(x), p_{\sim A}(x)]\!]\}$; furthermore, in $\nu^\bullet \cup \mu^\bullet$, $p_A, p_{\sim A}$ have been painted green: $p_A(t) \mapsto p_A(t)$, $p_{\sim A}(t) \mapsto p_{\sim A}(t)$.

It may be useful to use a specific pair of colours for each cut. Then a vehichle with two cuts may be written $\mathcal{V} \cup \mathcal{C} \cup \mathcal{C}$. As to normalisation, there are three possibilities:

1. Identify green $=$ blue, magenta $=$ yellow and normalise.
2. Normalise the cuts blue/yellow, then the residual cuts green/magenta.
3. Normalise the cuts green/magenta, then the residual cuts blue/yellow

The Church-Rosser property will equate (in any possible sense) these three possibilities. From this, it will be possible to show the *compositionality* of cut, hence to develop various functional, i.e., category-theoretic, interpretations.

The Church-Rosser property also says that one pair of colours is enough, at least for theoretical considerations.

3.1.4 Gabarits (I)

We must now make sense of the lower part of the proof-net, the one dealing with the $\parr, \otimes$ and Cut links. The main problem is to give a precise definition of the switching discipline leading to the correctness condition. Indeed, to each switch, we shall associate an *ordeal*, i.e., a coloured galaxy. The finite set of all these ordeals is called the *gabarit*.

We already defined the unary functions $p_A(x)$ for each formula and subformula of the proof-net. We now introduce $q_A(x) := p_A(\mathtt{g} \cdot x)$, where $\mathtt{g}$ is yet another constant. The replacement of p_A with q_A in the context of gabarits is due to the fact that $p_{A\otimes B}(x)$ is not disjoint from $p_A(x) := p_A(\mathtt{1} \cdot x)$, whereas $q_{A\otimes B}(x)$ and $q_A(x)$ are disjoint. In other terms, the q_A provide disjoint locations for the formulas occurring in the lower part of the proof-net.

Given a proof-net of conclusions Γ, a switch L/R of its $\parr$-links induces an *ordeal*, namely the coloured galaxy made of the following stars:

$\otimes$: if $A\otimes B \notin \Gamma$, $[\![q_{A\otimes B}(x), q_A(x), q_B(x)]\!]$; if $A\otimes B \in \Gamma$, $[\![q_{A\otimes B}(x), q_A(x), q_B(x)]\!]$.

$\parr_L$: if $A\parr B \notin \Gamma$, $[\![q_{A\parr B}(x), q_A(x)]\!]$ and $[\![q_B(x)]\!]$; if $A\parr B \in \Gamma$, $[\![q_{A\parr B}(x), q_A(x)]\!]$ and $[\![q_B(x)]\!]$. In terms of graphs, $[\![q_B(x)]\!]$ « terminates » all $[\![q_B(t)]\!]$.

$\parr_R$: if $A\parr B \notin \Gamma$, $[\![q_{A\parr B}(x), q_B(x)]\!]$ and $[\![q_B(x)]\!]$; if $A\parr B \in \Gamma$, $[\![q_{A\parr B}(x), q_B(x)]\!]$ and $[\![q_A(x)]\!]$.

Cut: $[\![q_A(x), q_{\sim A}(x)]\!]$.

Let $\mathcal{V}^q := \mathcal{V}[\mathtt{g}{\cdot}x]$, and let $\mathcal{V}^q$ be $\mathcal{V}^q$ painted `green`; let us select an ordeal $\mathcal{S}$. Then the galaxy $\mathcal{V}^q \cup \mathcal{S}$ strongly normalises and its normal form is $\{[\![q_A(x)\ ;\ A \in \Gamma]\!]\}$. This is the *correctness criterion* for proof-nets, obviously necessary, indeed sufficient.

3.1.5 Gabarits (II)

Let us turn our attention towards an exotic multiplicative, namely the « linear affine » implication $A \twoheadrightarrow B$. « $\twoheadrightarrow$ » yields a purely multiplicative second-order reduction of additives: $A \oplus B := \forall X((A \multimap X) \twoheadrightarrow ((B \multimap X) \twoheadrightarrow X))$[14].

Indeed, $A \twoheadrightarrow B$ is an intuitionistic implication without reuse of premises; this is why it interests us. The associated disjunction $A \ltimes B := {\sim}A \twoheadrightarrow B$ is problematic in terms of gabarits. Indeed, the $\ltimes$-link:

$$\frac{[A] \quad B}{A \ltimes B}$$

is problematic: the premise A (written $[A]$ for this reason) might be absent, hence the switch « L » is hazardous, since it may destroy everything in that case. On the other hand, we cannot content ourselves with the sole « R », hence the idea of a *virtual switch*, i.e., a way of replacing the missing switch with something else.

Virtual switches are inspired from the proof by de Naurois and Mogbil [dNM11] of the NL complexity of multiplicative proof-nets; improving the idea of *contractibility* introduced by Danos [Dan90], the authors show that it is enough to switch $\parr$ on one side, e.g., always « R ». In our case, the virtual switch remains weaker than the actual switching « L » of the $\parr$[15]; this is why I made up this marginal implication as an example where virtual switches do matter.

A *virtual switch* is a set $[\![t; u_1, \dots, u_n]\!]$ (or $[\![t; u_1, \dots, u_n]\!]$), with a distinguished *root* (t or t). $u_1, \dots, u_n$ must be pairwise disjoint; each variable occuring in t (or t) must still occur in the u_i.

Our notion of ordeal is modified as follows: besides its stars, the ordeal contains a galaxy of virtual switches. Typically, in the case of $A \ltimes B$, besides $[\![q_{A\ltimes B}(x), q_B(x)]\!]$ (or $[\![q_{A\ltimes B}(x), q_B(x)]\!]$) and $[\![q_A(x)]\!]$ we shall add the virtual switch $[\![q_{A\ltimes B}(x); q_A(x)]\!]$ (or $[\![q_{A\ltimes B}(x); q_A(x)]\!]$).

Some condition as to the various virtual switches must be asked: consider the unique correct diagram in $\mathcal{V}^q \cup \mathcal{S}$, and let us unify it, so as to get a galaxy $\mathcal{G}$. For each virtual switch $[\![t; u_1, \dots, u_n]\!]$ (or $[\![t; u_1, \dots, u_n]\!]$, say $[\![t; u_1, \dots, u_n]\!]$), consider all rays obtained by unification from some u_i; since $u_i\theta = u_i\theta'$ implies $t\theta = t\theta'$, each such ray « comes from » a specific instanciation of t, its « root ». We require that:

1. If $u_i\theta \in \mathcal{G}$, then its root $t\theta$ occurs in $\mathcal{G}$.
2. $u\theta$ is « upwards connected » to $t\theta$, i.e., the connection does not transit through the vertex $t_1\theta_1 = t\theta$. In the case of a virtual switch $[\![t; u_1, \dots, u_n]\!]$, this condition is empty.

[14] Instead of $\forall X((A \multimap X) \Rightarrow ((B \multimap X) \Rightarrow X))$.
[15] The actual presence of the premise A is dubious; what is not the case in [dNM11].

For each $t\theta$ (or $t\theta$, say $t\theta$) in $\mathcal{G}$), we can consider the set $\mathcal{G}_{t\theta}$ of all rays standing in between $t\theta$ and some $u_i\theta'$ with root $t\theta$ (i.e., s.t. $t\theta = t\theta'$) including extremities; $\mathcal{G}_{t\theta}$ is this a sort of tree, rooted in $t\theta$. We then require that:

3. If $\mathcal{G}_{t_1\theta_1} \cap \mathcal{G}_{t_2\theta_2} \neq \emptyset$, then either $\mathcal{G}_{t_1\theta_1} \subset \mathcal{G}_{t_2\theta_2}$ or $\mathcal{G}_{t_2\theta_2} \subset \mathcal{G}_{t_1\theta_1}$.

Virtual switches are well-adapted to weakening, since they cope with uncertainty as to the presence of a specific premise. Moreover, since u_i may contain variables not in t, there is no limitation as to the number of $u_i\theta'$ rooted in a given $t\theta$: the extra variables thus account for contraction.

3.1.6 Normalisation

The normalisation of a proof-net is indeed the normalisation of the union $\mathcal{V} \cup \mathcal{C}$ of its vehicle and its cuts, namely the galaxy
$\mathcal{C} := \{[\![p_B(x), p_{\sim B}(x)]\!]\}$, B ranging over all cut formulas of the proof-net.

3.2 Exponentials

3.2.1 The problem

The technology of *virtual switches* allows one to solve the major pending questions related to exponentials. However, we are still unable to cope with expressions like $?A \otimes ?B$ (dually, $!A \parr !B$). Indeed, we shall restrict to *intuitionistic implication*: $A \Rightarrow B := !A \multimap B$, i.e., limit the use of exponentials to the configurations $?A \parr B$ and $!A \otimes B$, where B does not begin with $?, !$.

This restriction can be implemented by saying that $?A, !A$ are sort of *pre-formulas* than can only be used in sequents. Sequents will assume the forms: $\vdash \Gamma, ?\Delta$ ($\Gamma \neq \emptyset$) or $\vdash \Gamma, ?\Delta, !A$. A more ambitious setting would distinguish three *polarities*, namely *standard, perennial, co-perennial*, so as to allow a less spartan socialisation for exponentials, typically $!A \oplus !B$[16], together with the appropriate notion of proof-net in the spirit of Laurent's *polarised nets* [Lau03]. In this paper, we content ourselves with a solution *a minima*.

What is definitely missing is the neutral element $\mathbf{1}$, treated as a formula: $!A \otimes \mathbf{1}$ would then be a standard formula and behave like the usual exponential. $\mathbf{1}$ is definable in second order logic as $\forall X(X \multimap X)$; hence if we are not too picky with the *subformula property*, i.e., if we are only *epidictic*, we shall gain access to the *standard* exponential.

3.2.2 Proof-nets

The rules of sequent calculus are the usual ones. As to proof-nets, the rules of natural deduction:

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \Rightarrow B} \qquad\qquad \dfrac{\begin{array}{cc}\vdots & \vdots \\ A & A \Rightarrow B\end{array}}{B}$$

conveniently topsy-turvied, become the two exponential links, « $?\parr$ » and « $!\otimes$ »:

[16] $A \vee B := !A \oplus !B$ is the linear translation of intuitionistic implication.

$$\frac{[A] \quad B}{?A \parr B}\ ?\parr \qquad\qquad \frac{A \quad B}{!A \otimes B}\ !\otimes$$

As usual, $[A]$ denotes an unspecified number of « occurrences » of A.

3.2.3 Vehicles (I)

p_A, p_B are defined from $p_{?A \parr B}$ (or $p_{!A \otimes B}$) in the familiar way. The novelty is that $p_A(x)$ will be replaced with $p_A(x \cdot y)$; the extra variable y copes with repetitions, like the tensorisation with I in the first GoI [Gir89].

$\otimes$-rule: if the proof π of $\vdash ?\Gamma, !A \otimes B, \Delta$ has been obtained from proofs ν of $\vdash ?\Gamma, A$ and μ of $\vdash B, \Delta$, then we modify $\nu^\bullet$ into ν°:
$p_A(t) \mapsto p_A(t \cdot y), p_B(t \cdot u) \mapsto p_B(t \cdot (u \cdot y))$ $(B \in \Gamma), p_C(t) \mapsto p_C(t \cdot y)$. Then define $\pi^\bullet := \nu^\circ \cup \mu^\bullet$, with $p_A(x) := p_{!A \otimes B}(\mathtt{l} \cdot x)$, $p_B(x) := p_{!A \otimes B}(\mathtt{r} \cdot x)$.

Weakening: if the proof π of $\vdash \Gamma, ?A$ has been obtained from the proof ν of $\vdash \Gamma$, then $\pi^\bullet := \nu^\bullet$.

Derelection: if the proof π of $\vdash \Gamma, ?A$ has been obtained from the proof ν of $\vdash \Gamma, A$, then $\pi^\bullet$ is obtained from $\nu^\bullet$ by means of the replacement $p_A(t) \mapsto p_A(t \cdot \mathtt{d})$, where $\mathtt{d}$ is a constant.

Contraction: if the proof π of $\vdash \Gamma, ?A$ has been obtained from the proof ν of $\vdash \Gamma, ?A', ?A"$, then $\pi^\bullet$ is obtained from $\nu^\bullet$ by means of the replacement $p_{A'}(t \cdot u) \mapsto p_A(t \cdot (\mathtt{l} \cdot u))$, $p_{A"}(t \cdot u) \mapsto p_A(t \cdot (\mathtt{r} \cdot u))$. Contraction thus merges various « occurrences » of $?A$, kept disjoint by adequate choices of the components u in the $p_A(x \cdot u)$. Due to the $\otimes$-rule, u may involve variables $y_1, \ldots, y_n$, where n is the nesting of $?A$ in terms of boxes.

$?\parr$: if the proof π of $\vdash \Gamma, ?A \parr B$ has been obtained from the proof ν of $\vdash \Gamma, ?A, B$, define $\pi^\bullet := \nu^\bullet$, with $p_A(x) := p_{?A \parr B}(\mathtt{l} \cdot x)$, $p_B(x) := p_{?A \parr B}(\mathtt{r} \cdot x)$.

3.2.4 Vehicles (II)

We should first have a *synthetic* look at the various p_A occuring in a proof-net. In the multiplicative case, the p_A were defined *unary* functions:
$p_A(x) := p_{A \otimes B}(\mathtt{l} \cdot x), p_B(x) := p_{A \otimes B}(\mathtt{r} \cdot x)$, etc.

Formulas in a net now receive a *depth* measuring their exponential *nesting*: if $!A \otimes B$ (or $?A \parr B$) is of depth n, then A is of depth $n+1$. The conclusions of a net, including the cuts, are at depth 0.

A formula A of depth n induces a $n+1$-ary defined function $p_A(x; y_n, \ldots, y_1)$; the y_i stand for possible duplicating instructions occuring in the various depths, the deepmost being y_1. To the conclusions $C \in \Gamma$ we associate unary functions $p_C(x)$, the same with cut formulas. Given any C and its associated $p_C(x; y_n, \ldots, y_1$, we define functions for the immediate subformulas of C.

$C = A \otimes B \,/\, A \parr B$ **:** $p_A(x; y_n, \ldots, y_1) := p_C(\mathtt{l} \cdot x; y_n, \ldots, y_1)$ and $p_B(x; y_n, \ldots, y_1) := p_C(\mathtt{r} \cdot x; y_n, \ldots, y_1)$.

$C = !A \otimes B \,/\, ?A \parr B$ **:** $p_A(x; y_{n+1}, y_n, \ldots, y_1) := p_C(\mathtt{l} \cdot (x \cdot y_{n+1}); y_n, \ldots, y_1)$ and $p_B(x; y_n, \ldots, y_1) := p_C(\mathtt{r} \cdot x; y_n, \ldots, y_1)$.

The p_A represent the subformulas of the conclusions, including cut formulas. Now, due to contraction, several « occurrences » of the same A will occur. The vehicle thus appears as as made of stars $[\![p_A(x;t_m,\dots,t_1),p_B(x;u_n,\dots,u_1)]\!]$: they correspond to axiom links whose precise location is given by the t_j,u_j, following the analytical description given above. Of course, those p_A standing above a cut must be painted green.

3.2.5 Gabarits

We define $q_A(x;y_n,\dots,y_1) := p_A(\mathfrak{g}\cdot x;y_n,\dots,y_1)$; the q_A are pairwise disjoint.

The ordeal consists in the following stars, depending upon a switching L/R of the links $⅋$ and $!\otimes$; the switch « R » of $!\otimes$ severs the implicit exponential box above A which will thus be destroyed.

$\otimes$: if $A\otimes B\notin\Gamma$, $[\![q_{A\otimes B}(x;y_n,\dots,y_1),q_A(x;y_n,\dots,y_1),q_B(x;y_n,\dots,y_1)]\!]$;
if $A\otimes B\in\Gamma$, $[\![q_{A\otimes B}(x),q_A(x),q_B(x)]\!]$.

$⅋_L$: if $A⅋B\notin\Gamma$, $[\![q_{A⅋B}(x;y_n,\dots,y_1),q_A(x;y_n,\dots,y_1)]\!]$ and $[\![q_B(x;y_n,\dots,y_1)]\!]$;
if $A⅋B\in\Gamma$, $[\![q_{A⅋B}(x),q_A(x)]\!]$ and $[\![q_B(x)]\!]$.

$⅋_R$: if $A⅋B\notin\Gamma$, $[\![q_{A⅋B}(x;y_n,\dots,y_1),q_B(x;y_n,\dots,y_1)]\!]$ and $[\![q_A(x;y_n,\dots,y_1)]\!]$;
if $A⅋B\in\Gamma$, $[\![q_{A⅋B}(x),q_B(x)]\!]$ and $[\![q_A(x)]\!]$.

Cut: $[\![q_A(x),q_{\sim A}(x)]\!]$ (cuts are at depth 0).

$!\otimes_L$: if $!A\otimes B\notin\Gamma$, $[\![q_{!A\otimes B}(x;y_n,\dots,y_1),q_A(x;x,y_n,\dots,y_1),q_B(x;y_n,\dots,y_1)]\!]$;
if $!A\otimes B\in\Gamma$, $[\![q_{!A\otimes B}(x),q_A(x;x),q_B(x)]\!]$.

$!\otimes_R$: if $!A\otimes B\notin\Gamma$, $[\![q_{!A\otimes B}(x;y_n,\dots,y_1),q_B(x;y_n,\dots,y_1)]\!]$;
if $!A\otimes B\in\Gamma$, $[\![q_{!A\otimes B}(x),q_B(x)]\!]$. Notice the disparition of q_A, which amounts at a physical destruction of everything linked to A (its « box »).

$?⅋$: if $?A⅋B\notin\Gamma$, $[\![q_{?A⅋B}(x;y_n,\dots,y_1),q_B(x;y_n,\dots,y_1)]\!]$ and $[\![q_A(x;y_{n+1},y_n,\dots,y_1)]\!]$;
if $?A⅋B\in\Gamma$, $[\![q_{?A⅋B}(x),q_B(x)]\!]$ and $[\![q_A(x;y_1)]\!]$.

« $?⅋$ » involves no switching: it corresponds to a « $⅋$ » switched on its only reliable premise, B. By the way, there is no reasonable way to link the conclusion with the premises $[A]$. On the other hand, $[\![q_A(x,;y_{n+1},y_n,\dots,y_1)]\!]$ « terminates » all $[A]$ independently of their number and their locations. Due to the absence of right switches, virtual switches must be used:

The virtual switches are thus the following:

- $[\![q_{A⅋B}(x;y_n,\dots,y_1);q_B(x;y_n,\dots,y_1)]\!]$ (or $[\![p_{A⅋B}(x),q_B(x)]\!]$).
- $[\![q_{?A}(x;y_n,\dots,y_1;q_A(x;y_{n+1},y_n,\dots,y_1)]\!]$ or $[\![q_{?A}(x);q_A(x,y_1)]\!]$.

By the way, observe the loss of linearity involved with exponentials: the left switch for $!\otimes$ involves the repetition (= contraction) of the variable x in $q_A(x;x,y_n,\dots,y_1)$, the first one so far; the virtual (= left) switch for $?⅋$ involves the disparition (= weakening) of the variable y_{n+1} of $q_A(x;y_{n+1},y_n,\dots,y_1)$.

3.2.6 Open questions

Here the questions, starting with the stability of correctness under normalisation, sequentialisation, are less obvious than in the multiplicative case.

What about *light logics*? The only intuition so far comes from [Gir11a] which makes use of the hyperfinite factor, an algebra with no « atoms », hence no dereliction. Without coming back to vN algebras, is it possible to make reasonable restrictions on the *rays* so as to achieve similar phenomenons, e.g., the refusal of dereliction for « want of atoms »?

3.3 Additives

3.3.1 The problem

The delicate question is that of *superposition* of contexts: the &-rule of sequent calculus calls for a mingling of a proof of $\vdash \Gamma, A$ and a proof of Γ, B. The original « solution » in terms of *boxes* is a typical example of cooking recipe, for one must devise various *commutative conversions* for cuts involving a « side door » of the box. Much better is my solution [Gir96] in terms of *weighted proof-nets*, whose counterpart is the *dialectal* GoI of [Gir95]. Hughes and van Glabbeek [HvG03] proposed an alternative solution based on *slices*; the exponential size of such proofs is problematic (see section 1.2.2), but the real problem is that their criterion is too extraterritorial in the light of « questions ».

I will propose here a non dialectal approach: no boxes, no weights, no dialect. Indeed, each axiom link occurring in $\vdash \Gamma, A$ (resp. $\vdash \Gamma, B$) will be *tethered* to A (resp. B), so that we shall be able to tell the original « slices ». The only drawback is that normalisation will produce aberrant axiom links tethered to both A and B. A sort of *garbage collection* must thus be designed to erase those links.

3.3.2 Tethers

The *tethers* $\flat$ and $\sharp$ are purely *locative* variants of ? and !. Sequents will now assume one of the forms: $\vdash \Gamma, ?\Delta, \flat\Lambda$ ($\Gamma \neq \emptyset$), $\vdash \Gamma, ?\Delta, \flat\Lambda, !A$ or $\vdash \flat\Lambda, \sharp A$

$$\frac{}{\vdash A, \sim A, \flat C_1, \ldots, \flat C_k} \qquad\qquad \frac{}{\vdash \sharp A, \flat C_1, \ldots, \flat C_k}$$

The $\flat B_i$ replace the *weights* of familiar additive proof-nets; $\sharp A$ is used to weight the premise A of $A \oplus B$. The tethers $\flat B$ will usually be repeated many times, with distinct locations; such repetitions are created by the normalisation procedure.

The only structural rule for tethers is contraction:

$$\frac{\vdash \Gamma, \flat A, \flat A}{\vdash \Gamma, \flat A}\ (\mathrm{C})$$

Moreover, in binary rules, the tethered contexts must be the same:

$$\frac{\vdash \Gamma, \flat\Lambda, A \quad \Delta, \flat\Lambda, B}{\vdash \Gamma, \Delta, \flat\Lambda, A \otimes B} \qquad\qquad \frac{\vdash \Gamma, \flat\Lambda, A \quad \Delta, \flat\Lambda, B}{\vdash \Gamma, \Delta, \flat\Lambda, !A \otimes B}$$

The sequent rules for additives now write as:

$$\frac{\vdash \flat\Lambda, \sharp A \quad \vdash \Gamma, \flat\Lambda, A}{\vdash \Gamma, \flat\Lambda, A \oplus B} \qquad \frac{\vdash \flat\Lambda, \sharp B \quad \vdash \Gamma, \flat\Lambda, B}{\vdash \Gamma, \flat\Lambda, A \oplus B}$$

$$\frac{\vdash \Gamma, \flat\Lambda, \flat A, A \quad \vdash \Gamma, \flat\Lambda, \flat B, B}{\vdash \Gamma, \flat\Lambda, A \,\&\, B}$$

3.3.3 Proof-nets

Besides the axioms, the additive links have four premises:

$$\frac{A \quad \sim A \quad \flat C_1 \ \ldots \ \flat C_k}{\sharp A \quad A \quad \sharp B \quad B} \qquad \frac{A \quad \flat C_1 \ \ldots \ \flat C_k}{\flat A \quad A \quad \flat B \quad B}$$

$$A \oplus B \qquad\qquad A \,\&\, B$$

However, in the $\oplus$-link, it is possible to use only two premises, $\sharp A, A$ or $\sharp B, B$.

3.3.4 Vehicles, analytic version

Identity axiom: interpreted by $\{[\![p_A(x), p_A(\sim x), p_{\flat C_1}(x \cdot \mathtt{d}), \ldots, p_{\flat C_k}(x \cdot \mathtt{d})]\!]\}$.

Tether axiom: interpreted by $\{[\![p_{\sharp A}(x; y), p_{\flat C_1}(x; y), \ldots, p_{\flat C_k}(x; y)]\!]\}$.

&-rule: if the proof π of $\vdash \Gamma, \flat\Lambda, A \,\&\, B$ has been obtained from proofs ν of $\vdash \Gamma, \flat A, A$ and μ of $\vdash \Gamma, \flat\Lambda, \flat B, B$, then $\pi^\bullet := \nu^\bullet \cup \mu^\bullet$, with:
$p_A(x) := p_{\flat A}(x) := p_{A\&B}(\mathtt{l}' \cdot x), p_{A\&B}(\mathtt{l} \cdot x)$,
$p_{\flat B}(x) := p_{A\&B}(\mathtt{r}' \cdot x), p_B(x) := p_{A\&B}(\mathtt{r} \cdot x)$.

$\oplus_L$-rule: if the proof π of $\vdash \Gamma, \flat C_1, \ldots, \flat C_k, A \oplus B$ has been obtained from proofs ν of $\vdash \flat\Lambda, \sharp A$ and μ of $\vdash \Gamma, \flat\Lambda, A$ and, then $\pi^\bullet := \nu^\bullet \cup \mu^\bullet$, with:
$p_{\sharp A}(x) := p_{A\oplus B}(\mathtt{l}' \cdot x)$, $p_A(x) := p_{A\oplus B}(\mathtt{l} \cdot x)$.

$\oplus_R$-rule: symmetrical.

As to tethered axioms, the choice $p_{\flat C_i}(x \cdot \mathtt{d})$ is quite arbitrary; $p_{\flat C_i}(t_i \cdot u_i)$ would do the same, provided $t_i \cdot u_i$ has exactly one variable, x. In particular, through normalisation, the tethered axioms will assume this general form.

3.3.5 Gabarits

A switching L/R of all with &-links must be selected, as well as a switching LL/LR/RL/RR of the $\oplus$-links; in that case we content ourselves To each ordeal we add the following stars:

$\oplus_{LL}$: if $A \oplus B \notin \Gamma$, $[\![q_{A\oplus B}(x; y_n, \ldots, y_1), q_{\sharp A}(x; x, y_n, \ldots, y_1), q_A(x; y_n, \ldots, y_1)]\!]$ and $[\![q_{A\oplus B}(x; y_n, \ldots, y_1), q_{\sharp B}(x; x, y_n, \ldots, y_1), q_B(x; y_n, \ldots, y_1)]\!]$;
if $A \oplus B \in \Gamma$, $[\![q_{A\oplus B}(x), q_{\sharp A}(x; x), q_A(x)]\!]$ and $[\![q_{A\oplus B}(x), q_{\sharp B}(x; x), q_B(x)]\!]$.

$\oplus_{LR}$: if $A \oplus B \notin \Gamma$, $[\![q_{A\oplus B}(x; y_n, \ldots, y_1), q_{\sharp A}(x; x, y_n, \ldots, y_1), q_A(x; y_n, \ldots, y_1)]\!]$ and $[\![q_{A\oplus B}(x; y_n, \ldots, y_1), q_B(x; y_n, \ldots, y_1)]\!]$;
if $A \oplus B \in \Gamma$, $[\![q_{A\oplus B}(x), q_{\sharp A}(x; x), q_A(x)]\!]$ and $[\![q_{A\oplus B}(x), q_B(x)]\!]$.

$\oplus_{RL}$: if $A \oplus B \notin \Gamma$, $[\![q_{A\oplus B}(x; y_n, \dots, y_1), q_A(x; y_n, \dots, y_1)]\!]$
and $[\![q_{A\oplus B}(x; y_n, \dots, y_1), q_{\sharp B}(x; x, y_n, \dots, y_1), q_B(x; y_n, \dots, y_1)]\!]$;
if $A \oplus B \in \Gamma$, $[\![q_{A\oplus B}(x), q_A(x)]\!]$ and $[\![q_{A\oplus B}(x), q_{\sharp B}(x; x), q_B(x)]\!]$.

$\oplus_{RR}$: if $A \oplus B \notin \Gamma$, $[\![q_{A\oplus B}(x; y_n, \dots, y_1), q_A(x; y_n, \dots, y_1)]\!]$
and $[\![q_{A\oplus B}(x; y_n, \dots, y_1), q_B(x; y_n, \dots, y_1)]\!]$;
if $A \oplus B \in \Gamma$, $[\![q_{A\oplus B}(x), q_A(x)]\!]$ and $[\![q_{A\oplus B}(x), q_B(x)]\!]$.

$\&_L$: if $A \,\&\, B \notin \Gamma$, $\{q_{A\&B}(x; y_n, \dots, y_1), q_A(x; y_n, \dots, y_1)\}$
and $\{q_{\flat A}(x; y_{n+1}, y_n, \dots, y_1)\}$; if $A\&B \in \Gamma$, $\{q_{A\&B}(x), q_A(x)\}$ and $\{q_{\flat A}(x; y_1)\}$.

$\&_R$: if $A \,\&\, B \notin \Gamma$, $\{q_{A\&B}(x; y_n, \dots, y_1), q_B(x; y_n, \dots, y_1)\}$
and $\{q_{\flat B}(x; y_{n+1}, y_n, \dots, y_1)\}$; if $A\&B \in \Gamma$, $\{q_{A\&B}(x), q_B(x)\}$ and $\{q_{\flat B}(x; y_1)\}$.

Together with the virtual switches $\{q_{A\&B}(x; y_n, \dots, y_1); q_{\flat A}(x; y_{n+1}, y_n, \dots, y_1)\}$ (or $\{q_{A\&B}(x); q_{\flat A}(x; y_1)\}$) in the case of $\&_L$, $\{q_{A\&B}(x; y_n, \dots, y_1); q_{\flat B}(x; y_{n+1}, y_n, \dots, y_1)\}$ (or $\{q_{A\&B}(x); q_{\flat B}(x; y_1)\}$) in the case of $\&_R$.

3.3.6 Open questions

Additives modify the general pattern in several ways:

- We are no longer *isometric*. Typically the vehicles involved in the &-rule involve a real superposition of the context. In the same way, the ordeals of the $\oplus$ superpose two choices.
- When normalising, this superposition will enable one to mingle the various slices, with a completely hazardous result.

A theory of normalisation, far more interesting than the sole *sequentialisation*, should be developed in that case. A central notion would be that of *incarnation*: a link in the vehicle is *incarnated* when it occurs in some correct diagram w.r.t. some ordeal. Links involving both tethers $\flat A$ and $\flat B$ of $A \,\&\, B$ are thus not incarnated. The problem therefore reduces at finding the way of restricting normalisation to the incarnation of the normal form, without making it a cooking recipe: they have been so far avoided!

3.4 Additive neutrals

3.4.1 The problem

The additive neutral $\top$ is extremely delicate to treat, but only because of the category-theoretic *diktats* which force identifications to the point that one cannot even write its rules. However, there is an easy solution, namely use the second order definition $\top := \exists X X$. Second order definitions should be avoided when possible, but this is not a dogma, only a matter of common sense: second order quantification may hide cuts. For instance, replace the cut:

$$\frac{\vdash \Gamma, A \qquad \vdash \sim A, \Delta}{\vdash \Gamma, \Delta}$$

with

$$\dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \sim A, \Delta}{\vdash \Gamma, A \otimes \sim A, \Delta}}{\vdash \Gamma, \exists X (X \otimes \sim X), \Delta}$$

So that the formula $\exists X(X \otimes \sim X)$ « swallows » the cut. This is why the second order translation $\bot := \exists X(X \otimes \sim X), \mathbf{1} := \forall X(X \Rightarrow X)$ of the multiplicative neutrals is fishy.

This absorbtion is made possible because X occurs both positively and negatively in $\exists X(X \otimes \sim X)$. But this is not the case for $\exists XX$. I thus propose:

$$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, \top}$$

3.4.2 Proof-nets

There is not much to object to the link:

$$\dfrac{A}{\top}$$

except a local loss of the subformula property.

This local loss is in turn easy to fix: replace A (a formula) with the gabarit for A, now seen as part of the vehicle. A gabarit being a combinatoric, logic-free artifact whose only aim is to connect $\top$ in a nice way with its context, there is no longer any mention of the formula A! The bad news is that the gabarit expells us from the category-theoretic paradise in which $\top$, as a terminal element must have a unique proof: indeed, the category-theoretic quotient basically consists in forgetting this gabarit.

What we propose is not a paradise — i.e., a sort a perfect jail —, it is a space of freedom in which certain foundational discussions may be reopened, typically the aporia of Π^0_1 formulas. Taking the gabarits into account, a proof of $\forall n \top$ can become extremely complex! But this is the topic of another paper.

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