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Geometric Arbitrage Theory and Market Dynamics

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Abstract

We have embedded the classical theory of stochastic finance into a differential geometric framework called *Geometric Arbitrage Theory* and show that it is possible to:

- Write arbitrage as curvature of a principal fibre bundle.
- Parameterize arbitrage strategies by its holonomy.
- Extend the Fundamental Theorem of Asset Pricing by a differential homotopic characterization for both complete and not complete arbitrage free markets.
- Characterize geometric arbitrage theory by five principles and show they they are compatible with the classical theory of stochastic finance.
- Derive for a closed market the equilibrium solution for market portfolio and dynamics in the cases where:
  - Arbitrage is allowed but minimized.
  - Arbitrage is not allowed.
- Prove that the no-free-lunch-with-vanishing-risk condition is equivalent to the existence of a utility maximizer.

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1 Introduction

This paper develops a conceptual structure - called *Geometric Arbitrage Theory or GAT* - embedding the classical stochastic finance into a stochastic differential geometric framework. The main contribution of this approach consists of modelling markets made of basic financial instruments together with their term structures as principal fibre bundles. Financial features of this market - like no arbitrage (or better no-free-lunch-with-vanishing-risk) and equilibrium - are then characterized in terms of standard differential geometric constructions - like curvature - associated to a natural connection in this fibre bundle or to a stochastic Lagrangian structure that can be associated to it.

Several research areas can benefit from the GAT approach:

- Risk management, with the development of a consistent scenario generators reducing the complexity of the market, while maintaining the fundamental connections between financial instruments and allowing for a reconciliation of econometric forecasting with

SDEs techniques. See Smith and Speed ([SmSp98]) and Farinelli ([Fa08]).

- Pricing, hedging and statistical arbitrage, with the development of generalized Black-Scholes equations accounting for arbitrage and the computation of positive arbitrage strategies in intraday markets. See Farinelli and Vazquez ([FaVa09]).

Principal fibre bundle theory has been heavily exploited in theoretical physics as the language in which laws of nature can be best formulated by providing an invariant framework to describe physical systems and their dynamics. These ideas can be carried over to mathematical finance and economics. A market is financial-economic system that can be described by an appropriate principle fibre bundle. A principle like the invariance of market laws under change of numéraire can be seen then as gauge invariance. The fact that gauge theories are the natural language to describe economics was first proposed by Malaney and Weinstein in the context of the economic index problem ([Ma96], [We06])[1]. Ilinski (see [Il00], [Il01] and all his papers cited therein) and Young ([Y099]) proposed to view arbitrage as the curvature of a gauge connection, in analogy to some physical theories. Independently, Cliff and Speed ([SmSp98]) further developed Flesaker and Hughston seminal work ([FlHu96]) and utilized techniques from differential geometry (indirectly mentioned by allusive wording) to reduce the complexity of asset models before stochastic modelling. Perhaps due to its borderline nature lying at the intersection between stochastic finance and differential geometry, there was almost no further mathematical research, and the subject, unfairly considered as an exotic topic, remained confined to econophysics, (see [FeJi07] and [Mo09]). We would like to demonstrate that Geometric Arbitrage Theory can be given a rigorous mathematical background and can bring new insights to mathematical finance by looking at the same concepts from a different perspective. That for we will utilize the formal background of stochastic differential geometry as in Schwartz ([Schw80]), Elworthy ([El82]), Eméry ([Em89]), Hackenbroch and Thalmaier ([HaTh99]), Stroock ([St00]) and Hsu ([Hs02]).

This paper is structured as follows. In section 2, after an introductory review of classical stochastic finance, the foundations of Geometric Arbitrage Theory are explained. GAT allows to give an interpretation of arbitrage as curvature of a principal fibre bundle representing the

[1]This was unknown to the author of this paper during the preparation of its first draft.

market and enables to define the quantity of arbitrage associated to a market or to a self-financing strategy. The no-free-lunch-with-vanishing-risk (or NFLVR for short) condition has the interpretation of a continuity equation satisfied by value density and current of the market, as fluid density and current in the hydrodynamics of an incompressible flow. The maximization of (infinitesimal) expected utility is proven to be equivalent to (NFLVR). Section 3 provides a guiding example for a market whose asset prices are Itô processes. In section 4 the connections between mathematical finance and differential topology are analyzed. Homotopic equivalent self-financing arbitrage strategies can be parameterized by the Lie algebra of the holonomy group of the principal fibre bundle. The no-free-lunch-with-vanishing-risk condition is seen to be equivalent in the general case to the triviality of the local holonomy group, and to the triviality of the global holonomy group in the case of complete markets. This is a differential-homotopic extension of the Fundamental Theorem of Asset Pricing. In section 5 we express the market model in terms of a stochastic Lagrangian system, whose dynamics is given by the stochastic Euler-Lagrange Equations. Symmetries of the Lagrange function can be utilized to derive first integrals of the dynamics by means of the stochastic version of Nöther's Theorem. Equilibrium and non-equilibrium solutions are explicitly computed. Section 6 concludes.

2 Geometric Arbitrage Theory Fundamentals

2.1 Stochastic Prelude and Market Model

In this subsection we will briefly summarize the classical set-up. which will be rephrased in subsection (2.7) in differential geometric terms. We basically follow [HuKe04] and the ultimate reference [DeSc08].

We assume continuous time trading and that the set of trading dates is $[0, +\infty[$. This assumption is general enough to embed the cases of finite and infinite discrete times as well as the one with a finite horizon in continuous time.

The uncertainty is modelled by a filtered probability space $(\Omega, \mathcal{A}, P)$, where P is the statistical (physical) probability measure, $\mathcal{A} = \{\mathcal{A}_t\}_{t\in[0,+\infty[}$ an increasing family of sub-σ-algebras of $\mathcal{A}_\infty$ and $(\Omega, \mathcal{A}_\infty, P)$ is a probability space. The filtration $\mathcal{A}$ is assumed to satisfy the usual

conditions, that is

- right continuity: $\mathcal{A}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0, +\infty[$.
- $\mathcal{A}_0$ contains all null sets of $\mathcal{A}_\infty$.

The market consists of finitely many assets indexed by $j = 1, \ldots, N$, whose prices with respect to the 0th asset, the cash, are given by the vector valued semimartingale $S : [0, +\infty[\times \Omega \to \mathbf{R}^N$ denoted by $(S_t)_{t\in[0,+\infty[}$ adapted to the filtration $\mathcal{A}$. The stochastic process $(S^j_t)_{t\in[0,+\infty[}$ describes the price of the jth asset. The cash evolves according to $S^0_t = \exp(\int_0^t du\, r^0_u)$, where the predictable semimartingale $(r^0_t)_{t\in[0,+\infty[}$ represents the continuous interest rate provided by the cash account: one always know in advance what the interest rate on the own bank account is, but this can change from time to time. We remark that there is no need to assume that asset prices are positive. If we want to renormalize the prices by choosing another asset as reference, i.e. by making it to our numéraire, then this asset must have a strictly positive price process.

We assume that there are no transaction costs and that short sales are allowed. The filtration $\mathcal{A}$ is not necessarily generated by the price process $(S_t)_{t\in[0,+\infty[}$: other sources of information than prices are allowed. All agents have access to the same information structure, that is to the filtration $\mathcal{A}$.

A strategy is a locally bounded and predictable stochastic process $x : [0, +\infty[\times \Omega \to \mathbf{R}^N$ describing the portfolio holdings. The stochastic process $(x^j_t)_{t\in[0,+\infty[}$ represents the number of pieces of jth asset portfolio held by the portfolio as time goes by. Remark that the Itô stochastic integral

$$\int_0^t x \cdot dS = \int_0^t x_u \cdot dS_u, \tag{1}$$

and the Stratonovich stochastic integral

$$\int_0^t x * dS := \int_0^t x \cdot dS + \frac{1}{2}\int_0^t d[x, S] = \int_0^t x_u \cdot dS_u + \frac{1}{2}\int_0^t d[x, S]_u \tag{2}$$

are well defined for this choice of integrator (S) and integrand (x). Thereby, the bracket $[\cdot, \cdot]$ denotes the quadratic covariation of two processes. For details we refer to Appendix A in

[Em89], which summarizes Chapter VII of the authoritative [DeMe80]. An admissible strategy is a strategy x for which the Itô integral $\int_0^t x \cdot dS$ is a.s. t-uniformly bounded from below.

The portfolio value is the process $\{V_t\}_{t\in[0,+\infty[}$ defined by

$$V_t := V_t^{x_t} := x_t \cdot S_t. \tag{3}$$

An admissible strategy x is said to be self-financing if and only if the portfolio value at time t is given by

$$V_t = V_0 + \int_0^t x_u \cdot dS_u. \tag{4}$$

This means that the portfolio gain is the Itô integral of the strategy with the price process as integrator: the change of portfolio value is purely due to changes of the base assets' values. The self-financing condition can be rewritten in differential form as

$$dV_t = x_t \cdot dS_t. \tag{5}$$

As pointed out in [BjHu05], if we want to utilize the Stratonovich integral to rephrase the self-financing condition, while maintaining its economical interpretation (which is necessary for the subsequent constructions of mathematical finance), we write

$$V_t = V_0 + \int_0^t x_u * dS_u - \frac{1}{2}\int_0^t d[x,S]_u \quad \text{or, equivalently,} \quad dV_t = x_t * dS_t - \frac{1}{2}d[x,S]_t. \tag{6}$$

An arbitrage for the market model is a self-financing strategy, such that one of the following condition holds for some horizon $T > 0$:

- $P[V_0 < 0] = 1$ and $P[V_T \geq 0] = 1$,
- $P[V_0 \leq 0] = 1$ and $P[V_T \geq 0] = 1$ with $P[V_T > 0] > 0$.

In [DeSc08] (Chapter 9) the no arbitrage condition is given a topological characterization. In view of the fundamental theorem of asset pricing, the no-arbitrage (NA) condition is substituted by a stronger condition, the so called no-free-lunch-with-vanishing-risk (NFLVR).

Definition 1. *Let $(S_t)_{t\in[0,+\infty[}$ be a semimartingale and $(x_t)_{t\in[0,+\infty[}$ and admissible strategy.*

We denote by $(x \cdot S)_{+\infty} := \lim_{t \to +\infty} \int_0^t x_u \cdot dS_u$*, if such limit exists, and by* K_0 *the subset of* $L^0(\Omega, \mathcal{A}_\infty, P)$ *containing all such* $(x \cdot S)_{+\infty}$*. Then we define*

- $C_0 := K_0 \setminus L^0_+(\Omega, \mathcal{A}_\infty, P)$.
- $C := C_0 \cap L^\infty(\Omega, \mathcal{A}_\infty, P)$.
- $\bar{C}$*: the closure of* C *in* L^∞ *with respect to the norm topology.*

The market model satisfies

- *the* ***no-arbitrage condition (NA)*** *if and only if* $C \cap L^\infty(\Omega, \mathcal{A}_\infty, P) = \{0\}$*, and*
- *the* ***no-free-lunch-with-vanishing-risk (NFLVR)*** *if and only if* $\bar{C} \cap L^\infty(\Omega, \mathcal{A}_\infty, P) = \{0\}$.

Delbaen and Schachermayer proved in 1994 (see [DeSc08] Chapter 9.4)

Theorem 2 (**Fundamental Theorem of Asset Pricing in Continuous Time**)**.** *Let* $(S_t)_{t \in [0,+\infty[}$ *be a locally bounded semimartingale. There is an equivalent martingale measure* P^* *for* S *if and only if the market model satisfies the (NFLVR).*

This is a generalization for continuous time of the Dalang-Morton-Willinger Theorem proved in 1990 (see [DeSc08], Chapter 6) for the discrete time case, where the (NFLVR) is relaxed to the (NA) condition. The Dalang-Morton-Willinger Theorem generalizes to arbitrary probability spaces the Harrison and Pliska Theorem (see [DeSc08], Chapter 2) which holds true in discrete time for finite probability spaces.

An equivalent alternative to the martingale measure approach for asset pricing purposes is given by the pricing kernel (state price deflator) method.

Definition 3. *Let* $(S_t)_{t \in [0,+\infty[}$ *be a semimartingale describing the price process for the assets of our market model. The positive semimartingale* $(\beta_t)_{t \in [0,+\infty[}$ *is called* ***pricing kernel (or state price deflator)*** *if and only if* $(\beta_t S_t)_{t \in [0,+\infty[}$ *is a* P*-martingale.*

As shown in [HuKe04] (Chapter 7), the existence of a pricing kernel is equivalent to the existence of an equivalent martingale measure

Theorem 4. *Let $(S_t)_{t\in[0,+\infty[}$ be a locally bounded semimartingale. The process S admits an equivalent martingale measure P^* if and only if there is a pricing kernel β.*

2.2 Derivatives of Stochastic Processes

One of the main contribution of this paper is to reformulate what has been summarized in the preceding subsection in a natural geometric language. In stochastic differential geometry one would like to lift the constructions of stochastic analysis from open subsets of $\mathbf{R}^N$ to N dimensional differentiable manifolds. To that aim, chart invariant definitions are needed and hence a stochastic calculus satisfying the usual chain rule and not Itô's lemma. (cf. [HaTh99], Chapter 7). That is why we will be mainly concerned in the following by stochastic integrals and derivatives meant in the sense of *Stratonovich* and not of *Itô*.

Definition 5. *Let I be a real interval and $Q = \{Q_t\}_{t\in I}$ be a continuous vector valued stochastic process on the probability space $(\Omega, \mathcal{A}, P)$ adapted to the filtration $\{\mathcal{A}_t\}_{t\in I}$ of the σ-algebra $\mathcal{A}_\infty$. Assuming that the following limits exist,* ***Nelson's stochastic derivatives*** *are defined as*

$$\boxed{\begin{array}{l} DQ_t := \lim\limits_{h\to 0^+} \mathbb{E}\left[\dfrac{Q_{t+h}-Q_t}{h} \mid \mathcal{A}_t\right] \qquad D_*Q_t := \lim\limits_{h\to 0^+} \mathbb{E}\left[\dfrac{Q_t-Q_{t-h}}{h} \mid \mathcal{A}_{t-h}\right] \\ \\ \mathcal{D}Q_t := \dfrac{DQ_t + D_*Q_t}{2}. \end{array}} \tag{7}$$

Let $\mathcal{S}^1(I)$ the set of all processes Q such that $t \mapsto Q_t$, $t \mapsto DQ_t$ and $t \mapsto D_*Q_t$ are continuous mappings from I to $L^2(\Omega, \mathcal{A}_\infty)$.
Let $\mathcal{C}^1(I)$ the completion of $\mathcal{S}^1(I)$ with respect to the norm

$$\|Q\| := \sup_{t\in I} \left(\|Q_t\|_{L^2(\Omega,\mathcal{A}_\infty)} + \|DQ_t\|_{L^2(\Omega,\mathcal{A}_\infty)} + \|D_*Q_t\|_{L^2(\Omega,\mathcal{A}_\infty)}\right). \tag{8}$$

We remark that the stochastic derivatives D and $\mathcal{D}$ correspond to Itô's and, respectively, to Stratonovich's derivative via the corresponding integrals ([Ne08]). The process space $\mathcal{C}^1(I)$ contains all diffusions.

2.3 Gauges

We are going to introduce a more general representation of the asset model introduced in section 2.1, which better suits to the arbitrage modelling task.

Definition 6. *A **gauge** is an ordered pair of two $\mathcal{A}$-adapted real valued semimartingales (D, P), where*

- *$D = \{D_t\}_t$ is called **deflator** and*
- *$P = \{P_{t,s}\}_{t,s}$, which is called **term structure**, is considered as a stochastic process with respect to the time t, termed **valuation date**. The parameter $s \geq t$ is referred as **maturity date**. The following properties must be satisfied a.s. for all t, s such that $s \geq t \geq 0$:*

  (i) $P_{ts} > 0$,

  (ii) $P_{tt} = 1$,

  (iii) $P_{ts} = o(\exp(-s)) \quad (s \to +\infty)$.

Deflators and term structures can be considered *outside the context of fixed income.* An arbitrary financial instrument is mapped to a gauge (D, P) with the following economic interpretation:

- Deflator: D_t is the value of the financial instrument at time t expressed in terms of an "universal currency" taken as absolute reference over time and asset classes. If we choose $D_t^j := S_t^j \quad (j = 1, \dots N)$ for the asset model introduced in subsection 2.1, then this "universal currency" becomes automatically the cash account, the 0-th asset.
- Term structure: P_{ts} is the value at time t (expressed in units of deflator at time t) of a synthetic zero coupon bond with maturity s delivering one unit of financial instrument at time s.

We point out that there is no unique choice for deflators and term structures describing an asset model. If a set of deflators qualifies, then we can multiply every deflators in it by the same

positive semimartingale to obtain another suitable set of deflators. Of course term structures have to be modified accordingly. If we want to make an asset to our numéraire, then we have to make sure that the corresponding deflator is a strictly positive stochastic process.

Example 7. *Stock Index*
Let us consider a total return stock index, where the dividends are reinvested.

- D_t = *stock index value at time* t.
- P_{ts} = *price of a forward on the stock index issued at time* t *maturing at time* s *expressed in terms of* D_t.

Example 8. *Zero Bonds*
Let us consider a family of maturing zero bonds.

- D_t = *value of a zero bond maturing at time* t = *value of one unit of cash at time* t.
- P_{ts} = *price of a zero bond issued at time* t *and delivering one unit of cash at time* s *expressed in terms of* D_t.

Deflators typically represent for a currency the time evolution of inflation or deflation. Quotients of deflators are exchange rates.

Example 9. *Exchange Rates*

$$\frac{D_t^{USD}}{D_t^{CHF}} = FX_t^{CHF \to USD}. \tag{9}$$

2.4 Absence of Arbitrage and Positive State Price Deflator

We want to study what it means for our asset model described in terms of gauges to be arbitrage free. As we see from theorem 2 and 4 the no-free-lunch-with-vanishing-risk property is equivalent to the existence of a positive state price deflator, that is of a positive semimartingale $\beta = \{\beta_t\}_t$ such that the market value at time t of a contingent claim X at time $s > t$ is

$$\frac{1}{\beta_t}\mathbb{E}_t[\beta_s X], \tag{10}$$

where $\mathbb{E}_t$ denotes conditional expectation. Since prices are expressed in units of the deflator to which they relate the formula writes

$$D_t P_{ts} = \frac{1}{\beta_t}\mathbb{E}_t[\beta_s D_s]. \tag{11}$$

Therefore we have proved the following

Theorem 10. *The asset model satisfies the no-free-lunch-with-vanishing-risk condition if and only if there exists a positive stochastic process $\beta = \{\beta_t\}_t$ such that for all assets*

$$\boxed{P_{ts} = \frac{\mathbb{E}_t[\beta_s D_s]}{\beta_t D_t}} \tag{12}$$

holds.

This is another important feature of this model building: once the state price deflator is established, *it suffices to model deflators, term structures follow by conditional expectation* and the no-free-lunch-with-vanishing-risk condition is satisfied. This approach is similar to the *actuarial pricing* which is of course equivalent to the well known *risk neutral pricing*, as shown in [HePl06] Chapters 9.2 and 9.4.

After the state price deflator has been established, deflators can be rescaled as

$$D \longrightarrow \beta D \tag{13}$$

and equation (12) writes

$$P_{ts} = \frac{\mathbb{E}_t[D_s]}{D_t}. \tag{14}$$

2.5 Tautological Embedding of Classical No Arbitrage Asset Models

We want to demonstrate how a classical asset model like the one presented in subsection 2.1 can be rewritten to fit into the new framework under the no-free-lunch-with-vanishing-risk assumption. Let us consider an asset with its stochastic cashflow intensity $c : [0, +\infty[\times\Omega \to \mathbf{R}$. For example, a bond is identified with its future coupons and nominal and a stock is identified

with all its future dividends. The process $c = \{c_t\}_{[0,+\infty[}$ is adapted to the filtration $\mathcal{A}$ and represents the asset cashflow pattern. The deflator value at time t is defined as

$$D_t := \mathbb{E}_t^* \left[\int_t^{+\infty} dh\, c_h \exp\left(- \int_t^h du\, r_u^0 \right) \right] = \int_t^{+\infty} dh\, \mathbb{E}_t^{*,h}[c_h] d_{t,h}^0 = \frac{1}{\beta_t} \mathbb{E}_t \left[\int_t^{+\infty} dh\, c_h \beta_h \right], \tag{15}$$

where $d_{t,s}^0 := \mathbb{E}_t^* \left[\exp\left(- \int_t^s du\, r_u^0 \right) \right]$ denotes the price of a default risk free zero bond at time t with maturity s for the reference currency, $\mathbb{E}_t^*$ denotes the risk neutral conditional expectation, $\mathbb{E}_t^{*,s}$ for $s \geq t$ the s-forward neutral conditional expectation, and β the state price deflator. The term structure can be defined as

$$P_{ts} := \frac{\mathbb{E}_t[\beta_s D_s]}{\beta_t D_t}. \tag{16}$$

For assets with a cashflow pattern with compact support in time (e.g. a coupon bond with maturity T), deflators vanish beyond the last cashflow (e.g. $D_t \equiv 0$ for $(t > T)$).

2.6 Gauge Transforms

We want now to introduce transforms of deflators and term structures in order to group gauges containing the same (or less) stochastic information. That for, we will consider *deterministic* linear combinations of assets modelled by the same gauge (e. g. zero bonds of the same credit quality with different maturities).

Definition 11. *Let $\pi : [0, +\infty[\longrightarrow \mathbf{R}$ be a deterministic cashflow intensity (possibly generalized) function. It induces a* ***gauge transform*** *$(D, P) \mapsto \pi(D, P) := (D, P)^\pi := (D^\pi, P^\pi)$ by the formulae*

$$\boxed{D_t^\pi := D_t \int_0^{+\infty} dh\, \pi_h P_{t,t+h} \qquad P_{t,s}^\pi := \frac{\int_0^{+\infty} dh\, \pi_h P_{t,s+h}}{\int_0^{+\infty} dh\, \pi_h P_{t,t+h}}.} \tag{17}$$

A gauge transform is well defined if and only if the integrals are convergent, which is the case if $\limsup_{h\to\infty} \exp\left(\frac{\log(|\pi_h|)}{h} \right) \leq 1$.

A gauge transform with positive cashflows always maps a gauge to another gauge. A generic gauge transform does not, since the positivity of term structures is not a priori preserved. Therefore, when referring to a generic gauge transform, it is necessary to specify its domain of

definition, that is the set of gauges which are mapped to other gauges.

We can use gauge transforms to construct portfolios of instruments already modelled by a known gauge.

Example 12. *Coupon Bonds*
Let (D, P) the gauge describing the family of zero bonds. To model a family of straight coupon bonds with coupon rate g and term to maturity T let us choose the (generalized) cashflow intensity function

$$\pi_t := \sum_{s=1}^{T-1} g\delta_{t-s} + (1+g)\delta_{t-T}. \tag{18}$$

Thereby δ denotes the Dirac-delta generalized function.

- *D_t^π = value at time t of a coupon bond issued at time t.*
- *P_{ts}^π = price of a synthetic zero bond issued at time t and delivering at time s a coupon bond (issued at time s), expressed in terms of D_t^π.*

Proposition 13. *Gauge transforms induced by cashflow vectors have the following property:*

$$((D,P)^\pi)^\nu = ((D,P)^\nu)^\pi = (D,P)^{\pi * \nu}, \tag{19}$$

where $*$ denotes the convolution product of two cashflow vectors or intensities respectively:

$$(\pi * \nu)_t := \int_0^t dh\, \pi_h \nu_{t-h}. \tag{20}$$

The convolution of two non-invertible gauge transform is non-invertible. The convolution of a non-invertible with an invertible gauge transform is non-invertible.

Definition 14. *An invertible gauge transform is called **non-singular**. Two gauges are said to be in **same orbit** if and only if there is a non-singular gauge transform mapping one onto the other. A singular gauge transform π defines a partial ordering $(D,P) \succ (D^\pi, P^\pi)$ in the set of gauges. (D,P) is said to be in a **higher orbit** than (D^π, P^π).*

It is therefore possible to construct gauges in a lower orbit from higher orbits, but not the

other way around. Orbits represent assets containing equivalent information. For every orbit it s

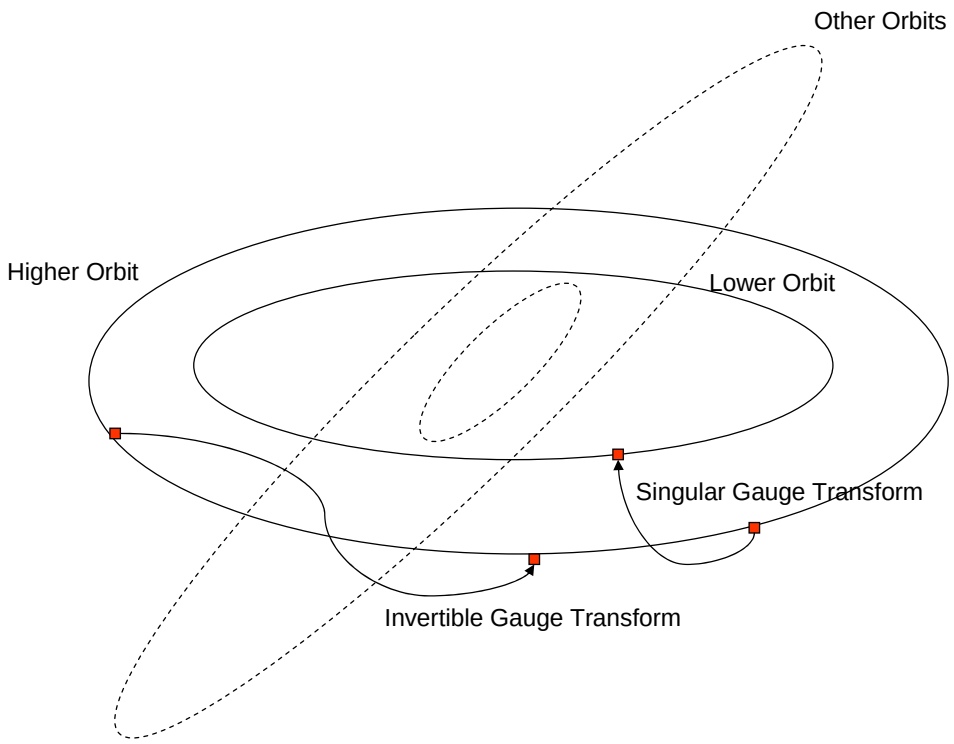


Figure 1: Gauge Transforms

Definition 15. *A gauge (D, P) with term structure $P = \{P_{ts}\}_{ts}$ satisfies the* ***positive interest condition*** *if and only if for all t the function $s \mapsto P_{ts}$ is strictly monotone decreasing. Such a gauge is said to be* ***positive****. A gauge not satisfying this property is termed* ***principal gauge****. The term structure can be written as a functional of the* ***instantaneous forward rate*** *f defined as*

$$\boxed{f_{t,s} := -\frac{\partial}{\partial s} \log P_{t,s}, \quad P_{t,s} = \exp(-\int_t^s f_{t,h} dh).} \tag{21}$$

and

$$\boxed{r_t := \lim_{s \to^+} f_{t,s}} \tag{22}$$

is termed ***short rate****.*

We see that the positive interest condition is satisfied if and only if $f_{ts} > 0$ for all t, s, $s \geq t$. The positive interest condition is associated with the *storage requirement*. Whenever it

is always more valuable to get a piece of a financial object today than in the future, then it should be modelled with a gauge satisfying the positive interest condition. Examples are: non perishable goods, currencies, price indices for equities and real estates, total return indices. Examples of financial quantities not satisfying the positive interest condition and thus reflecting items which are not storable, are: inflation indices, short rates, dividend indices for equities, rental indices for real estates.

Definition 16. *The cash flow intensity* $[-1] := -\delta'$*, first derivative of the Dirac delta generalized function, defines the* ***short rate transform****,*

$$\begin{aligned} D_t^{[-1]} &= D_t \int_0^{+\infty} dh \ - \delta'_h P_{t,t+h} = -D_t r_t \\ P_{t,s}^{[-1]} &= \frac{\int_0^{+\infty} dh \ - \delta'_h P_{t,s+h}}{\int_0^{+\infty} dh \ - \delta'_h P_{t,t+h}} = \frac{f_{ts}}{r_t} P_{ts} \end{aligned} \tag{23}$$

while the cash flow intensity $[+1] := \Theta$*, Heavyside function, defines the* ***perpetuity transform***

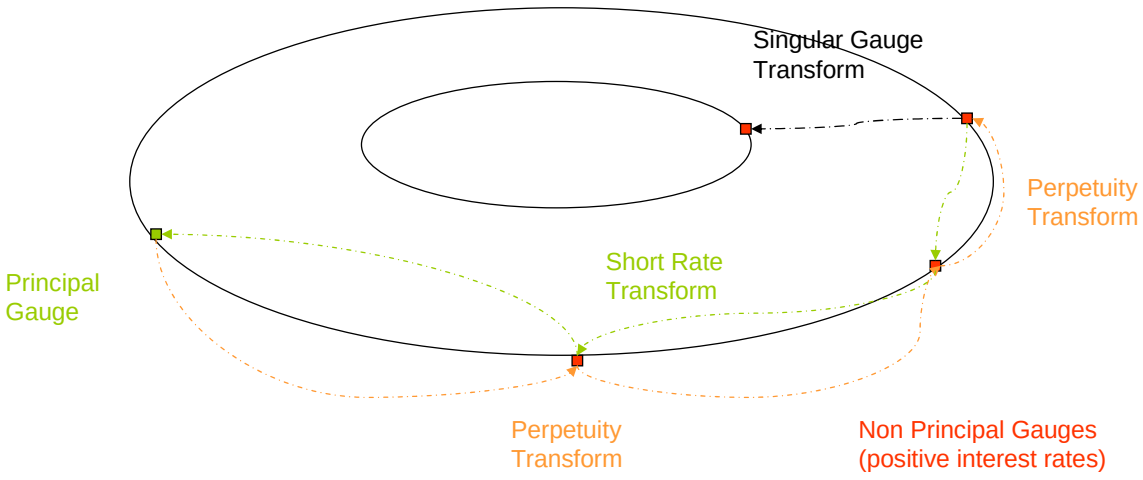


Figure 2: Short Rate and Perpetuity Transforms

Notation 17. *Repeated application of perpetuity and short rate transforms are given by:*

$$\begin{aligned}
[0] &:= \delta\text{: Dirac delta generalized function} \\
[+1] &:= \Theta\text{: Heavyside function} \\
[+n]_t &:= \frac{t^{n-1}}{(n-1)!} \quad (n \geq 2) \\
[-1]_t &:= -\delta'\text{: first derivative of Dirac delta generalized function} \\
[-n]_t &= (-1)^n \delta^{(n)}\text{: n-th derivative of Dirac delta generalized function} \quad (n \geq 2)
\end{aligned} \tag{25}$$

Thereby, for any integers m, n one has $[m] * [n] = [m+n]$ (cf. [Ho03] Chapter IV).

The short rate and the perpetuity transform are inverse to another, as one can see from proposition 13 and $[+1] * [-1] = [0]$. The short rate transform can be applied only to a positive gauge producing a gauge which possibly does not satisfy the positive interest rate condition. The perpetuity transform is a gauge transform that can be applied to any gauge producing always a positive gauge.

Proposition 18. *A gauge satisfies the positive interest condition if and only if it can be obtained as the perpetuity transform of some other gauge.*

The positive interest condition is not an easy requirement for a stochastic model of a gauge.

Example 19. *Fixed Income, Equity and Real Estate Gauges*

2.7 Arbitrage Theory in a Differential Geometric Framework

Now we are in the position to rephrase the asset model presented in subsection 2.1 in terms of a natural geometric language. That for, we will unify Smith's and Ilinski's ideas to model a simple market of N base assets. In Smith and Speed ([SmSp98]) there is no explicit differential geometric modelling but the use of an allusive terminology (e.g. gauges, gauge transforms). In Ilinski ([Il01]) there is a construction of a principal fibre bundle allowing to express arbitrage in terms of curvature. Our construction of the principal fibre bundle will differ from Ilinski's one

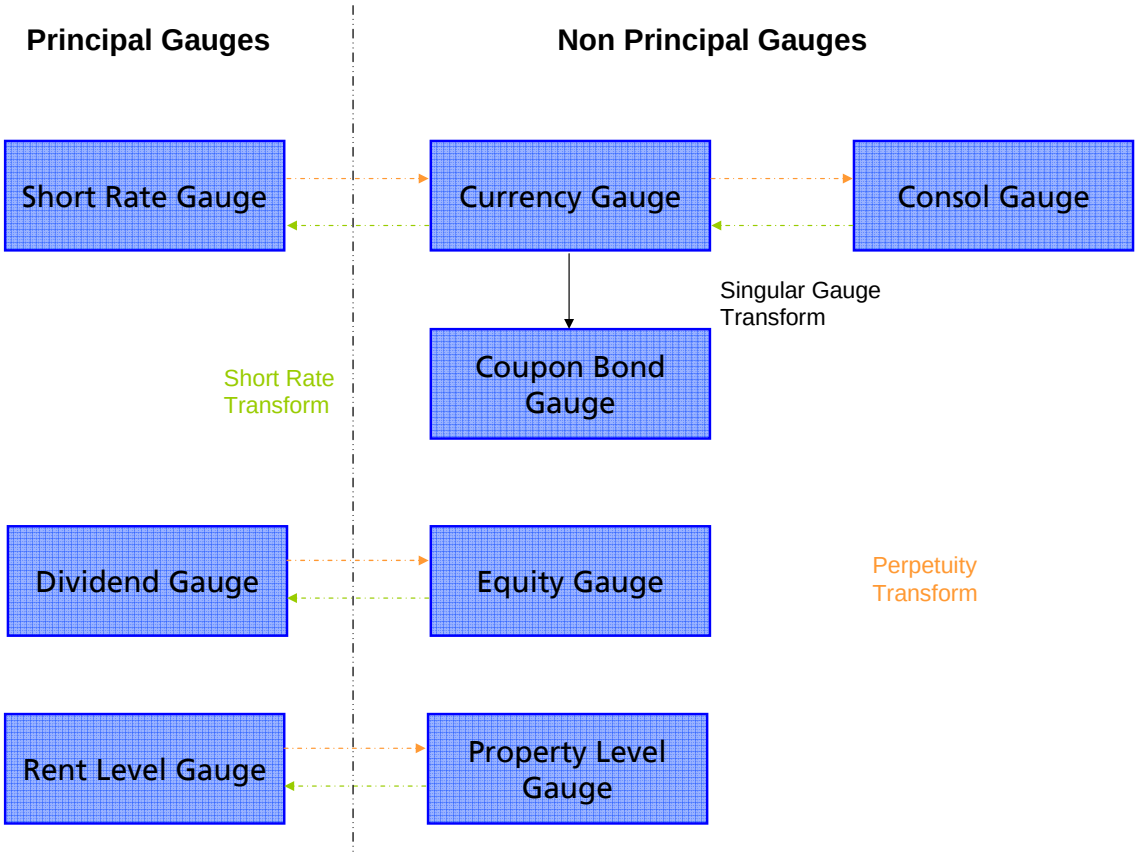


Figure 3: Gauges

in the choice of the group action and the bundle covering the base space. Our choice encodes Smith's intuition in differential geometric language.

In this paper we explicitly model no derivatives of the base assets, that is, if derivative products have to be considered, then they have to be added to the set of base assets. The treatment of derivatives of base assets is tackled in ([FaVa09]). Given N base assets we want to construct a portfolio theory and study arbitrage. Since arbitrage is explicitly allowed, we cannot a priori assume the existence of a risk neutral measure or of a state price deflator. In terms of differential geometry, we will adopt the mathematician's and not the physicist's approach. The market model is seen as a principal fibre bundle of the (deflator, term structure) pairs, discounting and foreign exchange as a parallel transport, numéraire as global section of the gauge bundle, arbitrage as curvature. The Ambrose-Singer theorem allows to parameterize arbitrage strategies as element of the Lie algebra of the holonomy group. The no-free-lunch-

with-vanishing-risk condition is proved to be equivalent to a zero curvature condition or to a continuity equation allowing for an hydrodynamics study of arbitrage flows.

2.7.1 Market Model as Principal Fibre Bundle

AAs a concise general reference for principle fibre bundles we refer to Bleecker's book ([Bl81]). More extensive treatments can be found in Dubrovin, Fomenko and Novikov ([DuFoNo84]), and in the classical Kobayashi and Nomizu ([KoNo96]). Let us consider a market with N assets in continuous time. A general portfolio at time t is described by the vector of nominals $x \in \mathbf{R}^N =: X$. The gauge corresponding to the j-th asset is, for $j = 1, \ldots, N$,

$$(D^j, P^j) = ((D^j_t)_{t\in[0,+\infty[}, (P^j_{ts})_{s\geq t}), \tag{26}$$

where D^j denotes the deflator and P^j the term structure. This can be written as

$$P^j_{t,s} = \mathbb{E}_t \left[\exp\left(- \int_t^s r^j_u du \right) \right], \tag{27}$$

where r^j is the short rate process for the j-th asset. Remark that the expectation is taken with respect to the statistical measure P and that r^j_t is *not* locally risk-free. We do not know whether or not a risk neutral measure exists. For a portfolio with nominals $x \in \mathbf{R}^N$ we define

$$\boxed{D^x_t := \sum_{j=1}^N x_j D^j_t \qquad r^x_t := \sum_{j=1}^N \frac{x_j D^j_t}{\sum_{j=1}^N x_j D^j_t} r^j_t \qquad P^x_{t,s} := \mathbb{E}_t \left[\exp\left(- \int_t^s r^x_u du \right) \right].} \tag{28}$$

The image space of all possible strategies reads

$$\boxed{M := \{(x,t) \in \mathbf{R}^N \times [0,+\infty[\}.} \tag{29}$$

In subsection 2.6 cashflow intensities and the corresponding gauge transforms were introduced. They have the structure of an Abelian semigroup

$$\boxed{G := \mathcal{E}'([0,+\infty[,\mathbf{R}) = \{F \in \mathcal{D}'([0,+\infty[) \mid \mathrm{supp}(F) \subset [0,+\infty[\text{ is compact}\},} \tag{30}$$

where the semigroup operation on distributions with compact support is the convolution (see [Ho03], Chapter IV), which extends the convolution of regular functions as defined by formula (20).

Definition 20. *The* ***market fibre bundle*** *is defined as the fibre bundle of gauges*

$$\boxed{\mathcal{B} := \{({D^\pi}_t{}^x, {P^\pi}_{t\cdot}{}^x) | (x,t) \in M, \pi \in G^*\}.} \tag{31}$$

The cashflow intensities defining invertible transforms constitute an Abelian group

$$\boxed{G^* := \{\pi \in G | \text{ it exists } \nu \in G \text{ such that } \pi * \nu = [0]\} \subset \mathcal{E}'([0, +\infty[, \mathbf{R}).} \tag{32}$$

From proposition 13 we obtain

Theorem 21. *The market fibre bundle $\mathcal{B}$ has the structure of a G^*-principal fibre bundle given by the action*

$$\begin{aligned} \mathcal{B} \times G^* &\longrightarrow \mathcal{B} \\ ((D,P),\pi) &\mapsto (D,P)^\pi = (D^\pi, P^\pi) \end{aligned} \tag{33}$$

The group G^* acts freely and differentiably on $\mathcal{B}$ to the right.

2.7.2 Cashflows as Sections of the Associated Vector Bundle

By choosing the fiber $V := \mathbf{R}^{]-\infty,+\infty[}$ and the representation $\rho : G \to \mathrm{GL}(V)$ induced by the gauge transform definition, and therefore satisfying the homomorphism relation $\rho(g_1 * g_2) = \rho(g_1)\rho(g_2)$, we obtain the associated vector bundle $\mathcal{V}$. Its sections represents cashflow streams - expressed in terms of the deflators - generated by portfolios of the base assets. If $v = (v_t^x)_{(x,t)\in M}$ is the *deterministic* cashflow stream, then its value at time t is equal to

- the deterministic quantity v_t^x, if the value is measured in terms of the deflator D_t^x,
- the stochastic quantity $v_t^x D_t^x$, if the value is measured in terms of the "universal currency" (e.g. the cash account for the choice $D_t^j := S_t^j$ for all $j = 1, \ldots, N$).

In the general theory of principal fibre bundles, gauge transforms are bundle automorphisms preserving the group action and equal to the identity on the base space. Gauge transforms of $\mathcal{B}$ are naturally isomorphic to the sections of the bundle $\mathcal{B}$ (See theorem 3.2.2 in [Bl81]). Since G^* is Abelian, right multiplications are gauge transforms. Hence, there is a bijective correspondence between gauge transforms and cashflow intensities admitting an inverse. This justifies the terminology introduced in definition 11.

2.7.3 Discounting and Foreign Exchange as Parallel Transport

Let us consider the projection of $\mathcal{B}$ onto M

$$\begin{aligned} p : \mathcal{B} \cong M \times G^* &\longrightarrow M \\ (x,t,g) &\mapsto (x,t) \end{aligned} \tag{34}$$

and its tangential map

$$T_{(x,t,g)}p : \underbrace{T_{(x,t,g)}\mathcal{B}}_{\cong \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^{[0,+\infty[}} \longrightarrow \underbrace{T_{(x,t)}M}_{\cong \mathbf{R}^N \times \mathbf{R}.} \tag{35}$$

The vertical directions are

$$\mathcal{V}_{(x,t,g)}\mathcal{B} := \ker\left(T_{(x,t,g)}p\right) \cong \mathbf{R}^{[0,+\infty[}, \tag{36}$$

and the horizontal ones are

$$\mathcal{H}_{(x,t,g)}\mathcal{B} \cong \mathbf{R}^{N+1}. \tag{37}$$

A connection on $\mathcal{B}$ is a projection $T\mathcal{B} \to \mathcal{V}\mathcal{B}$. More precisely, the vertical projection must have the form

$$\begin{aligned} \Pi^v_{(x,t,g)} : T_{(x,t,g)}\mathcal{B} &\longrightarrow \mathcal{V}_{(x,t,g)}\mathcal{B} \\ (\delta x, \delta t, \delta g) &\mapsto (0, 0, \delta g + \Gamma(x,t,g).(\delta x, \delta t)), \end{aligned} \tag{38}$$

and the horizontal one must read

$$\begin{aligned}\Pi^h_{(x,t,g)} : T_{(x,t,g)}\mathcal{B} &\longrightarrow \mathcal{H}_{(x,t,g)}\mathcal{B} \\ (\delta x, \delta t, \delta g) &\mapsto (\delta x, \delta t, -\Gamma(x,t,g).(\delta x, \delta t)),\end{aligned} \tag{39}$$

such that

$$\Pi^v + \Pi^h = \mathbf{1}_{\mathcal{B}}. \tag{40}$$

With the following choice of connection

$$\boxed{\Gamma(x,t,g).(\delta x, \delta t) := g\left(\frac{D^{\delta x}_t}{D^x_t} - r^x_t \delta t\right),} \tag{41}$$

the parallel transport in $\mathcal{B}$ has a beautiful financial interpretation:

Theorem 22 (**Geometric Interpretation of Discounting and Foreign Exchange**)**.**

- *Parallel transport along the nominal directions (x-lines) corresponds to a multiplication by an exchange rate.*
- *Parallel transport along the time direction (t-line) corresponds to a division by a stochastic discount factor.*

Recall that time derivatives needed to define the parallel transport along the time lines have to be understood in Stratonovich's sense. We see that the bundle is trivial, because it has a global trivialization, but the connection is not trivial.

Proof. Let us consider a curve $\gamma(\tau) = (x(\tau), t(\tau))$ in M for $\tau \in [\tau_1, \tau_2]$ and an element of the fiber over the starting point $g_1 \in p^{-1}(\gamma(\tau_1)) \cong G$. The parallel transport of g_1 along γ is the solution $g = g(\tau)$ of the first order differential equation

$$\begin{cases} \Pi^v_{(x(\tau),t(\tau),g(\tau))}(\mathcal{D}x(\tau), \mathcal{D}t(\tau), \mathcal{D}g(\tau)) = 0 \\ g(\tau_1) = g_1, \end{cases} \tag{42}$$

which in our case writes

$$\begin{cases} \mathcal{D}g(\tau) = -g(\tau)\left(\frac{D_{t(\tau)}^{\mathcal{D}x(\tau)}}{D_{t(\tau)}^{x(\tau)}} - r_{t(\tau)}^{x(\tau)}\mathcal{D}t(\tau)\right) \\ g(\tau_1) = g_1, \end{cases} \tag{43}$$

Recall that the time derivative $\mathcal{D}$ is Nelson's derivative corresponding to the Stratonovich integral, see subsection 2.2. Now, if γ is a nominal direction, then $t(\tau) \equiv t$ and $\mathcal{D}t(\tau) \equiv 0$. Thus

$$\begin{cases} \mathcal{D}g(\tau) = -g(\tau)\frac{\sum_{j=1}^N \mathcal{D}x_j(\tau)D_t^j}{\sum_{j=1}^N x_j(\tau)D_t^j} \\ g(\tau_1) = g_1, \end{cases} \tag{44}$$

which means

$$g(\tau) = g_1\frac{\sum_{j=1}^N x_j(\tau_1)D_t^j}{\sum_{j=1}^N x_j(\tau)D_t^j} \tag{45}$$

corresponding to a multiplication by an exchange rate at time t from portfolio $x(\tau_1)$ to portfolio $x(\tau)$.

If γ is the time direction, then $x(\tau) \equiv x$, $t(\tau) = \tau$ and $\mathcal{D}x(\tau) \equiv 0$, $\mathcal{D}t(\tau) \equiv 1$. Thus

$$\begin{cases} \mathcal{D}g(\tau) = g(\tau)r_\tau^x \\ g(\tau_1) = g_1, \end{cases} \tag{46}$$

which means

$$g(\tau) = g_1 \exp\left(\int_{\tau_1}^{\tau} r_u^x du\right) \tag{47}$$

corresponding to a division by the stochastic discount rate for portfolio x from time τ_1 to time τ. □

Remark 23. *Following Ilinski's idea ([Il01]), the choice of this particular connection is motivated by the fact that it allows to encode foreign exchange and discounting as parallel transport. When the first draft of this paper was completed, the author discovered that Malaney and Weinstein ([Ma96]) already introduced a connection in the deterministic case in the context of selffinancing basket of goods for divisa indices. Very recently, [FaVa09] have elaborated a stochastic version of the Malaney-Weinstein connection proving its equivalence with the connection defined in (41).*

2.7.4 Weakly Differentiable Market Model

We continue to reformulate the classic asset model introduced in subsection 2.1 in terms of stochastic differential geometry.

Definition 24. *A **weakly differentiable market model** for N assets is described by N gauges which are weakly differentiable with respect to the time variable. More exactly, for all $t \in [0, +\infty[$ and $s \geq t$ there is an open time interval $I \ni t$ such that the deflators $D_t := [D_t^1, \ldots, D_t^N]^\dagger$ and the term structures $P_{t,s} := [P_{t,s}^1, \ldots, P_{t,s}^N]^\dagger$ are in $\mathcal{C}^1(I, \mathbf{R}^N)$ as processes in t and parameter s. The short rates are defined by $r_t := \lim_{s \to t^-} \log P_{ts}$.*

*A strategy is a curve $\gamma : I \to X$ in the portfolio space parameterized by the time. This means that the allocation at time t is given by the vector of nominals $x_t := \gamma(t)$. We denote by $\bar{\gamma}$ the lift of γ to M, that is $\bar{\gamma}(t) := (\gamma(t), t)$. A strategy is said to be **closed** if it it represented by a closed curve. An element of $\mathcal{C}^1(I, X)$ is termed **weakly differentiable strategy**. A $\mathcal{C}^1$**-admissible strategy** is predictable and weakly differentiable.*

In general the allocation can depend on the state of the nature i.e. $x_t = x_t(\omega)$ for $\omega \in \Omega$. Unless otherwise specified strategies will always be $\mathcal{C}^1$-admissible for an appropriate time interval.

Definition 25. *A $\mathcal{C}^1$-admissible strategy is said to be $\mathcal{C}^1$**-self-financing** if and only if*

$$\mathcal{D}(x_t \cdot D_t) = x_t \cdot \mathcal{D}D_t + [x_t, D_t] \quad or \qquad \mathcal{D}x_t \cdot D_t = [x_t, D_t], \tag{48}$$

almost surely.

For the reminder of this paper unless otherwise stated we will deal only with weakly differentiable market models, weakly differentiable strategies, and, when necessary, with weakly differentiable state price deflators.

2.7.5 Numéraire as Global Section of the Bundle of Gauges

If we want to make an arbitrary asset or, more generally, a portfolio defined by the nominal vector x^{Num} to our numéraire, we have to renormalize all deflators by an appropriate gauge transform $\pi^{\mathrm{Num},x}$ so that:

- The portfolio value is constantly over time normalized to one:

$$D_t^{x^{\mathrm{Num}},\pi^{\mathrm{Num}}} \equiv 1. \tag{49}$$

- All other assets' and portfolios' are expressed in terms of the numéraire:

$$D_t^{x,\pi^{\mathrm{Num}}} = \mathrm{FX}_t^{x\to x^{\mathrm{Num}}} := \frac{D_t^x}{D_t^{x^{\mathrm{Num}}}}. \tag{50}$$

It is easily seen that the appropriate choice for the gauge transform π^{Num} making the portfolio x^{Num} to the numéraire is given by the global section of the bundle of gauges defined by

$$\boxed{\pi_t^{\mathrm{Num},x} := \mathrm{FX}_t^{x\to x^{\mathrm{Num}}}.} \tag{51}$$

Of course such a gauge transform is well defined if and only if the numéraire deflator is a never vanishing stochastic process. It is customary, without loss of generality to assume that the numéraire is strictly positive.

2.7.6 Arbitrage as Curvature

The Lie algebra of G is

$$\mathfrak{g} = \mathbf{R}^{[0,+\infty[} \tag{52}$$

and therefore commutative. The $\mathfrak{g}$-valued connection 1-form writes as

$$\chi(x,t,g)(\delta x,\delta t) = \left(\frac{D_t^{\delta x}}{D_t^x} - r_t^x \delta t\right) g, \tag{53}$$

or as a linear combination of basis differential forms as

$$\boxed{\chi(x,t,g) = \left(\frac{1}{D_t^x}\sum_{j=1}^{N} D_t^j dx_j - r_t^x dt\right) g.} \tag{54}$$

The $\mathfrak{g}$-valued curvature 2-form is defined as

$$R := d\chi + [\chi, \chi], \tag{55}$$

meaning by this, that for all $(x, t, g) \in \mathcal{B}$ and for all $\xi, \eta \in T_{(x,t)}M$

$$R(x, t, g)(\xi, \eta) := d\chi(x, t, g)(\xi, \eta) + [\chi(x, t, g)(\xi), \chi(x, t, g)(\eta)]. \tag{56}$$

Remark that, being the Lie algebra commutative, the Lie bracket $[\cdot, \cdot]$ vanishes. After some calculations we obtain

$$\boxed{R(x, t, g) = \frac{g}{D_t^x} \sum_{j=1}^{N} D_t^j \left(r_t^x + \mathcal{D} \log(D_t^x) - r_t^j - \mathcal{D} \log(D_t^j) \right) dx_j \wedge dt,} \tag{57}$$

and can prove following results which characterizes arbitrage as curvature.

Theorem 26 (**No Arbitrage**). *The following assertions are equivalent:*

(i) The market model satisfies the no-free-lunch-with-vanishing-risk condition.

(ii) The local holonomy group $\mathrm{Hol}^0(\chi)$ *of the principal fibre bundle* $\mathcal{B}$ *is trivial.*

(iii) The curvature form R *vanishes everywhere on* $\mathcal{B}$*.*

(iv) There exists a positive semimartingale $\beta = \{\beta_t\}_t$ *such that deflators and short rates satisfy for all portfolio nominals and all times the condition*

$$\boxed{r_t^x = -\mathcal{D} \log(\beta_t D_t^x).} \tag{58}$$

(v) There exists a positive semimartingale $\beta = \{\beta_t\}_t$ *such that deflators and term structures satisfy for all portfolio nominals and all times the condition*

$$\boxed{P_{ts}^x = \frac{\mathbb{E}_t[\beta_s D_s^x]}{\beta_t D_t^x}.} \tag{59}$$

We have utilized the concept of holonomy, which is the group generated by the parallel transport along closed curves. We distinguish the local from the global case.

Definition 27. *The **holonomy group** based at $b \in \mathcal{B}$ is defined as*

$$\boxed{\mathrm{Hol}_b(\chi) := \{g \in G \mid b \text{ and } b.g \text{ can be joined by an horizontal curve in } \mathcal{B}\}.} \tag{60}$$

*The **local holonomy group** based at $b \in \mathcal{B}$ is defined as*

$$\boxed{\mathrm{Hol}_b^0(\chi) := \{g \in G \mid b \text{ and } b.g \text{ can be joined by a contractible horizontal curve in } \mathcal{B}\}.} \tag{61}$$

If M and $\mathcal{B}$ are connected, then holonomy and local holonomy depend on the base point b only up to conjugation. In this paper we will always assume connectivity for both M and $\mathcal{B}$ and therefore drop the reference to the basis point b, with the understanding that the definition is good up to conjugation.

Proof of theorem 26:

(ii)⇔(iii): The bundle is trivial. The assertion is then a standard result in differential geometry (the Ambrose-Singer theorem), see f.i. [KoNo96] Chapters II.4 and II.8.

(iii)⇒(iv):

$$R \equiv 0 \Rightarrow \mathcal{D}\log(D_t^x) + r_t^x = \mathrm{Const}_t, \tag{62}$$

where Const_t depends only on the time t but not on the portfolio x. By writing $\mathrm{Const}_t = \log(\beta_t)$ for a positive semimartingale $\beta = \{\beta_t\}_t$ assertion (iv) is proved.

(iii)⇐(iv): trivial.

(iv)⇔(v): (v) is the integral version of (iv), which is the differential version of (v).

(i)⇔(v): This has been shown in section (2.4).

□

2.8 No Arbitrage Condition, Flows and Continuity Equation

The geometric language introduced enlightens similarities between the asset model on one side and hydro- or electrodynamics on the other. The counterpart of a liquid or charge flow in

physics is a value flow in mathematical finance. The associated continuity equation is satisfied if and only if the no-free-lunch-with-vanishing-risk condition is fulfilled.

Definition 28. *Let $M_t(y) := \{x|(x,t) \in M\, x \leq y\}$ be the set of all possible portfolios at time t bounded from above by the portfolio y. The* ***log value current*** *for the market model is defined as a vector field on M by*

$$\boxed{J(x,t) := \int_{M_t(x)} du^N\, r_t\, u,} \tag{63}$$

where $r_t := [r_t^1, \ldots, r_t^N]^\dagger$ and $r_t\, u$ the componentwise multiplication. Let $\beta = \{\beta_t\}_t$ be a positive semimartingale. The β-scaled ***log value density*** *for the market model is defined on M as*

$$\boxed{\rho^\beta(x,t) := \log(\beta_t D_t^x).} \tag{64}$$

The results of the preceding subsection can be reformulated in terms of a *continuity* equation analogously to classical electrodynamics (cf. [Ja98], 5.1, p. 175) or continuum mechanics (cf. [Ši02], 3.3.1, pp. 67-68).

Theorem 29 (**Continuity Equation**)**.** *The market model satisfies the no-free-lunch-with-vanishing-risk-condition if and only if there exists a positive semimartingale β such that one of following equations is satisfied:*

$$\boxed{\begin{aligned} &\mathcal{D}\rho^\beta + \operatorname{div}_x J = 0 \\ &\mathcal{D}\int_{X_{t_0}} dx^N \rho^\beta = -\oint_{X_{t_0}} dn \cdot J. \end{aligned}} \tag{65}$$

The first expression is the differential version of the continuity equation and the second the integral one, which must hold text for any $1-$codimensional submanifold X_{t_0} lying in the hyperplane $t \equiv t_0$.

The integral version of the continuity equation has a beautiful financial interpretation: a market satisfies the no-free-lunch-with- vanishing-risk condition if and only if the log total value change of any submarket is due to the log value current flow through its boundary.

Proof. It is an application of the vector field divergence definition:

$$\operatorname{div}_x J(x,t) = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} J(x,t) = r_t^x = -\mathcal{D}\log(\beta_t D_t^x) \tag{66}$$

Gauss' theorem proves the integral version. □

The left hand side expression in the differential version of the continuity equation (65) is a natural candidate for a local arbitrage measure. The link with the definition of arbitrage given in the preceding section is given by

Proposition 30 (**Curvature Formula**). *Let R be the curvature, ρ^β the log value density and J the log value current. Then, the following quality holds for any positive semimartingale β:*

$$\boxed{R(x,t,g) = gdt \wedge d_x \left[\mathcal{D}\rho^\beta + \operatorname{div}_x J\right] = gdt \wedge d_x \left[\mathcal{D}\log D_t^x + r_t^x\right].} \tag{67}$$

Proof. We develop the expression for the curvature as:

$$\begin{aligned}
R(x,t,g) &= gd\left(\frac{\partial \log(D_t^x)}{\partial x}\cdot dx - r_t^x dt\right) = \\
&= gd\left(\frac{\partial \rho^\beta}{\partial x}\cdot dx - \operatorname{div}_x J dt\right) = \\
&= gd\left(\sum_{j=1}^{N} \frac{\partial \rho^\beta}{\partial x_j} dx_j - \operatorname{div}_x J dt\right) = \\
&= g\left(\sum_{1\le i<j\le N} \frac{\partial^2 \rho^\beta}{\partial x_i \partial x_j} dx_i \wedge dx_j + \mathcal{D}\frac{\partial \rho^\beta}{\partial x_j} - \sum_{i=1}^{N} \frac{\partial \operatorname{div}_x J}{\partial x_j} dt \wedge dx_j\right) = \\
&= g \sum_{1\le i,j\le N} \frac{\partial}{\partial x_j}\left(\mathcal{D}\rho^\beta + \operatorname{div}_x J\right) dt \wedge dx_j = \\
&= gdt \wedge \left\{\sum_{1\le i,j\le N} \frac{\partial}{\partial x_j}\left(\mathcal{D}\rho^\beta + \operatorname{div}_x J dx_j\right)\right\} = \\
&= gdt \wedge d_x\left[\mathcal{D}\rho^\beta + \operatorname{div}_x J\right] = gdt \wedge d_x\left[\mathcal{D}\log D_t^x + r_t^x\right].
\end{aligned} \tag{68}$$

□

Corollary 31 (**No Arbitrage Revisited**). *The following assertions are equivalent:*

(i) The market model satisfies the no-free-lunch-with-vanishing-risk condition.

(ii) The curvature form R vanishes everywhere on $\mathcal{B}$.

(iii) There exist a positive semimartingale β for which the continuity equation is satisfied.

2.9 Price and Return Dynamics in the No-Arbitrage Case

Theorem 26 formula for the short rate reads

$$r_t^x = -\mathcal{D}\log(\beta_t D_t^x). \tag{69}$$

We solve this equation with respect to β as:

$$\mathcal{D}\log(\beta_t) = -r_t^x - \mathcal{D}\log(D_t^x). \tag{70}$$

For any admissible portfolio x this is a first order (stochastic) differential equation, whose solution writes

$$\beta_t = \beta_0 \frac{D_0^x}{D_t^x} \exp\left(-\int_0^t du\, r_u^x\right). \tag{71}$$

If we choose asset i as numéraire, we can assume $D_t^i \equiv 1$ and the state price deflator writes, since $D_t^x = \mathrm{FX}_t^{i\to x} D_t^i$, as

$$\beta_t = \beta_0 \frac{\mathrm{FX}_0^{i\to x}}{\mathrm{FX}_t^{i\to x}} \exp\left(-\int_0^t du\, r_u^i\right). \tag{72}$$

Therefore we have proved the following

Proposition 32. *If the market model satisfies the no-free-lunch-with vanishing risk condition, then the* ***portfolio instantaneous total return***

$$\boxed{Ret_t^x := \lim_{h\to 0^+} \mathbb{E}_{t-h}\left[\frac{D_{t+h}^x - D_{t-h}^x}{2hD_{t-h}^x}\right] = \mathcal{D}\log D_t^x.} \tag{73}$$

satisfies the equation

$$Ret_t^x = -\mathcal{D}\log\beta_t - r_t^x. \tag{74}$$

2.10 Arbitrage and Utility

Let us now consider a utility function, that is a real C^2-function of a real variable, which is strictly monotone increasing (i.e. $u' > 0$) and concave (i.e. $u'' < 0$). Typically, a market participant would like to maximize the expected utility of its wealth at some time horizon. Let us assume that he (or she) holds a portfolio of synthetic zero bonds delivering at maturity base assets and that the time horizon is infinitesimally near, that is that the utility of the instantaneous total return has to be maximized. The portfolio values read as:

- At time $t-h$: $D^x_{t-h}P^x_{t-h,t+h}$.
- At time t: $D^x_t P^x_{t,t+h}$.
- At time $t+h$: D^x_{t+h}.

The instantaneous return can therefore be computed as :

$$\lim_{h\to 0^+} \mathbb{E}_{t-h}\left[\frac{D^x_{t+h} - D^x_{t-h}P^x_{t-h,t+h}}{2hD^x_{t-h}P^x_{t-h,t+h}}\right] = \mathcal{D}\log(D^x_t) + r^x_t. \tag{75}$$

Definition 33 (**Utility of Instantaneous Synthetic Bond Portfolio Return**). *The expected utility maximization problem writes in its infinitesimal version at time t as*

$$\boxed{\max_{\substack{x\in\mathbf{R}^N \\ D^x_t = w_t}} u\left(\mathcal{D}\log(D^x_t) + r^x_t\right),} \tag{76}$$

where the linear expression $D^x_t = w_t$ represents a budget constraint.

Now we can formulate the main result of this subsection.

Theorem 34. *A market satisfies the no-free-lunch-with-vanishing-risk condition if and only if at least one agent maximizes its infinitesimal expected utility.*

This result can be seen as the natural generalization of the corresponding result in discrete time, as theorem 3.3 in [FoeSc04], see also [Ro94]. Since it concerns a finite dimensional optimization, it is less ambitious than Bellini's, Frittelli's and Schachermayer's results for

infinite dimensional optimization problems in continuous time, see theorem 22 in [BeFr02] and theorem 2.2 in [Scha01].

Proof. The Lagrange principal function associated to the maximization problem writes

$$\Phi(x,\lambda) := u\left(\mathcal{D}\log(D_t^x) + r_t^x\right) - \lambda(D_t^x - w_t), \tag{77}$$

where λ denotes the Lagrange multiplier. An extremum has to satisfy the system of equations

$$\begin{cases} \frac{\partial\Phi}{\partial x} = & u'\left(\mathcal{D}\log(D_t^x) + r_t^x\right)\frac{1}{D_t^x}\left[\mathcal{D}\log(D_t) - \mathcal{D}\log(D_t^x) + r_t - r_t^x\right]D_t - \lambda D_t & = 0 \\ \frac{\partial\Phi}{\partial\lambda} = & (-D_t^x - w_t) & = 0. \end{cases} \tag{78}$$

This means for all $j = 1, \ldots, N$

$$u'\left(\mathcal{D}\log(D_t^x) + r_t^x\right)\frac{1}{D_t^x}\left[\mathcal{D}\log(D_t^j) - \mathcal{D}\log(D_t^x) + r_t^j - r_t^x\right] = \lambda. \tag{79}$$

This again means that for all $i, j, i \neq j$:

$$\mathcal{D}\log(D_t^i) + r_t^i = \mathcal{D}\log(D_t^j) + r_t^j = \text{Const}, \tag{80}$$

where Const can be written as $-\mathcal{D}\log\beta_t$ for an appropriate positive semimartingale $\beta = \{\beta_t\}_{t\geq 0}$. Therefore: if the maximization problem has a solution, then the equation system has a solution, implying by theorem 26 the no-free-lunch-with-vanishing-risk condition for the market. Conversely, if the no-free-lunch-with-vanishing-risk condition is satisfied, the equation system must have a solution by theorem 26. Since (76) is a concave problem, this must be a global maximum.

□

3 A Guiding Example

We want now to construct an example to demonstrate how the most important geometric concepts of section 2 can be applied. Given a filtered probability space $(\Omega, \mathcal{A}, P)$, where P is the statistical (physical) probability measure, we assume that all processes introduced in this example are are adapted to the filtration $\mathcal{A} = \{\mathcal{A}_t\}_{t\in[0,+\infty[}$ satisfying the usual conditions. Let us consider a market consisting of $N+1$ assets labeled by $j = 0, 1, \ldots, N$, where the 0-th asset is the cash account utilized as a numéraire. Therefore, as explained in the introductory subsection 2.1, it suffices to model the price dynamics of the other assets $j = 1, \ldots, N$ expressed in terms of the 0-th asset. As vector valued semimartingale for the price process $S : [0, +\infty[\times\Omega \to \mathbf{R}^N$, we chose the multidimensional Itô-process given by

$$dS_t = S_t(\alpha_t dt + \sigma_t dW_t), \tag{81}$$

where

- $(W_t)_{t\in[0,+\infty[}$ is a standard P-Brownian motion in $\mathbf{R}^K$, for some $K \in \mathbf{N}$, and,
- $(\sigma_t)_{t\in[0,+\infty[}$, $(\alpha_t)_{t\in[0,+\infty[}$ are $\mathbf{R}^{N\times K}$-, and respectively, $\mathbf{R}^N$- valued locally bounded predictable stochastic processes.

The processes α and σ generalize drift and volatility of a multidimensional geometric Brownian motion. Therefore, we have modelled assets satisfying the zero liability assumptions like stocks, bonds and commmodities. The solution of the SDE (81) can be obtained by means of Itô's lemma and reads

$$S_t = S_0 \exp\left(\int_0^t \left(\alpha_u - \frac{1}{2}\operatorname{Tr}(\sigma_u^\dagger \sigma_u)\right) du + \int_0^t \sigma_u dW_u\right), \tag{82}$$

where integration and exponentiation are meant componentwise. To define the corresponding deflators to meet definition 6, we can just set

$$D := S. \tag{83}$$

In order to construct term structures representing future contracts on the assets, we pass by the definition of their short rates as in definition 15, assuming that they follow the multidimensional Itô-process

$$dr_t = a_t dt + b_t dW_t, \tag{84}$$

where W is multidimensional P-Brownian motion introduced above and $(b_t)_{t\in[0,+\infty[}$, $(a_t)_{t\in[0,+\infty[}$ are $\mathbf{R}^{N\times K}$-, and respectively, $\mathbf{R}^N$- valued locally bounded predictable stochastic processes, the drift and the instantaneous volatility of the multidimensional short rate. The solution of the SDE (84), again by Itô's lemma is

$$r_t = r_0 + \int_0^t \left(a_u - \frac{1}{2}\operatorname{Tr}(b_u^\dagger b_u)\right) du + \int_0^t b_u dW_u. \tag{85}$$

Term structures are defined via

$$P_{t,s} := \mathbb{E}_t\left[\exp\left(-\int_t^s r_u du\right)\right]. \tag{86}$$

At time t, the price of synthetic zero bonds delivering at time s one unit of the base asset j is

$$\bar{S}_t^j := S_t P_{t,s}^j. \tag{87}$$

This means that we have constructed N gauges

$$(D^j, P^j) \quad j = 1, \ldots, N, \tag{88}$$

satisfying definition 6. Moreover, if drifts α, a and volatilities σ, b satisfy appropriate regularity assumptions, then we have a weakly differentiable market model as in definition 24 with nominal space $X = \mathbf{R}^N$ and base manifold $M = \mathbf{R}^N \times [0, +\infty[$. The dynamics of asset prices, short rates and term structures read

$$\begin{aligned} S_t^x &= S_0^x \exp\left(\int_0^t \left(\alpha_u^x - \frac{1}{2}\operatorname{Tr}(\sigma_u^{x\dagger}\sigma_u^x)\right) du + \int_0^t \sigma_u^x dW_u\right) \\ r_t^x &= r_0^x + \int_0^t \left(a_u^x - \frac{1}{2}\operatorname{Tr}(b_u^{x\dagger} b_u^x)\right) du + \int_0^t b_u^x dW_u \\ P_{t,s}^x &= \mathbb{E}_t\left[\exp\left(-\int_t^s r_u^x du\right)\right], \end{aligned} \tag{89}$$

for any nominals $x \in \mathbf{R}^N$. Volatilities σ_u^x, b_u^x and drifts α_u^x, a_u^x have to be chosen as appropriate functions of x. The curvature of the market principal fibre bundle $\mathcal{B}$ can be computed with proposition 30:

$$\begin{aligned} R(x,t,g) &= gdt \wedge d_x \left(\mathcal{D} \log S_t^x + r_t^x\right) = \\ &= gdt \wedge d_x \left(\alpha_u^x - \frac{1}{2} \operatorname{Tr}(\sigma_u^{x\dagger} \sigma_u^x) + r_0^x + \right. \\ &\qquad \left. + \int_0^t \left(a_u^x - \frac{1}{2} \operatorname{Tr}(b_u^{x\dagger} b_u^x) \right) du + \sigma_t^x \frac{W_t}{2t} + \int_0^t b_u^x dW_u \right). \end{aligned} \tag{90}$$

Thereby, we have utilized

$$\mathcal{D} W_t = \frac{W_t}{2t}. \tag{91}$$

The zero curvature condition is equivalent to

$$\begin{aligned} &\alpha_u^x - \frac{1}{2} \operatorname{Tr}(\sigma_u^{x\dagger} \sigma_u^x) + r_0^x + \int_0^t \left(a_u^x - \frac{1}{2} \operatorname{Tr}(b_u^{x\dagger} b_u^x) \right) du = C_t^1 \\ &\sigma_t^x \frac{W_t}{2t} + \int_0^t b_u^x dW_u = C_t^2, \end{aligned} \tag{92}$$

where C^1 and C^2 are semimartingales which do not depend on x. Inserting equation (92) into the expression for the short rate in equation (89) allows us to compute the term structure as

$$\begin{aligned} P_{t,s}^x &= \mathbb{E}_t \left[\exp\left(- \int_t^s r_u^x du \right) \right] = \\ &= \mathbb{E}_t \left[\exp\left(\int_t^s \left(\alpha_u^x - \frac{1}{2} \operatorname{Tr}(\sigma_u^{x\dagger} \sigma_u^x) \right) du + \int_t^s \sigma_u^x dW_u \right) \exp\left(- \int_t^s (C_u^1 + C_u^2) du \right) \right] = \\ &= \mathbb{E}_t \left[\frac{S_s^x}{S_t^x} \frac{\beta_s}{\beta_t} \right], \end{aligned} \tag{93}$$

where we have introduced the positive semimartingale

$$\beta_t := \exp\left(- \int_0^t (C_u^1 + C_u^2) du \right). \tag{94}$$

Equation (93) can be rewritten as

$$\beta_t S_t^x P_{t,s}^x = \mathbb{E}_t \left[\beta_s S_s^x P_{s,s}^x \right], \tag{95}$$

meaning that for the price of the synthetic zero bond

$$\bar{S}_t^{x,T} := S_t^x P_{t,T}^x, \tag{96}$$

the process $\{\beta_t \bar{S}_t^{x,T}\}_{t\in[0,T[}$ is a P-martingale for all maturities $T \in [0, +\infty[$. Therefore, β is a pricing kernel and the no-free-lunch-with-vanishing-risk condition is satisfied. Conversely, from (NFLVR) one can infer the vanishing of the curvature. We have thus rediscovered theorem 26.

4 Market Model and Differential Topology

Till now we have not considered any concrete asset dynamics for the market model. In our differential geometric framework a market dynamics should be specified as an infinite dimensional stochastic process for deflators $D_t := [D_t^1, \ldots, D_t^N]^\dagger$ and term structures $P_{ts} := [P_{ts}^1, \ldots, P_{ts}^N]^\dagger$ or equivalently as $2N-$dimensional stochastic process for deflators and short rates $r_t := [r_t^1, \ldots, r_t^N]^\dagger$. Inspired by [Il01], we modify Ilinski's five principles characterizing the market dynamics to obtain

Definition 35 (Principles of Market Dynamics).

- ***(A1) Intrinsic Uncertainty:*** *Deflators, term structures and short rates are random variables:*

$$D = D(\omega), \quad P = P(\omega), \quad r = r(\omega), \tag{97}$$

where $\omega \in \Omega$ represents a state of nature and $(\Omega, \mathcal{A}, P)$ is a probability space.

- ***(A2) Causality :*** *Time dynamics of deflators, term structures and short rates depend on their history such that future events can be influenced by past events only. Formally, we assume the existence of a filtration $\{\mathcal{A}_t\}_{t\geq 0}$ of the σ-algebra $\mathcal{A}_\infty$ such that D_t, r_t and $\{P_{ts}\}_{s\geq t}$ are $\mathcal{A}_t$-measurable.*

- ***(A3) Gauge Invariance:*** *Assume that deflator-term-structure stochastic process for $(D, P) = \{(D_t, P_{ts})\}_{t\geq 0, s\geq t}$ satisfies the equation*

$$f(D, P) = 0 \tag{98}$$

almost surely. Thereby, f denotes a possibly stochastic function, that is $f = f(\omega, D, P)$. Then, for every invertible gauge transform π the equation

$$f(D^{\pi}, P^{\pi}) = 0 \tag{99}$$

must hold almost surely as well.

- ***(A4) Minimal Arbitrage:*** *The most likely configurations of the random connections among deflators and term structures (or short rates) are those minimizing the arbitrage for the market portfolio strategy for almost every state of the nature $\omega \in \Omega$.*
- ***(A5) Extension Consistency :*** *The theory has to contain stochastic finance theory.*

We will try to realize this program in the rest of this paper. We remark that our framework already fulfills principles $(A1)$ and $(A2)$. In section 5 we will obtain $(A3)$, $(A4)$ and $(A5)$.

4.1 Arbitrage Action as Differentiable Homotopic Invariant

We will investigate what happens to arbitrage properties of self-financing strategies in case of smooth deformations. We will assume for the remaining of this paper that the space of allocations X is connected.

Definition 36. *Two $\mathcal{C}^1$-self-financing strategies $\gamma_{1,2} : [0,1] \to X$ with the same start and end points (i.e. $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$)are said to be **$\mathcal{C}^1$-homotopic** if and only if one is a smooth deformation of the other. This means that there exists a $\mathcal{C}^1$-function $\Gamma : [0,1]^2 \to X$ such that $\gamma_1 = \Gamma(0, \cdot)$ and $\gamma_2 = \Gamma(1, \cdot)$. A $\mathcal{C}^1$-self-financing strategy is said to be **$\mathcal{C}^1$-contractible** if it is null-$\mathcal{C}^1$-homotopic that is $\mathcal{C}^1$-homotopic to a point. The relation **$\mathcal{C}^1$-homotopy** is an equivalence relation in the set of $\mathcal{C}^1$-self-financing strategies and its quotient space, that is the set of all equivalence classes, is called the first **$\mathcal{C}^1$-self-financing differentiable fundamental group** of the portfolio space and denoted by $\Pi_1(X, D)$. The market is said to be **simply connected** in the case of a trivial first fundamental group or equivalently if and only if every closed $\mathcal{C}^1$-self-financing strategy is contractible.*

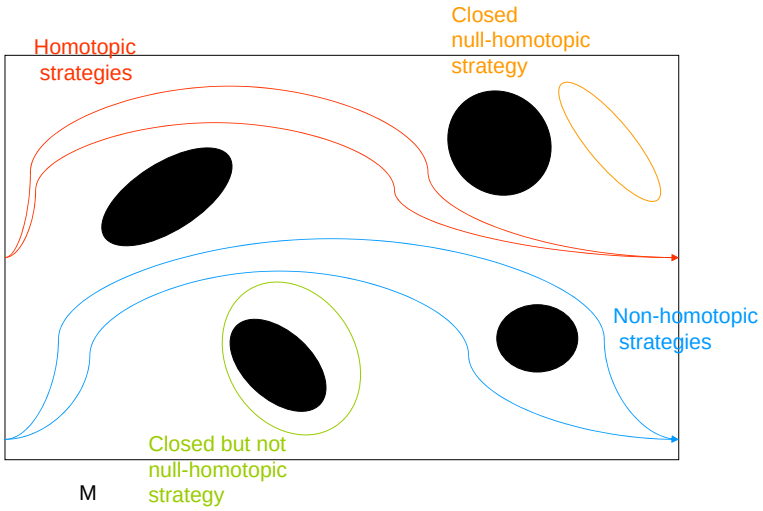


Figure 4: Differentiable Homotopy

Definition 37. *Let β be a positive semimartingale. The **arbitrage action** of the $\mathcal{C}^1$-strategy γ is defined as*

$$A^{\beta}(\gamma; D, r) := \int_{\gamma} ds \left\{ \mathcal{D}\rho^{\beta} + \mathrm{div}_x J \right\}. \tag{100}$$

Thereby $ds := |\mathcal{D}x(t)|dt$ denotes the arc length line element. By reparameterizing the strategy with respect to the arc-length parameter $\tau(t) := \int_0^t |\mathcal{D}x(u)|du$, then one obtains

$$|\mathcal{D}_{\tau} x(\tau)| \equiv 1, \tag{101}$$

and

$$A^{\beta}(\gamma; D, r) = \int_0^1 dt \left\{ \mathcal{D} \log(\beta_t D_t^{x_t}) + r_t^{x_t} \right\}. \tag{102}$$

In this subsection we will always assume that all $\mathcal{C}^1$-non-null-homotopic strategies are parameterized by arc-length parameter which we will continue to denote by t instead of τ.

Theorem 38 (**Arbitrage Action Formula for Self-financing Strategies**). *Let $\gamma : [0,1] \to X$ be a $\mathcal{C}^1$-self-financing strategy and $d_{01}^{\gamma} := \exp(-\int_0^1 r_u^{x_u} du)$ the stochastic discount factor. Then*

$$A^{\beta}(\gamma; D, r) = \log \left(\frac{\beta_1 D_1^{x_1}}{\beta_0 D_0^{x_0} d_{01}^{\gamma}} \right) \tag{103}$$

almost surely.

Proof. We rewrite the arbitrage action as

$$A^{\beta}(\gamma; D, r) = \int_{\gamma} ds \, \{\mathcal{D} \log(\beta_t D_t^{x_t}) + r_t^{x_t}\} = \int_{\gamma} ds \, \mathcal{D} \log(\beta_t D_t^{x_t}) + \int_{\gamma} ds \, r_t^{x_t} = \\ = \log\left(\frac{D_1^{x_1}\beta_1}{D_0^{x_0}\beta_0}\right) - \log(d_{01}^{\gamma}) = \log\left(\frac{\beta_1 D_1^{x_1}}{\beta_0 D_0^{x_0} d_{01}^{\gamma}}\right) \tag{104}$$

and the proof is completed. □

Lemma 39 (**Differentiable Homotopy Invariance Property of Stochastic Discount Factor for Self-financing Strategies**). *The stochastic discount factor satisfies $d_{01}^{\gamma} \equiv 1$ for any $\mathcal{C}^1$-contractible self-financing strategy γ. The stochastic discount factor is a $\mathcal{C}^1$-homotopy invariant, that is for any $\mathcal{C}^1$-homotopy class of $\mathcal{C}^1$-self-financing strategies $[\gamma] \in \Pi_1(X, D)$ the following statement holds a.s.*

$$\gamma_{1,2} \in [\gamma] \Rightarrow d_{01}^{\gamma_1} = d_{01}^{\gamma_2} = d_{01}^{\gamma}. \tag{105}$$

Proof. Let γ be a closed strategy. Then, for the stochastic discount factor we have:

$$-\log(d_{01}^{\gamma}) = \int_0^1 du \, r_u^{x_u} = \int_{\gamma} ds \, r_t^x = \int_{\gamma} ds \, \operatorname{div}_x J = \int_{\partial\gamma} dn \cdot J = 0, \tag{106}$$

since $\partial\gamma = \emptyset$. The lemma follows. □

Corollary 40 (**Differentiable Homotopy Invariance Property of Action for Self--financing Strategies**). *The arbitrage action $A^{\beta}(\gamma)$ vanishes for any $\mathcal{C}^1$-contractible self-financing strategy γ. The arbitrage action is a $\mathcal{C}^1$-homotopy invariant, that is for any $\mathcal{C}^1$-homotopy class of self-financing strategies $[\gamma] \in \Pi_1(X, D)$ the following statement holds a.s.*

$$\gamma_{1,2} \in [\gamma] \Rightarrow A^{\beta}(\gamma_1; D, r) = A^{\beta}(\gamma_2; D, r) = A^{\beta}(\gamma; D, r). \tag{107}$$

We can now give an alternative definition of arbitrage strategy end extend it to positive and negative arbitrage.

Definition 41. *A $\mathcal{C}^1$-self-financing strategy γ is said to be a **positive, zero, negative** β-**arbitrage strategy** if and only if the arbitrage action $A^{\beta}(\gamma)$ is a.s. positive, zero, negative,*

respectively.

If $\beta \equiv 1$, a zero arbitrage strategy is of course a no-arbitrage strategy in the usual sense. A strategy x is a negative β-arbitrage strategy if and only if $-x$ is a positive β-arbitrage strategy.

Corollary 42 (**Arbitrage Strategies**). *A $\mathcal{C}^1$-self-financing strategy γ is a positive, zero, negative β-arbitrage strategy if and only if*

$$\beta_1 D_1^{x_1}(\omega) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \beta_0 D_0^{x_0}(\omega) d_{01}^{\gamma}(\omega) \qquad a.s. \tag{108}$$

4.2 Parametrization of Strategies and Differential-Topological Version of the Fundamental Arbitrage Pricing Theorem

Now we will investigate the relationships between homotopy for self-financing strategies and holonomy of the connection for the principal fibre bundle describing the market model. Let us assume that we have fixed a numéraire by choosing an appropriate global cross section g^{Num} of the gauge bundle $\mathcal{B}$. From the Ambrose-Singer Theorem (see [St82], theorem VII.1.2.) we now that the Lie algebra $\mathfrak{hol}(\chi)$ of the holonomy group $\mathrm{Hol}(\chi)$ is spanned by the values of the curvature form. More exactly, assuming that both M and $\mathcal{B}$ are connected, for any $b \in \mathcal{B}$

$$\mathfrak{hol}(\chi) = \mathrm{Span}\left(\{R(c)(\eta,\xi) \mid \eta, \xi \in T_c\mathcal{B}, \text{ for an horizontal curve } \gamma \text{ joining } b \text{ to } c\}\right). \tag{109}$$

Theorem 43 (**Holonomic Parametrization of Self-financing Strategies**). *Let M and $\mathcal{B}$ be connected. The Lie algebra $\mathfrak{hol}(\chi)$ parameterizes all $\mathcal{C}^1$-homotopic self-financing strategies in the market model in the following sense: it exists a group isomorphism*

$$\phi : \mathfrak{hol}(\chi) \to \Pi_1(X, D) \tag{110}$$

mapping

- $0 \in \mathfrak{hol}(\chi)$ *to the equivalence class of no-free-lunch-with-vanishing-risk strategies which are null-$\mathcal{C}^1$-contractible.*

- *Non trivial elements of $\mathfrak{hol}(\chi)$ to different equivalence classes of $\mathcal{C}^1$-self-financing arbitrage strategies with the same start and end points.*

This theorem allows us to count the different $\mathcal{C}^1$-self-financing strategies which are equivalent in terms of arbitrage. In fact the Lie algebra $\mathfrak{hol}(\chi)$ is mapped iso- and diffeomeorphically to the holonomy Lie group $\mathrm{Hol}(\chi)$ by the exponential map. Therefore it follows

Corollary 44 (**Number of Different Equivalent Self-Financing Strategies**)**.** *The maps*

$$\mathrm{Hol}(\chi) \xrightarrow{Exp} \mathfrak{hol}(\chi) \xrightarrow{\phi} \Pi_1(X, D) \tag{111}$$

are group isomorphisms and manifold diffeomorphisms. In particular

$$|\Pi_1(X, D)| = |\mathrm{Hol}(\chi)| \,. \tag{112}$$

This corollary can be rephrased by saying that the foliation by holonomy leaves of the market principal fiber bundle $\mathcal{B}$ are in bijective correspondence with the equivalence classes of $\mathcal{C}^1$-self-financing arbitrage strategies with the same start and end points.

The results seen so far in this section corroborate the belief that there is a deep relationship between the market topology and the no-free-lunch-with-vanishing-risk condition. As a matter of fact we can extend the Fundamental Theorem of Asset Pricing to

Theorem 45 (**Differential-Topological FTAP**)**.** *The following assertions are equivalent:*

- *The market model satisfies the no-free-lunch-with-vanishing-risk condition.*
- *It exists at least one probability measure with respect to which market prices expressed in terms of a numéraire are martingales.*
- *It exists at least a state price deflator pricing the assets.*
- *The market curvature vanishes.*
- *The market local holonomy is trivial.*

- *There exists a positive semimartingale and every point in the market has a neighborhood such that the arbitrage action vanishes for all closed strategies lying in that neighborhood.*

which has a corollary in the case of complete markets

Corollary 46 (**Differential-Topological FTAP for Complete Markets**). *The following assertions are equivalent:*

- *The market model satisfies the no-free-lunch-with-vanishing-risk condition and is complete.*
- *It exists a unique probability measure with respect to which market prices expressed in terms of a numéraire are martingales.*
- *It exists a unique state price deflator pricing the assets.*
- *The market global holonomy is trivial.*
- *There exists a unique positive semimartingale such that the arbitrage action vanishes for all closed strategies.*
- *The market is simply connected.*

The interpretation of these results is that for a complete market an arbitrage possibility can only exist, if the market topology is not trivial. That is true if and only if there are restrictions in the nominal space acting as a topological obstruction, preventing *every* $\mathcal{C}^1$-self-financing closed strategy from being contractible.

Example 47 (**Pension Funds' Market**).
Let us consider a market whose agents are all pension funds in the world. The asset side of a pension fund is subject to several regulatory constraints. Beside the usual no short sales constraints, there are mixed linear constraints limiting the part of allocation in specific currencies, in regional markets, in the fixed-income or equity market and so on. These restrictions translate into hyperplanes cutting the nominal space X and the market M. A situation as in Figure 5 can occur, where the pension fund restrictions determine a "hole", that is, a non-trivial first fundamental group: the strategy α is contractible but strategy γ not.

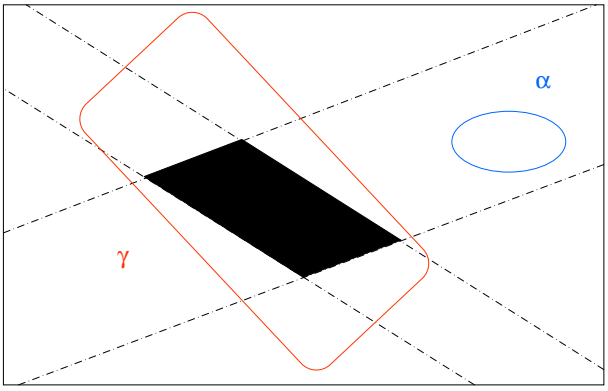


Figure 5: Pension Funds' Market

5 Lagrangian Theory of a Closed Market

5.1 Hamilton Principle and Lagrange Equations

We will now enforce the minimal arbitrage principle $(A4)$ for the market portfolio strategy to derive the market dynamics for deflators and short rates. There exists an interaction between market portfolio and market dynamics: the market portfolio allocation is determined by the choices of all market participants and influence therefore the asset values. These -on their part- will influence the choices of the market participants and therefore the market portfolio. Everything happens simultaneously in time. Summarizing:

$$\boxed{\begin{array}{c}\text{Market dynamics for deflators and short rates } (D,r)=(D_t,r_t)\\ \uparrow\downarrow \\ \text{Market Portfolio Strategy } x=x_t\end{array}} \tag{113}$$

If the market is closed, that is, if there is no external leverage, the market strategy must be self-financing. We denote by x_t the dynamics of the market portfolio and by (D_t, r_t) that of the deflators and short rates. By principle $(A4)$ these dynamics are characterized by the fact that they must be a. s. minimizer of the arbitrage action on the set of all self-financing strategies which are candidates for the market portfolio. Remark that we do not make any assumptions whether the market portfolio strategy allows for arbitrage or not, we just assume that it is

a minimizer. Since the arbitrage action is constant on every homotopic equivalence class, we have to factor out the equivalent self-financing strategies. After we have passed from curves to their homotopy classes, we set up the minimization problem. To perform formal calculation we first introduce the Hamilton-Lagrange formalism of classical mechanics and follow chapter 3 of the beautiful [Ar89]. We first treat the deterministic case. We embed the homotopy class of market portfolio strategy, deflators and short rate dynamics into one-parameter family homotopy classes, deflators and short rates. Then we pass to stochastic processes by means of the techniques introduced in [CrDa07].

Definition 48. *Let γ be the market $\mathcal{C}^1$-strategy, $[\gamma]$ its $\mathcal{C}^1$-homotopy class and $\delta[\gamma]$, δD, δr be perturbations of the market strategy $\mathcal{C}^1$-homotopy class, deflators' and short rates' dynamics. The **variation** of $([\gamma, D, r])$ with respect to the given perturbations is the following one parameter family:*

$$\epsilon \longmapsto ([\gamma]_\epsilon, D_\epsilon, r_\epsilon) := ([\gamma], D, r) + \epsilon(\delta[\gamma], \delta D, \delta r). \tag{114}$$

*Thereby, the parameter ϵ belongs to some open neighborhood of $0 \in \mathbf{R}$. The **arbitrage action** on the $\mathcal{C}^1$-homotopy class with respect to a positive semimartingale β can be consistently defined by the arbitrage action of a representant*

$$A^\beta([\gamma]; D, r) := A^\beta(\gamma; D, r), \tag{115}$$

and the first variation of the arbitrage action as

$$\delta A^\beta([\gamma]; D, r) := \frac{d}{d\epsilon} A^\beta([\gamma]_\epsilon; D_\epsilon, r_\epsilon) \mid_{\epsilon:=0} . \tag{116}$$

Definition 49. *Let us introduce the notation $q := (x, D, r)$ and $q' := (x', D', r')$ for two vectors in $\mathbf{R}^{3N}$. The **Lagrangian (or Lagrange function)** is defined as*

$$L(q, q') := L(x, D, r, x', D', r') := \left|x'\right| \frac{x \cdot (D' + rD)}{x \cdot D}. \tag{117}$$

Lemma 50. *The arbitrage action for an $\mathcal{C}^1$-homotopy class of a self-financing strategy γ is*

the integral of the Lagrange function along the $\mathcal{C}^1$-strategy:

$$A^\beta([\gamma]; D, r) = \int_\gamma dt\, L(q_t, q'_t) + \log\frac{\beta_1}{\beta_0} = \int_\gamma dt\, L(x_t, D_t, r_t, x'_t, D'_t, r'_t) + \log\frac{\beta_1}{\beta_0}. \tag{118}$$

A fundamental result of classical mechanics allows to compute the extrema of the arbitrage action in the *deterministic* case as the solution of a system of ordinary differential equations.

Theorem 51 (**Hamilton Principle**). *Let us denote the derivative with respect to time as $\frac{d}{dt} =: \prime$ and assume that all quantities observed are deterministic. The local extrema of the arbitrage action satisfy the Lagrange equations under the self-financing constraints*

$$\delta A^\beta([\gamma]; D, r) = 0 \text{ for all perturbations } \delta[\gamma], \delta D, \delta r \Leftrightarrow \begin{cases} \frac{d}{dt}\frac{\partial L}{\partial q'} - \frac{\partial L}{\partial q} = 0 \\ x'_t \cdot D_t = 0 \end{cases} \tag{119}$$

for all strategies in $[\gamma]$

5.2 Stochastic Lagrangian Systems

In this subsection we briefly summarize those contents of Cresson and Darses ([CrDa07]) needed in this paper. Cresson and Darses follow previous works of Yasue ([Ya81]) and Nelson ([Ne01]).

Definition 52. *Let $L = L(q, q')$ be the Lagrange function of a deterministic Lagrangian system. The dynamics is given by the Euler-Lagrange equations*

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial q'} - \frac{\partial L}{\partial q} = 0 \qquad (EL),} \tag{120}$$

meaning by this that the deterministic solution $q = q_t$ satisfies

$$\frac{d}{dt}\frac{\partial L}{\partial q'}\left(q_t, \frac{dq_t}{dt}\right) - \frac{\partial L}{\partial q}\left(q_t, \frac{dq_t}{dt}\right) = 0. \tag{121}$$

*The formal **stochastic embedding of the Euler-Lagrange equations** is obtained by the formal substitution*

$$S : \frac{d}{dt} \longmapsto \mathcal{D}, \tag{122}$$

and allowing the coordinates of the tangent bundle to be stochastic

$$\boxed{\mathcal{D}\frac{\partial L}{\partial q'} - \frac{\partial L}{\partial q} = 0 \qquad \textit{(SEL)},} \tag{123}$$

meaning by this that the stochastic solution $Q = Q_t$ satisfies

$$\mathcal{D}\frac{\partial L}{\partial q'}(Q_t, \mathcal{D}Q_t) - \frac{\partial L}{\partial q}(Q_t, \mathcal{D}Q_t) = 0. \tag{124}$$

Definition 53. *Let $L = L(q, q')$ be the Lagrange function of a deterministic Lagrangian system on a time interval I. Set*

$$\Xi := \left\{ Q \in \mathcal{C}^1(I) \mid \mathbb{E}\left[\int_I |L(Q_t, \mathcal{D}Q_t)|dt\right] < +\infty \right\}. \tag{125}$$

The action functional associated to L defined by

$$\boxed{\begin{array}{rl} F: & \Xi \longrightarrow \mathbf{R} \\ & Q \longmapsto \mathbb{E}\left[\int_I L(Q_t, \mathcal{D}Q_t)dt\right] \end{array}} \tag{126}$$

is called ***stochastic analogue of the classic action****.*

For a sufficiently smooth Lagrangian L a necessary and sufficient condition for a stochastic process to be a critical point of the action functional F_I is the fulfillment of the stochastic Euler-Lagrange equations (SEL): see theorem 7.1 page 54 in [CrDa07]. Moreover we have the following

Lemma 54 (Coherence). *The following diagram commutes*

$$\boxed{\begin{array}{ccc} L(q_t, q_t') & \xrightarrow{S} & L(Q_t, \mathcal{D}Q_t) \\ \Big\downarrow{\scriptstyle \textit{Critical Action Principle}} & & \Big\downarrow{\scriptstyle \textit{Stochastic Critical Action Principle}} \\ (EL) & \xrightarrow[S]{} & (SEL) \end{array}} \tag{127}$$

5.3 Arbitrage Dynamics For Deflators, Short Rates And Market Portfolio

By means of the stochastization procedure (see subsection 5.2) we can extend theorem 51 to the stochastic case:

Theorem 55 (**Stochastic Hamilton Principle**). *Let all quantities observed be stochastic and denote Nelson's stochastic derivative with respect to time as $\mathcal{D}$. The local extrema of the expected arbitrage action satisfy the Lagrange equations under the $\mathcal{C}^1$-self-financing constraints*

$$\delta\mathbb{E}_0[A^\beta([\gamma];D,r)]=0 \text{ for all perturbations } \delta[\gamma],\delta D,\delta r \Leftrightarrow \begin{cases} \mathcal{D}\frac{\partial L}{\partial q'}-\frac{\partial L}{\partial q}=0 \\ (\mathcal{D}x_t)\cdot D_t=[x_t,D_t] \\ \text{for all strategies in } [\gamma] \end{cases} \tag{128}$$

almost surely.

Before we tackle the problem of solving the stochastic Euler-Lagrange equations, we remark that they satisfy the gauge invariance principle ($A3$). As a matter of fact the Lagrange function definition is invariant with respect to a coordinate change in the tangent bundle $T\mathcal{B}$, that is we can write

$$L:T\mathcal{B}\to\mathbf{R}. \tag{129}$$

We will first solve the deterministic Euler-Lagrange equations and then construct a stochastic solution by adding appropriate perturbations with zero mean. More exactly, we write the stochastic optimal solution as the sum of the deterministic one and a zero mean perturbation $\delta x,\delta D,\delta r\in\mathcal{C}^1$ (see subsection 5.2) satisfying the conditions given by (131).

$$\begin{aligned} x_t&=\mathbb{E}_0[x_t]+\delta x_t \\ D_t&=\mathbb{E}_0[D_t]+\delta D_t \\ r_t&=\mathbb{E}_0[r_t]+\delta r_t, \end{aligned} \tag{130}$$

whereas

$$\begin{array}{llll} \mathbb{E}_0[\delta x_t] = 0, & \mathcal{D}\delta x_t = 0, & \delta x_t \cdot \delta D_t = 0, & \mathbb{E}_0[x_t] \cdot \delta r_t \delta D_t = 0 \\ \mathbb{E}_0[\delta D_t] = 0, & \mathcal{D}\delta D_t = 0, & \mathbb{E}_0[x_t] \cdot \delta D_t = 0, & (\mathbb{E}_0[r_t]\mathbb{E}_0[D_t]) \cdot \delta x_t = 0 \\ \mathbb{E}_0[\delta r_t] = 0, & \mathcal{D}\delta r_t = 0, & \mathbb{E}_0[D_t] \cdot \delta x_t = 0, & \delta x_t \cdot (\delta r_t \delta D_t) = 0 \\ [\delta x_t, \delta D_t] = 0. & & & \end{array} \tag{131}$$

We remark the Lagrange function satisfies $L(q, q') = L(\mathbb{E}_0[q], \mathbb{E}_0[q'])$ for any $q = (x, D, r)$ satisfying conditions (130) and(131).

Now we will compute for the arbitrage case the deterministic solution of the Lagrange equations under the self-financing constraints, which explicitly written out read

$$\begin{cases} D_t(x_t \cdot D_t') - x_t \cdot (D_t r_t + D_t')(x_t \cdot D_t' + x_t' \cdot D_t)x_t' + \\ + x_t \cdot D_t(-D_t' + (x_t \cdot (r_t D_t' + D_t r_t' + D_t'') + x_t' \cdot (D_t r_t + D_t'))x_t' + \\ + x_t \cdot (D_t r_t + D_t')x_t'' = 0 \\ (x_t \cdot D_t)x_t' - (x_t \cdot D_t + x_t \cdot D_t' + x_t' \cdot D_t)x_t = 0 \\ x_t' \cdot D_t = 0 \end{cases} \tag{132}$$

After a long and tedious calculation involving many substitutions and the solutions of a system of first order linear inhomogeneous ordinary differential equations with non-constant coefficients (see Appendix A), one obtains

Theorem 56 (**Arbitrage Market Dynamics**). *Let us assume that the initial condition for the market portfolio is not zero. Then, the orthogonal complement to* $\mathrm{Span}(x_0)$ *in* $\mathbf{R}^N$ *has an orthonormal basis* $v_1, \ldots, v_{N-1}$ *and the operator*

$$g(D, x_0) := \sum_{j=1}^{N-1} (D_0 \cdot v_j)v_j + \frac{D_0' \cdot x_0}{|x_0|^2} x_0 \tag{133}$$

is well defined. If $x_0 = 0$*, (that is in the quite strange case that all market participants at the initial time have no investments at all), we define the operator* g *as:*

$$g(D, 0) :\equiv D_0. \tag{134}$$

then the solution for the stochastic Euler-Lagrange equations satisfies a.s.

$$
\begin{aligned}
x_t &\equiv x_0 + \delta x_t \\
D_t &= e^{-t} g(x_0, D_0) + \delta D_t \\
r_t &= \exp\left(\int_1^t \frac{x_0 \cdot D_0' \left(3 - e^{-u} x_0 \cdot D_0'\right) + 2e^u}{x_0 \cdot D_0' \left(x_0 \cdot (e^{-u} g(x_0, D_0)) - 1\right)} \, du \right) \cdot \\
&\quad \cdot \exp\left(- \int_1^0 \frac{x_0 \cdot D_0' \left(3 - e^{-u} x_0 \cdot D_0'\right) + 2e^u}{x_0 \cdot D_0' \left(x_0 \cdot (e^{-u} g(x_0, D_0)) - 1\right)} \, du \right) x_0 \cdot (D_0 r_0) + \\
&\quad - \exp\left(\int_1^t \frac{x_0 \cdot D_0' \left(3 - e^{-u} x_0 \cdot D_0'\right) + 2e^u}{x_0 \cdot D_0' \left(x_0 \cdot (e^{-u} g(x_0, D_0)) - 1\right)} \, du \right) \cdot \\
&\quad \cdot \int_1^0 \exp\left(-v - \int_1^v \frac{x_0 \cdot D_0' \left(3 - e^{-u} x_0 \cdot D_0'\right) + 2e^u}{x_0 \cdot D_0' \left(x_0 \cdot (e^{-u} g(x_0, D_0)) - 1\right)} \, du \right) \cdot \\
&\quad \cdot \left(x_0 \cdot g(x_0, D_0) - \frac{x_0 \cdot D_0'}{x_0 \cdot (e^{-v} g(x_0, D_0)) - 1} \right) dv + \\
&\quad \exp\left(\int_1^t \frac{x_0 \cdot D_0' \left(3 - e^{-u} x_0 \cdot D_0'\right) + 2e^u}{x_0 \cdot D_0' \left(x_0 \cdot (e^{-u} g(x_0, D_0)) - 1\right)} \, du \right) + \\
&\quad + \int_1^t \exp\left(-v - \int_1^v \frac{x_0 \cdot D_0' \left(3 - e^{-u} x_0 \cdot D_0'\right) + 2e^u}{x_0 \cdot D_0' \left(x_0 \cdot (e^{-u} g(x_0, D_0)) - 1\right)} \, du \right) \cdot \\
&\quad \cdot \left(x_0 \cdot g(x_0, D_0) - \frac{x_0 \cdot D_0'}{x_0 \cdot (e^{-v} g(x_0, D_0)) - 1} \right) dv + \delta r_t,
\end{aligned}
\tag{135}
$$

where $\delta x, \delta D$ and δr are processes satisfying condition (131).

In particular, we observe that the market strategy minimizing the arbitrage is a dynamic allocation, with statistical expectation constant over time and the deflators decay in time with inverse exponential speed.

5.4 Symmetries, Conservation Laws and No-Arbitrage Dynamics

In this subsection we continue the study of the analogies between finance and classical mechanics with the Hamilton-Lagrange formalism and introduce the concept of symmetries of the asset model described by a Lagrangian on the principle fibre bundle. By chapter 4 of [Ar89] we will derive from symmetries conservation laws which hold true in the general case of an asset model allowing arbitrage or not. The short rates follow by the no-arbitrage principle by deriving the logarithm of the solution for deflators. Asset values and instantaneous total returns follow by formula (74).

Definition 57 (**Market Symmetry**). *A bundle map $h : \mathcal{B} \to \mathcal{B}$ is called a symmetry of the market described by $(\mathcal{B}, L)$ if and only if*

$$L(T_b h.B) = L(B) \quad \text{for all } B \in T_b\mathcal{B}\,, \text{ for all } b \in \mathcal{B}. \tag{136}$$

Example 58 (**Linear or Affine Symmetries**).
We represent term structures by their short rate, so that $b = (x, D, r)$ and $h = h(x, D, r)$. Since h in this example is an affine function, its tangential map is the linear part of h.

- ***Rotation:*** *$h(x, D, r) := (Sx, SD, r)$, where S is a orthogonal matrix, i.e. $S \in O(N)$.*
- ***Nominal Translation:*** *$h(x, D, r) := (x + s, D, r)$, where s is a vector, i.e. $s \in \mathbf{R}^N$.*
- ***Deflator Translation:*** *$h(x, D, r) := (x, D + s, r)$, where s is a vector, i.e. $s \in \mathbf{R}^N$.*
- ***Deflator Dilation:*** *$h(x, D, r) := (x, (1 + s)D, r)$, where s is a real number, i.e. $s \in \mathbf{R}$.*

These examples all fulfill the definition of market symmetry, as one can see from the Lagrange function

$$L(\underbrace{x, D, r}_{=q}, \underbrace{x', D', r'}_{=q'}) = \left|x'\right| \frac{x \cdot (D' + rD)}{x \cdot D}. \tag{137}$$

The connection between symmetries and conservation laws in classical mechanics can be restated for our market model.

Theorem 59 (**Noether**). *Assume that all quantities observed are deterministic. Let $\{h_\epsilon\}_\epsilon$ be a one-parameter group of market symmetries $h_\epsilon : \mathcal{B} \to \mathcal{B}$ for the market described by $(\mathcal{B}, L)$. Then, the dynamics of market the portfolio, deflators and short rates have a first integral $I : T\mathcal{B} \to \mathbf{R}$. This means that there is a function, which writes*

$$I(q, q') = \frac{\partial L}{\partial q'} \frac{dh_\epsilon(q)}{d\epsilon} \mid_{\epsilon=0}, \tag{138}$$

such that

$$\frac{d}{dt} I(q_t, q_t') = 0 \tag{139}$$

where $q = q_t$ is the solution of the deterministic Euler-Lagrange equations.

By means of the stochastization procedure as explained in subsection5.2, we can extend the preceding theorem to the stochastic case (see [CrDa07] page 60):

Theorem 60 (**Stochastic Version of Noether's Result**). *Let all quantities observed be stochastic and* $\{h_\epsilon\}_\epsilon$ *be a one-parameter group of market symmetries* $h_\epsilon : \mathcal{B} \to \mathcal{B}$ *for the market described by* $(\mathcal{B}, L)$*. Then, the dynamics of market the portfolio, deflators and short rates have a first integral* $I : T\mathcal{B} \to \mathbf{R}$*. This means that there is a function, which writes*

$$I(q, q') = \mathbb{E}_0 \left[\frac{\partial L}{\partial q'} \frac{dh_\epsilon(q)}{d\epsilon} \mid_{\epsilon=0} \right], \tag{140}$$

such that

$$\frac{d}{dt} I(Q_t, \mathcal{D}Q_t) = 0 \tag{141}$$

where $Q = Q_t$ *is the solution of the stochastic Euler-Lagrange equations.*

In classical mechanics Noether's theorem is applied to a N point particle system to derive the so called 10 conservation laws for energy, momentum, angular momentum, and center of mass of an isolated system, i.e. with no external forces acting on its particles. Now we will will derive conservation laws from the symmetries of a market with N assets and no external leverage. That for, we have to extend the symmetries of our example to one parameter groups.

Example 61 (**One Parameter Group of Linear or Affine Symmetries**).

- ***Rotations' Group:*** $h_\epsilon(x, D, r) := (S_\epsilon x, S_\epsilon D, r)$*, where* S_ϵ *is a one parameter group of orthogonal matrices, i.e.* $S_\epsilon \in O(N)$*, and* $S_0 = I_N$*, the identity matrix in* N *dimensions. Therefore,* $h_0(x, D, r) = (x, D, r)$*. The first integral is*

$$I = \mathbb{E}_0 \left[\frac{\partial L}{\partial q'} \Xi [x, D, 0]^\dagger \right], \tag{142}$$

  where $\Xi := \mathrm{diag}(\xi, \xi, I_{N+1})$ *is a* $3N \times 3N$ *matrix and* ξ *is a* $N \times N$ *antisymmetric matrix. By theorem (60) we obtain* $N + 1$ *non-trivial first integrals:*

$$\begin{cases} \mathbb{E}_0 \left[\xi \left(\frac{x_t \cdot (\mathcal{D}D_t + r_t D_t)}{x_t \cdot D_t} \frac{\mathcal{D}x_t}{|\mathcal{D}x_t|} \right) \right] \equiv \textit{ Time Constant} \\ \mathbb{E}_0 \left[\xi \left(\frac{x_t}{x_t \cdot D_t} \right) \right] \equiv \textit{ Time Constant.} \end{cases} \tag{143}$$

- ***Nominal Translations' Group:*** *$h_\epsilon(x, D, r) := (x + \epsilon s, D, r)$, where s is a vector in $\mathbf{R}^N$, so that $h_0(x, D, r) = (x, D, r)$. The first integral is*

$$I = \mathbb{E}_0 \left[\frac{\partial L}{\partial q'} [e, 0, 0]^\dagger \right], \tag{144}$$

where $e := [1, \ldots, 1] \in \mathbf{R}^{1 \times N}$. By theorem (60) we obtain one non-trivial first integral:

$$\mathbb{E}_0 \left[\frac{x_t \cdot (\mathcal{D}D_t + r_t D_t)}{x_t \cdot D_t} \frac{e \cdot \mathcal{D}x_t}{|\mathcal{D}x_t|} \right] \equiv \textit{ Time Constant.} \tag{145}$$

- ***Deflator Translations' Group:*** *$h_\epsilon(x, D, r) := (x, D + \epsilon s, r)$, where s is a vector in $\mathbf{R}^N$, so that $h_0(x, D, r) = (x, D, r)$. The first integral is*

$$I = \mathbb{E}_0 \left[\frac{\partial L}{\partial q'} [0, e, 0]^\dagger \right]. \tag{146}$$

By theorem (60) we obtain one non-trivial first integral:

$$\mathbb{E}_0 \left[\frac{e \cdot x_t}{x_t \cdot D_t} \right] \equiv \textit{ Time Constant.} \tag{147}$$

- ***Deflator Dilations' Group:*** *$h_\epsilon(x, D, r) := (x, (1 + \epsilon)D, r)$, so that $h_0(x, D, r) = (x, D, r)$. The first integral is*

$$I = \mathbb{E}_0 \left[\frac{\partial L}{\partial q'} [0, D, 0]^\dagger \right]. \tag{148}$$

By theorem (60) we obtain one trivial first integral, the 1-constant function:

$$\mathbb{E}_0 \left[\frac{x_t \cdot D_t}{x_t \cdot D_t} \right] \equiv 1. \tag{149}$$

We can summarize the results in the preceding example in the following

Corollary 62. *The dynamics of deflators and market portfolio in a closed market must satisfy*

the following system of equations:

$$\begin{cases} \frac{d}{dt}\mathbb{E}_0\left[\frac{x_t\cdot(\mathcal{D}D_t+r_tD_t)}{x_t\cdot D_t}\frac{\mathcal{D}x_t}{|\mathcal{D}x_t|}\right]=0 \\ \frac{d}{dt}\mathbb{E}_0\left[\frac{x_t}{x_t\cdot D_t}\right]=0 \\ \frac{d}{dt}\mathbb{E}_0\left[\frac{x_t\cdot(\mathcal{D}D_t+r_tD_t)}{x_t\cdot D_t}\frac{e\cdot\mathcal{D}x_t}{|\mathcal{D}x_t|}\right]=0 \\ \frac{d}{dt}\mathbb{E}_0\left[\frac{e\cdot x_t}{x_t\cdot D_t}\right]=0 \end{cases} \tag{150}$$

In the no-free-lunch-with-vanishing-risk case we have $\mathcal{D}D_t + r_tD_t \equiv 0$ *and the systems reduces to*

$$\begin{cases} \frac{d}{dt}\mathbb{E}_0\left[\frac{x_t}{x_t\cdot D_t}\right]=0 \\ \frac{d}{dt}\mathbb{E}_0\left[\frac{e\cdot x_t}{x_t\cdot D_t}\right]=0 \end{cases} \tag{151}$$

We already provided an arbitrage dynamics in theorem 56 and do not expect other additional insights from these equations in this case. Therefore, we will concentrate now on the arbitrage free closed market and solve this system of first order ordinary differential equations together with the self-financing condition:

$$\begin{cases} \frac{d}{dt}\mathbb{E}_0\left[\frac{x_t}{x_t\cdot D_t}\right]=0 \\ \frac{d}{dt}\mathbb{E}_0\left[\frac{e\cdot x_t}{x_t\cdot D_t}\right]=0 \\ (\mathcal{D}x_t)\cdot D_t=[x_t,D_t]. \end{cases} \tag{152}$$

The deterministic counterpart of system (152) reads

$$\begin{cases} \frac{d}{dt}\left(\frac{x_t}{x_t\cdot D_t}\right)=0 \\ \frac{d}{dt}\left(\frac{e\cdot x_t}{x_t\cdot D_t}\right)=0 \\ \frac{dx_t}{dt}\cdot D_t=0. \end{cases} \tag{153}$$

In order to solve the stochastic case (152) we write the stochastic optimal solution as the sum of the deterministic one, that is the solution of (153), and a zero mean perturbation $\delta x, \delta D \in \mathcal{C}^1$ (see subsection 5.2) satisfying the conditions given by (155).

$$\begin{aligned} x_t &= \mathbb{E}_0[x_t]+\delta x_t \\ D_t &= \mathbb{E}_0[D_t]+\delta D_t \end{aligned} \tag{154}$$

whereas

$$\begin{array}{llll} \mathbb{E}_0[\delta x_t] = 0, & \mathcal{D}\delta x_t = 0, & \delta x_t \cdot \delta D_t = 0, & \mathbb{E}_0[D_t] \cdot \delta x_t = 0 \\ \mathbb{E}_0[\delta D_t] = 0, & \mathcal{D}\delta D_t = 0, & \mathbb{E}_0[x_t] \cdot \delta D_t = 0, & (\mathbb{E}_0[x_t] \cdot \mathbb{E}_0[D_t])\delta x_t = 0 \\ \delta x_t \cdot e = 0, & [\delta x_t, \delta D_t] = 0. & & \end{array} \tag{155}$$

Theorem 63 (**No Arbitrage Market Dynamics**)**.** *In a closed market satisfying the no-free-lunch-with-vanishing-risk condition, the dynamics for market portfolio strategy and deflators have constant expectations over time. More exactly the following identity holds a.s.*

$$\begin{aligned} x_t &= x_0 + \delta x_t \\ D_t &= D_0 + \delta D_t, \end{aligned} \tag{156}$$

where $\delta x, \delta D$ are processes satisfying condition (155). Every portfolio has risk neutral expected constant value:

$$\mathbb{E}_0^*[D_t^x] \equiv D_0^x \quad \textit{for all portfolios } x \in \mathbf{R}^N. \tag{157}$$

Recall that the cash account has been made to the numéraire.

The proof of theorem (63) is postponed to Appendix B. Equation (157) is a standard result in mathematical finance, see f.i proposition 1.16 in [FoeSc04] for the discrete time case and equation (5.2.27) in [Sh00] for the continuous time case. In particular, we see that the theory developed satisfies the extension consistency axiom (A5) for it contains the already known stochastic finance result for the no arbitrage case.

6 Conclusion

By introducing an appropriate stochastic differential geometric formalism the classical theory of stochastic finance can be embedded into a conceptual framework called *Geometric Arbitrage Theory*, where the market is modelled with a principal fibre bundle, arbitrage corresponds to its curvature and arbitrage strategies to its holonomy. The Fundamental Theorem of Asset Pricing is given a differential homotopic characterization for both complete and not complete arbitrage free markets. The market dynamics is seen to be the solution of stochastic Euler-

Lagrange equations for a choice of the Lagrangian allowing to express Hamilton's principle of minimal action as the minimal expected arbitrage principle, an extension of the no-arbitrage principle. Explicit equilibrium and non-equilibrium solutions are provided for a closed market.

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A Proof of Theorem 56

Proof. The proof consists in applying theorem 55 to the solution of the system of ordinary differential equations (132), which can be extended to a solution of (128) by means of formula (130) under the restrictions (131). All we need to do now is to solve (132). After having inserted the third equation of the system into the first two we obtain:

$$\begin{cases} (x_t \cdot D_t')D_t - (x_t \cdot D_t)D_t' + \big[-(x_t \cdot (D_t r_t))(x_t \cdot D_t') - (x_t \cdot D_t')^2 + \\ +(x_t \cdot D_t)(x_t \cdot (r_t D_t')) + (x_t \cdot D_t)(x_t \cdot D_t r_t') + (x_t \cdot D_t)(x_t \cdot D_t'') + \\ +(x_t \cdot D_t)(x_t' \cdot (D_t r_t)) + (x_t \cdot D_t)(x_t' \cdot D_t')\big]\, x_t' + [(x_t \cdot D_t r_t) + (x_t \cdot D_t')]x_t'' = 0 \\ (x_t \cdot D_t)x_t' - [x_t \cdot D_t + x_t \cdot D_t']x_t = 0 \\ x_t' \cdot D_t = 0. \end{cases} \tag{158}$$

The second equation can be written as

$$x_t' = \left[1 + \frac{x_t \cdot D_t'}{x_t \cdot D_t}\right] x_t. \tag{159}$$

From equation (159) we can obtain expressions for x_t' and x_t' in terms of x_t, which, once inserted into the first equation of (158), lead to

$$\begin{cases} (x_t \cdot D_t)^2 (x_t \cdot D_t') D_t - (x_t \cdot D_t)^3 D_t' + \big[-(x_t \cdot (D_t r_t))(x_t \cdot D_t')^2 + \\ +(x_t \cdot (r_t D_t)(x_t \cdot D_t'') + 3(x_t \cdot (D_t r_t))(x_t \cdot D_t') + (x_t \cdot D_t')(x_t \cdot D_t'') + \\ +3(x_t \cdot D_t')^2 \big] x_t = 0 \\ x_t' = \left[1 + \frac{x_t \cdot D_t'}{x_t \cdot D_t} \right] x_t \\ x_t' \cdot D_t = 0. \end{cases} \tag{160}$$

After having introduced the function $y_t := \log(x_t \cdot D_t)$, equation (159) becomes

$$\frac{x_t'}{x_t} = 1 + y_t', \tag{161}$$

which, once integrated, leads to

$$x_t = x_0 \exp(t + y_t - y_0) \quad \text{or} \quad \frac{x_t}{x_t \cdot D_t} = \frac{x_0}{x_0 \cdot D_0} \exp(t), \tag{162}$$

and, after scalar multiplication with D_t, D_t', and D_t'',

$$\begin{aligned} D_t \cdot x_0 &= D_0 \cdot x_0 \exp(-t) \\ D_t' \cdot x_0 &= -(D_t \cdot x_0) \\ D_t'' \cdot x_0 &= D_t \cdot x_0. \end{aligned} \tag{163}$$

Now we insert (162) into the first equation of (160) to obtain

$$\begin{aligned} &- D_t - D_t' + \Big\{ f_t^2 \big[-(x_t \cdot D_t)^2 (x_0 \cdot D_t) - (x_t \cdot D_t)^2 (x_0 \cdot D_t)(x_0 \cdot (r_t D_t' + r_t' D_t)) + \\ &+ (x_t \cdot D_t)^2 (x_0 \cdot (D_t r_t) - 2(x_t \cdot D_t)(x_0 \cdot D_t)(x_0 \cdot (D_t r_t)) + 3(x_t \cdot D_t)(x_0 \cdot D_t)^2 \big] + \\ &- f_t^3 (x_t \cdot D_t)(x_0 \cdot (D_t r_t) + f(t)(x_t \cdot D_t)) \big[1 + r_t D_t' + r_t' D_t + D_t'' + D_t' + D_t r_t + \\ &\frac{x_0 \cdot (D_t r_t)}{x_0 \cdot D_t} + \frac{x_0 \cdot (D_t r_t - D_t)}{x_0 \cdot D_t} \big] \Big\} x_0 = 0, \end{aligned} \tag{164}$$

where we introduced the function

$$f_t := x_t \cdot (r_t D_t - D_t'). \tag{165}$$

With the short hand

$$z_t := x_t \cdot D_t, \tag{166}$$

equation, and after having inserted (163), equation (164) becomes

$$\begin{aligned} &\frac{D_t' + D_t}{f_t z_t} + \Big\{ \frac{x_0 \cdot ((r_t D_t)') + 3(r_t D_t) - D_t}{z_t} + f_t \left[-z_t(x_0 \cdot D_t) + \right. \\ &- z_t(x_0 \cdot D_t)(x_0 \cdot (r_t D_t' + r_t' D_t)) + z_t(x_0 \cdot (D_t r_t)) + \\ &\left. -(x_0 \cdot D_t)(x_0 \cdot D_t)(x_0 \cdot (D_t r_t)) + 3(x_0 \cdot D_t)^2 \right] + f_t^2 (x_0 \cdot (D_t r_t)) \Big\} x_0 = 0. \end{aligned} \tag{167}$$

Let now $v \in \mathrm{Span}(x_0)^\perp$. Scalar multiplication of equation 167 with v yields to

$$v \cdot D_t = v \cdot D_0 \exp(-t). \tag{168}$$

Because of (163), scalar multiplication of equation 167 with $x_0 \in \mathrm{Span}(x_0)$ leads to

$$\begin{aligned} &x_0 \cdot ((r_t D_t)') + 3(r_t D_t) - D_t + f_t z_t \left[-z_t(x_0 \cdot D_t) - z_t(x_0 \cdot D_t)(x_0 \cdot (r_t D_t' + r_t' D_t)) + \right. \\ &\left. + z_t(x_0 \cdot (D_t r_t)) - (x_0 \cdot D_t)(x_0 \cdot D_t)(x_0 \cdot (D_t r_t)) + 3(x_0 \cdot D_t)^2 \right] + f_t^2 (x_0 \cdot (D_t r_t)) = 0. \end{aligned} \tag{169}$$

With the substitutions

$$w_t := x_0 \cdot (r_t D_t) \quad \text{and} \quad m_t := x_0 \cdot D_t, \tag{170}$$

equation (169) becomes

$$(1 - f_t z_t m_t) w_t' = (-3 - f_t z_t^2 + f_t z_t m_t - f_t^2 z_t) w_t + (m_t + f_t z_t^2 m_t - 3 f_t z_t m_t^2), \tag{171}$$

which is a linear first order inhomogeneous ordinary differential equation in w_t with variable coefficients.

The first equation in (163) and equation (168) can be summarized as

$$D_t = g(D_0, x_0)\exp(-t), \tag{172}$$

where g is defined by (133) and (134). This closed expression for vector of deflators can be inserted into equation (171) to obtain

$$(1 - m_t)w_t' + \left(3 + \frac{2}{z_t} - z_t\right) w_t = m_t + z_t - m_t^2. \tag{173}$$

Now we substitute equation (162) into the third equation of (159) and obtain

$$z_t' + z_t = 0, \tag{174}$$

whose solution is $z_t = (x_0 \cdot D_0')\exp(-t)$. Therefore, after the substitution into equation 162, the market portfolio is found:

$$x_t \equiv x_0. \tag{175}$$

Being now z_t and m_t known, equation (173) can be solved, which allows us to find the following closed expression for the vector of short rates:

$$\begin{aligned}
r_t = {} & \exp\left(\int_1^t \frac{x_0 \cdot D_0'\,(3 - e^{-u}x_0 \cdot D_0') + 2e^u}{x_0 \cdot D_0'\,(x_0 \cdot (e^{-u}g(x_0, D_0)) - 1)}\,du\right) \cdot \\
& \cdot \exp\left(-\int_1^0 \frac{x_0 \cdot D_0'\,(3 - e^{-u}x_0 \cdot D_0') + 2e^u}{x_0 \cdot D_0'\,(x_0 \cdot (e^{-u}g(x_0, D_0)) - 1)}\,du\right) x_0 \cdot (D_0 r_0) + \\
& - \exp\left(\int_1^t \frac{x_0 \cdot D_0'\,(3 - e^{-u}x_0 \cdot D_0') + 2e^u}{x_0 \cdot D_0'\,(x_0 \cdot (e^{-u}g(x_0, D_0)) - 1)}\,du\right) \cdot \\
& \cdot \int_1^0 \exp\left(-v - \int_1^v \frac{x_0 \cdot D_0'\,(3 - e^{-u}x_0 \cdot D_0') + 2e^u}{x_0 \cdot D_0'\,(x_0 \cdot (e^{-u}g(x_0, D_0)) - 1)}\,du\right) \cdot \\
& \cdot \left(x_0 \cdot g(x_0, D_0) - \frac{x_0 \cdot D_0'}{x_0 \cdot (e^{-v}g(x_0, D_0)) - 1}\right) dv + \\
& \exp\left(\int_1^t \frac{x_0 \cdot D_0'\,(3 - e^{-u}x_0 \cdot D_0') + 2e^u}{x_0 \cdot D_0'\,(x_0 \cdot (e^{-u}g(x_0, D_0)) - 1)}\,du\right) + \\
& + \int_1^t \exp\left(-v - \int_1^v \frac{x_0 \cdot D_0'\,(3 - e^{-u}x_0 \cdot D_0') + 2e^u}{x_0 \cdot D_0'\,(x_0 \cdot (e^{-u}g(x_0, D_0)) - 1)}\,du\right) \cdot \\
& \cdot \left(x_0 \cdot g(x_0, D_0) - \frac{x_0 \cdot D_0'}{x_0 \cdot (e^{-v}g(x_0, D_0)) - 1}\right) dv
\end{aligned} \tag{176}$$

The proof is completed.

□

B Proof of Theorem 63

Proof. The proof reduces to the solution of the deterministic system (153). The solution of the stochastic system (152) is given by (154) under the restrictions (155). System (152) can be rewritten as

$$\begin{cases} x_t'(x_t \cdot D_t) - x_t(x_t' \cdot D_t + x_t \cdot D_t') = 0 \\ (x_t' \cdot e)(x_t \cdot D_t) - (x_t \cdot e)(x_t' \cdot D_t) = 0 \\ x_t' \cdot D_t = 0, \end{cases} \tag{177}$$

from which we infer

$$\begin{cases} x_t'(x_t \cdot D_t) = 0 \\ (x_t' \cdot e)(x_t \cdot D_t) = 0 \\ x_t' \cdot D_t = 0, \end{cases} \tag{178}$$

Therefore, for any initial condition x_0

$$\begin{cases} x_t \equiv x_0 \\ \frac{x_t}{x_t \cdot D_t} \equiv 0, \end{cases} \tag{179}$$

This means that $x_t \equiv x_0$ and $D_t \equiv D_0$ and the proof is completed.

□

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