Chapter 4
Semiotics in Mathematics Education: Topological Foundations and Diagrammatic Methods

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Abstract The question of mathematical pedagogy depends on the perceptual and intellectual capacities of teachers and students on the one hand and on the intrinsic demands for abstract understanding and rigorous formal proof on the other. The chapter sketches a semiotic sequence from metaphysics through category theory to topology to applied topology; and revisits the philosophies of Plato, Deleuze and others to elucidate the relevant mathematical problematics. While mathematics is intrinsically caught up in the dialectic of sense and idea, edusemiotics takes this distinctive feature of conceptual knowledge and learning into account. The use of diagrams as a semiotic tool is shown to be an essential component of any mathematics teaching and learning. An edusemiotic approach to processes of teaching and learning mathematics demonstrates that topological concepts of continuity and free variation support a diagrammatic framework for experimenting with and appropriating mathematical knowledge. This framework, consistent with the intuitive approach and formal notation of category theory, helps cultivate both 'upward' and 'downward' transits between abstract and concrete domains.

Introduction

In *Mathematics as Sign*, philosopher Brian Rotman (2000) links the semiotics of material inscription (writing), creative fiction (imagination) and iterated memory (counting) in order to develop an approach to mathematical practice that remains sensitive to both the concrete modes of mathematical reasoning employed by embodied human subjects and the uniquely abstract status of the objects that mathematics investigates. Rotman shows how formal and material technologies determine the historical modes of the 'mathematics-world interface' and how semiotic theory provides a unified and synthetic approach that does justice to both of these necessary sides of mathematical activity. To treat mathematics from a

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semiotic perspective as Rotman does is to take a relatively unorthodox point of view with respect to mainstream philosophy of mathematics, one pioneered in the early 20th century by Charles S. Peirce (2010). As the work by Peirce and Rotman demonstrates, conceiving of fundamental problems in philosophy of mathematics in semiotic terms has the distinct advantage of situating mathematical truth and cognition in a quite general space of human sign-usage common to multiple disciplines and types of practices, thus opening natural connections between mathematics and other fields of human endeavor: from science and engineering to art and religion. In particular, the semiotic approach to mathematics provides a way to think the continuity of substantive questions of mathematical truth on the one hand with inquiries into the social context of its propagation and use on the other.

The present chapter examines the issues that arise when considering mathematical teaching and learning in the broader context of edusemiotic pedagogy (Semetsky and Stables 2014). It begins by distinguishing between the strictly mathematical problem of logico-formal foundations and the pedagogical problem of cognitive foundations. The usage of diagrams as both an educative tool and a formal notation indicates certain lines of research for how these otherwise distinct problems might be bridged. In particular, from an edusemiotic point of view it is the phenomenologically rich topological features of diagrams that are most useful for encouraging an explorative and creative environment for learning and applying mathematics. However, this emphasis on topo-phenomenology raises the problem of the specifically abstract and non-phenomenological character of mathematical cognition. A sketch of a possible educational sequence passing from philosophical metaphysics through the abstract foundational domain of category theory to topology and applied topology shows how the use of diagrams may respond to this problem by enveloping in a unified manner both phenomenological and formal considerations in mathematical edusemiotics.

Pedagogical Foundations in Mathematics

We are concerned with the foundations of mathematical learning. By stressing the term *foundation* we are confronted with the homonymous term as used in the philosophical problem of the foundations of mathematics (Marquis 2009). In this more common usage, the ‘foundations’ of mathematics, if existing at all, are the ultimate terms of recourse for grounding mathematical truth in the formal structures of proof and for obtaining general constructions within which various distinct mathematical domains may be modeled. The two most important contenders for the status of foundations are set theory, based ultimately on discrete structures, and category theory, based on holistic systems of relations. This chapter uses edusemiotics as a conceptual framework, but as will emerge below, seemingly distant concerns with purely mathematical foundations are not completely separable from the ways in which mathematics is effectively learned, taught, and applied.

One of the core ideas of edusemiotics is that background metaphors of what teaching is and how it works enter in subtle ways into the concrete practices that actually implement teaching and learning and that revisions of these metaphors in favor of better ones may elicit improved understanding (Pesce 2014; see also Danesi's chapter in this handbook). A common educational metaphor is that of direct transmission. Teaching is considered, traditionally, as communicating a message. But of course a message is only transmitted if a common code is shared between sender and receiver. What if the core of a certain type of teaching is precisely to instill not just a code but a unique *type* of code that might be incommensurate with other semiotic codes previously understood by the student? In this case a better metaphor would be that of learning as a kind of apprenticeship akin to magnetism:

> When an external magnetic field is applied to a ferromagnetic substance, the item "absorbs" some of the external field: when the external field is removed, the magnet will maintain some of the structure that was imposed on it via the field – it becomes magnetized. The substance changes magnetic phase. This change in phase of a ferromagnetic material is sometimes referred to as memory, because it resembles the learning process with which each of us is familiar. Ferromagnets are like entities that learn, in that the changes wrought in them via the "teaching" episodes outlast the duration of the training process (Thalos 2013, p. 98).

Mathematical learning from this somewhat non-standard point of view would be best understood as an apprenticeship in abstract relations. In a given pedagogical context the corresponding problem of mathematical foundations would be less about establishing formal grounds for expressing theorems and proofs in terms of a single mathematical language than of establishing an experiential and cognitive baseline that could be assumed common to all the students involved. As classroom contexts vary, so too would the relevant 'baseline foundations' of pedagogical support. Context is considered a 'situational variable' in edusemiotics. This is evident in the case of different levels of instruction, as for the assumed differences in students' skills and competencies in different topics. Yet this kind of difference is also important for distinct student constituencies engaged in one and same mathematical topic. Introductory calculus courses at an inner-city high school, an elite liberal arts college and an adult-education community college evening course might require different pedagogical approaches based on what is assumed to be common knowledge even under the hypothesis that exactly the same material is being taught and the same assessment tools utilized in these courses.

One of the aspects of mathematics that makes it unusual, if not unique among core curricular disciplines, is that in principle the background knowledge required to grasp mathematical concepts and techniques extends no further than the most fundamental structures of experience. Such indifference to history, language, and culture is part of the source of mathematics' universality. That's why mathematics is uniquely positioned among human semiotic codes in terms of its ease of communication across different cultures. Nonetheless, in actual classroom practice vast differences emerge at all levels of learning and grasping new concepts. Mathematical truth is not meant to be dependent on anything other than its form,

but in teaching mathematics one must instill the very passage from the experiential richness of content to the purity of form.

This disjunction of content and form is not only a mathematical issue, but has been one of the core features of the Western metaphysical tradition at least since Plato. Importantly, it is precisely this feature of mathematics that originally suggested to Plato the alignment of mathematics with the universality of philosophical truth. Not only are mathematical truths eternal from a Platonist perspective: importantly, they serve as the ultimate model for all intelligible structure that may be found in the sensible world. Mathematics is thus not only the paradigm but also the transcendental condition of all truth. Its status is not only universal (true for everyone) but ubiquitous (true everywhere). It might seem then that in a decidedly (for the most part) post-Platonic epoch the grounding of truth in mathematics would have become a mere historical curiosity. Yet if anything the importance of mathematics in its purely formal power has only increased, and the basic Platonic problem of abstract form is all the more foregrounded in its contemporary technological employments. For instance, Rotman (2008) emphasizes the importance of new computer technologies in mathematical study and the rise of computational models, simulations and digital mathematics.

From our present perspective, the relevant cross-section of this rich and multidimensional problem is how it impacts mathematical edusemiotics. On the one hand, the mathematics teacher is faced with an embarrassment of riches. At least at introductory levels, easily intuited examples from a great variety of different domains are readily available, and arithmetic can be taught with any materials that may be counted. Euclidean geometry quite naturally arises from consideration of the basic shapes taught in early childhood: circle, square, triangle. Calculus lends itself to application in a wide variety of scientific disciplines as diverse as physics and economics, and elementary examples drawn from these disciplines are often useful ways to introduce new concepts and techniques.

On the other hand, mathematics instruction quickly moves beyond the realm of intuition, and examples may then come in short supply. It is perhaps for this reason that so many students seem to 'hit a wall' in their mathematical development, beyond which no amount of study enables them to move forward. The proper aim of mathematics pedagogy is a certain familiarity and flair for working with abstractions. In elementary and applied mathematics, it is primarily a matter of developing skills in the back and forth movement between abstract structures and concrete models. In 'pure' and typically more advanced mathematical topics, the space of mathematical abstraction is investigated in its own right. But the cognitive act of abstraction permeates mathematical study in a way that distinguishes it from all other areas of learning. So while the problem of finding a common baseline of shared knowledge in the mathematics classroom does not pose the same kinds of challenges that confront the teaching and learning of history or literature, it reappears in a different register as the problem of finding a common capacity, a skill at performing a certain type of cognitive act. One of the greatest challenges for mathematics pedagogy is to draw attention to this very act of abstraction, which is by definition non-representational. It is one thing to teach and to learn via examples;

it is something else entirely to teach and to learn how examples as such exemplify. This understanding involves a passage from the concrete to the conceptual, common to mathematics and philosophy.

Diagrammatics and Phenomenology

From an edusemiotic perspective, the specific difficulties in the teaching and learning of abstract objects may be expressed in terms of the variable degrees of iconicity of the signs used in communicating about such objects. In Peirce's familiar trichotomy of signs, an *icon* is "a sign of which the character that fits it to become a sign of the sort that it is, is simply inherent in it as a quality of it" (Peirce 1998, p. 306). An icon is a type of sign that signifies by virtue of what it is and the particular properties it has, as distinguished from *indices* which are signs related to their objects by means of causal connections—an index is "a sign which is fit to serve as such by virtue of being in a real reaction with its object"—while *symbols* signify by means of conventional rule or habitual regularity: a symbol is "a sign which is fit to serve as such simply because it will be so interpreted" (Peirce 1998, pp. 306–307). An icon expresses the very qualities that it signifies in its object, a typical example being any figurative image. Unlike the words 'NUAGE' (French) or 'CLOUD' (English), none of which looks anything like a cloud, a child's appropriately rounded doodle shares at least some relevant perceptual characteristics with actual clouds. For this reason, such a doodle is an icon, while the culturally specific linguistic signs—words as symbols—are not.

Iconic signs are essential tools for grasping abstract forms instantiated in experiential data. Central in this regard is the prominent usage of diagrams throughout mathematics pedagogy. In Peircean semiotics, a diagram is an iconic representation of relations that through manipulation and experimentation becomes capable of generating new knowledge concerning its represented object (Stjernfelt 2007; Gangle 2016). Thus a diagram is an epistemically oriented sign that builds relevant relational aspects of its semantics into its own syntactical form. Peirce (2010) insists that diagrammatic thinking is at the very heart of mathematical method and mathematical reasoning is in essence the discovery of necessary consequences entailed by arbitrary hypotheses made manifest through experimentation on diagrams. The intelligibility of Peirce's view depends upon the breadth he accords to the notion of diagrams. Diagrams are not only sketches drawn on paper or some other two-dimensional surface, but are instantiated systems of relations that share epistemically relevant structural or qualitative features with their intended object domain. As Peirce points out, even the most apparently non-iconic forms of mathematical notation such as syntactical strings representing algebraic equations still rely at some level of abstraction upon relational homologies between what they represent (the notational sign) and what is represented (the mathematical structure at issue).

One of the strengths of iconic diagrams is their pedagogical utility. Because they rely on relations of iconicity linking the sign to its object, the semiotic structure of a diagram includes *within itself* both the problem and the partial solution of how to communicate concerning abstract objects! The abstract and typically underdetermined system of relations that constitutes the object of a diagram is given in an evident and immediately realized way by the instantiated relations of the diagram *per se*. Yet the proper understanding and use of the diagram attends not to the particular qualities of the instantiated relations, but only to their general character as relevant to a given interpretation. For instance, the sketch of a parabola on a board certainly has a *particular* size and shape, but understood correctly it represents not *these* characteristics but rather the invariant symmetry and differentiable curvature *common to any and all* parabolas. As Stjernfelt (2007) has demonstrated, Peirce's conception of diagrammatic experimentation bears remarkable similarity to Husserl's phenomenological method of eidetic intuition via controlled imaginative variation and thus establishes a possible line of connection between semiotic and phenomenological approaches to cognition and learning.

The pedagogical utility of diagrams is not limited to attaining the abstract and general by way of the concrete and intuitible. Just as the abstract nature of mathematical objects allows them to range universally across disparate concrete domains, the relatively concrete character of the diagrammatic sign allows it to vary in its significance across multiple abstract spaces of interpretation. Importantly, the way that one and the same diagram may serve as an iconic sign in different ways in independent contexts is not simply a consequence of its abstract character. It is rather a function of the interpretative selection of which features are to be considered relevant and what they are chosen to represent in whatever context happens to be at issue. This selection is at once dependent upon the concrete character of the diagram and equally the shared space of the diagram together with its community of interpreters. The fundamental type of relation in play is thus not that between abstract generality and concrete particularity (types and tokens), but rather of *mappings* between concrete structures and possible models of those structures that vary in a compositionally ordered way from parts to whole across interpretative contexts (Caterina and Gangle 2015). In a context of a particular classroom, this selection is typically a matter of communal deliberation and agreement. Even the authoritative stipulation of a teacher (e.g., "All of these points are equidistant from this one here") depends upon a general context of agreement in which such stipulations can become meaningful. A more semiotically-aware teacher might instead ask: "Can we all agree that these points here may be understood to be equidistant from this one?"

As an embodied, sensually rich activity, mathematics is essentially semiotic and diagrammatic, a fact that is especially evident at the most elementary stages of learning and understanding. The basic cognitive practices at the root of mathematical thinking are typically engaged through the semiotics of counting. This approach privileges arithmetic as the 'natural' basis of mathematics from which more advanced topics might be built up in turn. From a strictly semiotic point of view, it is worth noting that the practice of counting is primarily visual. It requires

both distinguishing objects and keeping track of them. The closely related semiotics of rhythm—for instance in music and dance—is in general insufficient for developing arithmetical skills because the bodily dynamics of repetition here take precedence over the 'stack memory' required by addition and subtraction. Good drummers do not necessarily make good arithmeticians. They just keep counting the same beats over and over! When we count, we fix the discrete objects of a world.

It is helpful to compare standard pedagogical models with less familiar and thus less habitual ones, thus situating the standard models in a larger and more variable space of possibility. In this case, in contrast to counting it is useful to consider the semiotics of *covering* as a possible base model for conceiving mathematical learning. In contrast to the predominantly visual character of counting, the sensible correlate of covering is primarily tactile. To slide one's finger along a surface is to trace a continuous line by covering a singular path across it. This is not to say that there are not visual metaphors for covering. Indeed, a highly significant aspect of the visual field is that non-transparent objects typically 'cover' whatever is behind them. Consider the phenomena of shadows.

The abstract relations of counting correspond structurally to the mathematics of arithmetic and algebra. Algebraic relations link discrete objects of a mathematical domain operatively in order to generate other objects via these operations themselves. The abstract relations of covering, on the other hand, correspond more closely to topology. Topological spaces are naturally expressible in terms of mathematical 'coverings' of open sets by collections of other open sets (Hatcher 2001). From this perspective, the key characteristics of mathematical objects are their intersections and overlaps, their continuities and invariances under controlled deformations rather than their fixed identities.

The relations of covering and the corresponding topological mathematics, because of their natural connections to all of the aspects of human embodiment (not only visual representations of objects), seem to be an especially appropriate and underappreciated foundation for mathematical pedagogy. In this regard, the work of Sha Xin Wei is an important touchstone. Sha's explicit focus is not pedagogical, but it concerns especially the relationships between lived experience, which edusemiotics indeed posits as one of its important characteristics, and mathematical models. For Sha, mathematics is organized not inductively in terms of formal abstraction from the concrete details and multiplicities of ordinary experience, but rather abductively as the continually self-correcting motility of the embodied and active human subject in space (cf. Semetsky 2015). Sha's core insight is that whereas mathematics and the formal codings prevalent in computers and other technological media are structured discretely, the experiential matrix of space and time, sensual qualities and manipulated objects is essentially continuous:

> As I lift my hand to pick up a cup or to wave hello, it does not jerk discontinuously from point to point and moment to moment, at least not in ordinary experience. And it is ordinary experience in its boundless density with which I am concerned ethico-aesthetically, technically, and conceptually. Of course, for reasons of efficiency and economy, we have built logical and algebraic systems of discrete representation of various ontological strata of the

> world (phyletic taxonomies, the periodic table, grammar), and processes and institutions that discipline practices to these fixed representations. But the practical challenge I set is to see how to articulate continuous processes in our art and our technology, given how continuous, material process is pervasive in our lived experience (Sha 2013, p. 123).

In other words, Sha's program is to find modes of continuous representation that accord most naturally with the continuities of space, time and quality inherent to everyday experience. While Sha's concerns are primarily with technological interfaces and aesthetic environments disposed to creative exploration, these very concerns indicate the relevance of his work to the learning of mathematics. Again, the development of cognitive facility with abstraction and abstractions requires a semiotically rich, habit-forming training in a multitude of experiential passages from the material to the formal, from the sensual to the structural and back.

The learning of mathematics requires not only a phenomenologically appropriate environment for gaining intuitions of mathematical concepts. It also requires mastery of the conceptual leap beyond initial intuition and the intellectual grasp of abstract structures as dynamically determined in their own native environment. It is here that the edusemiotic needs of mathematics students (and all of us, even the most gifted professional mathematicians, remain students of an abstract universe of thought only the tiniest fragment of which has been explored) may be partially met by turning to the other abstract intellectual discipline: philosophy.

In *A Thousand Plateaus* Deleuze and Guattari (1987) complain of typical formalist linguistics that it is "not abstract enough" (p. 90). The point is not that in their view the formal characteristics of linguistic theory remain too dependent on sensual intuition or are insufficiently technical. It is rather that the specifically philosophical character of linguistics—its power to make of the regional science of language a mode of access to universal problematics of ontology and conceptual creation—resides not in its formalization but rather in its concrete affective and pragmatic character. Only an analysis in terms of the 'abstract machine' as a feature of language captures what is of greatest philosophical interest. Similarly, Deleuze's earlier work *Difference and Repetition* in its detailed conceptual study and philosophical generalization of differential calculus implies that mathematics in its own way fails to be sufficiently abstract (Deleuze 1994). Paradigmatic in this regard from an edusemiotic perspective are the field extension theorems proved by Galois, which Deleuze describes as a radical reversal in the problem-solution relation:

> The theory of problems is completely transformed and at last grounded, since we are no longer in the classic master-pupil situation where the pupil understands and follows a problem only to the extent that the master already knows the solution and provides the necessary adjunctions. For […] the group of an equation does not characterise at a given moment what we know about the roots, but the objectivity of what we do not know about them. Conversely, this non-knowledge is no longer a negative or an insufficiency but a rule or something *to be learnt* which corresponds to a fundamental dimension of the object (Deleuze 1994, p. 180).

Much of the motivation for Deleuze's overall philosophical project stems from a principled rejection of phenomenology—at least in its canonical Husserlian form—and

the requirement of a form of thought that would exceed its merely representational empirical dimension. What reappears in Deleuze from the present point of view is the undeniable force of the Platonic problematic within philosophy *at least with respect to mathematics*. Mathematical truth and the philosophical relevance of mathematics go beyond the quotidian application of mathematical form to experiential content. The long-standing dichotomies of form and content, abstract and concrete, immanence and transcendence, or idea and materiality that have governed philosophical conceptions of mathematical truth at least since Plato—to say nothing of auxiliary problems in metaphysics, ethics and other areas of philosophy—still remain in force in the contemporary philosophical context, and more importantly continue to privilege in one way or another the form-, abstraction-, and idea- poles of these dichotomies themselves, even if the understanding of these dichotomies has been radically transformed.

The upshot for the edusemiotics of mathematics is that encouraging semiotic training in the passage from the concrete and sensuous to the abstract and formal is a necessary but not sufficient condition for truly grasping the internal dynamics of mathematical truth. Equally important is the logically-dual passage from the abstract to the concrete, from the ideal to the material. It is in this movement, paradoxically, that a dichotomy of categories can be overcome because of establishing a semiotic 'double-sided' relation. If following the 'upward' arrow oriented towards pure mathematical structure must undoubtedly become a habitual cognitive act of students apprenticed to mathematics, so too must learning to trace in thought and practice the oppositely oriented 'downward' arrow from structures to models and from abstract forms to material realizations.

From Metaphysics to Applied Topology: An Edusemiotic Proposal

If the fundamental problematics in play are those of the dialectic of phenomenological sense and abstract mathematical truth as well as the coordination of intuitive models with axiomatic structures, it seems clear that any concern with the semiotics of mathematical pedagogy must engage these problematics first and foremost from the side and with the prejudicial bias—relatively—of phenomenological sense and illustrative intuition. Classrooms are concrete places and students of mathematics at all levels, but particularly at undergraduate levels, require concrete interactions with teachers, diagrams and exemplary cases in order to come to the cognitive facility with abstract structures defining mathematical understanding. Yet the genesis of more highly determined structures out of less determined ones as specializations of the latter, such as building concrete models of various systems of axioms, represents in an important sense the most natural order of mathematical truth as well as the proper form of cognitive understanding capable of grasping that truth. So the edusemiotics of mathematics appears to be confronted with a sort of deadlock or

double bind. The proper direction of mathematical understanding moves in principle from the abstract and general to the concrete and particular, yet the usual semiotic path of mathematical learning proceeds from the sensuously concrete to the merely formal. Iconicity must instead be cultivated in *both* directions. What follows is an outline of what teaching from the abstract to the concrete (specializing or concretizing) rather than from the concrete to the abstract (abstracting or generalizing) might actually mean.

Consider an introductory undergraduate college-level course in topology. Topology studies the structures of spaces that remain invariant under continuous deformations, such as the stretching and bending of a flexible material (but not its puncturing, gluing or tearing). It is for this reason that topology is often described colloquially as rubber sheet geometry. Of course, the mathematics of topology extends far beyond the obvious intuitions of rubber sheets. Historically, topology developed out of the study of the behavior of continuous functions in local neighborhoods of chosen points in the functions' domains. A residue of this historical trajectory remains active in the semiotics of its standard pedagogy. Topology is usually taught as a generalization of the behavior of continuous functions in local regions of a space, as if it naturally—and not only historically—grew out of the differential calculus.

However, as in many, if not most, mathematical fields, the modern formalism and axiomatic treatment of topology conceives of its subject matter independently of its historical origins and in advanced topics and cutting-edge research agendas is often focused on those aspects of topology that reach beyond the restrictions of spaces defined by point-sets; restrictions which in the contemporary context appear as artificial limits on the genuine scope of topological mathematics. This is part of why the question of mathematical foundations is important. Whatever the actual status of such foundations, such as how exactly mathematics may be grounded in set theory or category theory, it is clear at any rate that the historical development of mathematics has not typically been 'downward' in abstraction from more to less foundational disciplines but rather 'upward' from more specialized and concrete domains to more abstract and general ones. Pedagogy typically follows history in this regard. Is it possible to trace the opposite movement edusemiotically?

Consider the sequence presented in Fig. 4.1 on the top left. From 'top' to 'bottom' one passes from metaphysics to category theory and from there to topology and then finally to applied topology, represented by such diverse applications as robotics, experimental art environments, data analysis and economics. This represents a proposed sequence of study that would encourage students to grasp the downward movement from more to less abstract domains from the very beginning, thus proceeding in the opposite direction from the typical one of generalization and abstraction. The sequence begins with the abstract domain of metaphysics. What place is there for metaphysics in a mathematics classroom? It would be inappropriate and counterproductive to attempt to treat metaphysics in its full range and scope, as this would be to defer indefinitely the purpose that semiotically governs the learning process. Instead, it is a matter here of introducing

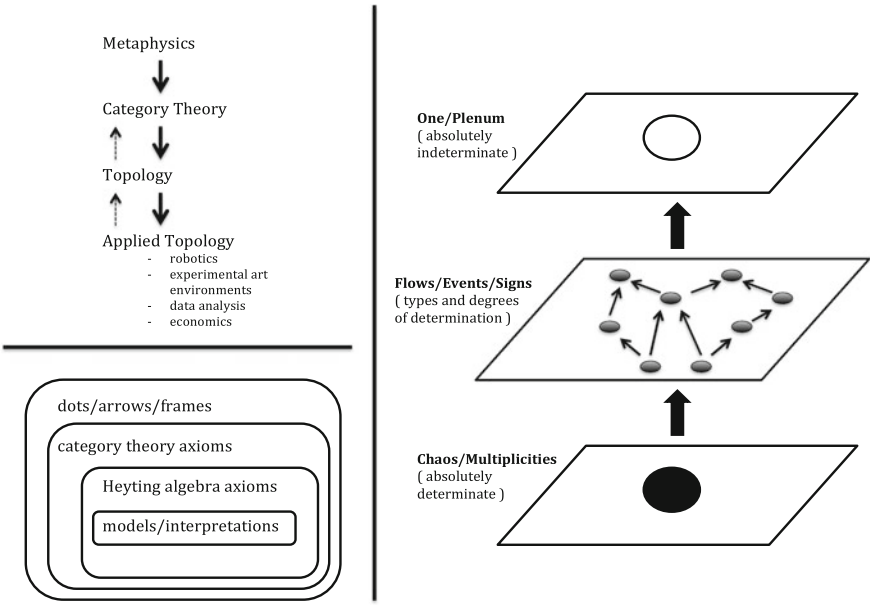


Fig. 4.1 Progressive determinations from metaphysics to applied topology

merely some elementary notions that might give rise to certain basic shared understandings in the classroom.

The diagram on the right in Fig. 4.1 portrays three levels of determination represented as distinct 'frames' or 'planes.' The plane at the bottom represents the absolutely determinate, the wholly fixed. For instance, common sense notions of reality often assume that there is an ultimate way that things are, defined for any particular instant of time. In the image of classical physics given by Newtonian mechanics, for example, every object in a system at a given time T has a wholly determinate position and momentum. Whether or not this view is in fact correct (it almost assuredly is not!), it is very useful to have in hand a shared diagram for representing and experimenting with this idea clearly and succinctly. In Fig. 4.1 it is pictured simply as a single dot, which iconically represents the non-variable structure of a determinate unity, for instance, a deterministic chaotic system. One level up, the middle plane is intended to represent systems of whatever type involving various degrees of determination, for example systems of flows, causes, events, signs and so forth. According to the conventions introduced and agreed upon by the teacher and students in the classroom, such systems may be pictured with diagrams of dots and arrows, where dots represent states or elements of whatever system is at issue and arrows represent determinations of one such state by another. Finally, the top plane represents the polar opposite of the bottom plane. Whereas the bottom plane pictures the absolutely determinate, the top plane portrays the absolutely indeterminate. Here, the 'same' diagram (but the real identity of

a diagram, as discussed above, includes the specific context of its interpretation and use) represents the pure metaphysical plenum, an absolute potentiality for determination that totally lacks any internal differentiation or articulate structure. For example, in Peirce's system of Existential Graphs the role of the uppermost plane is played by the blank Sheet of Assertion, the sheer determinability of a given universe of discourse being prior to any actual statement made within it.

In the context of a philosophy course and a philosophy classroom, the degree of metaphysical clarity, and specificity invoked here, would surely be insufficient. Just three ordered levels: absolutely determinate and absolutely indeterminate at the extremes, with the 'mixed state' of determination and indetermination situated between them. To most philosophers this probably looks like a pasteboard caricature of bad Platonic metaphysics! Yet it is precisely the pedagogical utility of such a diagrammatic caricature *for mathematical ends* that is here in question. Having established this shared 'notation' for sketching some intuitive notions of metaphysical determination—a process that would most likely take up a class period or two—it would then subsequently become possible to refine and specialize this notation for properly mathematical purposes. In particular, the intuitional pump will have been primed, so to speak, for understanding in a concrete diagrammatic way the otherwise highly abstract axiomatic basis for the foundational mathematics of category theory. In passing from metaphysics to category theory, one simply refines in this way what is meant by 'determination' by means of introducing *formal* rules governing the interaction and identity of the dots and arrows previously used in an *informal* fashion as the notation for metaphysics. In fact, the axioms of category theory are simple to formulate in this way (Lawvere and Schanuel 2009). They may be easily grasped by students of nearly any level of training—even children—with only a modicum of explanation. Of course the full ramifications of these axioms remain only implicit!

Yet having introduced the axioms of category theory in this way (and leaving aside any further development of category theory as such), it then becomes relatively straightforward to further refine these rules and requirements so that they then become the elementary axioms of topology understood as lattices of open sets subject to the basic restrictions that characterize complete Heyting algebras. To be sure, this step takes a bit more pedagogical work. Lattices must first be categorically defined via meets and joins understood as limits, and the intuitive notions of inclusion and implication must be formalized accordingly. But note that these mathematical concepts have already been 'prepared' by the far more informal diagrams of metaphysical determination and the less numerous and more intuitive dot-and-arrow axioms of categories. The extreme generality and abstraction of category theory is thus smoothly transformed into the more specific, yet still abstract, domain of topological spaces. From here, the more traditional back and forth movement between the formal axiomatic approach with its concomitant theorems and constructions on the one hand and the various regional applications of these latter in statistics, robotics, communications-systems design, etc., on the other becomes a repetition of an already established pattern of successive levels of abstraction and determination. The relation between pure and applied topology thus

appears not as an absolute dualism between formal structures and material instances, but instead as itself just one case of an iterated sequence of relative 'abstract-concrete' bipolar structures (sign-relations) that each determine the various levels or stages of a much richer and more differentiated domain that envelops ordinary experience, mathematical structures and philosophical concepts.

The primary advantage of this approach is that by the time students are ready to engage the relationship between the more abstract and more concrete aspects of topology, they are already thoroughly familiar in a diagrammatically, cognitively and habitually embodied way with the practice of transit from one space of diagrammatic and conceptual determination to another, the reversible passage back and forth from the relatively abstract to the relatively concrete. What appears in the sequence from metaphysics to applied topology is thus the reciprocity of two 'pathways' in mathematics learning: an upward movement of abstraction and a correlative downward path of concretion or realization. Thus the mathematical notation of category theory functions in this schema in a twofold role, both informal and formal. Dots and arrows are highly iconic modes of representation that very intuitively and naturally depict systems of objects and relations relatively stripped of their irrelevant properties and reduced to their selected structure. But they are also the basic elements out of which the formal diagrammatic notation of category theory may be built up.

Sha considers category theory as an ambient mathematical environment for modeling the topological dynamics of phenomenological experience, but he rejects it as being "a *description*, not a mode of articulation of material" and as saying "nothing about the dynamics of physical, or living, affective material" (Sha 2013, p. 112). But it seems that this point of view does not consider the possibility that the standard diagrammatic notation for category theory might *also* serve as an informal phenomenologically-intuitive method for tracking degrees of determination at multiple levels. It is here that the formal and the material at once intersect and remain conceptually distinct as a genuine paradox pertinent to edusemiotics.

Conclusion

The results of the preceding analysis may be summed up in the following sequence of claims:

(1) An essentially Platonic philosophical problem remains in force throughout nearly all aspects of mathematical pedagogy. Mathematical ideas are intrinsically caught up in the dialectic of sense and idea, and mathematical edusemiotics indeed takes this distinctive feature of conceptual knowledge and learning into account.
(2) The use of diagrams as a semiotic tool is an essential component of any mathematics teaching and learning.

(3) For both semiotic and phenomenological reasons, the topological features of diagrams as correlates of experiential continuities are likely to be especially effective in cultivating mathematical understanding.
(4) Ironically enough, the very pedagogical effectiveness that makes the use of diagrams and topo-phenomenological interfaces so important for mathematical edusemiotics risks obscuring the specifically abstract character of mathematical objects and the modes of knowing that correspond to them.
(5) The 'two paths' of mathematical edusemiotics—'upward' from concrete to abstract and 'downward' from abstract to concrete—are both necessary for genuine mathematical comprehension.
(6) The mathematics of category theory appears nicely situated for addressing the pedagogical challenges of both paths in an intuitively diagrammatic yet nonetheless potentially rigorous formal way.

In conclusion, then, it should be clear that the edusemiotics of mathematics is not wholly exterior to the subject matter and fundamental questions of mathematics itself. Mathematics is not a mere instrument—of whatever degree of complexity—to be applied mechanically to human interests and worldly ends. Mathematics is, like every serious intellectual endeavor, a project of simultaneously personal, political and cosmic-universal significance. The ultimate aim of the edusemiotics of mathematics is, or ought to be, to clarify how these dimensions or strata of mathematical experience exfoliate from the primary data of human ordinariness to encompass broader domains of social and natural reality and finally the most fundamental aspects of the physical cosmos as well as the possibly somewhat more extensive domain of the purely intelligible (cf. Semetsky 2013). Part of what we humans always do, what we must do and yet what we must always still learn to do is to negotiate the interface—at its heart a semiotic one—between the concrete and the abstract, the sensible and the intelligible. The conclusion to be drawn from the above example of topology may seem a counterintuitive one: it can be better from a strictly pedagogical perspective to begin at least in some cases from more abstract structures and to conceive of mathematics primarily in terms of the successive specializations of structures to increasingly differentiated, that is, individuated domains. Plato may have been right after all about the incommensurable duality of sense and form. Yet the fundamental problem in this respect, from the point of view of mathematical edusemiotics, may be not so much looking for any escape from the materiality of the Cave but rather of strategically cultivating formal methodologies of projection, materialization, and descent.

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