Iconicity and abduction: a categorical approach to creative hypothesis-formation in Peirce's existential graphs

GIANLUCA CATERINA* and ROCCO GANGLE†, *Department of Mathematics and Department of Humanities, Endicott College, 376 Hale Street, Beverly, MA 01915, USA*

Abstract

In this work, the iconic nature of abductive inference in Peirce's Existential Graphs (EG) is recast into the framework of Heyting algebras. The creative structure of abduction emerges then in a topological fashion as we produce a generalized version of Peirce's EG using a categorical approach, drawing from the tools of intuitionist logic.

Keywords: Peirce, abduction, Existential Graphs, category theory, Heyting algebras, Badiou, iconicity.

> The hypothesis of the mathematician is always the conception of a system of relations. In order that they may be reasoned about mathematically, these relations must be conceived as embodied in some kind of objects; but the character of the objects, apart from the relations, is utterly immaterial. They are always made as bare, skeleton-like, or diagrammatic as possible. With mathematicians not born blind, they are always visual objects of the simplest kind, such as dots, or lines, or letters, and the like. The mathematician often passes from one mode of embodiment to another. Such a change is no change in the hypothesis but only in the diagrammatic embodiment of the hypothesis. The hypothesis itself consists in the system of relations alone.
>
> —C.S. PEIRCE, 'ON THE LOGIC OF QUANTITY' (1895)

1 Introduction

In his well known early essay 'Some Consequences of Four Incapacities,' C.S. Peirce critiques the dominant Cartesian assumptions of modern philosophy and lays out the key principles for the alternative that Peirce himself proposes, above all the principle of an intrinsic relation between a semiotic conception of thinking and a pragmatic notion of truth as the formation and testing of hypotheses. This link between signs (triadic relations) and hypotheses (abductive inferences) will continue to prove crucial throughout Peirce's later oeuvre. One main point Peirce makes with respect to Descartes concerns the Cartesian method of doubt. In opposition to Descartes's radical conception of systematic skepticism which is meant to clear the way for an unshakeable ground of absolute certainty, Peirce [12] maintains that '[w]e cannot begin with complete doubt. We must begin with all the prejudices which we actually have when we enter upon the study of philosophy. These prejudices are not to be dispelled by a maxim, for they are things which it does not occur to us can be questioned' (p. 40). In contrast to Descartes, Peirce insists that doubt and certainty, as well as all other modes of thinking, remain essentially situated within a definite context. No one thinks purely

*E-mail: gcaterin@endicott.edu
†E-mail: rgangle@endicott.edu

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'in general'. Thought is always a specific response to a more or less definite problem that arises in a determinate context. Thus, Peirce advises, 'Let us not pretend to doubt in philosophy what we do not doubt in our hearts' (ibid.).

Peirce's notion of determinately situated inquiry has profound implications for the study and theorization of abductive inference. Abduction is the process whereby in the course of inquiry, in the wake of some unexplained phenomenon or surprising event, it becomes necessary to produce a hypothesis (one as good as possible) that would account for the phenomenon in question. One of the most interesting and controversial questions surrounding abduction continues to be whether or to what extent this process of abductive inference may itself be formalized. This question has been posed very explicitly by several authors and in different contexts: Aliseda [1] and Walton [17] focus on the logical formalism of abduction, whereas Magnani [10] emphasizes the role of abduction in cognitive science and its interdisciplinary connections. It is now generally accepted that while abduction can be modelled within specific frameworks, there is no single adequate model of abductive inference. More importantly, it appears there can be none, since the creative aspect of abduction remains intrinsically open to unforeseeable possibilities.

One of the reasons for the deadlock in the construction of formal models of abductive inference has been the unquestioned assumption of a clear distinction between syntax and semantics in formal logic. This distinction is so fully ingrained in contemporary logic that it would seem almost to pass beyond questioning. Yet so long as syntax and semantics are held to be independent, the work of formal logicians on abduction often repeats the Cartesian error that Peirce was at pains to criticize. A formal syntax is meant to abstract away from any determinate content, i.e. any particular semantics, much as Cartesian doubt is meant to generalize skepticism across all possible beliefs. As soon the problem of creative hypothesis-formation is posed, one must range across all possible hypotheses in a similar fashion. The problem, to be sure, is that in such cases there is no well-defined, pre-statable set of *all* beliefs or *all* hypotheses (real possibility exceeds and outstrips actuality and representability in principle). The construction of a hypothesis thus at times requires a creative act, a 'guess' that is not merely the selection of one among a set of pre-given candidates. Any problem for which an abductive inference is truly necessary is necessarily an actual and thus determinately particular problem. There is no problem of abduction in general; there are only particular problems that emerge within specific trains of concrete inquiry. Such concreteness precludes the radical bracketing of semantics.

If the formalization of abductive inference is to be capable of tracking the situatedness of inquiry, it will be advantageous for the above reasons to work within a framework in which syntax and semantics are not held entirely distinct. How might such a framework be structured? A logical formalism could be developed that would be at once a syntax and a semantics for that syntax, in other words, that would possess the semiotic character of *iconicity* (see [16]).

The line of thought just sketched may suggest at least part of the motivation for Peirce's own development of an iconic logical formalism in his system of Existential Graphs (EG). This three-tier system (alpha, beta and gamma) corresponding to propositional, first-order predicative, and higher-order and modal logics, respectively, was considered by Peirce himself to be his greatest achievement and continues to generate scholarly interest and study today. Our concern in what follows is to draw out the consequences of positioning the problem of the formalization of abduction within the context of iconic logical notation and its concomitant blurring of the distinction between syntax and semantics. In particular, we will focus on Peirce's EG-alpha and the problem of abductive amplification within the binarism of propositional logic, in effect the introduction of an intermediate truth-value into a binary system. We will first show how the iconic nature of the representation of logical relations in EG-alpha allows for the distinguishing of part-whole Gestalts that correspond to shapes of possible inference. Different levels of visual inspection of the graphs correspond to finer and

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coarser scales of logical analysis. Then in a second step, we will show how these shapes themselves may be ordered naturally through the mathematics of partial orders. The category-theoretic treatment of partial orders allows these iconic relations themselves to be represented iconically. This suggests a broad and highly general research platform for the investigation of abductive inference, one that is capable of integrating previous formal work into a unified context and opening up unforeseen possibilities through its iconic nature.

2 Peirce's EG-alpha

Despite their status as the chef d'oeuvre of Peirce's philosophy in the estimation of Peirce himself, the EG have received only intermittent attention even among philosophers otherwise well-versed in Peirce's mathematical logic and semiotics, not to mention the wider scholarly community. Notable contemporary exceptions are the logician Shin [15], the mathematician and knot theorist Kauffmann [7] and Zalamea [19], all of whom judge Peirce's graphs to possess relevance and importance that go far beyond the narrow concerns of Peirce specialists. Recent work by Brady and Trimble [5] has recontextualized the graphs within the mathematical framework of category theory. The present work proceeds broadly in tandem with this research program. For the sake of clarity, we will restrict ourselves to the first level of the graphs, EG-alpha, which corresponds to propositional logic. The insights gained at this level, however, translate readily to both EG-beta and EG-gamma.

Our primary concern is to show the usefulness of a category-theoretical framework for investigating the iconic and abductive characteristics of the EG. Our main idea is to use the natural ordering of deductions from one graph to another to induce a second-order level of iconicity that can be represented by lattices and a fortiori by Category Theory. The powerful category-theoretical tool of functors (structure-preserving maps) provides a rigorous mathematical framework for representing such iconic relations at the second level. In particular, we will show that the structures emerging at the second level of iconicity provide surprising links to topology and open up connections between EG-alpha and intuitionistic logic in a new way.

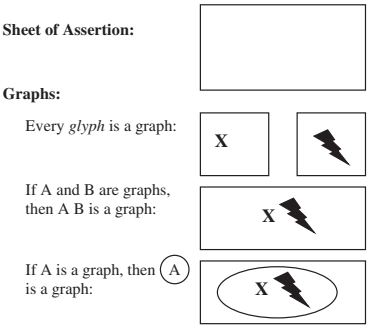
With this approach in view, we now turn to the basic elements of Peirce's EG-alpha and present a schematic introduction for those unfamiliar with the graphs. Besides the rigorous mathematical presentation in [5], detailed but more accessible treatments may be found in Roberts [14] and Shin [15].

The objects with which we will be dealing are graphs (in Peirce's sense). To emphasize their iconic qualities, we introduce a new concept, that of *glyph*, in order to define the class of such graphs in a somewhat more general way than Peirce. This will help to make the particular way they blend syntax and semantics more explicit.

The class of graphs may be characterized through the following recursive definition (readers should note both the similarities and the differences between this approach and the more standard definitions of wffs in formal systems):

1. The blank sheet (called the Sheet of Assertion or SA) is both the site on which graphs are *scribed* and is itself a graph (called the empty graph).
2. A *glyph* is any reproducible image (an iconic symbol) scribed on part of the SA.
3. Glyphs may be enclosed, along with a local area surrounding them (a neighbourhood on the SA) that may or may not include other glyphs, by a closed curve called a *sep*. It is convenient to construct these generally as ovals or circles. These seps may not intersect glyphs, nor may they intersect one another. They may, however, be nested with any number of glyphs and seps scribed in the areas or enclosures they distinguish.

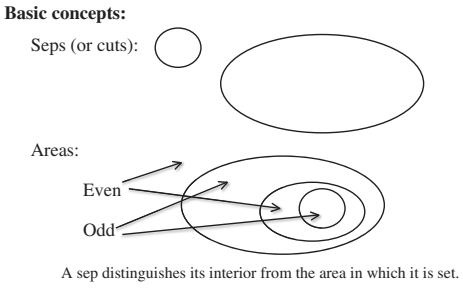
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Note that we diverge here from the standard treatments, in particular from Brady and Trimble, in allowing glyphs themselves to possess intrinsic structure (i.e. they are iconic visual figures rather than mere formal indices or labels). When glyphs are reduced to labelled points, they lose any iconicity.

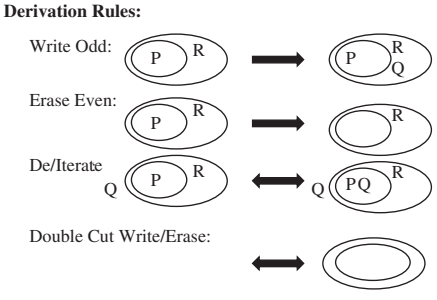
As characterized in this way, the graphs already possess a rich, natural structure even prior to any semantic interpretation. This structure is precisely what gives them their iconicity. If we abstract from the inscriptions of the glyphs and attend only to the adjacency and nesting of the cuts, the structure that is manifest has been systematically studied by Spencer Brown as an iconic arithmetic and algebra of considerable range.

Let us first notice that the graphs separate regions in the sheet of assertion in even and odd enclosed regions, as the following picture shows.



A sep distinguishes its interior from the area in which it is set.

We may collect in the set **G** all the graphs that may be constructed according to the recursive rules just given. This set, although it has no internal structure of its own—merely a collection of elements—possesses latent structure that will serve as the basis for Peirce's transformation rules. A full treatment of these rules may be found in [14]. As a reminder to the reader, examples of all of the rules are provided in an iconic way below, where P, Q and R represent (and are) glyphs:



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The correspondence between EG-alpha and propositional calculus (with seps interpreted as the *negation* of their contents and the inscription of multiple graphs on the same area as their logical *conjunction*) have been exhaustively investigated by Zeman [21] and more recently in a much broader context by Brady and Trimble [5]. The iconic quality of Peirce's calculus, however, should not be reduced to the formalism of logic, since it does possess, as shown by several authors, including the knot theorist Kauffmann [7], a truly topological nature. The derivation rules themselves may be characterized in a combinatorial-topological nature, in a such way that the dual nature of a graph as equally syntactical and semantical emerges naturally. The following result provides one example of this, showing how the derivation rules may be applied locally (in a topological sense) in such a way that local transformations can be embedded in larger structures.

THEOREM 2.1
Let G' be a graph which can be derived from a graph G. If the graph G is inscribed in an evenly enclosed area of another graph $\hat{G}$, then the graph $\hat{G}'$, obtained by replacing, in $\hat{G}$, G with G', is derivable from $\hat{G}$.

PROOF. If G' is derivable from G, then there is a sequence of graphs

$$G = G_0, G_1, \ldots, G_n = G',$$

where G_i is obtained via a single application of one of the above rules from G_{i-1}. Each element of any such sequence can be lifted to a correspondent element in the sequence of graphs

$$\hat{G} = \hat{G}_0, \hat{G}_1, \ldots, \hat{G}_n = \hat{G}',$$

where by $\hat{G}_i$ we denote the graph $\hat{G}$ with G in it replaced by the correspondent G_i

For each step in the sequence, say from the i-th to the $(i+1)$-th either the *parity* (i.e. the evenness or oddness of the enclosures of a graph) matters or it does not in determining if $\hat{G}_{i+1}$ can be derived from $\hat{G}_i$ in agreement with the transformation rules. If it does not (De/Iterate, Double Cut), then $\hat{G}_{i+1}$ can be trivially derived from $\hat{G}_i$. If it does matter (Write Odd, Erase Even), then again $\hat{G}_{i+1}$ can be derived from $\hat{G}_i$. For otherwise we should conclude that G was enclosed in an oddly enclosed area (because only then would the parity of the relevant enclosures in $\hat{G}_i$ be different from their correspondent enclosures in G_i) which contradicts our assumption. ∎

This theorem offers one instance of how certain logical properties of the graphs (syntactical relations) may be expressed as topological transformations (dynamical semantics). More generally, the topological syntax embedded in the transformation rules makes EG-alpha formally equivalent to classical propositional logic when glyphs are taken to represent distinct propositions [21]. In the next section we aim at embedding the graphs in an even more general and formally rigorous framework in which any such intrinsic relation between syntax and semantics in the graphs may be modelled and investigated.

3 EG-alpha as a category

The derivation rules depend upon a variety of relational structures among the elements of the set **G**. These relations can be formalized by considering the elements of the unstructured set **G** as the objects of a category $\mathcal{G}$. A category is a simple, highly general mathematical structure, consisting of only two components: objects and relations (arrows between objects). An elementary, yet rigorous

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introduction to Category Theory can be found in [8], while we refer to [9] for a more comprehensive treatment.

We define categorial objects and arrows in the category $\mathcal{G}$ as follows:

- *Objects*: well-defined graphs (elements of G)
- *Arrows*: given any two objects (i.e. two graphs) g_1 and g_2, an arrow exists $g_1 \rightarrow g_2$ iff g_2 may be derived from g_1 via finite application of the transformation rules given above.

It is an easy matter to check that $\mathcal{G}$ satisfies the axioms that define a category, namely

1. *Composition*: Given two arrows $f : G \rightarrow G'$, $g : G' \rightarrow G''$ there always exists a unique arrow $g \cdot f : G \rightarrow G''$ (sequences of applications of the transformation rules can be composed);
2. *Identity*: For any graph G there always exists a unique arrow $1_G : G \rightarrow G$ such that $1_G \cdot f = f = f \cdot 1_G$ for any arrow f (every graph is derivable from itself by applying no transformation rule at all);
3. *Associativity*: Given three arrows $f : G \rightarrow G'$, $g : G' \rightarrow G''$, $h : G'' \rightarrow G'''$, then the composition $(h \cdot (g \cdot f))$ is the same as $((h \cdot g) \cdot f)$ (sequences of applications of the transformation rules are associative).

Details of these axioms can be found in Mac Lane [9] and elsewhere. Because of the way we have defined our category $\mathcal{G}$, it is clear that there is at most one arrow from any graph g_1 to any other graph g_2. Even if there are multiple ways to derive g_2 from g_1 through different sequences of application of the transformation rules, this does not imply multiple arrows from g_1 to g_2. Like the binary structure of the graphs themselves (every glyph of a graph is either inside or outside any given area, *tertium non datur*), either an arrow exists from g_1 to g_2 or not. This structure in turn implies that $\mathcal{G}$ is a *pre-order*.

We now use various category-theoretical tools to investigate the internal structure of $\mathcal{G}$. The most important tool we employ is that of a functor. A *functor* is a 'relation-preserving' map between two categories. More precisely, given two categories $\mathcal{C}$ and $\mathcal{D}$ a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a function which takes any object $C \in \mathcal{C}$ to the object $F(C) \in \mathcal{D}$ and takes any arrow f between two objects $C, C' \in \mathcal{C}$ into an arrow $F(f) : F(C) \rightarrow F(C')$. We also require that:

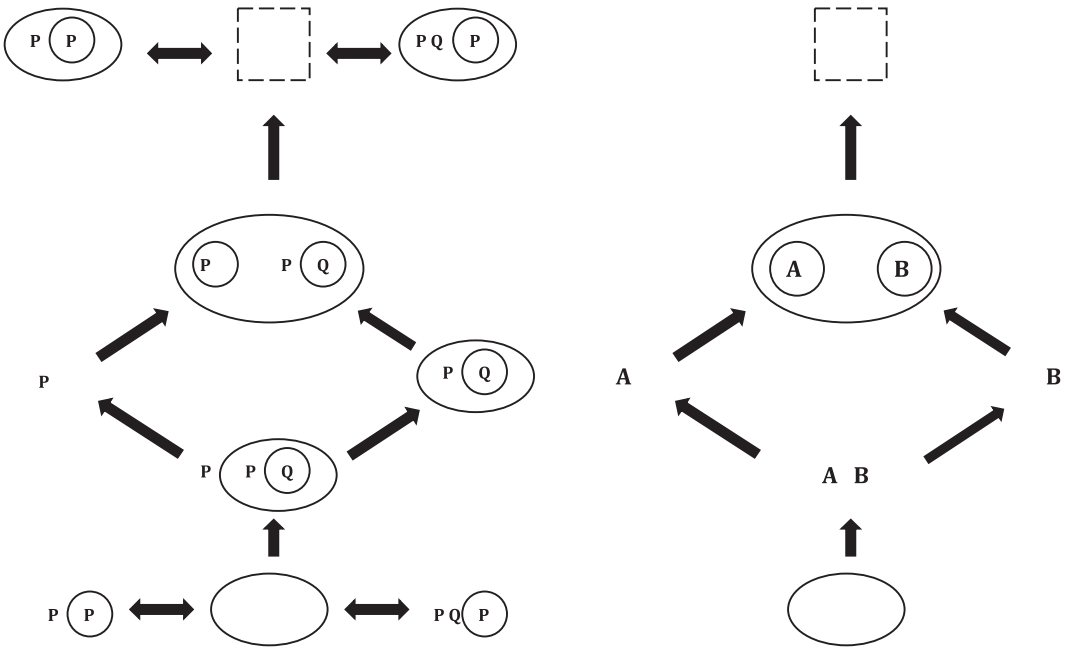
1. Composition of arrows is preserved;
2. Identity maps are mapped to identity maps ($F(1_C) = 1_{F(C)}$) for any object $C \in \mathcal{C}$.

We will be interested in both endofunctors and exofunctors of $\mathcal{G}$.[1] An important instance of the former may be given in the following way: we first define an equivalence class on G, by declaring two graphs equivalent if they can be homeomorphically transformed into each other. This is just to say that the only relevant structural features of the graphs are those concerning the combinatorial relations between enclosures and glyphs. For a more formal treatment, see [5]. We then associate each class with a single graph that is an element of that class, using the axiom of choice. This association gives rise to a natural endofunctor $f : \mathcal{G} \rightarrow \mathcal{G}$ that 'shrinks' $\mathcal{G}$ to an equivalent subcategory of itself in which every class of graphs that are 'the same' are represented by a single graph. We may in a similar fashion define another class of functors that reduce $\mathcal{G}$ from a pre-order to a *partial order* (thus collapsing syntactically equivalent graphs).

[1] We introduce these terms according to the following straightforward definitions: an *endofunctor* of a category C is a functor $C \rightarrow C$. An *exofunctor* of C is any functor $X \rightarrow C$ or $C \rightarrow X$ where C is not equal to X.

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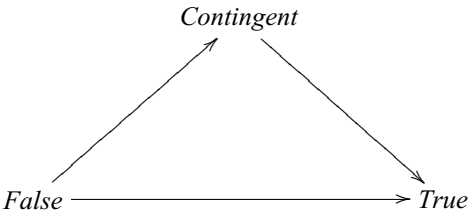
Such endofunctors correspond to various scales of viewing the graphs, various Gestalts of semantics and syntax. The functorial nature of the transformation ensures that the logical relations of derivation are always preserved. The following pair of images displays the basic idea.



The lattice-structure of $\mathcal{G}$ exhibits various interesting properties, many of which may be defined by exofunctors into and out of $\mathcal{G}$ that correspond to determinate structural aspects of EG-alpha. For instance, a derivation of g_2 from g_1 corresponds to a functor $2 \rightarrow \mathcal{G}$, and any sequence of derivations corresponds to a functor $N \rightarrow \mathcal{G}$. Most powerfully, perhaps, it provides an iconic intuition of what could be called the flow of truth involved in any such natural deduction system. As the diagrams above show, once what is called the *skeleton category* of $\mathcal{G}$ has been constructed, the minimal element of the lattice (in category-theoretical terms the *initial object*) may be identified with what Peirce calls the 'pseudo-graph,' namely the graph with only an empty cut on the sheet of assertion which represents the value 'False'. By the same token, the maximal element of the lattice (the *terminal object* in the category) represents the value 'True' and may be identified with the empty sheet of assertion itself. All intermediate values G such that $False \rightarrow G \rightarrow True$, where $G \neq False$, $G \neq True$, represent graphs that are true under certain conditions, namely those conditions which ensure that what they assert is in fact true. In this way, the three most general 'roles' in the lattice—minimum value, intermediate value, maximum value—correspond to the modalities of impossibility, contingency and necessity, respectively (as concerns the satisfiability of the graphs at issue). Indeed, to represent this in an iconic fashion, there is a natural class of functors $X : \mathcal{G} \rightarrow \mathcal{G}$ that takes the empty sheet of assertion (and all graphs isomorphic to it) to itself, takes the pseudo-graph (and all graphs isomorphic to it) to itself and all intermediate graphs to some single chosen graph that is neither the empty sheet nor the pseudo-graph. The three-valued collapses of $\mathcal{G}$ that result from these functors are all isomorphic to the category 3, the 'triangle' category that has the same structure—the same iconic 'shape'—as

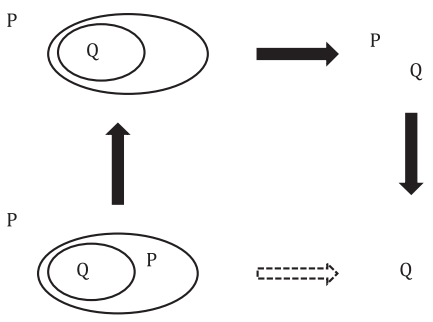
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Peirce's philosophical conception of the triadic sign.

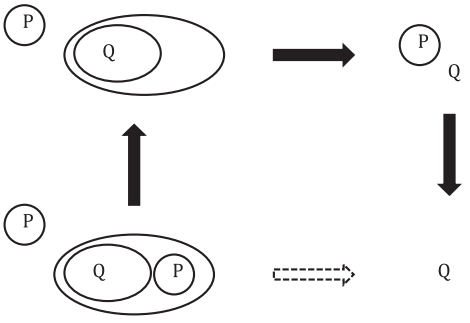


Such syntax-preserving maps within $\mathcal{G}$ may be constructed so as to identify and compare visual or iconic Gestalts within the graphs. For instance, consider a representation of the sequence of transformations in EG-alpha that corresponds to the logical rule of *modus ponens* as compared to those corresponding to *disjunctive syllogism* (in each case, the sequence: deiterate, erase double cut, erase even):

Example—Modus Ponens:



Example—Disjunctive Syllogism:



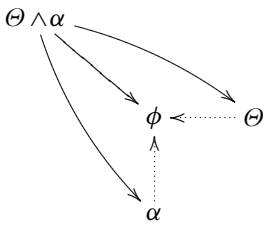
The usefulness of considering the system of EG-alpha within the framework of category theory consists in the fact that the iconic relations that enable one graph's 'looking like' another graph (roughly, showing the same visual Gestalt) to correspond to actual common logical properties they share are representable by structures within the category $\mathcal{G}$ that may be made manifest through purely categorical constructions, such as the functors indicated above. These categorical relations themselves may be shown to possess further, sometimes surprising, iconic properties, especially when they are represented by intuitive tree-like diagrams. What this means is that the diagrammatic expression of the second-order iconicity that emerges naturally in $\mathcal{G}$ is able to process both *inherent* and *imputed* relations in the graphs, iconic semantics and algebraic syntax. Peirce summarizes these complementary kinds of diagrammatic relations as follows:

> The diagrams in which the hypotheses are embodied are of two kinds. In the one kind the parts of the diagram are seen in the visual image to have the relations supposed. In the other kind of diagrams, the parts have shapes to which conventions or rules are attached, by means of which the supposed relations are attributed, or imputed, to the parts of the diagrams. Geometrical figures are diagrams of the inherential kind, while algebraical formulae are diagrams of the imputations kind ("On the Logic of Quantity" p. 46–47 in [13]).

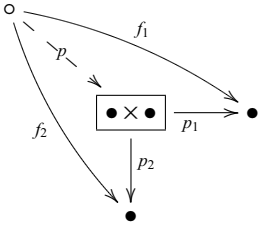
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4 Adjointness and formalizable abduction in EG-alpha

Abduction, in the framework of classical propositional logic, has been investigated thoroughly by Aliseda [1] and others. In that context, one straightforward way to model abduction is by taking a set of propositions Θ (the underlying theory), a 'surprising' proposition ϕ that must be accounted for, and then aiming to identify an optimal proposition (or set of propositions) α, such that when α is adjoined to Θ then ϕ can be deduced from $\Theta \wedge \alpha$. It is also reasonable to request that ϕ cannot be deduced directly from either Θ or α alone. This simple model admits of many interesting variations and restrictions (see again [1] for details). In what follows we show how a schematic representation of this 'standard model' of abduction fits in the category-theoretical framework and discuss how the notion of adjoint functor is able to capture a fundamental duality intrinsic in abductive inference. Let us begin by giving a diagram which illustrates the standard model introduced above (here the absence of an arrow between two objects is indicated by a dotted arrow):



In this way, abduction may be recast in terms of a determinate diagrammatic shape within our category $\mathcal{G}$, a shape in which Θ and ϕ are given and candidates for α may vary across all possible objects which 'fit' into the shape in the way displayed. The optimality property mentioned above (what makes α the *best explanation* for ϕ) then corresponds to one of the most fundamental and useful concepts in category theory, namely the *universal mapping property* (UMP). A clear illustration of a UMP can be given by considering a different and more general construction, the *product* of two objects in a given category:



This diagram iconically represents the following: given a fixed shape of arrows (in this case a 'fork' of two arrows coming out of any single object—represented here by the open circle—and going into two fixed objects) we look for the optimal candidate, i.e. the special object that is capable of representing all such objects, which it does by having the property of all such candidates relating to it in a unique way. In other words, there is exactly one arrow from each of the possible candidates into this optimal one. We represent the unicity of such a relation here with a dashed arrow.[2]

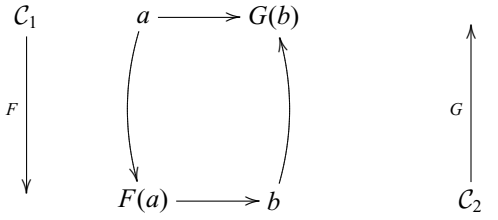
[2]Unicity is needed when dealing with categories more general than pre-orders, in which multiple arrows between objects are allowed to exist.

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The existence of objects with UMPs may, in certain cases, be represented solely through functors. A functor can be seen as a generalization of the concept of function: functors not only take objects to objects, but their action also preserves relations between objects in the translation from the 'domain' category to the 'codomain' category. This movement opens up a rich universe of relation that cannot be seen at the level of functions.

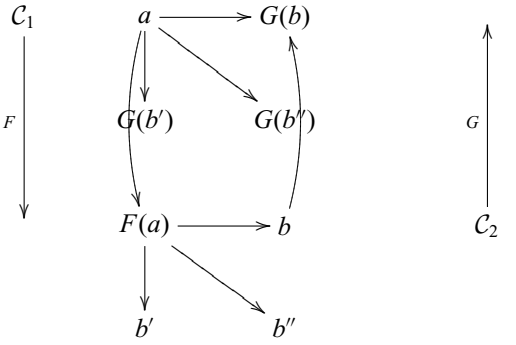
One of the most fundamental of such relations is called 'adjointness', and it concerns the relations between two functors. The formal definition of adjointness can be found in any standard Category Theory textbook (see, for instance, [9]). Although a thorough discussion of adjointness goes well beyond the scope of our article, we believe that it is important to outline the general features of such a construction, with the goal of showing how well it fits the relational framework sought in Peirce's works for conceiving abduction.

Given two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ and two functors $F:\mathcal{C}_1 \longrightarrow \mathcal{C}_2$ $G:\mathcal{C}_2 \longrightarrow \mathcal{C}_1$, we can illustrate adjointness by the following diagram:



As the above picture shows, the main idea behind adjointness is to constrain the images $G(b) \in \mathcal{C}_1$ of the elements $b \in \mathcal{C}_2$ that are related (by an arrow!) to the image $F(a)$ of an element $a \in \mathcal{C}_1$ to be related to the 'original' element $a \in \mathcal{C}_1$! This beautiful movement has an intrinsic orientation, which justifies the fact that we say that F is the *left adjoint* of G.[3]

The diagram below aims to further reinforce this concept:

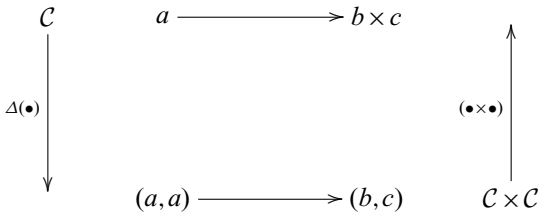


For any category $\mathcal{C}$ for which the product of any two objects exists, this very fact is itself instantiated in a higher-order relation between two functors $\Delta:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ (here Δ is the *diagonal functor* $\Delta(x)=(x,x)$) and $\times:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ (here $\times$ represents the product). In other words, the existence of a specific type of adjunction based on a given category guarantees that the inner structure of this

[3]There is a second condition, called *naturality*, that F and G need to obey in order to generate adjointness, which has to do with the UMP. It is a rather technical condition, which, for sake of clarity we omit in this article.

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category coheres with respect to a certain UMP.

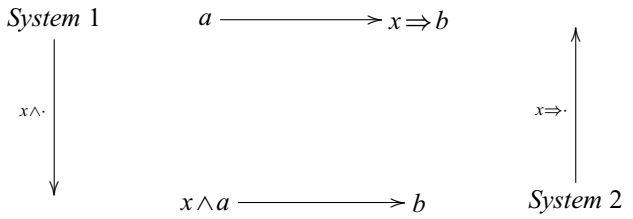


The role of adjoint functors in abductive inference has been investigated by Arzarello *et al.* in [2]. They present two different way to attack abduction, which can be summarized by the following:

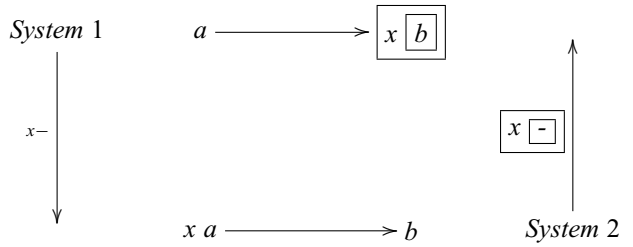
- $\Theta \vdash ? \longrightarrow \phi$
- $\Theta \wedge ? \longrightarrow \phi$

As they point out, 'In both cases abduction plays an essential role in reversing the course of thought (from ascending to descending control); in the second case the use of counterexamples seems to be at the origin of a more involved course of thought, which we have called the logic of not' [2].

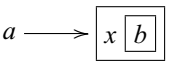
The duality that they identify, which is the duality between the propositional connectives $\wedge$ and $\Rightarrow$ corresponds to the fact that $\wedge$ is the right adjoint of $\Rightarrow$, as the following diagram shows:



What is more striking, however, is how the above relations translate into an adjointness in EG-alpha, as shown below:

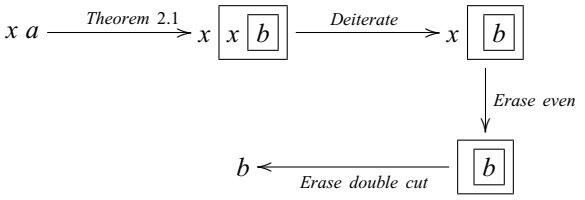


In order to prove that the above is an adjoint square, we first suppose that

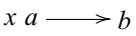


Then, using the result from the previous section along with Peirce's rules, we have that

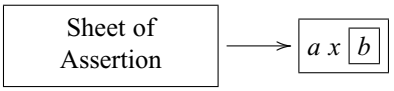
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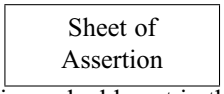


Viceversa, suppose that

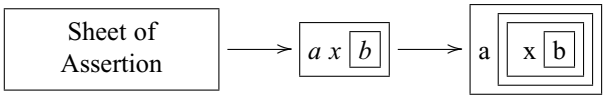


This also means that

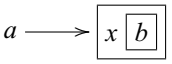


where 

denotes the (empty) sheet of assertion corresponding to 'True'. By inscribing a double cut in the graph as shown below, we get



which corresponds to the statement



The richness of these iconic functorial relations—which, with respect to the iconic relations of the graphs themselves function as meta-iconic representations—opens a wide-ranging field for the visual presentation and intuitive comprehension of many complex structures within propositional logic.

Although the iconic nature of these categorical relationships allows them to be represented in a more or less immediately intuitive way, the formal machinery that is needed to work out such notions in rigorous detail can become quite intricate and forbidding for all but the most intrepid logicians and mathematicians. Some of the most fundamental elements of this work has already been done, particularly through the groundbreaking efforts of Zalamea [20] and Brady and Trimble [5], but much remains still to be constructed, studied and mathematically grounded in this regard. Part of the power of the iconicity here is that highly complex and abstract logical notions may be investigated and various hypotheses corresponding to these constructed at an initially intuitive level that is often accessible even to general academic audiences. The rigorous formal framework of category theory may then be utilized to test and ultimately ground these motivating intuitions. This entire process may itself be 'captured,' i.e. at once recorded and represented—at least potentially—through sequences of graphs that represent various subcategories and functors iconically. We turn now to one particular iconic aspect of these lattices that corresponds to a well-grounded and well-studied area in intuitionistic logic: Heyting algebras.

5 Heyting algebras, intuitionism and creative abduction

The pragmatic perspective conveyed by a categorical interpretation of EG-alpha as generating iconicity at a second diagrammatic level immediately opens up a whole new universe of abductive opportunities. In particular, we can move beyond the information carried by one single graph and start

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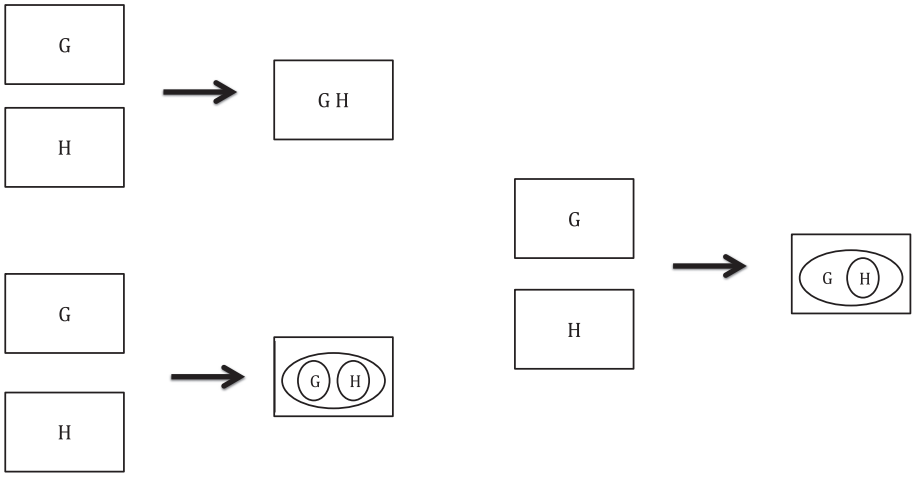
looking at the structure of organized subsets of $\mathcal{G}$. There are two main reasons why we believe that this is an important shift in perspective. In the first place, in agreement with the self-interpretative nature of graphs, it is desirable to think of $\mathcal{G}$ as a lattice of truth-values: although Brady and Trimble already made this connection implicitly by proving the correspondence of EG-alpha with Boolean algebras, we show that there is a natural topology on EG-alpha leading to an extension to algebras that are more general than Boolean—Heyting algebras—that play a fundamental role in Alan Badiou's latest work [3, 4]. As a result of this process, we can outline abductive inference in a much more 'flexible' way, leaving space for the 'preservation of ignorance' [18] which does not seem to be grasped by locking abduction into an algorithmic framework.

Let S be a set endowed with three binary operations $\wedge, \vee$ and $\Rightarrow$ and two special elements $\top$ and $\bot$. S is an *Heyting algebra* if the following hold:

1. $g \to \top$ for any $g \in S$
2. $g \wedge h \to h$
3. $g \wedge h \to g$
4. $g \to h$ and $g \to k$ implies $g \to h \wedge k$
5. $\bot \to g$ for any $g \in S$
6. $g \to g \vee h$
7. $h \to g \vee h$
8. $h \to g$ and $k \to g$ implies $h \wedge k \to g$
9. $g \to (h \Rightarrow k)$ iff $g \wedge h \to k$

We also define the *complement* of g as $\neg g = (g \Rightarrow \bot)$. Heyting Algebras are extensively used in Intuitionistic Logic because they do not require the law of the excluded middle to hold. That is, in general, $\neg(\neg g)$ need not be equal to g. For further details see [11].

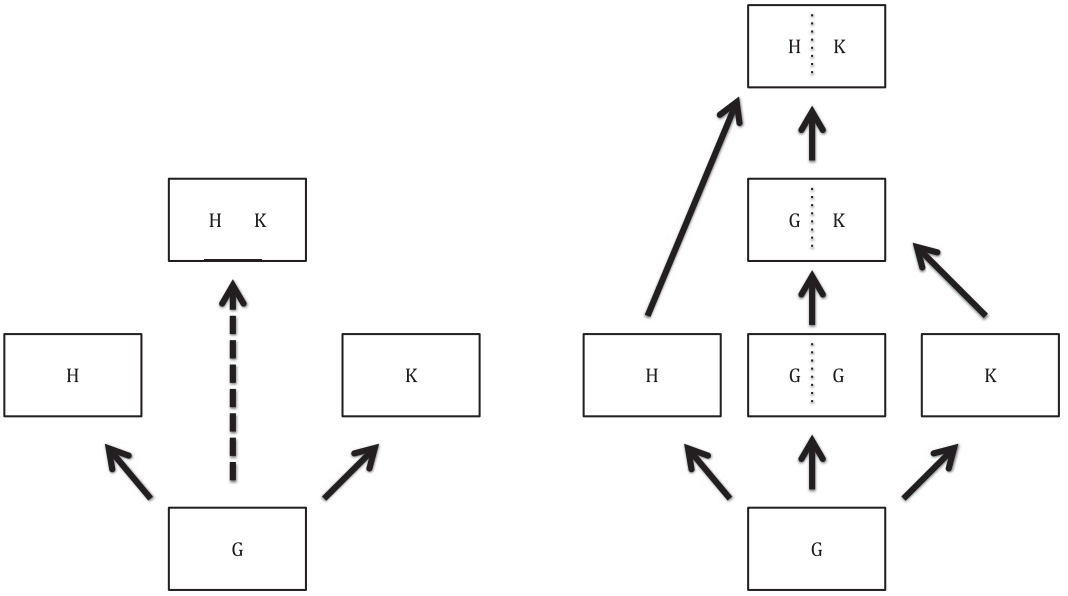
It can be easily shown that $\mathcal{G}$ is an Heyting algebra. All of the above operations and relative conditions can be translated into EG-alpha as interpreted within $\mathcal{G}$. In particular, the operations $\wedge, \vee, \Rightarrow$ are expressed by functors $G \times G \longrightarrow G$. These functors are illustrated iconically below:



That $\mathcal{G}$ (or a suitably collapsed $\mathcal{G}$) meets condition 4, for instance, may be shown easily. Every graph G may be 'doubled' to form the graph GG (this itself is demonstrated through application of write double cut, iteration, erase double cut). Then using Theorem 2.1 above twice transforms GG

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to *HK*. The diagrams below present the intuitive idea (the dotted arrow represents what needs to be 'derived' from the two given arrows taken together).



Condition 9 was already illustrated in the previous section. All the other conditions may be demonstrated in similar fashion. In a sense, this is a trivial point because it has been shown that $\mathcal{G}$ is equivalent to the theory of Boolean Algebras [5] and Boolean Algebras are a subset of Heyting Algebras. What is interesting is that the conditions for being a Heyting algebra may be expressed by functors with an immediate iconic representation of the algebraic structure of $\mathcal{G}$.

Everything we have done so far has taken place in a Boolean context. Yet having translated the graphs into a categorical framework opens up a new set of perspectives that might not have been otherwise evident. This is precisely the naturally abductive character of iconic signs [16]. In particular, we can extend our notion of 'object' from single graphs to structured subsets of $\mathcal{G}$. As we will show below, this simple move corresponds to a well-established notion of open sets of a topological space, which has been extensively used by logicians to create models of truth alternate to the Boolean ones.

Let us first define the *upper sets* of $\mathcal{G}$.

For any $g \in \mathcal{G}$ we define $g\uparrow$ as

$$g\uparrow = \{h \in \mathcal{G} : g \to h\},$$

that is, intuitively, $g\uparrow$ represents the collection 'of the futures' of g. We now declare a subset U of G an *upper closed set* if $g\uparrow \subseteq U$ for any $g \in U$.

LEMMA 5.1
The set $UC(\mathcal{G})$ of all upper closed subsets of $\mathcal{G}$ is a topological space.

PROOF. The denumerable union of upper closed sets is an upper closed set and the finite intersection of upper closed sets is an upper closed set. Moreover, the empty set and $\mathcal{G}$ are also upper closed sets. ∎

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6 Conclusion: from Peirce to Badiou

We have seen how a category-theoretical approach to EG-alpha lends itself to second-order iconic expression in terms of the category $\mathcal{G}$ and the functors and adjunctions that emerge within it. We have also seen how this framework can in turn open new abductive possibilities, in particular through the use of upper sets and the topology on $\mathcal{G}$ that they induce. We conclude by noting that this approach to EG-alpha not only enriches Peirce's system itself, but even more importantly allows it to communicate naturally with a variety of other formal and iconic systems, in particular with the powerful and wide-ranging formal ontology and phenomenology developed by the contemporary philosopher Alain Badiou.

Badiou's ambitious project of constructing a self-contained ontology based entirely on Mathematics was first outlined in *Being and Event*, where he uses Zermelo-Fraenkel Set Theory and Paul Cohen's technique of forcing to define a concept of truth able to explain, among other things, the apparent paradox of naming the indiscernible and to shed light on the epistemic mechanics of the scientific method and its specific inflections of abductive inference [6].

Following *Being and Event*, Badiou—in a pattern that follows very closely the development of Mathematics in the second half of the last century—moves towards a 'phenomenological' supplement to his set-theoretical ontology. Inspired by recent work in category theory, particularly the Ω-sets used in topos theory, in [4] Badiou adheres to a pragmatism that gives primacy to relations, rather than to objects. Every conceivable world is endowed with a function of appearing, whose intrinsic coherence allows for the natural emergence of an order of progressively determined truth that translates back into the cohesive nature of the world itself.

A crucial role in Badiou's construction is played by the Transcendental T, which essentially is the codomain of the appearing function. Such a transcendental is simply an order structure into which the ontological status of the world as a pure multiplicity can be translated and cohesively 'situated'. Mathematically, T is required to have the structure of a Heyting Algebra, which as we saw above may be generated in multiple ways by extracting certain organized subsets of $\mathcal{G}$. Given that the structure of $\mathcal{G}$ is built on topological properties of the plane, we find remarkable that various models for Badiou's transcendental may be constructed in EG-alpha. In particular, this opens up the broad field of abductive exploration of sheaves over various subcategories of $\mathcal{G}$.

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