Game semantics in string diagrams

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Abstract

A dialogue category is a symmetric monoidal category equipped with a notion of tensorial negation. We establish that the free dialogue category is a category of dialogue games and total innocent strategies. The connection clarifies the algebraic and logical nature of dialogue games, and their intrinsic connection to linear continuations. The proof of the statement is based on an algebraic presentation of dialogue categories inspired by knot theory, and a difficult factorization theorem established by rewriting techniques.

1 Introduction

Tensorial logic. Tensorial logic is a primitive logic of tensor and negation which refines linear logic by relaxing the hypothesis that linear negation is involutive. Its purpose is to resolve in this way the traditional gap between game semantics and linear logic, by unifying both of them with the monadic theory of linear continuations, see [22] for details. The logic emerged from the discovery that the linear distributivity law of linear logic

$$A \otimes (B \parr C) \quad \longrightarrow \quad (A \otimes B) \parr C \tag{1}$$

is the "shadow" of a more primitive distributivity law existing in any logic (typically intuitionistic logic) equipped with a notion of conjunction $\otimes$ and of negation. This distributivity law may be written as

$$A \otimes [B, C] \quad \longrightarrow \quad [A \otimes B, C] \tag{2}$$

where the n-ary connective $[A_1, ..., A_n]$ is syntactic sugar for

$$[A_1, ..., A_n] \quad = \quad \neg\,(\,\neg A_1 \,\otimes\, ...\, \otimes\, \neg A_n\,).$$

This observation was the sparkle which lead to the introduction of tensorial logic, whose sequent calculus is recalled in the Appendix. Interestingly, the distributive law (2) holds in tensorial logic as an instance of a more general distributivity law, defined as follows:

$$A \otimes [B, C_1, \ldots, C_n] \quad \longrightarrow \quad [A \otimes B, C_1, \ldots, C_n]$$

Observe that one recovers in the case $n = 0$ the well-known principle which makes the double negation monad a strong monad:

$$A \otimes \neg\neg B = A \otimes [B] \quad \longrightarrow \quad [A \otimes B] = \neg\neg(A \otimes B).$$

These considerations indicate the existence of a unified theory of linear logic and continuations which could benefit from the two approaches. One purpose of the paper is precisely to substantiate this idea, by exhibiting an algebraic presentation by generators and relations of the proofs of tensorial logic. In particular, we will establish in the course of the paper that all the various distributivity laws are generated by the following principle

$$A \otimes (X \multimap B) \quad \longrightarrow \quad X \multimap (A \otimes B) \tag{3}$$

where $X \multimap B$ is syntactic sugar for

$$X \multimap B \quad = \quad \neg\,(X \,\otimes\, \neg\, B\,)$$

This principle (3) should be seen as a parametric variant of the tensorial strength, which is recovered when X is the unit of the tensor product. Note that the principle is provable in tensorial logic by the derivation tree below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}}{A\,,\, B \;\vdash\; A \otimes B}\,\text{Tensor}}{\neg\,(\,A \otimes B\,)\,,\, A\,,\, B \;\vdash}\,\text{Negation}}{\neg\,(\,A \otimes B\,)\,,\, A \;\vdash\; \neg\, B}\,\text{Negation} \quad \dfrac{}{X \vdash X}}{X\,,\, \neg\,(\,A \otimes B\,)\,,\, A \;\vdash\; X \otimes \neg\, B}\,\text{Tensor}}{X\,,\, \neg\,(\,A \otimes B\,)\,,\, A\,,\, \neg\,(X \otimes \neg\, B\,) \;\vdash}\,\text{Negation}}{A\,,\, \neg\,(X \otimes \neg\, B\,) \;\vdash\; \neg\,(\,X \otimes \neg\,(\,A \otimes B\,)\,)}\,\text{Negation}}{A \otimes \neg\,(X \otimes \neg\, B\,) \;\vdash\; \neg\,(\,X \otimes \neg\,(\,A \otimes B\,)\,)}\,\text{Tensor}$$

Tensorial proof-nets. It appears that this unified view of linear logic and continuations leads to a compelling notion of proof-net for tensorial logic, which lies at the converging point of:

- multiplicative proof-nets in linear logic,
- innocent strategies in game semantics,
- string diagrams in low dimensional algebra.

In order to devise this graphical notation for tensorial proofs, we start from the algebraic approach to proof theory initiated by Lambek [13] and promoted in his book with Scott [14] using the notion of cartesian-closed category. We briefly recall this notion, and explain how to relate it to string diagrams using ideas coming from knot theory.

Cartesian closed categories. The key idea inspired by Lawvere's categorical semantics is to think of the λ-calculus as a syntactic presentation of a *category with structure*. Typically, Lambek established that the free cartesian closed category $\mathbf{free\text{-}ccc}(\mathscr{X})$ on a given category $\mathscr{X}$ has:

- the formulas of minimal logic (constructed with the binary constructors $\times$ and $\Rightarrow$ together with the conjunctive unit 1) with atoms given by the objects of the category $\mathscr{X}$,
- the simply-typed λ-terms of type $A \Rightarrow B$ as morphisms from the formula A to the formula B, considered modulo a suitable notion of equivalence between λ-terms (that is, β and η conversion and composition of maps between atoms in $\mathscr{X}$).

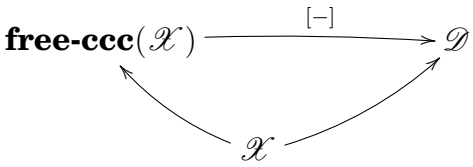
This result enables to construct denotational models of the simply-typed λ-calculus in a simple and conceptual way. Consider the full and faithful functor

$$\mathscr{X} \quad \longrightarrow \quad \mathbf{free\text{-}ccc}(\mathscr{X})$$

which transports every object of $\mathscr{X}$ to the corresponding atomic formula in $\mathbf{free\text{-}ccc}(\mathscr{X})$. The theorem established by Lambek states that every functor $\mathscr{X} \longrightarrow \mathscr{D}$ to a cartesian-closed category $\mathscr{D}$ extends to a structure-preserving functor

$$[-] \quad : \quad \mathbf{free\text{-}ccc}(\mathscr{X}) \quad \longrightarrow \quad \mathscr{D}$$

making the diagram commute:



This functor $[-]$ is moreover unique up to structure-preserving isomorphism. The benefit of the construction is that the functor $[-]$ transports every simply-typed λ-term P to a denotation $[P]$ in the cartesian closed category $\mathscr{D}$ – that is, an invariant of the λ-term P modulo $\beta\eta$-conversion.

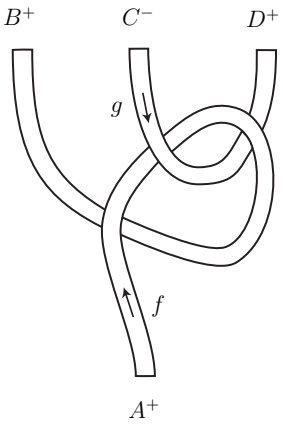
Ribbon categories. This story on cartesian closed categories and intuitionistic logic is well-known among the logicians and the computer scientists interested in the semantics of proofs and programs. What is less known is that the very same story occurs in a quite different part of mathematics – knot theory and representation theory of quantum groups. There, Turaev defined the notion of a *ribbon category* as a monoidal category equipped with combinators for braiding and U-turns, satisfying a series of expected equations, see for instance [10, 24, 5, 19] for details. Then, one establishes a similar coherence theorem, which states that the free ribbon category on a category $\mathscr{X}$ has

- sequences $(A_1^{\epsilon_1}, \ldots, A_n^{\epsilon_n})$ of signed objects of $\mathscr{X}$ as objects, where each A_i is an object of the category $\mathscr{X}$, and each ϵ_i is either $+$ or $-$,
- oriented ribbon tangles as morphisms, colored by morphisms of $\mathscr{X}$, and considered modulo topological deformation.

So, a typical morphism

$$(A^+) \quad \longrightarrow \quad (B^+, C^-, D^+)$$

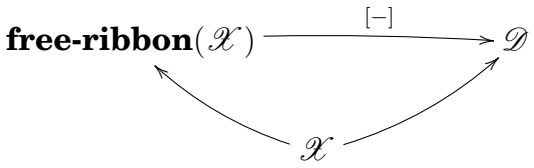
in the category **free-ribbon**$(\mathscr{X})$ looks like this



where $f : A \longrightarrow B$ and $g : C \longrightarrow D$ are morphisms in the category $\mathscr{X}$. Now, consider the full and faithful functor

$$\mathscr{X} \quad \longrightarrow \quad \textbf{free-ribbon}(\mathscr{X})$$

which transports every object A of the category $\mathscr{X}$ to the signed sequence (A^+) in the category **free-ribbon**$(\mathscr{X})$. Then, just as in the case of the free cartesian closed category, every functor from the category $\mathscr{X}$ to a ribbon category $\mathscr{D}$ extends to a structure-preserving functor $[-]$ which makes the diagram below commute:



The point is that, once properly oriented and colored, every topological knot P defines a morphism $P : I \longrightarrow I$ from the tensorial unit $I = ()$ to itself in the category **free-ribbon**$(\mathscr{X})$. From this follows that the image $[P]$ defines a invariant of the knot P modulo topological deformation. This method enables in particular to establish that the Jones polynomial $[P]$ associated to a knot P defines indeed a topological invariant, see [10] for details.

Dialogue categories. We have just pointed out a similarity between the conceptual tools developed in order to study topological knots and logical proofs. A key difference however is that the definition of **free-ccc**$(\mathscr{X})$ is based on a symbolic calculus, the simply-typed λ-calculus, instead of a topological notation, as in the definition of **free-ribbon**$(\mathscr{X})$. At this point, it is worth asking oneself whether the notion of cartesian closed category could

be fine-tuned and replaced by a notion closer to ribbon categories... whose stringy nature could be then uncovered.

It appears that this question leads again to tensorial logic — and more specifically to the corresponding notion of dialogue category. A dialogue category is defined as a symmetric monoidal category $\mathscr{D}$ equipped with an exponentiating object $\bot$. Recall that

Definition 1 (exponentiating object) *An exponentiating object in a symmetric monoidal category $\mathcal{C}$ is an object $\bot$ equipped for every object a with an object $a \multimap \bot$ and with a bijection*

$$\mathscr{D}(a \otimes b, \bot) \quad \cong \quad \mathscr{D}(b, a \multimap \bot)$$

natural in the object b.

Definition 2 (dialogue categories) *A dialogue category is a symmetric monoidal category equipped with an exponentiating object.*

Hence, the notion of dialogue category reflects tensorial logic and its two logical connectives:

- the linear conjunction $\otimes$ and its unit I,
- the linear negation $a \mapsto \neg\, a$ defined as $a \multimap \bot$

in exactly the same way as the notion of cartesian closed category reflects (minimal) intuitionistic logic. A typical example of dialogue category is provided by the category of vector spaces (of finite and infinite dimension) on a base field k, with $\bot = k$. In that case, the negation of a vector space V is its dual vector space V^* of linear forms from V to k. Another example of dialogue category is provided by the category Set of sets and functions equipped with a set $\bot$ chosen as return type. More generally, any symmetric monoidal closed category $\mathscr{D}$ equipped with an arbitrary object $\bot$ as return type. Another important class of example is provided by $*$-autonomous categories with the dualizing object $\bot$ as return type. Note that a $*$-autonomous category may be seen as a degenerate dialogue category where the canonical map $a \longrightarrow \neg\neg\ a$ is invertible.

Dialogue games. The notion of innocent strategy introduced by Hyland and Ong in dialogue games [8] provides us with a clean abstract and combinatorial notion of proof (and program) which may be formulated in three different styles, with clear transcriptions between them:

- the traditional syntax of λ-terms, because an innocent strategy is generated by a (possibly infinite) Bohm tree, whose branches describe the so-called *Proponent views* of the innocent strategy,
- the dynamic semantics provided by game semantics, because an innocent strategy is a particular kind of interactive strategy which plays according to its current view of the play, rather than according to the play itself,
- the static semantics provided by relational semantics, because an innocent strategy is positional in asynchronous games [16] and thus entirely described by its set of halting positions.

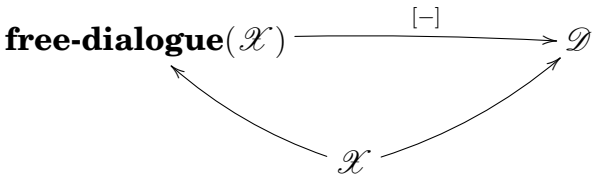
One main observation of our work is that game semantics should be understood as a *syntax* of tensorial proofs, expressed at the same time graphically as a flow of control. As a matter of fact, we show how to apply the algebraic principles of knot theory to this flow of control, and to characterize the free dialogue category **free-dialogue**$(\mathscr{X})$ generated by a category $\mathscr{X}$ in the same way as one characterizes the free ribbon category **free-ribbon**$(\mathscr{X})$.

Namely, we will exhibit a series of basic logical principles of tensorial logic (or generating proofs) providing a logical counterpart to the topological generators of framed tangles (braidings and U-turns) appearing in the construction of the free ribbon category **free-ribbon**$(\mathscr{X})$. Using this decomposition of tensorial proofs in basic blocks, we establish that the category **free-dialogue**$(\mathscr{X})$ has

- dialogue games as objects, with objects of the category $\mathscr{X}$ considered as atomic games,
- total innocent strategies as morphisms, where the terminal moves between atomic games are colored by a morphism of $\mathscr{X}$.

We believe that this coherence theorem for dialogue category provides a firm algebraic foundation to the traditional "full completeness theorems" of game semantics obtained in [1, 8] and to the connection between innocent strategies and linear continuations stressed in [21, 22]. This result also demonstrates that game semantics is a syntax as much as a semantics,

since every functor $\mathscr{X} \longrightarrow \mathscr{D}$ to a dialogue category induces a structure-preserving functor $[-]$ which makes the diagram below commute:



In particular, the coherence theorem establishes that *every* dialogue category $\mathscr{D}$ provides a semantic interpretation of innocent strategies. A typical illustration of this phenomenon is provided by the dialogue category $\mathscr{D}$ of vector spaces (of a finite or infinite dimension) on a field k, which provides a denotational semantics of tensorial logic, where every innocent strategy σ of a dialogue game A is interpreted as a vector $[\sigma]$ in the appropriate vector space $[A]$. This shows that game theoretical structures appear in somewhat unexpected fields of mathematics.

Similarly, the category $\mathscr{D}$ of coherence space with $\bot = 1$ is a $*$-autonomous category and thus a dialogue category: the coherence theorem enables us to extract the static content of an innocent strategy σ of the dialogue game A by transporting it to a clique $[\sigma]$ in the appropriate coherence space $[A]$.

Plan of the paper. We recall the notion of innocent strategy in dialogue games in § 2. We then study in § 3 the logic of pure negation (that is, tensorial logic without its tensor product) and establish that it is generated by the adjunction of negation with itself. This uncovers the unit η and counit ε of this self-adjunction as our two first generators of tensorial proof-nets. Once dialogue categories have been reformulated as dialogue chiralities in § 4, we describe in § 5 a pair of distributivity laws $\kappa_{⅋}$ and $\kappa_{⊗}$, alternatively seen as parametric strengths, which provide the second pair of generators of tensorial proof-nets. We conclude our algebraic presentation in § 6 with two generators ***axiom*** and ***cut*** reflecting the axioms and cuts of tensorial logic. We then establish in § 8 our main technical result: a factorization theorem, which states that every sequence of generators factors as a sequence of Opponent combinators: ε, $\kappa_{⊗}$, ***cut*** followed by a sequence of Proponent combinators: η, $\kappa_{⅋}$, ***axiom***, providing a purely combinatorial account of innocence. The coherence theorem is then deduced from this factorization property in § 9. We describe the related works in § 10.

2 Dialogue games and innocent strategies

In this section, we define the category of dialogue games and innocent strategies generated by a category $\mathscr{X}$ of atoms. The definition follows the pattern of [16] except for the careful treatment of the atomic moves reflecting the objects and morphisms of the underlying category $\mathscr{X}$.

Dialogue games. A dialogue game is a quadruple

$$A \quad = \quad (M_A, \leq_A, \lambda_A, o_A)$$

consisting of:

- a finite forest $(M_A, \leq_A)$ whose nodes are called the *moves* of the game,
- a function $\lambda_A : M_A \longrightarrow \{-1, +1\}$ which associates to every move a *polarity* $+1$ (for the Player moves) or -1 (for the Opponent moves),
- a function $o_A : M_A \longrightarrow \mathbf{obj}(\mathscr{X}) + \{\bullet\}$ which associates to every move a *label* consisting of an object of the category $\mathscr{X}$ or the tag $\bullet$.

A move labeled by $\bullet$ is called a regular move, whereas a move labeled by an object of $\mathscr{X}$ is called an atomic move. Three conditions are moreover required on a dialogue game:

- the nodes at the root of the forest are required to be Opponent moves,
- the forest is required to be bipartite: the polarities of nodes are alternating on every branch,
- the atomic moves are required to be at the leaves of the forest.

Note that the converse property does not hold: a move at the leaves of the forest is not required to be atomic.

Positions. A position of the game is a downward closed set of nodes of the forest. Every position may be thus seen as a partial exploration of the forest, starting from the roots.

The asynchronous graph. Every dialogue game A induces a graph $\mathscr{G}_A$:

- its nodes are the positions $x, y \in \mathscr{D}_A$,
- its edges $m : x \longrightarrow y$ are the moves verifying $y = x \uplus \{m\}$, where $\uplus$ denotes disjoint union, or equivalently, that $y = x \cup \{m\}$ and that the move m is not an element of x.

This graph $\mathscr{G}_A$ is called the *asynchronous graph* of the dialogue game A. We write $s : x \longrightarrow\!\!\!\!\!\rightarrow y$ for a path

$$x \overset{m_1}{\to} x_1 \overset{m_2}{\to} \cdots \overset{m_{k-1}}{\to} x_{k-1} \overset{m_k}{\to} y$$

between two positions x and y.

Alternating paths. A path $m_1 \cdots m_k : x \longrightarrow\!\!\!\!\!\rightarrow y$ is *alternating* when:

$$\forall i \in \{1, ..., k-1\}, \quad \lambda_A(m_{i+1}) = -\lambda_A(m_i).$$

Legal plays. A *legal play* is a pair (s, φ) consisting of an alternating path s starting from the empty position $*_A$ in the asynchronous graph $\mathscr{G}_A$:

$$*_A \overset{m_1}{\to} x_1 \overset{m_2}{\to} \cdots \overset{m_{2k-1}}{\to} x_{2k-1} \overset{m_{2k}}{\to} x_{2k}$$

together with a word of length k consisting of the letter $\bullet$ or a morphism of the category $\mathscr{X}$, such that, for all $1 \leq i \leq k$:

$$\varphi(i) = \bullet \Rightarrow o(m_{2i-1}) = \bullet \text{ and } o(m_{2i}) = \bullet$$

$$\varphi(i) = f : A \longrightarrow B \quad \Rightarrow \quad o(m_{2i-1}) = B \text{ and } o(m_{2i}) = A.$$

In that last case, we find useful to write the morphism f as a double pointer from the move m_{2i} to the move m_{2i-1}, as follows:

$$s \;\cdot\; m_{2i-1} \;\cdot\; \overset{f}{\overleftarrow{m_{2i}}} \;\cdot\; t$$

It should be stressed that the double pointer does not mean that the move m_{2i} is justified by the move m_{2i-1}. This is never the case in fact, since an atomic move like m_{2i-1} is a leaf of the dialogue game, and thus does not justify any other move. The set of legal plays is noted P_A.

Strategy. A strategy σ is a set of legal plays such that:

- the strategy σ contains the empty play,
- σ is closed by even-length prefix:

$$\forall s \in P_A, \forall m, n \in M_A, \quad s \cdot m \cdot n \in \sigma \Rightarrow s \in \sigma,$$

- σ is deterministic: $\forall s \in P_A, \forall m, n_1, n_2 \in M_A,$

$$s \cdot m \cdot n_1 \in \sigma \text{ and } s \cdot m \cdot n_2 \in \sigma \quad \Rightarrow \quad n_1 = n_2.$$

One should be careful here with the fact that a play is defined as a path s equipped with an additional structure φ. In particular, determinism means that whenever the two plays



are element of the strategy σ, then the two moves n_1 and n_2 coincide, as well as the two morphisms f_1 and f_2 of the category $\mathscr{X}$.

View extraction. The binary relation $\overset{\text{OP}}{\rightsquigarrow}$ is defined as the smallest relation between alternating plays such that:

$$s_1 \cdot m \cdot n \cdot s_2 \quad \overset{\text{OP}}{\rightsquigarrow} \quad s_1 \cdot s_2$$

for every alternating play s_1 of even-length, every *nonempty* alternating path s_2, every Opponent move m which does not justify any move in s_2, and every Player move n which does not justify any move in s_2.

Player view. The relation $\overset{\text{OP}}{\rightsquigarrow}$ defines a noetherian and locally confluent rewriting system on alternating plays. By Newman's Lemma, the rewriting system is confluent. Thus, every alternating play s induces a unique normal form noted $\lceil\ s\ \rceil$ and called the *Player view* of s:

$$s \quad \overset{\text{OP}}{\rightsquigarrow} \quad s_1 \quad \overset{\text{OP}}{\rightsquigarrow} \quad \cdots \quad \overset{\text{OP}}{\rightsquigarrow} \quad s_k \quad \overset{\text{OP}}{\rightsquigarrow} \quad \lceil\ s\ \rceil.$$

Innocent strategies. A strategy σ is called *innocent* in a dialogue game A when for all plays $s, t \in \sigma$, for every Opponent move $m \in M_A$ and Player move $n \in M_A$, the following situation

$$s \cdot m \cdot n \in \sigma \ \text{ and } \ t \cdot m \in P_A \ \text{ and } \ \lceil\ s \cdot m\ \rceil = \lceil\ t \cdot m\ \rceil$$

implies that

$$t \cdot m \cdot n \in \sigma.$$

Once again, we should be careful about the additional structure φ equipping the path s. In particular, innocence means that the two legal plays



in the strategy σ are labeled by the same morphism

$$f \quad : \quad o(n) \quad \longrightarrow \quad o(m)$$

in the underlying category $\mathscr{X}$.

Total strategies. A strategy is total when every maximal play in the strategy contains all the moves of the underlying dialogue game A.

Tensor product. The tensor product of two dialogue games A and B is defined as the dialogue game

$$M_{A\otimes B} = M_A + M_B, \quad \lambda_{A\otimes B} = [\lambda_A, \lambda_B] \quad o_{A\otimes B} = [o_A, o_B].$$

In particular, the order $\leq_{A\otimes B}$ is defined as the disjoint union of $\leq_A$ and $\leq_B$. Note that the tensor product has a unit, provided by the dialogue game $\mathbf{1}$ with an empty set of moves.

Negation. The negation of a dialogue game A is defined as the dialogue game $\neg A$ where the polarities of moves have been reversed, lifted by a new initial move m played by Opponent is added:

$$M_{(\neg A)} = M_A + \{m\}, \qquad \lambda_{(\neg A)} = -\lambda_A, \qquad o_{(\neg A)} = o_A$$

together with $\lambda_{(\neg A)}(m) = -1$ and $o_{(\neg A)}(m) = \bullet$. In particular, the order $\leq_{\neg A}$ coincides with the order $\leq_A$ on the moves of A, and has the new move m as least element.

Remark. The careful reader must have noticed that the morphisms in the category $\mathscr{X}$ go in the reverse direction as the flow of control in the play. Typically, we will see very soon that the strategy associated to the ***cut*** link of tensorial logic plays the moves in this order:

$$\begin{array}{ccc} X \otimes \neg Y & \longrightarrow & \neg \mathbf{1} \\ & & m \\ & m & \\ & Y & \\ X & \overset{f}{\nearrow} & \end{array}$$

where the morphism $f : X \longrightarrow Y$ of the category $\mathscr{X}$ is indicated with a pointer from the atomic move X to the atomic move Y.

The category of dialogue games. The category of dialogue games generated by the category $\mathscr{X}$ is denoted **innocent**$(\mathscr{X})$ and defined as follows:

- its objects are the dialogue games with atoms in $\mathscr{X}$,
- its morphisms $A \longrightarrow B$ are the total innocent strategies of the game $A \multimap B$.

Here, the game $A \multimap B$ is defined as the game

$$M_{A \multimap B} = M_A + M_B, \quad \lambda_{A \multimap B} = [-\lambda_A, \lambda_B], \quad o_{A \multimap B} = [o_A, o_B]$$

where the order $\leq_{A \multimap B}$ is defined as the disjoint union of $\leq_A$ and $\leq_B$. Note that the polarities of the moves in the game A have been reversed. From this follows that the game $A \multimap B$ is not exactly a dialogue game, since the initial moves in A are Player moves, instead of Opponent moves. However, an innocent strategy σ in the game $A \multimap B$ may be defined as an innocent strategy in the underlying asynchronous graph $\mathscr{G}_{A \multimap B}$, where we are careful to ask that the legal plays in $A \multimap B$ start by an Opponent move (thus in the component B).

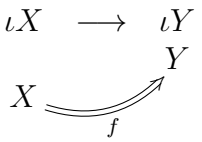
Now, let $\perp$ denote the dialogue game with a unique Opponent move labelled $\bullet$. Note that the morphisms from any dialogue game A to $\perp$ in the category $\mathbf{innocent}(\mathscr{X})$ are the same as the innocent strategies in the dialogue game $\neg A$. From this follows that:

Proposition 1 *The category* $\boldsymbol{innocent}(\mathscr{X})$ *equipped with the singleton game* $\perp$ *defines a dialogue category. Moreover, there is a functor*

$$\iota \quad : \quad \mathscr{X} \quad \longrightarrow \quad \boldsymbol{innocent}(\mathscr{X})$$

which transports

- *every object* X *to the dialogue game* ιX *consisting of a unique Opponent move labeled by the object* X*,*
- *every morphism* $f : X \longrightarrow Y$ *to the strategy*



One main purpose of the following sections is to establish that the category $\mathbf{innocent}(\mathscr{X})$ coincides with the free dialogue category generated by the category $\mathscr{X}$.

3 A self-adjunction

A first observation is that every exponentiating object $\perp$ in a symmetric monoidal category induces a pair of functors

$$L \ : \ \mathscr{C} \ \longrightarrow \ \mathscr{C}^{\,op} \qquad\qquad R \ : \ \mathscr{C}^{\,op} \ \longrightarrow \ \mathscr{C}$$

transporting an object a into its negation $a \multimap \bot$ in the opposite category. The series of natural bijections

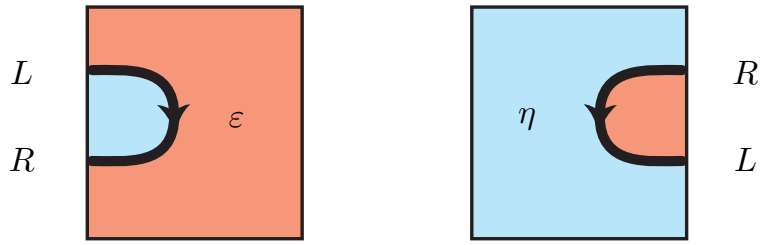
$$\begin{array}{lcll} \mathscr{C}(a, R(b)) & \cong & \mathscr{C}(a, b \multimap \bot) & \\ & \cong & \mathscr{C}(a \otimes b, \bot) & \\ & \cong & \mathscr{C}(b, a \multimap \bot) & \cong \quad \mathscr{C}^{op}(L(a), b) \end{array}$$

establishes moreover that the functor L is left adjoint to the functor R. The starting point of our work is the observation that the adjunction $L \dashv R$ of negation with itself plays in proof theory the same role as duality in knot theory. In order to establish this correspondence, we need to shift from string diagrams in monoidal categories to string diagrams in 2-categories, taking advantage of Street's algebraic presentation of adjunctions $L \dashv R$ performed in the language of 2-categories [25], see also [17]. At this point, we strongly advise the reader to read Selinger's nice introduction to string diagrams [24] and more specifically his Section 8 explaining the specific case of 2-categorical string diagrams.

Now, recall that every adjunction $L \dashv R$ induces a monad $R \circ L$ and a comonad $L \circ R$. In Street's presentation, the two generators of the adjunction $L \dashv R$ are the unit η of the monad $R \circ L$ and the counit ε of the comonad $L \circ R$:

$$\varepsilon : L \circ R(b) \longrightarrow b \qquad\qquad \eta : a \longrightarrow R \circ L(a)$$

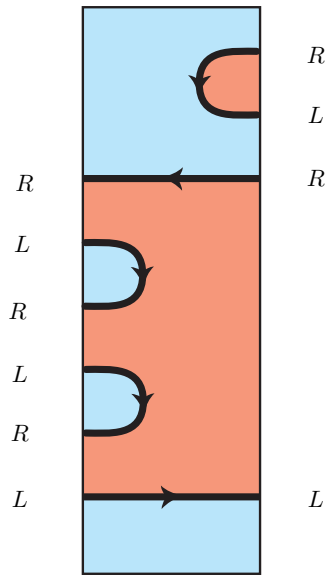
These two generators of the adjunction $L \dashv R$ are nicely depicted in the language of string diagrams for 2-categories:



The main novelty of these string diagrams compared to the familiar string diagrams of monoidal categories is that they are multi-coloured:

- the blue surface represents the category $\mathscr{C}$,
- the red surface represents the opposite category $\mathscr{C}^{op}$,
- the negation functors R and L are depicted as oriented strings separating a blue surface from a red surface.

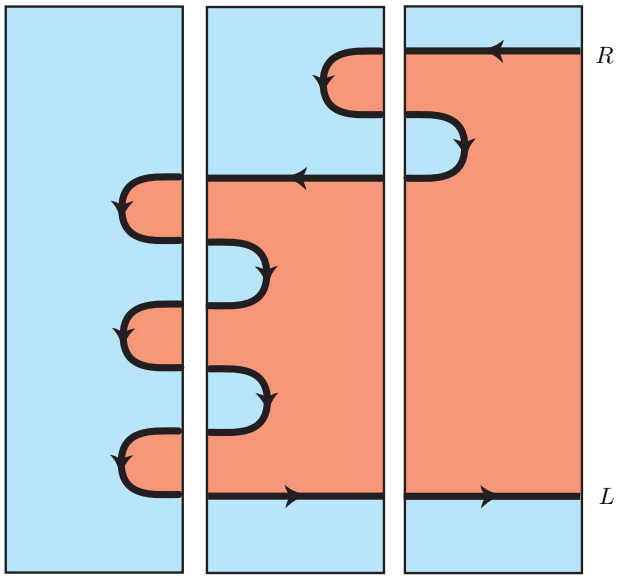
So, a typical proof from an atomic formula A negated six times to the same atomic formula A negated four times looks like this:



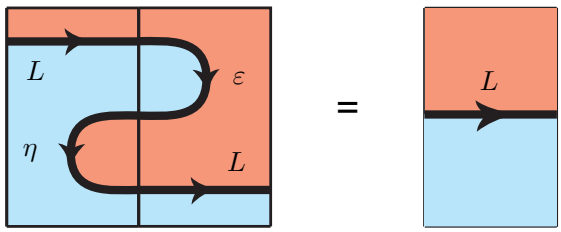
The correspondence with game semantics is striking at this point: if one reads the diagram topdown, and understands the orientation of the negation strands as the intended flow of control, then it appears that the flow of negations R and L described by the string diagram implements an interactive strategy playing the negation functors R as input tokens (or Opponent moves) and the negation functors L as output tokens (or Proponent moves). The fact that each functor R and L shifts from the category $\mathscr{C}$ to the category $\mathscr{C}^{op}$ and conversely captures the basic intuition that a negation is a turn in the dialogue game, where control is exchanged between players.

Composition of proofs is then performed by gluing the diagrams in the

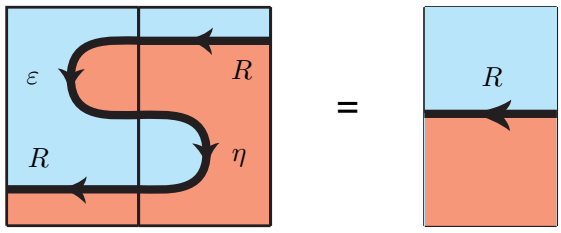
expected way:



The evaluation mechanism of game semantics (sometimes called interaction plus hiding) is then implemented by the pair of triangular laws regulating the generators η and ε in the algebraic presentation of adjunctions, see [25]. Diagrammatically speaking, these relations are the counterpart of the "zig-zag" relation for dual pairs, and thus describe the topological deformation of a curve into a straight line:



(4)



(5)

These two relations imply in particular that the composite proof depicted above is equal to the generator η. An important observation is that all the

proofs of this pure logic of negation are generated by the unit η and the counit ε of the adjunction $L \dashv R$. This means that this most elementary fragment of logic is entirely described by the algebraic fact that negation is adjoint to itself.

4 Dialogue categories as dialogue chiralities

Every monoidal category $\mathscr{C} = (\mathscr{C}, \otimes, I)$ induces a monoidal category noted

$$\mathscr{C}^{\,op(0,1)} \quad = \quad (\ \mathscr{C}^{\,op(1)}\ ,\ \otimes^{op(0)}\ ,\ I\ )$$

with the opposite category $\mathscr{C}^{\,op(1)} = \mathscr{C}^{\,op}$ as underlying category, and with the tensor product $\otimes^{op(0)}$ defined by permuting the two components of the original tensor product:

$$a \ \otimes^{op(0)} \ b \quad = \quad b \ \otimes \ a.$$

Dialogue categories are governed by a chirality principle which says that every map in a dialogue category $\mathscr{C}$ may be seen at the same time as a map in the opposite category $\mathscr{C}^{\,op(0,1)}$. This idea leads to the following notion of *refraction chirality* which enables to manipulate the two categories $\mathscr{C}$ and $\mathscr{C}^{\,op(0,1)}$ on a symmetric and equal footing, see [18] for details. A *refraction chirality* is defined as a pair of symmetric monoidal categories

$$(\mathscr{A}, \oslash, \mathbf{true}) \qquad\qquad (\mathscr{B}, \otimes, \mathbf{false})$$

equipped with a symmetric monoidal equivalence

$$(-)^* \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}^{\,op(0,1)} \quad : \quad {}^*(-)$$

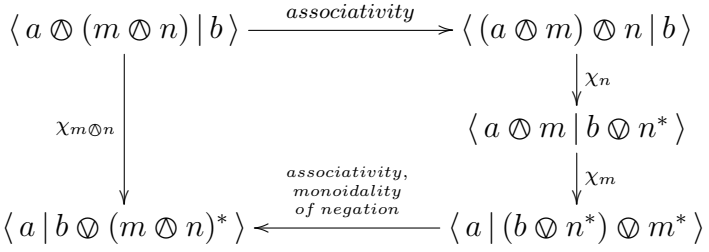
a distributor, or categorical bimodule

$$\langle\, - \,|\, - \,\rangle \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set$$

and a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, a \oslash m \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, b \otimes m^* \,\rangle$$

natural in a and b, making the diagram below commute:



A dialogue chirality is then defined as a refraction chirality whose distributor comes from an adjunction $L \dashv R$ in the sense that

$$\langle\, a \,|\, b \,\rangle \quad = \quad \mathscr{A}(a, R(b)).$$

Note that every dialogue category $\mathscr{C}$ induces a dialogue chirality $\mathcal{G}(\mathscr{C})$ where

$$\mathscr{A} \quad = \quad \mathscr{C} \qquad\qquad \mathscr{B} \quad = \quad \mathscr{C}^{\,op(0,1)}$$

and the two complementary functors $(-)^*$ and ${}^*(-)$ are the identity functor on the monoidal category $\mathscr{C}$, and the adjunction $L \dashv R$ is deduced from the negation as in Section 3. At this point, it is not extremely difficult (although it should be done with care) to construct a suitable 2-category DiaCat of dialogue categories and a suitable 2-category DiaChi of dialogue chiralities, in such a way that this construction $\mathcal{G}$ defines a 2-dimensional equivalence between DiaCat and DiaChi. In particular, every dialogue chirality is equivalent in the 2-category DiaChi to the image $\mathcal{G}(\mathscr{C})$ of a dialogue category $\mathscr{C}$. In other words, every dialogue chirality may be "strictified" into a dialogue category, this fact establishing the notion of dialogue chirality as a relaxed but equivalent notion of dialogue category. So, we will generally confuse the two notions in the sequel, and think of a dialogue chirality as a properly symmetric way to formulate a dialogue category.

5 A chiral pair of distributivity laws

Reformulating dialogue categories as dialogue chiralities enables to apply the chirality principle, which states that every generator of tensorial logic in the category $\mathscr{A} = \mathscr{C}$ has a "mirror picture" in the category $\mathscr{B} = \mathscr{C}^{\,op}$, defining a dual generator. This chirality principle is nicely illustrated by the fact that the generator ε is the mirror picture in the category $\mathscr{B}$ of the generator η in the category $\mathscr{A}$. In the same way, the chirality principle enables to "replicate" the distributivity law (3) mentioned in the introduction

$$a_1 \otimes \neg\, (b \otimes \neg\, a_2\,) \;\longrightarrow\; \neg\, (\, b \otimes \neg (\, a_1 \;\otimes\; a_2\,)\,)$$

as a dual pair $(\kappa_{\parr}, \kappa_{\with})$ of distributivity laws, consisting of a "conjunctive" distributivity law $\kappa_{\parr}$ living in the category $\mathscr{A}$ and of a "disjunctive" distributivity law $\kappa_{\with}$ living in the category $\mathscr{B}$

$$\begin{array}{lcl} a_1 \parr R(L(a_2) \with b) & \xrightarrow{\kappa_{\parr}(a_1,a_2,b)} & R(L(a_1 \parr a_2) \with b) \\ L(R(b_1 \with b_2) \parr a) & \xrightarrow{\kappa_{\with}(b_1,b_2,a)} & b_1 \with L(R(b_2) \parr a) \end{array}$$

It is worth observing that:

- the conjunctive distributivity law $\kappa_{\parr}$ imports $a_1 \parr -$ *inside* the scope of the two negations R and L,
- the disjunctive distributivity law $\kappa_{\with}$ exports $b_1 \with -$ *outside* the scope of the two negations R and L.

On the other hand, the two distributivity laws $\kappa_{\parr}$ and $\kappa_{\with}$ import a conjunction $\parr$ inside a disjunction $\with$, thus explaining why both laws are identified when one collapses tensorial logic to linear logic by "removing" the negations.

Just as in the case of ribbon knots in ribbon categories, the 2-dimensional string diagrams of linear negation (in Section 3) describe particles in motion. In order to interpret the tensor product of tensorial logic, one needs to consider surface diagrams describing *formula trees* (rather than simply particles) in motion. These surface diagrams are performed in the traditional notation for monoidal (or in that case cartesian) 2-categories, see for instance Street [26]. By way of illustration, consider the sequent

$$\neg\neg\, a \;\otimes\; \neg\neg\, b \quad \vdash \quad \neg\neg\, (a \;\otimes\; b)$$

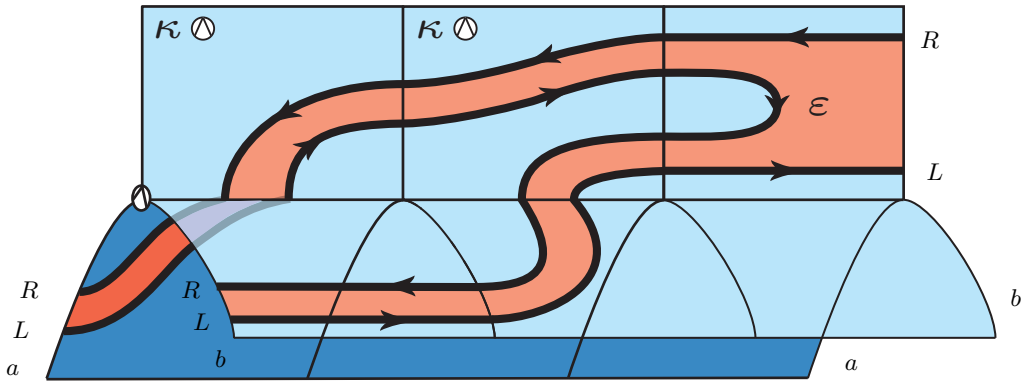
alternatively written as

$$RL(a) \parr RL(b) \quad \vdash \quad RL(a \parr b)$$

in the style of chiralities. The left-to-right proof of the sequent factors then as a composite of three basic combinators:

$$\begin{array}{lcl} RL(a) \parr RL(b) & \xrightarrow{\kappa_{\parr}} & RL(a \parr RL(b)) \\ & \xrightarrow{\kappa_{\parr}} & RLRL(a \parr b) \\ & \xrightarrow{\varepsilon} & RL(a \parr b) \end{array}$$

Accordingly, the proof is depicted in string diagrams as the composite of the three combinators, describing a formula tree in motion:



Note that the very same proof of tensorial logic may be formulated in the sequent calculus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a} \quad \dfrac{}{b \vdash b}}{a\,,\, b \vdash a \otimes b}\,\text{Tensor}}{b\,,\, \neg\,(a \otimes b)\,,\, a \vdash}\,\text{Negation}}{\neg\,(a \otimes b)\,,\, a \vdash \neg\, b}\,\text{Negation}}{a\,,\, \neg\neg\, b\,,\, \neg\,(a \otimes b) \vdash}\,\text{Negation}}{\neg\neg\, b\,,\, \neg\,(a \otimes b) \vdash \neg\, a}\,\text{Negation}}{\neg\,(a \otimes b)\,,\, \neg\neg\, a\,,\, \neg\neg\, b \vdash}\,\text{Negation}}{\neg\neg\, a\,,\, \neg\neg\, b \vdash \neg\neg\,(a \otimes b)}\,\text{Negation}}{\neg\neg\, a \otimes \neg\neg\, b \vdash \neg\neg\,(a \otimes b)}\,\text{Tensor}$$

The flow of negations appearing in the string diagram coincides with the game-theoretic interpretation of the derivation tree as an innocent strategy in dialogue games:

	$RL\,(a)$	$\parr$	$RL(b)$	$\vdash$	$RL\,(a \parr b)$
move 1.					R
move 2.	R				
move 3.	L				
move 4.			R		
move 5.			L		
move 6.					L

6 Axioms and cuts

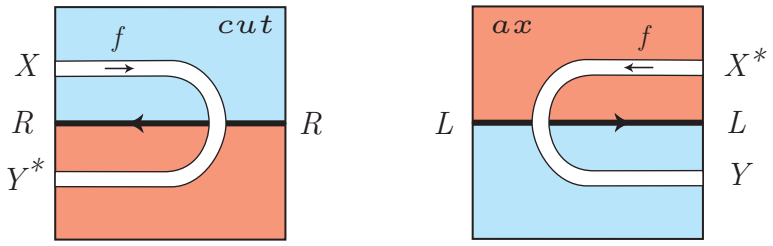
At this point, we describe the two generators ***cut*** and ***axiom*** corresponding to atomic axiom and cut rules in tensorial logic. Every morphism

$$f \quad : \quad X \quad \longrightarrow \quad Y \tag{6}$$

of the category $\mathscr{X}$ induces a pair of generators

$$\begin{array}{ccc} X \parr L(Y^* \otimes b) & \xrightarrow{cut_{f,b}} & Rb \\ La & \xrightarrow{ax_{f,a}} & X^* \otimes L(Y \parr a) \end{array}$$

which are depicted in the language of string diagrams as



Here, the white wire X on the category $\mathscr{A}$ (blue background) depicts the functor

$$a \mapsto X \parr a \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

whereas the white wire Y on the category $\mathscr{B}$ (red background) depicts the functor

$$b \mapsto Y \otimes b \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{B}.$$

Accordingly, given a morphism (6) in the underlying category $\mathscr{X}$, the arrow f tagged on the white wire in $\mathscr{A}$ (blue background) depicts the natural transformation

$$(X \parr -) \quad \Rightarrow \quad (Y \parr -) \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

whereas the arrow f tagged on the white wire in $\mathscr{B}$ (red background) depicts the natural transformation

$$(Y^* \otimes -) \quad \Rightarrow \quad (X^* \otimes -) \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{B}.$$

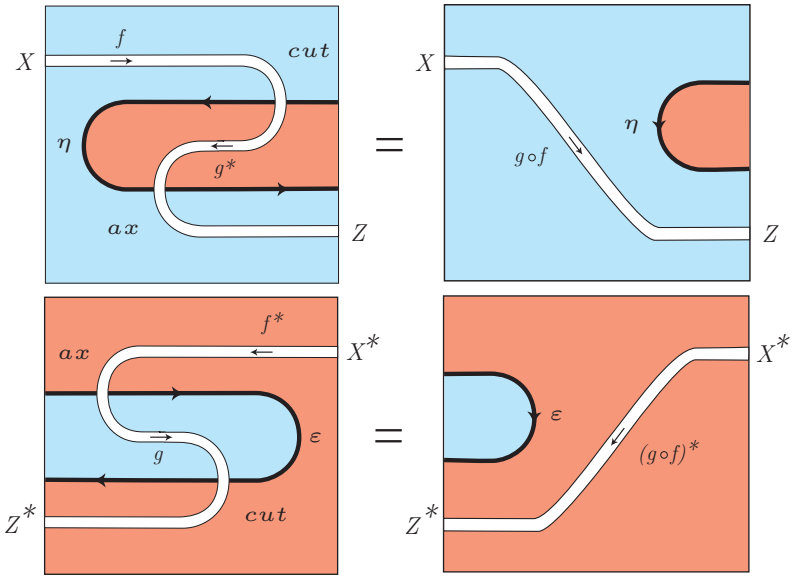
Note that the two generators ***cut*** and ***axiom*** are the chiral transcriptions of the sequent

$$X \ \otimes \ \neg \, (\, A \ \otimes \ Y) \quad \vdash \quad \neg \, A$$

which is provable in tensorial logic by the derivation tree

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{X \vdash Y}\, f}{A\ ,\ X \vdash A \otimes Y}\ \text{Tensor}}{A\ ,\ X\ ,\ \neg\,(\,A \otimes Y\,) \vdash}\ \text{Negation}}{A\ ,\ X \otimes \neg\,(\,A \otimes Y\,) \vdash}\ \text{Tensor}}{X \otimes \neg\,(\,A \otimes Y\,) \vdash \neg\, A}\ \text{Negation}$$

The interaction between the generators ***axiom*** and ***cut*** is regulated by two relations for each pair of morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in the category $\mathscr{X}$:



(7)

The generators ***axiom*** and ***cut*** together with the two relations above express that there exists a family of adjunctions

$$L\,(\,X \oslash -\,) \quad \dashv \quad R\,(\,X^* \oslash -\,)$$

natural in the object X. Note that the equality reduces to the cut-elimination rule of proof-nets in linear logic in the degenerate case when $L = R = id$.

7 The maps

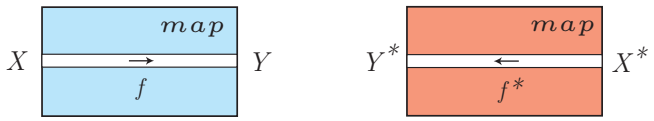
We describe the very last two generators of our algebra presentation of the free dialogue category **free-dialogue**$(\mathscr{X})$ on a category $\mathscr{X}$. The idea is that every morphism

$$f \quad : \quad X \quad \longrightarrow \quad Y$$

of the original category $\mathscr{X}$ induces a pair of generators:

$$X \xrightarrow{map(f)} Y$$
$$Y^* \xrightarrow{map(f^*)} X^*$$

depicted as follows in our string diagrams:



8 The Opponent-Proponent factorization theorem

In order to prove our coherence theorem for dialogue categories in Section 9, we establish that

Theorem 1 (Factorization) *Every sequence of generators picked among*

$$\varepsilon \quad \eta \quad \kappa_{\mathbb{O}} \quad \kappa_{\mathbb{I}} \quad \textbf{cut} \quad \textbf{axiom} \quad \textbf{map}$$

transforming a formula F into a formula G may be reorganized as a sequence of "Opponent" generators (or combinators)

$$\varepsilon \quad \kappa_{\mathbb{O}} \quad \textbf{cut}$$

transforming the formula F into a formula H_1, followed by a sequence of generators

$$\textbf{map}$$

transforming the formula H_1 into the formula H_2, followed by a sequence of "Proponent" generators (or combinators)

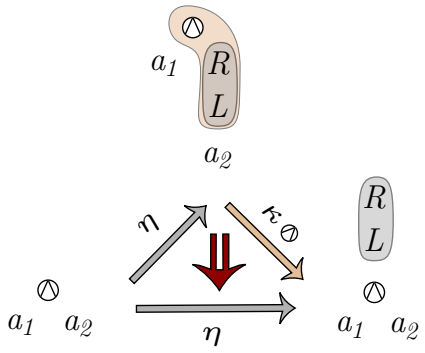
$$\eta \quad \kappa_{\mathbb{I}} \quad \textbf{axiom}$$

transforming the formula H_2 into the formula G.

The reorganization is achieved thanks to a family of basic rewrite rules $\Rightarrow$ which may be seen as the resolution of a critical pair in another term rewriting system $\mathcal{R}$ on tensorial formulas, whose rewriting rules are:

- the Opponent generators ε, $\kappa_{\parr}$ and ***cut***,
- the Proponent generators η, $\kappa_{\otimes}$ and ***axiom***, taken in the reverse direction: $\overline{\eta}$, $\overline{\kappa_{\otimes}}$ and $\overline{\textbf{\textit{axiom}}}$,
- the map generator ***map***.

By way of illustration, one of these rewriting rules orients the coherence law of the monadic strength, which appears in this way as a local confluence diagram for the critical pair between the two rewriting rules $\overline{\eta} : a_1 \parr RL(a_2) \to a_1 \parr a_2$ and $\kappa_{\parr} : a_1 \parr RL(a_2) \to RL(a_1 \parr a_2)$:



(8)

The overlapping patterns of the critical pair in the rewriting rules in $\mathcal{R}$ are colored in the term $a_1 \parr RL(a_2)$. The key observation is that the rewriting system $\mathcal{R}$ is strongly normalizing, at least when one removes the rewriting rule ***map***, and restricts it to the non trivial instances of the rewriting rules $\kappa_{\parr}$ and $\kappa_{\otimes}$. From this follows that the rewriting system $\Rightarrow$ itself is strongly normalizing, and thus reaches a normal form, which is a properly factorized sequence of transformations from F to G. This establishes our claim. To this, one should add the fact that all the rewriting rules $\Rightarrow$ between paths are commutative in a dialogue category – this point ensuring that the factorization procedure preserves the meaning of the morphism in any dialogue category.

9 The coherence theorem

Once the Opponent-Proponent factorization theorem is established for the proof $\pi : F \vdash G$, one may focus on the resulting sequence of "Opponent" generators from the formula F to the formula H_1, and show by a simple

combinatorial argument that every such sequence of generators factors as

$$F \xrightarrow{\kappa_{\mathbb{O}}} J_1 \xrightarrow{\textbf{\textit{cut}}} J_2 \xrightarrow{\varepsilon} H_1 \tag{9}$$

where each arrow is labeled by the name $\kappa_{\mathbb{O}}$, ***cut*** and ε of the generators appearing in it. Symmetrically, one establishes that the sequence of "Proponent" generators from H_2 to G factors as

$$H_2 \xrightarrow{\eta} K_2 \xrightarrow{\textbf{\textit{axiom}}} K_1 \xrightarrow{\kappa_{\mathbb{O}}} G \tag{10}$$

This decomposition of the proof $\pi : F \vdash G$ provides an algebraic counterpart to the notion of innocent strategy in dialogue games. A simple way to explain this connection with game semantics is to interpret the proof π as an innocent strategy σ in the dialogue category $\mathbf{innocent}(\mathscr{X})$ defined in Section 2. For notational convenience, we write the associated dialogue games in the same way as the formulas F and G. Now, the set of Proponent views of the strategy σ may be organized into a tree denoted $\lceil\ \sigma\ \rceil$ and called the *tree of Proponent views* of the strategy σ. Now, every Proponent view s of the strategy σ projects as

- an Opponent view $s \upharpoonright F$ on the dialogue game F,
- a Proponent view $s \upharpoonright G$ on the dialogue game G.

From this follows that the tree $\lceil\ \sigma\ \rceil$ induces a pair consisting of a tree of Opponent views $\lceil\ \sigma\ \rceil \upharpoonright F$ in the dialogue game F and a tree of Proponent views $\lceil\ \sigma\ \rceil \upharpoonright G$ in the dialogue game G. Moreover, every move in the dialogue games F and G appears once (and exactly once) in the tree of views, because the strategy σ is total. Hence, the tree $\lceil\ \sigma\ \rceil \upharpoonright F$ contains all the moves of F and the tree $\lceil\ \sigma\ \rceil \upharpoonright G$ contains all the moves of G. At this point, a good idea is to think of $\lceil\ \sigma\ \rceil \upharpoonright F$ and $\lceil\ \sigma\ \rceil \upharpoonright G$ as dialogue games themselves, rather than trees. The combined fact that $\lceil\ \sigma\ \rceil \upharpoonright G$ has the same moves as G, and that it a tree of Proponent views of G leads to the equation:

$$K_1 = \lceil\ \sigma\ \rceil \upharpoonright G$$

and symmetrically for the Opponent views:

$$J_1 = \lceil\ \sigma\ \rceil \upharpoonright F.$$

This justifies to think of the combinators $\kappa_{\mathbb{O}}$ and $\kappa_{\mathbb{O}}$ as the respective constructors of the Opponent and Proponent views of the strategy σ. The strategy τ interpreting the proof between J_1 and K_1 is simply the result of the decomposition of the tree of Player views $\lceil\ \sigma\ \rceil$ into the two dialogue games J_1

and K_1. In particular, every Proponent view of the strategy τ is a branch in $\lceil\ \sigma\ \rceil$, and thus projects as a branch of J_1 and as a branch of K_1.

This correspondence between the Opponent-Proponent decomposition of the proof $\pi : F \vdash G$ and the structure of Player views $\lceil\ \sigma\ \rceil$ of the associated innocent strategy σ is so tight that it is reversible: namely, every innocent strategy $\sigma : F \vdash G$ factors as (9) followed by (10) where J_1 and K_1 are defined as the dialogue games:

$$J_1 = \lceil\ \sigma\ \rceil \upharpoonright F \qquad\qquad K_1 = \lceil\ \sigma\ \rceil \upharpoonright G.$$

An easy combinatorial argument (by induction) establishes that the two strategies

$$F \longrightarrow J_1 \qquad\qquad K_1 \longrightarrow G$$

induced from σ are constructed by applying a series of combinators $\kappa_{\mathbb{O}}$ and $\kappa_{\mathbb{P}}$ respectively, and another easy combinatorial argument (by induction again) that the strategy

$$J_1 \longrightarrow K_1$$

decomposes as a sequence of ***cut***, ε, ***map***, η and ***axiom*** combinators. This establishes that every functor

$$\mathcal{F} \quad : \quad \mathscr{X} \quad \longrightarrow \quad \mathscr{D}$$

to a dialogue category $\mathscr{D}$ may be lifted as a functor

$$\mathcal{G} \quad : \quad \mathbf{innocent}(\mathscr{X}) \quad \longrightarrow \quad \mathscr{D}$$

by defining the functor $\mathcal{G}$ on the objects by induction on the dialogue games

- $\mathcal{G}(\neg A) = \neg\, \mathcal{G}(A)$
- $\mathcal{G}(A \otimes B) = \mathcal{G}(A) \otimes \mathcal{G}(B)$
- $\mathcal{G}(\iota X) = \mathcal{F}(X)$ for every object X of the category $\mathscr{X}$

and on the morphisms by decomposing every innocent strategy $\sigma : A \longrightarrow B$ into a sequence of basic combinators and then by transporting each of these combinators to the corresponding combinator in $\mathscr{D}$. This induces a morphism

$$\mathcal{G}(\sigma) \quad : \quad \mathcal{G}(A) \quad \longrightarrow \quad \mathcal{G}(B)$$

which is uniquely determined because the factorization of an innocent strategy is unique modulo the series of commutative diagrams considered in the proof of Opponent-Proponent factorization. From this follows that the functor $\mathcal{G}$ is properly defined, and is moreover a functor of dialogue categories. From this follows the main result of the paper:

Theorem 2 (Coherence theorem)
The category ***innocent***$(\mathscr{X})$ *of dialogue games and total innocent strategies is equivalent to the free dialogue category* ***free-dialogue***$(\mathscr{X})$ *generated by the category* $\mathscr{X}$*.*

It should be mentioned at this point that it would be also possible to establish the coherence theorem by more traditional techniques directly adapted from game semantics. The resulting proof would be simpler but much less informative. In particular, the alternative proof would not reveal the relationship between game semantics and string diagrams, nor the algebraic presentation of tensorial proofs by the generators and relations. This combinatorial and algebraic aspect would have remained concealed... Note that, on the other hand, we have recently used this alternative proof technique in order to establish a coherence theorem for a braided notion of dialogue categories, where proofs are manipulated in the same way as topological tangles, see [19, 20] for details.

10 Related works

The phenomenon of self-adjunction is mentioned by Kock [12] and stressed by Thielecke [27] in his work on continuations. The notion of dialogue category was inspired by the key observation by Hoffman and Streicher [6] and then Selinger [23] that every denotational model of classical logic (that is, the $\lambda\mu$-calculus) boils down to a continuation model. The formulation of dialogue categories as chiralities is inspired by Girard's work on polarities in LC [4] and its relationship to dialogue games noticed by Laurent [15], together with the correspondence between games, polarities and continuations originally investigated with Peter Selinger [21]. Note that similar ideas of symmetry appeared independently in the work by Cockett and Seely [2].

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Appendix. Tensorial logic. The formulas of tensorial logic are defined by the grammar

$$A, B \quad ::= \quad X \quad | \quad A \otimes B \quad | \quad \mathbf{1} \quad | \quad \bot$$

where X is an atom of the logic. A sequent of tensorial logic is a pair

$$A_1, \ldots, A_n \quad \vdash \quad B$$

consisting of a (possibly empty) sequence of formulas $A_1, ..., A_n$ called the hypothesis of the sequent, and of a (unique) formula B called the conclusion of the sequent. The derivation trees of tensorial logic are constructed by the following rules:

Axiom	$\dfrac{}{A \vdash A}$
Cut	$\dfrac{\Gamma \vdash A \qquad \Upsilon_1, A, \Upsilon_2 \vdash B}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B}$
Right negation	$\dfrac{A, \Gamma \vdash \bot}{\Gamma \vdash \neg A}$
Left negation	$\dfrac{\Gamma \vdash A}{\Gamma, \neg A \vdash \bot}$
Right $\otimes$	$\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$
Left $\otimes$	$\dfrac{\Upsilon_1, A, B, \Upsilon_2 \vdash C}{\Upsilon_1, A \otimes B, \Upsilon_2 \vdash C}$
Right 1	$\dfrac{}{\vdash 1}$
Left 1	$\dfrac{\Upsilon_1, \Upsilon_2 \vdash A}{\Upsilon_1, 1, \Upsilon_2 \vdash A}$
Exchange	$\dfrac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$

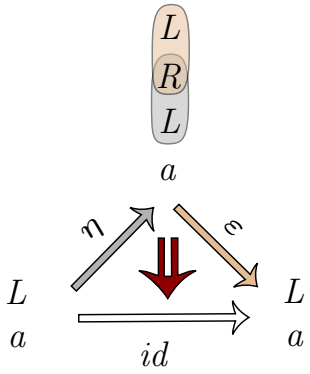
As in the case of minimal logic and free cartesian-closed categories generated by a category $\mathscr{X}$, it is natural to extend the logic (in that case, tensorial logic) with a rule:

$$\mathrm{Map}[f] \qquad \frac{}{X \vdash Y}$$

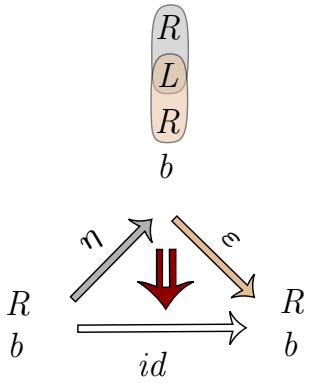
for each morphism $f : X \to Y$ in the category $\mathscr{X}$ one starts from.

Appendix. Coherence diagrams. We collect in this Appendix the relations of our algebraic presentation of the free dialogue category. Each of these relations may be seen as a specific coherence diagram of the underlying dialogue category. The proof of our coherence theorem in § 8 and § 9 is based on the observation that each of these coherence diagrams is a local confluence diagram for a critical pair of the rewriting system $\mathcal{R}$ described in § 8. The convention for colors is *light brown* for Opponent combinators and *light grey* for Proponent combinators. The orientation provided by the red 2-dimensional arrow expresses the direction in which the 2-dimensional rewriting system reorganizes (permutes) the Opponent combinators *before* the Proponent combinators in order to factorize the sequence of combinators.

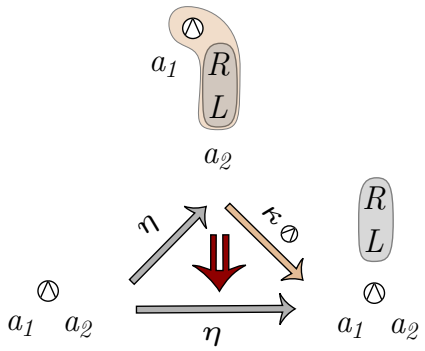
A. Unit vs. counit: triangular laws of an adjunction. The first triangular equation of adjunctions is formulated as a string diagram in Equation (4) in § 3.



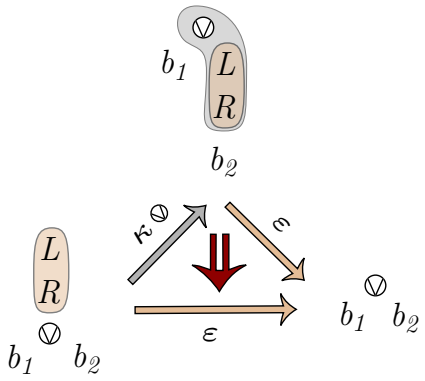
The second triangular equation of adjunctions is depicted in Equation (5) of § 3.



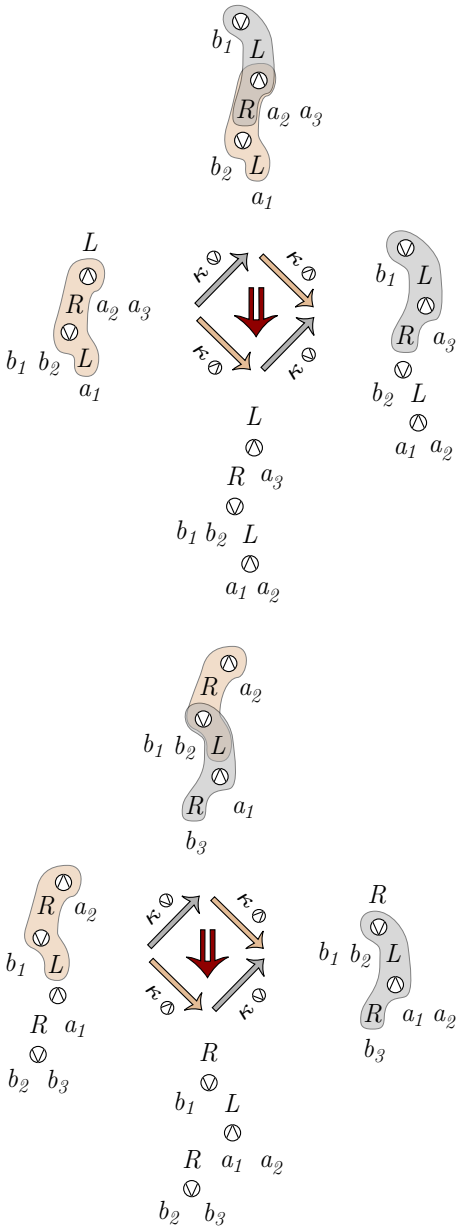
B. Unit vs. conjunctive strength: coherence laws of a monadic strength.



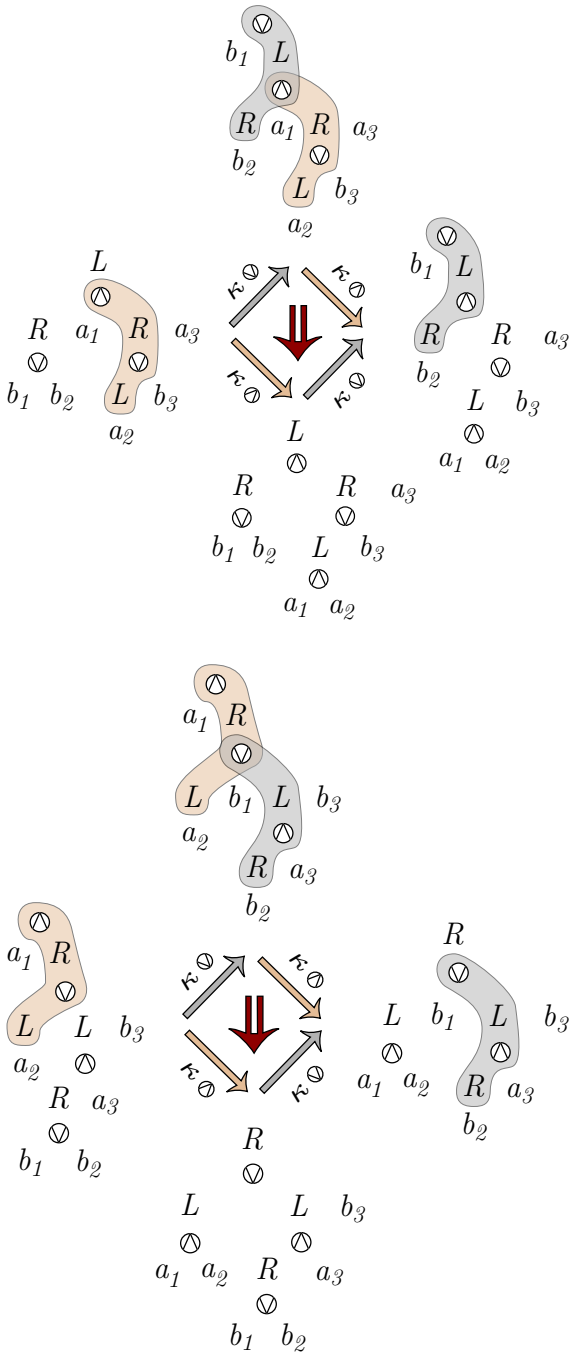
B-dual. Unit vs. conjunctive strength: coherence laws of a comonadic strength. The coherence diagram below are the dual of the coherence diagrams in B.



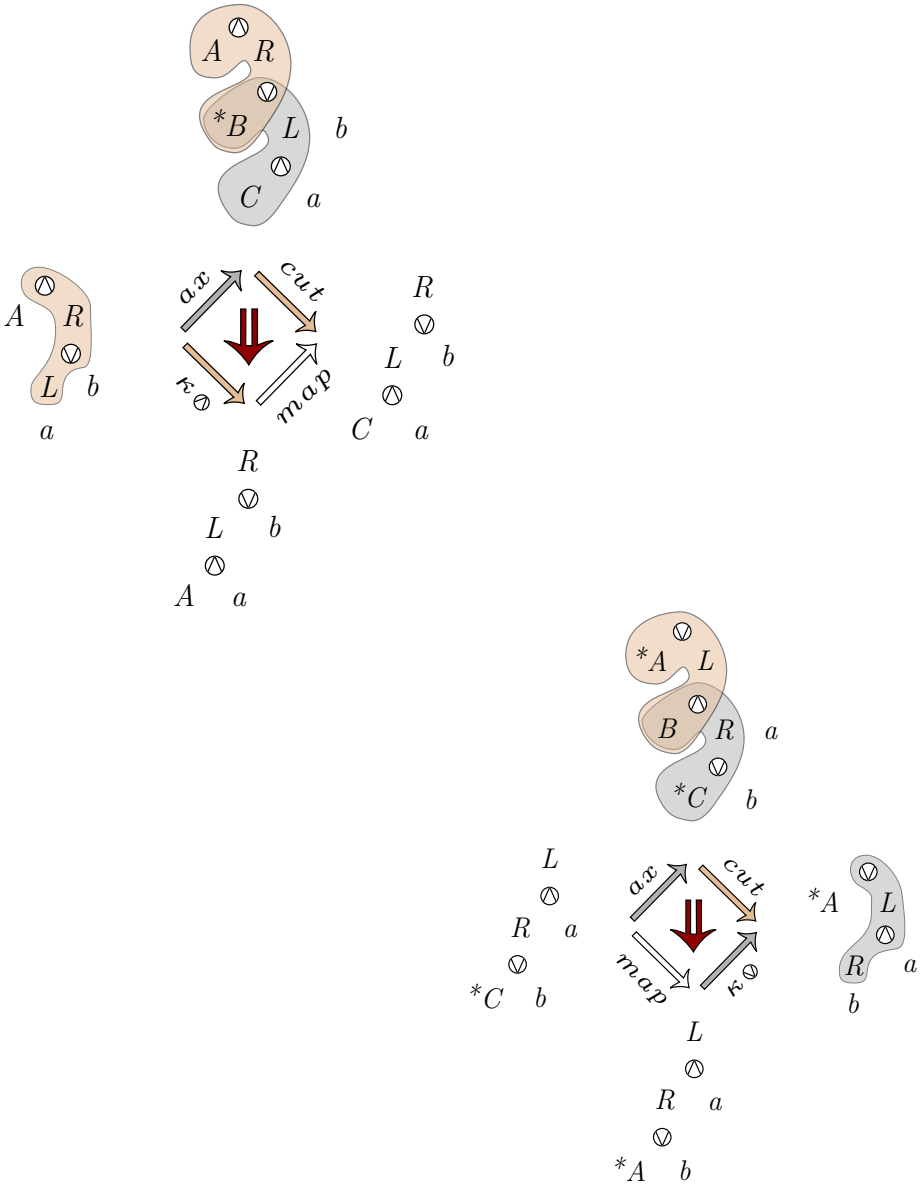
C. Disjunctive strength vs. conjunctive strength: square. A coherence diagram explaining how a disjunctive strength commutes with a conjunctive strength, followed by the dual coherence diagram.



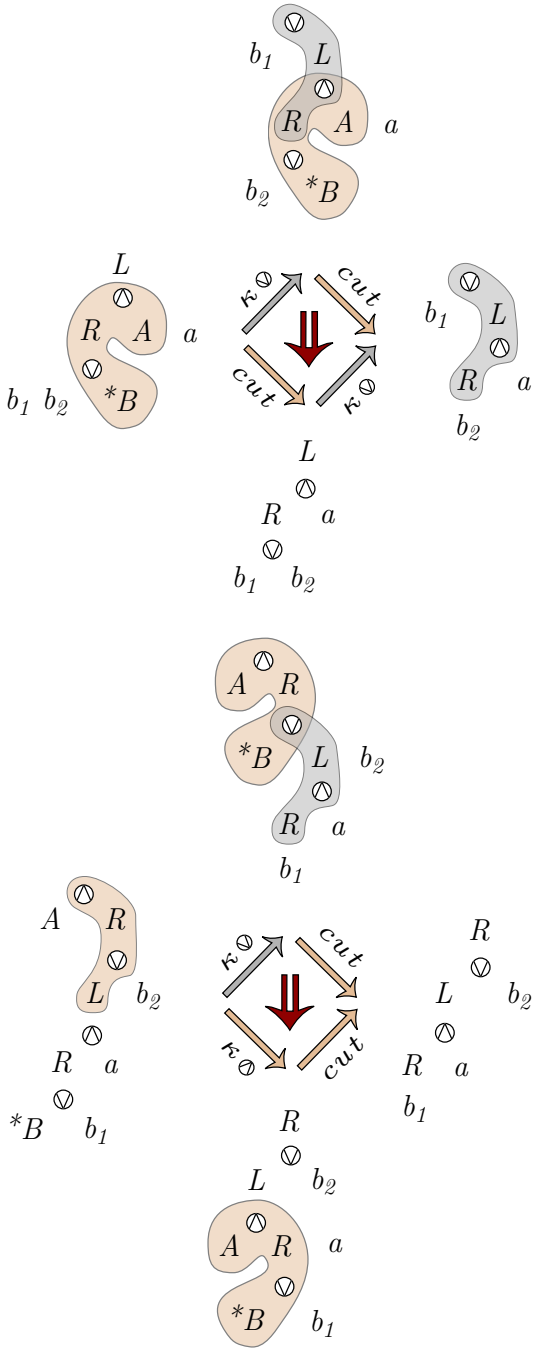
D. Disjunctive strength vs. conjunctive strength: square. Another coherence diagram for a disjunctive strength and a conjunctive strength, again followed by its dual coherence diagram.



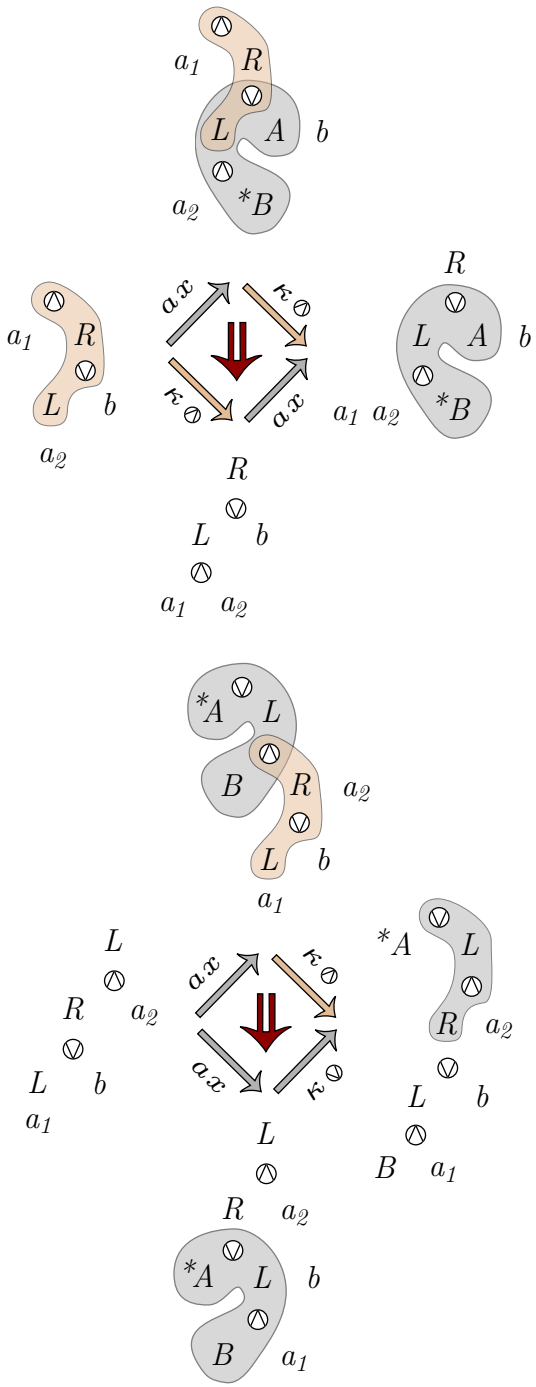
E. Axiom vs. cut. We give below the two coherence diagrams describing how an axiom and a cut interact. Quite interestingly, they are dual coherence diagrams. Note also that the first coherence diagram implies (together with the coherence diagram relating η with $\kappa_{\parr}$ in B) the first zig-zag equality formulated as string diagram in Equation (7) of § 6. Dually, the second coherence diagram implies (together with the coherence diagram relating ε with $\kappa_{\otimes}$ in B-dual) the second zig-zag equality in Equation (7) of § 6.



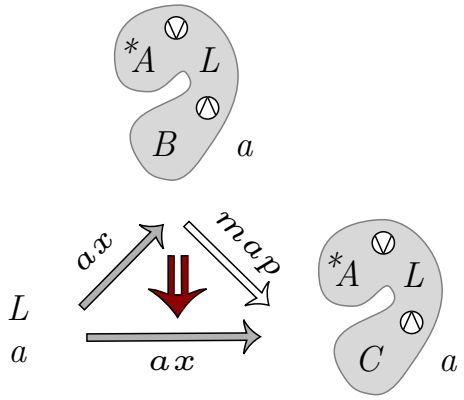
F. Disjunctive strength vs. cut. Two coherence diagrams between a disjunctive strength and an axiom.

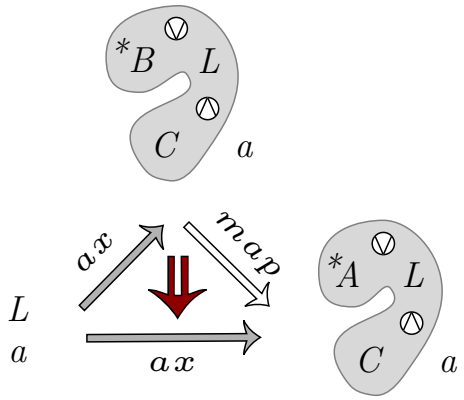


F-dual. Axiom vs. conjunctive strength. The two coherence diagrams below are the dual to the two coherence diagrams given in F.



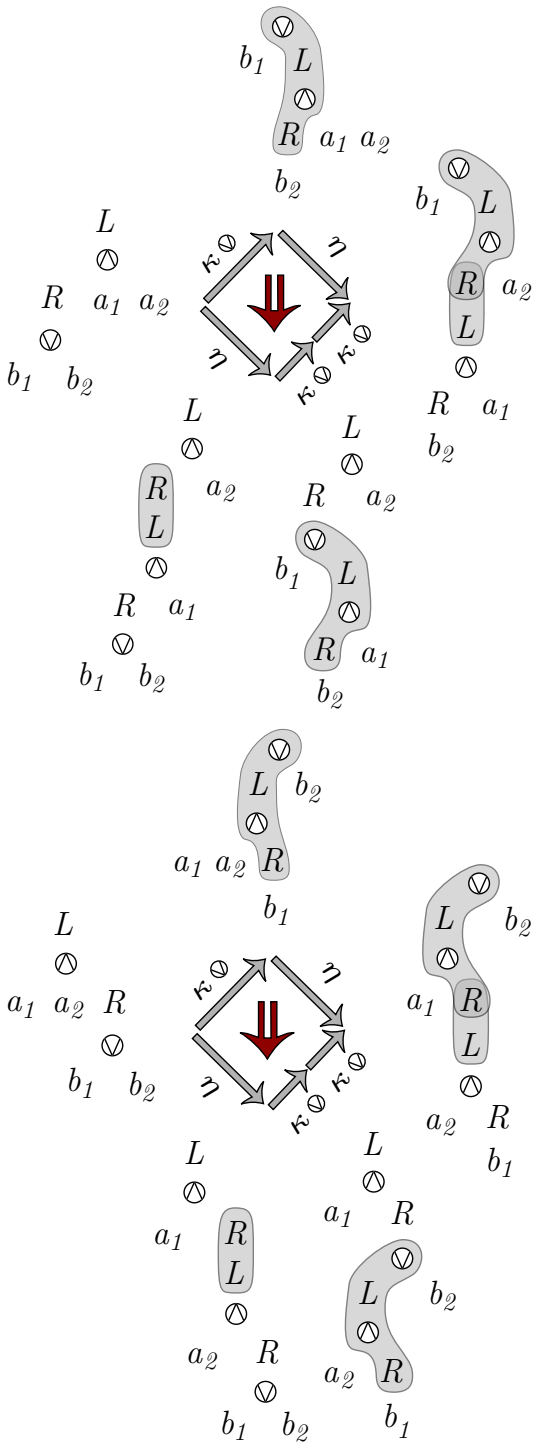
G. Axiom vs. map. This permutation rule describes the elementary interaction between an axiom link and a map link.





G-dual. Map vs. cut. Exactly the same diagram as in G, dualized.

H. Disjunctive strength vs. unit



I. Axiom vs. unit.

