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Virtual double categorical construction of a 2-dimensional model of differential linear logic

Soichiro Fujii (National Institute of Informatics)

Based on discussions with Keisuke Hoshino (RIMS, Kyoto University).

Special thanks to Zeinab Galal (RIMS, Kyoto University)
for pointers to the literature.

Context.

Model of Differential Linear Logic → Rel

Rel ⟲ ! ← free commutative monoid (co)monad

[Blute-Cockett-Seely, Differential categories, 2006]

(vertical) categorification ↓

2-dimensional model of Differential Linear Logic → Prof

Prof ⟲ ! ← free symmetric strict monoidal category pseudo (co)monad

[Fiore-Gambino-Hyland, Monoidal bicategories, differential linear logic, and analytic functors, 2025]

Kleisli ↘

Prof! — bicategory of generalized species of structures

[Joyal, Fiore-Gambino-Hyland-Winskel, ...]

Composition in Prof! generalizes substitution monoidal product on $[\mathbb{F}_{bij}, \mathbf{Set}] \cong \mathbf{Prof}!(1,1)$.

$\exists$ subst. mon. str. on $[\mathbb{F}_{id}, \mathbf{Set}]$, $[\mathbb{F}_{inj}, \mathbf{Set}]$, $[\mathbb{F}_{surj}, \mathbf{Set}]$, $[\mathbb{F}, \mathbf{Set}]$, ...

Q. Do these also arise from suitable Kleisli bicategories Prof! ?

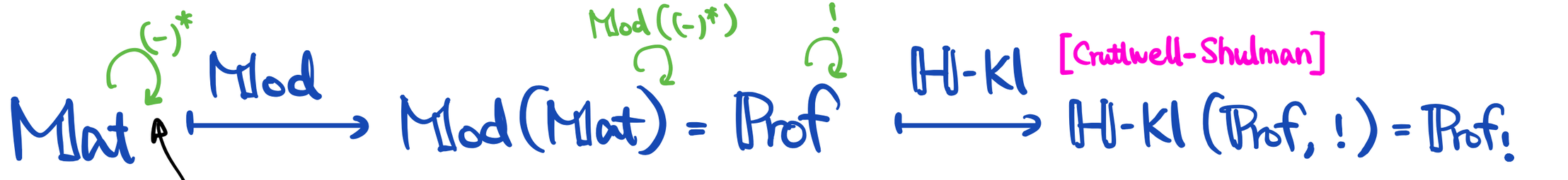
A. Yes – and they can be constructed via the theory of pseudo distributive laws.

[Tanaka-Power, Curien, Fiore-Gambino-Hyland-Winskel]

This talk.

Construction of various "(horizontal) Kleisli virtual double categories" $\mathbb{P}\mathrm{rof}_!$ containing $\mathbb{P}\mathrm{rof}_!$ (and hence substitution monoidal categories $([W, \mathrm{Set}], \bullet)$).

- No coherence axioms.
  (10 coherence axioms for pseudo dist. law)
- We construct a (v. dbl.) monad on a simple setting, and then let general machinery finish the rest of construction.



sets
functions
matrices
($\simeq$ spans)

categories
functors
profunctors

categories
functors
"generalized species"

$W \in \{\mathbb{F}_{\mathrm{bij}}, \mathbb{F}, \ldots\}$
(verbal category [Tronin, 2002] or
faithful cartesian club [Shulman, 2016])

$H(\mathbb{P}\mathrm{rof}_!)(1,1) \cong [W, \mathrm{Set}]$

Substitution monoidal structure

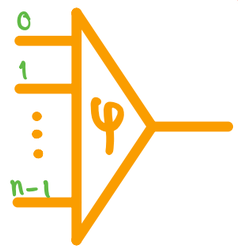
$\mathbb{F}$: category with

obj. finite ordinals $n = \{0, 1, \cdots, n-1\}$

mor. all functions

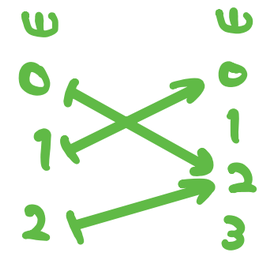
$X : \mathbb{F} \longrightarrow \mathbf{Set}$

$\varphi \in X_n$ means

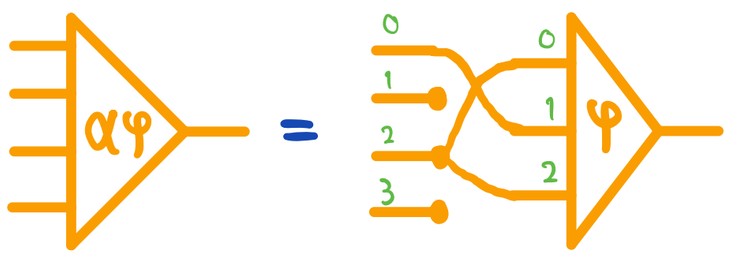


$\varphi(x_0, x_1, \cdots, x_{n-1})$
(n-ary term)

For $\alpha : 3 \longrightarrow 4$ in $\mathbb{F}$ and $\varphi \in X_3$,



$\alpha\varphi \in X_4$ is :

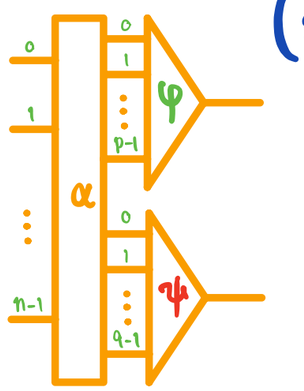


$(\alpha\varphi)(x_0, x_1, x_2, x_3) = \varphi(x_2, x_0, x_2)$

$[\mathbb{F}, \mathbf{Set}]$ has...

Day convolution induced by $(\mathbb{F}, +)$

$$(X * Y)_n = \int^{p,q \in \mathbb{F}} \mathbb{F}(p+q, n) \times X_p \times Y_q$$



$(\cong X_n \times Y_n)$

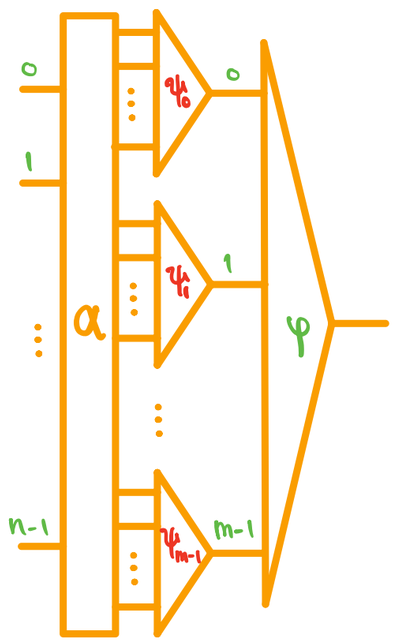
↳ Not true for other verbal cats.

"parallel composition"

Substitution monoidal product

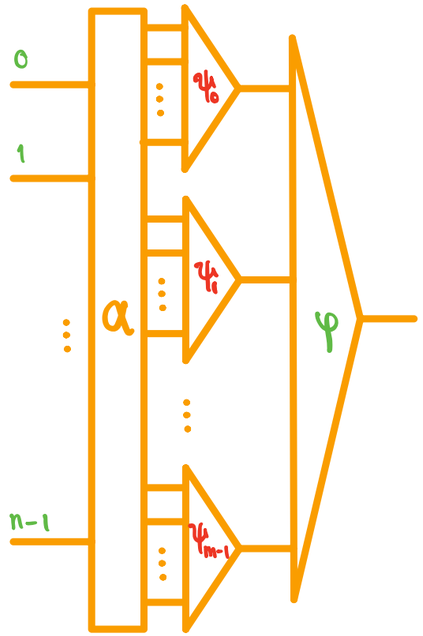
$$(X \bullet Y)_n = \int^{m \in \mathbb{F}} X_m \times (Y^{*m})_n$$

"sequential composition"



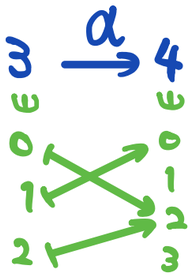
$$(X \bullet Y)_n = \int^{m \in \mathbb{F}} X_m \times (Y^{*m})_n$$

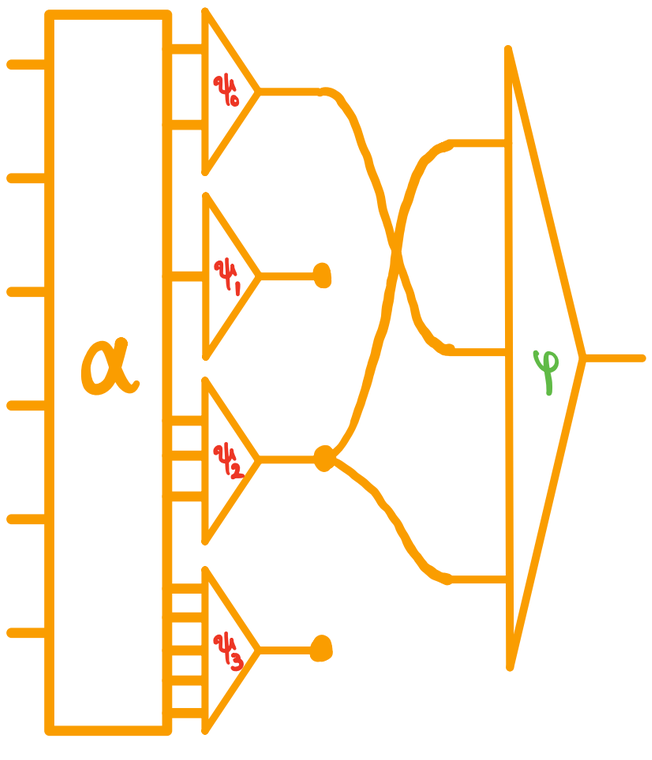
$$= \left(\sum_{m \in \mathbb{N}} X_m \times (Y^{*m})_n \right) / \sim$$



What is $\sim$?

For example,



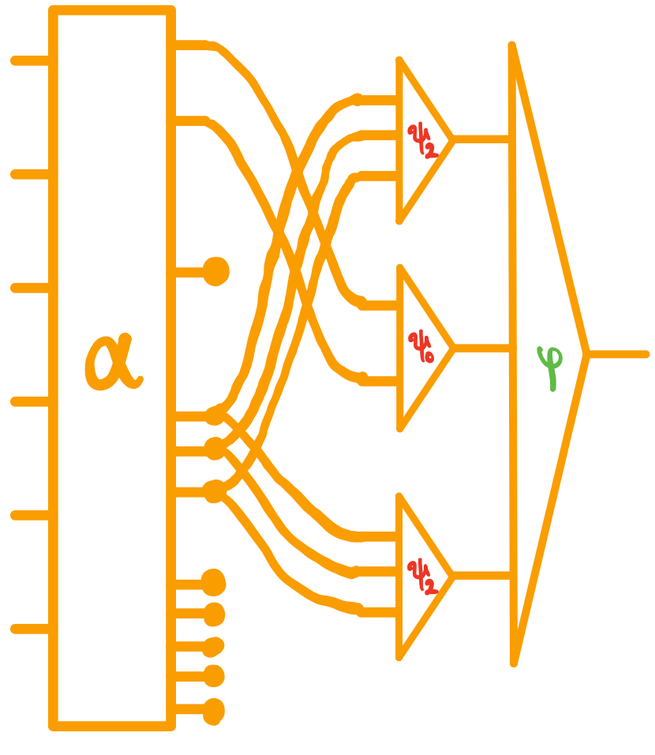


$$(X \bullet Y)_6$$

$$X_4 \times (Y^{*4})_6 \qquad X_3 \times (Y^{*3})_6$$

$$X_\alpha \times 1 \qquad 1 \times (Y^{*\alpha})_6$$

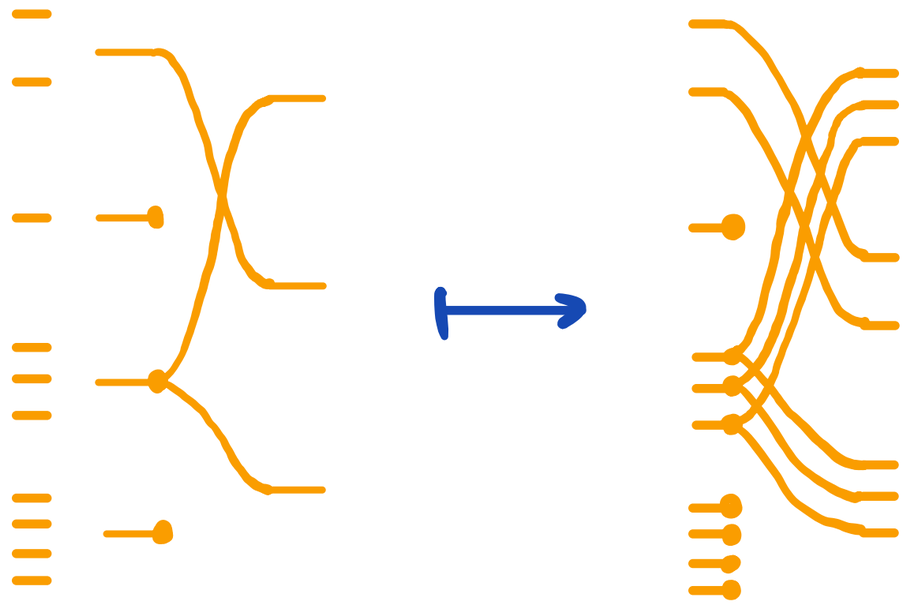
$$X_3 \times (Y^{*4})_6$$



We can perform this construction by replacing $\mathbb{F}$ with its wide subcategory $\mathbb{W} \subseteq \mathbb{F}$, provided that:

$ob(\mathbb{W}) = ob(\mathbb{F}) = \mathbb{N}$

- $\mathbb{W}$ is closed under $\mathbb{F}\times\mathbb{F} \xrightarrow{+} \mathbb{F}$ (so that $[\mathbb{W}, Set]$ has Day convolution product)
- $\mathbb{W}$ is closed under the "thickening" operation



Given $m \xrightarrow{\alpha} n$ in $\mathbb{W}$ and $\vec{k} = (k_0, k_1, \dots, k_{n-1}) \in \mathbb{N}^n$,

$$\sum_{i\in m} k_{\alpha(i)} \xrightarrow{\alpha * \vec{k}} \sum_{j\in n} k_j$$

is in $\mathbb{W}$.

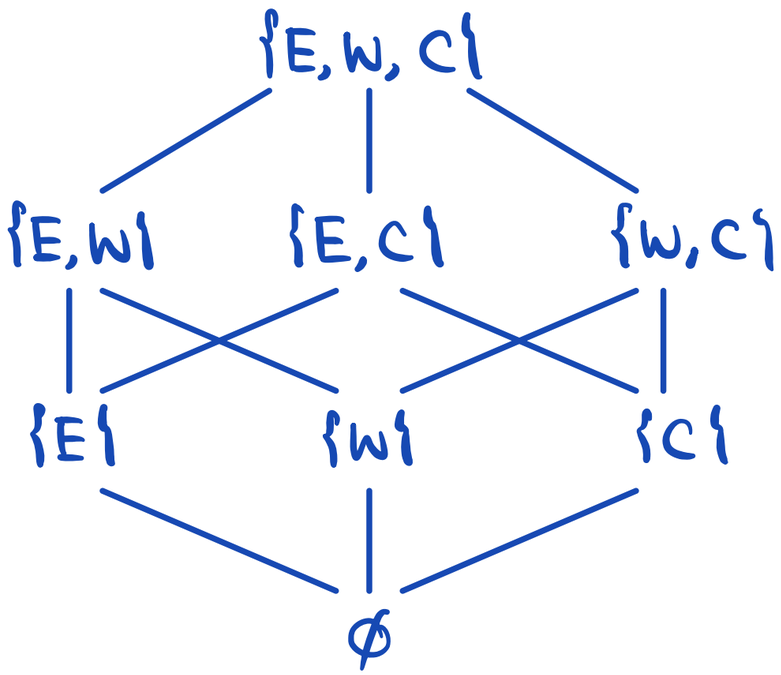
↑ Rearranges "blocks" according to α.

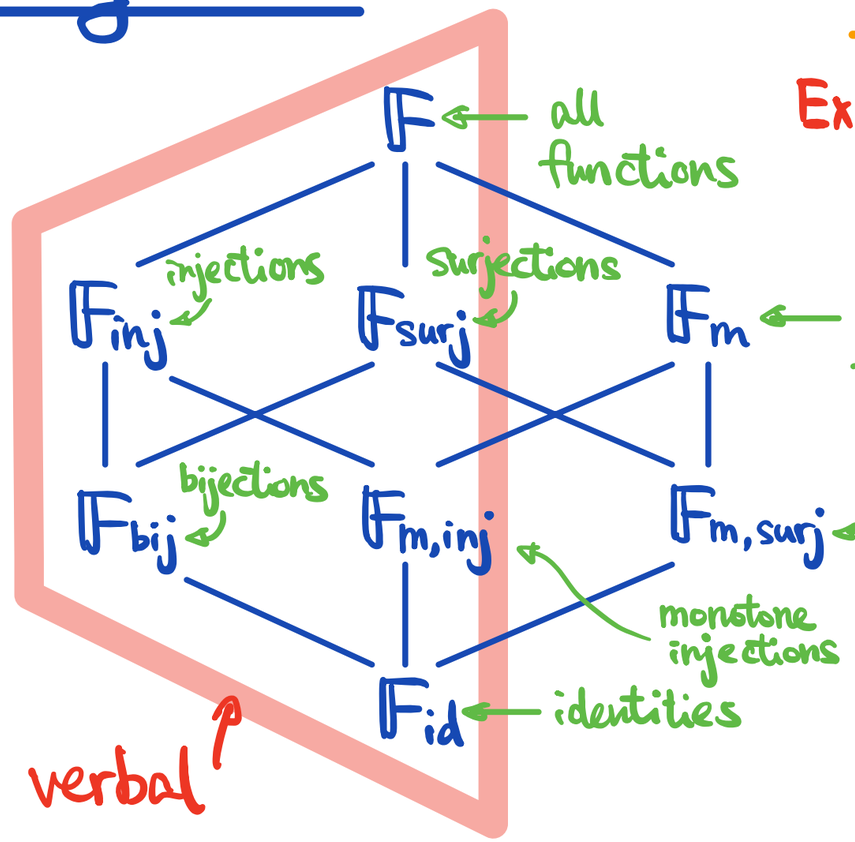
Such $\mathbb{W}$ are known as **verbal categories** or **faithful cartesian clubs**.

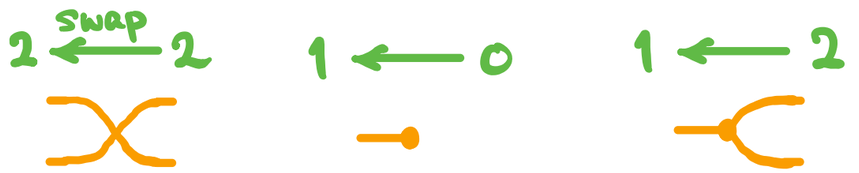
[Tronin, Abstract clones and operads, 2002]

[Shulman, Categorical logic from a categorical point of view, 2016]

Examples of verbal categories









(cf. [Curien, Operads, clones, and distributive laws, 2012])

Thm. [Lenke-Milius-Urbat, A unified treatment of substitution for presheaves, nominal sets, renaming sets, and so on, 2026]

$\lfloor {}^{\forall}\mathbb{W} \subseteq \mathbb{F}$: verbal cat., $[\mathbb{W}, \mathbf{Set}]$ has a substitution monoidal str. $\bullet$.

Monoids in $([\mathbb{W}, \mathbf{Set}], \bullet)$ = $\mathbb{W}$-operads [Tonin]

Intuitively, algebraic theory expressible with structural rules corresponding to $\mathbb{W}$.

E.g. $x \cdot y = y \cdot x$ requires E

$x = x \cdot x$ requires C

Thm. [Fujii-Hoshino]

$\forall W \subseteq \mathbb{F}$: verbal cat., $\exists$ virtual double cat. $\mathbb{P}\mathrm{rof}_!$ containing the underlying multicategory of $([W, \mathbf{Set}], \bullet)$ (as $\mathcal{H}\mathbb{P}\mathrm{rof}_!(1,1)$).

Sufficient to define e.g. W-operads.

[Tanaka-Power, Curien, Fiore-Gambino-Hyland-Winskel]

Classical substitution monoidal categories $[\mathbb{F}_{bij}, \mathbf{Set}]$, $[\mathbb{F}, \mathbf{Set}]$, ...	Kleisli bicategories $\mathbb{P}\mathrm{rof}_!$ relative to $\mathbb{F}_{bij}$, $\mathbb{F}$, ...
[Lenke-Milius-Urbat]	[Fujii-Hoshino]
Substitution mon. cat. $[W, \mathbf{Set}]$ for any verbal cat. W.	Kleisli virtual double categories $\mathbb{P}\mathrm{rof}_!$ relative to any verbal cat. W.

Virtual double categories

[Burroni, T-catégories, 1971] "multicatégorie"
[Leinster, Higher operads, higher categories, 2004] "fc-multicategory"
[Cruttwell-Shulman, A unified framework for generalized multicategories, 2010]

Def.

A virtual double category $\mathbb{D}$ consists of:

- objects $A, B, \dots$
- vertical morphisms $\begin{smallmatrix} A \\ \downarrow f \\ B \end{smallmatrix}, \dots$

] Form a category $\mathbb{D}_{vert}$ called the vertical category of $\mathbb{D}$.

- horizontal morphisms $A \overset{M}{\nrightarrow} B, \dots$ ← No composition operation for these!

- multicells

$$\begin{array}{ccccccc} A_0 & \overset{M_1}{\nrightarrow} & A_1 & \overset{M_2}{\nrightarrow} & \cdots & \overset{M_n}{\nrightarrow} & A_n \\ {\scriptstyle f}\downarrow & & \Downarrow \alpha & & & & \downarrow{\scriptstyle g} \\ B & & & & \xrightarrow[N]{} & & C \end{array}$$

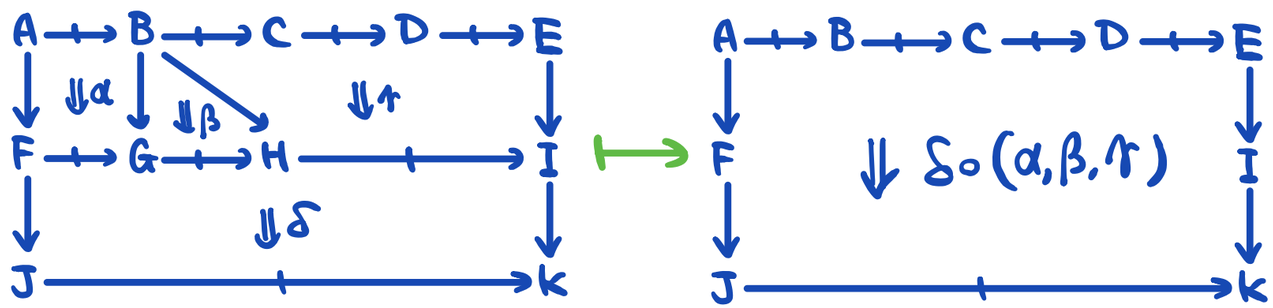
- identity cell

$$\begin{array}{ccc} A & \overset{M}{\nrightarrow} & B \\ \| & \Downarrow id_M & \| \\ A & \underset{M}{\nrightarrow} & B \end{array}$$

for each h.mor. $A \overset{M}{\nrightarrow} B$

- unital & associative substitution operation:

E.g.



Ex. [Leinster, Cruttwell-Shulman]

Mat

obj. sets

v.mor. functions

$\mathbb{M}\mathrm{at}_{\mathrm{vert}} = \mathrm{Set}$

h.mor. $\dfrac{A \overset{M}{\nrightarrow} B \text{ in } \mathbb{M}\mathrm{at}}{(M(a,b) \in \mathrm{Set})_{a \in A, b \in B}}$

multicell (suitable maps)

Ex. [Leinster, Cruttwell-Shulman]

Prof

obj. categories

v.mor. functors

$\mathbb{P}\mathrm{rof}_{\mathrm{vert}} = \mathrm{Cat}$

h.mor. $\dfrac{\mathcal{A} \overset{M}{\nrightarrow} \mathcal{B} \text{ in } \mathbb{P}\mathrm{rof}}{\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \xrightarrow{M} \mathrm{Set}}$

multicell (suitable nat. tr.)

Def. [Leinster, Cruttwell-Shulman]

$\mathbb{D}$: v. dbl. cat.

Then $\mathbb{M}\mathrm{od}(\mathbb{D})$: v. dbl. cat. with

obj. horizontal monoids in $\mathbb{D}$

$\left(A \overset{T}{\to} A,\ \begin{array}{ccccc} A & \overset{T}{\to} & A & \overset{T}{\to} & A \\ \| & & \Downarrow \mu & & \| \\ A & & \xrightarrow[T]{} & & A \end{array},\ \begin{array}{ccc} A & & A \\ & \overset{\Downarrow \eta}{=} & \\ A & \xrightarrow[T]{} & A \end{array}\right)$

v.mor. homomorphisms

$$\begin{array}{ccc} A & \overset{T}{\to} & A \\ f \downarrow & \Downarrow \omega & \downarrow f \\ B & \underset{S}{\to} & B \end{array}$$

$(A,T,\dots) \xrightarrow{(f,\omega)} (B,S,\dots)$

h.mor. bimodules $(A,T,\dots) \xrightarrow{(M,\lambda,P)} (B,S,\dots)$

$$\begin{array}{ccccc} A & \overset{T}{\to} & A & \overset{M}{\to} & B \\ \| & & \Downarrow \lambda & & \| \\ A & & \xrightarrow[M]{} & & B \end{array} \qquad \begin{array}{ccccc} A & \overset{M}{\to} & B & \overset{S}{\to} & B \\ \| & & \Downarrow P & & \| \\ A & & \xrightarrow[M]{} & & B \end{array}$$

$\mathbb{P}\mathrm{rof} = \mathbb{M}\mathrm{od}(\mathbb{M}\mathrm{at})$ $\quad \upsilon\mathbb{D}\mathrm{bl} \xrightarrow{\mathbb{M}\mathrm{od}} \upsilon\mathbb{D}\mathrm{bl}$: 2-functor ($\Rightarrow$ preserves monads)

Each verbal category $\mathbb{W} \subseteq \mathbb{F}$ induces a monad on Mat.

$\mathbb{W} \subseteq \mathbb{F}$ is a verbal category

$\iff$ $ob(\mathbb{W}) = ob(\mathbb{F}) = \mathbb{N}$

$\mathbb{W}$ closed under

- $\mathbb{F} \times \mathbb{F} \xrightarrow{+} \mathbb{F}$
- "thickening" = 

Mat

obj. sets

v.mor. functions

$\Big]$ $\mathrm{Mat}_{vert} = \mathrm{Set}$

h.mor. $A \xrightarrow{M} B$ in Mat

$(M(a,b) \in \mathrm{Set})_{a \in A, b \in B}$

$\mathrm{Mat} \xrightarrow{(-)^*} \mathrm{Mat}$

$(-)^*_{vert}$ = list monad on Set

obj. $A \longmapsto A^* \left(= \sum_{n \in \mathbb{N}} A^n\right)$

v.mor. $\begin{matrix} A \\ \downarrow f \\ B \end{matrix} \longmapsto \begin{matrix} A^* \\ \downarrow f^* \\ B^* \end{matrix}$

h.mor. $A \xrightarrow{M} B \longmapsto A^* \xrightarrow{M^*} B^*$

$\Big]$ μ

μ for any $A \xrightarrow{M} B$,

$$\begin{array}{ccc} A^{**} & \xrightarrow{M^{**}} & B^{**} \\ {\scriptstyle \mu_A}\downarrow & \Downarrow \mu_M & \downarrow{\scriptstyle \mu_B} \\ A^* & \xrightarrow[M^*]{} & B^* \end{array}$$

$$M^*((a_i)_{i \in m}, (b_j)_{j \in n}) = \sum_{\alpha \in \mathbb{W}(m,n)} \prod_{i \in m} M(a_i, b_{\alpha(i)}).$$

Defined using + and thickening.

$\mathbb{V}\mathrm{Dbl} \xrightarrow{\mathrm{Mod}(-)} \mathbb{V}\mathrm{Dbl}$: 2-functor

$\mathbb{M}\mathrm{at}$ ↻ $(-)^*$ $\longmapsto$ $\mathbb{P}\mathrm{rof}$ ↻ $!$ ← virtual double monad on $\mathbb{P}\mathrm{rof}$.

$\mathbb{P}\mathrm{rof} \xrightarrow{\ !\ } \mathbb{P}\mathrm{rof}$

obj. $\mathcal{A} \longmapsto !\mathcal{A}$

$$ob(!\mathcal{A}) = (ob\,\mathcal{A})^*$$
$$(!\mathcal{A})\left((a_i)_{i\in m}, (a'_j)_{j\in n}\right) = \sum_{\alpha\in W(m,n)} \prod_{i\in m} \mathcal{A}(a_i, a'_{\alpha(i)})$$

h.mor. $\mathcal{A} \xrightarrow{M} \mathcal{B} \longmapsto !\mathcal{A} \xrightarrow{!M} !\mathcal{B}$

$$(!M)\left((a_i)_{i\in m}, (b_j)_{j\in n}\right) = \sum_{\alpha\in W(m,n)} \prod_{i\in m} M(a_i, b_{\alpha(i)})$$

$!\mathcal{A}$:
- free strict mon. cat. over $\mathcal{A}$ $(W = \mathbb{F}_{id})$
- free symm. str. mon. cat. over $\mathcal{A}$ $(W = \mathbb{F}_{bij})$
- free fin. coprod. completion of $\mathcal{A}$ $(W = \mathbb{F})$

Also known as $\mathrm{Fam}(\mathcal{A})$

These v.dbl. monads also appear in [Cruttwell-Shulman], where they are constructed directly.

We simplify their constructions via $\mathrm{Mod}(-)$, and generalize these to arbitrary verbal categories W.

Horizontal Kleisli construction

Def. [Cruttwell-Shulman]

$\mathbb{D}$: v. dbl. cat.
T : v. dbl. monad on $\mathbb{D}$

The horizontal Kleisli v. dbl. cat. $H\text{-}Kl(\mathbb{D},T)$ is:

- obj. same as $\mathbb{D}$
- v.mor. same as $\mathbb{D}$

$\Big]\ H\text{-}Kl(\mathbb{D},T)_{vert} = \mathbb{D}_{vert}$

- h.mor. $\dfrac{A \xrightarrow{M} B \text{ in } H\text{-}Kl(\mathbb{D},T)}{A \xrightarrow{M} TB \text{ in } \mathbb{D}}$
- multicell (suitable def.)

In particular,

$H\,\mathbb{P}rof_!(\mathbb{T},\mathbb{T})$: underlying multicategory of the substitution monoidal category $[W, Set]$ of W-species.

$H\,\mathbb{P}rof_!(A,A)$: underlying multicategory of the substitution monoidal category $[A^{op} \times\, !A, Set]$ of A-colored W-species.

general machinery

$W \subseteq \mathbb{F}$ verbal cat. $\longmapsto$ $\mathbb{M}at$ ($(-)^*$) v.dbl. monad $\xrightarrow{\mathbb{M}od(-)}$ $\mathbb{P}rof$ ($!$) v.dbl. monad $\xrightarrow{H\text{-}Kl(-)}$ $\mathbb{P}rof_!$ v.dbl. cat. w/

- obj. categories
- v.mor. functors
- h.mor. $\dfrac{\dfrac{A \xrightarrow{M} B \text{ in } \mathbb{P}rof_!}{A \xrightarrow{M} !B \text{ in } \mathbb{P}rof}}{A^{op} \times\, !B \xrightarrow{M} Set}$ "generalized W-species".

[Cruttwell-Shulman]

simple construction

$$A \xrightarrow{M} B \longmapsto A^* \xrightarrow{M^*} B^*$$

$$M^*((a_i)_{i\in m},(b_j)_{j\in n}) = \sum_{\alpha \in W(m,n)} \prod_{i\in m} M(a_i, b_{\alpha(i)})$$