ELSEVIER

Theoretical Computer Science 278 (2002) 3–21

Theoretical Computer Science

www.elsevier.com/locate/tcs

Cartesian logic

Peter Freyd [1]

Department of Mathematics, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104-6395, USA

0. Introduction

The syntax that corresponds to cartesian categories (that is, categories with all finite limits) is the logic of unique existential assertions. Conjunction is the only connective. The only assertions are those that can made up from

$$A \,\mathtt{ue}\!> B$$

(pronounced "A YIELDS B") where A and B are conjunctions of atomic terms. Such an assertion holds for a given interpretation of the predicates symbols iff – in the elemental semantics – for every instantiation of the variables that validates A there is a *unique* instantiation of the remaining variables that validates B, e.g.:

Suppose that B is a binary predicate on a set.

$x = x \,\mathtt{ue}\!> Bxy$ says that B graphs a function from its first argument to its second.

$(x = x) \wedge (y = y) \,\mathtt{ue}\!> Bxy$ says that B holds for all pairs.

$z = z \,\mathtt{ue}\!> Bxy$ says that if the set is non-empty then B holds for just one pair of elements.

More formally, we describe such a syntax on a given set of SORTS as given by

1. a set of VARIABLES each of which has an assigned sort, there being an infinite number of variables for each sort;
2. a set of PREDICATE SYMBOLS, each of which has an assigned sort-word called its TYPE;
3. a TRUE predicate, $\top$ with the empty word as type (hence $\top \,\mathtt{ue}\!> Bxy$ says that B holds for just one pair of elements – in particular, the set is non-empty);
   and for each sort, α, an EQUALITY predicate of type $\alpha\alpha$.

We define Pw to be an ATOMIC TERM where ω be a sort-word, P a predicate symbol of type ω, and w a word of variables such that w_i is of sort ω_i each i.

[1] In deep appreciation for the splendid fest produced by John Power and Edmund Robinson. It was a great honor. And great fun!

E-mail address: pjf@saul.cis.upenn.edu (P. Freyd).

0304-3975/02/$ - see front matter © 2002 Published by Elsevier Science B.V.
PII: S0304-3975(00)00328-5

Any atomic term is a TERM and if A and B are terms, then so is $(A \wedge B)$. Typically, the outer pair of parentheses is omitted and the equality predicates are written in infix style: $x = y$.

A CARTESIAN THEORY can now be defined as a set of assertions of the form $A \,\mathtt{ue}\, > B$ where A and B are terms.

1. The categorical semantics of cartesian logic

Given a cartesian category we mean by an INTERPRETATION of a given language:

a choice of an object, $[\![\alpha]\!]$, for each sort α;

a choice of an object, $[\![P]\!]$, for each primitive predicate P; together with a jointly monomorphic sequence of maps

$$p_i : [\![P]\!] \to [\![\omega_i]\!],$$

where ω is the sort-word assigned as the type of P;

It is understood that $[\![\top]\!]$, the interpretation of $\top$, is the terminator (the empty product) and that the equality predicates are interpreted using only identity maps.

Convention. If x is a variable then $[\![x]\!]$ denotes $[\![\alpha]\!]$ where α is the sort assigned to x.

We extend a given interpretation of the language to an interpretation of terms. Given a term A its interpretation will be an object $[\![A]\!]$ together with a jointly monomorphic family of maps

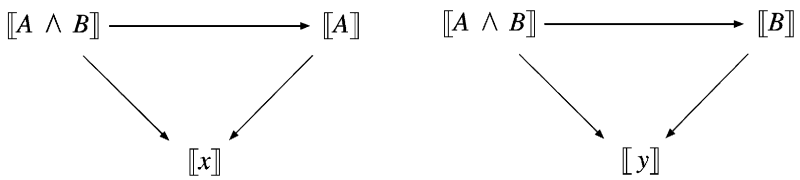
$$m_x : [\![A]\!] \to [\![x]\!],$$

one map for each variable that occurs in A.

Given interpretations of terms, $[\![A]\!]$ and $[\![B]\!]$, the interpretation $[\![A \wedge B]\!]$ comes equipped with a jointly monomorphic pair of maps

$$\ell_{A,B} : [\![A \wedge B]\!] \to [\![A]\!] \quad \text{and} \quad r_{A,B} : [\![A \wedge B]\!] \to [\![B]\!]$$

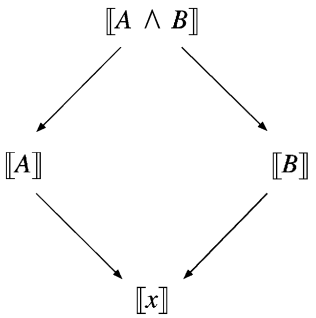
(left, right). For any variable, x, occurring in A or any variable, y, occurring in B, we have commuting diagrams:



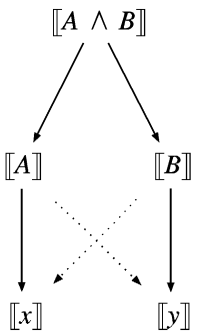
We take $[\![A \wedge B]\!]$ to be maximal with respect to these conditions. That is, the interpretation $[\![A \wedge B]\!]$ is the finite limit of the union of the interpretations $[\![A]\!]$ and $[\![B]\!]$ identifying objects only if they are named by the same variables. e.g.:

If A and B have no variables in common the interpretation $[\![A \wedge B]\!]$ is a product diagram.

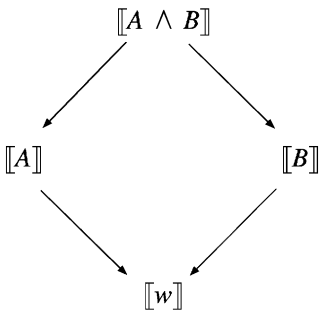
If A and B have just the variable x in common, the interpretation $[\![A \wedge B]\!]$ is a pullback diagram of the form:



If A and B have two variables in common, the interpretation $[\![A \wedge B]\!]$ is constructible as a finite limit diagram of the form



If A and B have several variables in common, the interpretation $[\![A \wedge B]\!]$ is constructible as an elaboration of the last diagram, alternatively as a pullback diagram of the form:



where w is a word that runs through the variables, once each, that occur in both A and B and $[\![w]\!]$ denotes the product

$$[\![w_1]\!] \times [\![w_2]\!] \times \cdots \times [\![w_n]\!].$$

What we have not discussed yet is the interpretation of atomic terms. If P is a predicate and w is a variable-word without repetitions then, of course, $[\![Pw]\!]=[\![P]\!]$ and for each i:

$$[\![Pw]\!] \overset{m_{w_i}}{\to} [\![w_i]\!] = [\![P]\!] \overset{p_i}{\to} [\![w_i]\!].$$

The problem arises when w has repetitions. In that case, we take $[\![Pw]\!]$ to be the maximal subobject of $[\![P]\!]$ so that the definitions of the maps m_{w_i} are consistent. That is, $[\![Pw]\!]$ is the intersection of all the pairwise equalizers of p_i and p_j one for each pair i, j such that $w_i = w_j$.

(The fiat may be avoided. Replace each atomic term with repeated variables with a conjunction of atomic terms each of which avoids such repetitions, to wit, the term obtained by using the same primitive predicate and whatever pairwise equalities are needed.)

Now for the key definition:

$$A \,\mathtt{ue}> B$$

is MODELED by an interpretation of a language in a cartesian category iff

$$\ell_{A,B} : [\![A \wedge B]\!] \to [\![A]\!]$$

is an isomorphism.

We will say that an interpretation MODELS A THEORY if every assertion therein is modeled.

We will show that for every cartesian theory there is a free model, that is, a cartesian category together with a model therein of the theory with the universal property that every model of the theory in any cartesian category is obtained via a finite-limit-preserving functor from the free model, unique up to natural equivalence. (And we will show that every cartesian category may be regarded as the free model of a cartesian theory.)

2. The rules of cartesian inference

Twenty-five years ago the author started listing these rules. Their remarkable inelegance caused him to conclude that he was staring at one of the very reasons for category theory. The notion of a cartesian category is much simpler than that of cartesian syntax. The language of cartesian categories has permeated mathematics as a whole, even among those who are convinced that there is nothing worth noting in the theory of categories (the usefulness of a language does not, in fact, imply the usefulness of its associated theory) and there is no chance that cartesian syntax is going to do any such thing.

The rules in question for all their inelegance are, it turns out, rather typical of the sort of rules one sees in theoretical computer science talks (usually via a brief glance

of a densely written transparency). The connection with logic programming gives a motivation not present from the purely mathematical side. The particular impetus that propelled the present effort arose from the necessity of getting a handle on the process of adjoining generic subobjects to objects in cartesian categories (the solution appears below).

Before encountering the rules consider that for the natural numbers it is the case that

$$\top \,\mathtt{ue}\!>\, (0 < x) \wedge (x < 2)$$

and

$$(0 < x) \wedge (x < 2)\,\mathtt{ue}\!>\, (0 < x);$$

but it is not the case that

$$\top \,\mathtt{ue}\!>\, (0 < x).$$

That is, $\mathtt{ue}>$ is not a transitive relation on terms.

The rules:

$[>]$:

$$A\,\mathtt{ue}\!>\, A.$$

$[\top]$:

$$A\,\mathtt{ue}\!>\, \top.$$

$[\wedge>]$:

$$A \wedge B\,\mathtt{ue}\!>\, A, \quad A \wedge B\,\mathtt{ue}\!>\, B.$$

$[>\wedge]$:

$$\frac{A\,\mathtt{ue}\!>\, B,\ A\,\mathtt{ue}\!>\, C}{A\,\mathtt{ue}\!>\, B \wedge C} \quad \textit{if all variables common to } B, C \textit{ occur in } A.$$

$[\mathtt{ue}>>]$:

$$\frac{A\,ue\!>\, B,\ B\,\mathtt{ue}\!>\, C}{A\,\mathtt{ue}\!>\, B \wedge C} \quad \textit{if all variables common to } A,\ C \textit{ occur in } B.$$

$[\mathtt{ue}>\mathtt{u}>]$:

$$\frac{A\,\mathtt{ue}\!>\, B}{A \wedge (B \wedge B')\,\mathtt{ue}\!>\, y = y'} \quad \begin{array}{l}\textit{if } y \textit{ occurs in } B \textit{ but not } A \textit{ and } y' \textit{ occurs}\\ \textit{in neither } A \textit{ nor } B \textit{ and } B' \textit{ is the result of}\\ \textit{replacing each } y \textit{ in } B \textit{ with } y'.\end{array}$$

$[=>]$:

$$A \wedge (y = y')\,\mathtt{ue}\!>\, A' \quad \begin{array}{l}\textit{if } A' \textit{ is the result of replacing an occurrence}\\ \textit{of } y \textit{ with } y'.\end{array}$$

$[>=]$:

$$A\,\mathtt{ue}> y'=y \quad \textit{if } y \textit{ occurs in } A \textit{ and } y' \textit{ does not.}$$

$[\mathtt{u}>\mathtt{e}>\mathtt{ue}>]$:

$$\frac{A\,\mathtt{u}> B,\ A\,\mathtt{e}> B}{A\,\mathtt{ue}> B}.$$

In the last rule we are using the following two definitions:

$A\,\mathtt{e}> B$ ("A WEAKLY YIELDS B") iff there is a term M such that $A\,\mathtt{ue}> M\wedge B$.

$A\,\mathtt{u}> B$ ("A PARTIALLY YIELDS B") iff for every variable y that occurs in B but not A it is the case that

$$A\wedge(B\wedge B')\,\mathtt{ue}> y=y'$$

where y' is a fresh variable and B' is the result of replacing all occurrences of y in B with y'. ("weak" and "partial" are established as operators on definitions: "weak" removes uniqueness conditions; "partial" removes existential conditions.)

In the elemental semantics, $A\,\mathtt{u}> B$ iff for every instantiation of the variables that validates A there is at most one instantiation of the remaining variables that validates B.

In the categorical semantics, $A\,\mathtt{u}> B$ iff $\ell_{A,B}$ is monic.

In the elemental semantics, $A\,\mathtt{e}> B$ iff for every instantiation of the variables validating A there is at least one instantiation of the remaining variables validating B.

In the categorical semantics, $A\,\mathtt{e}> B$ implies that $\ell_{A,B}$ is a split epi. In particular, in the free cartesian category (that we will construct), $\mathtt{e}> B$ iff $\ell_{A,B}$ is a slit epi. (The Cayley representation theorem will then yield that $\mathtt{e}> B$ iff for every interpretation in the category of sets it is the case that $\ell_{A,B}$ is a (plain) epi.)

Note that the rule named $[\mathtt{ue}>\mathtt{u}>]$ could have been stated as

$$\frac{A\,\mathtt{ue}> B}{A\,\mathtt{u}> B}$$

Two more definitions:

$A\,\mathtt{a}> B$ ("A IMPLIES B") iff $A\,\mathtt{ue}> B$ and all variables in B occur in A.

$A\equiv B$ ("A is EQUIVALENT with B") iff $A\,\mathtt{a}> B$ and $B\,\mathtt{a}> A$.

We will show below that if $A\equiv B$ then A and B are interchangeable in any assertion.

Lemma 1.

$$\frac{A\,\mathtt{a}> B,\ B\,\mathtt{a}> C}{A\,\mathtt{a}> C}.$$

Proof. From $A\,\mathtt{a}> B,\ B\,\mathtt{a}> C$ we may infer $A\,\mathtt{ue}> B\wedge C$ (the side condition on variables certainly holds). Since there are no new variables in C it is automatic that $A\,\mathtt{u}> C$ hence $[\mathtt{u}>\mathtt{e}>\mathtt{ue}>]$ gives $A\,\mathtt{ue}> C$. □

It is now easy to obtain the semi-lattice equations for $\wedge$ with respect to $\equiv$. That is:

Lemma 2. $\top \wedge A \equiv A$.

Lemma 3. $A \wedge A \equiv A$.

Lemma 4. $A \wedge B \equiv B \wedge A$.

Lemma 5. $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$.

Lemma 6.

$$\frac{A \mathtt{a}> B}{A \wedge C \mathtt{a}> B \wedge C}.$$

Proof. The transitivity of $\mathtt{a}>$ easily gives $A \wedge C \mathtt{a}> B$. □

Lemma 7.

$$\frac{A \equiv B}{A \wedge C \equiv B \wedge C}.$$

Proof. It is an immediate consequence of the monotonicity of $\mathtt{a}>$ with respect to $\wedge$. □

Note that $A \mathtt{a}> B$ iff $A \equiv A \wedge B$.

Turning attention to the rules for equality:

Lemma 8.

$$A \mathtt{a}> x = x \quad \textit{if } x \textit{ occurs in } A.$$

Proof. From Lemma 3 and $[= >]$ we may infer $x' = x \mathtt{a}> x = x$ since $(x' = x) \mathtt{a}> (x' = x) \wedge (x' = x) \mathtt{a}> (x = x)$. Given A let x' be a variable that does not occur in A. By $[> =]$ we have $A \mathtt{ue}> x' = x$ and by $[\mathtt{ue} \gg]$ we have $A \mathtt{ue}> (x' = x) \wedge (x = x)$ hence $A \mathtt{e}> x = x$. But $A \mathtt{u}> x = x$ is automatic since all the variables that appear in $x = x$ appear in A and we may conclude $A \mathtt{ue}> x = x$ by $[\mathtt{u}> \mathtt{e}> \mathtt{ue}>]$. □

Lemma 9.

$$x = x' \equiv x' = x.$$

Because,

$$x = x' \ \mathtt{a}> (x = x) \wedge (x = x') \ \mathtt{a}> x' = x.$$

Lemma 10.

$$(x = x') \wedge (x' = x'') \ \mathtt{a}> x = x''.$$

We will prove a sequence of alpha-rule lemmas that say, in effect, that the validity of an assertion doesn't depend on the particular variables used. Special care must be

taken, however, on whether the variables occur on one or both sides. Before we finish the sequence we will need the transitivity of $\mathtt{e}>$, which rule in turn needs some of the alpha-rule lemmas.

Lemma 11.

$$\frac{A\,\mathtt{a}>B}{A'\,\mathtt{a}>B'} \quad \textit{if } A' \textit{ (respectively } B') \textit{ is the result of replacing every occurrence of } x \textit{ with } x' \textit{ in } A \textit{ (resp. } B).$$

Proof. Each variable that occurs in B' occurs in A', hence it is automatic that $A'\,\mathtt{u}>B'$ and it suffices to infer $A'\,\mathtt{e}>B'$. For that we have $A'\,\mathtt{ue}>A'\wedge(x'=x)\,\mathtt{ue}>A$ hence $A'\,\mathtt{ue}>A\wedge A'\wedge(x'=x)$. From $A\wedge A'\wedge(x'=x)\,\mathtt{a}>A\,\mathtt{a}>B$ we may infer $A\wedge A'\wedge(x'=x)\quad\mathtt{a}>B\wedge A'\wedge(x'=x)$; hence, (using $[\mathtt{ue}\gg]$) $A'\,\mathtt{ue}>A\wedge B\wedge A'\wedge(x'=x)$. From $A\wedge B\wedge A'\wedge(x'=x)\,\mathtt{a}>B\wedge(x'=x)$ and $B\wedge(x'=x)\,\mathtt{a}>B'$ we have $A\wedge B\wedge A'\wedge(x'=x)\,\mathtt{a}>B'$ hence $A'\,\mathtt{ue}>A\wedge B\wedge A'\wedge(x'=x)\wedge B'$. Thus, $A'\,\mathtt{e}>B'$. □

Lemma 12.

$$\frac{A\,\mathtt{u}>B}{A'\mathtt{u}>B'} \quad \textit{if } A' \textit{ (respectively } B') \textit{ is the result of replacing every occurrence of } x \textit{ with } x' \textit{ in } A \textit{ (resp. } B).$$

Proof. It is a special case of the last lemma. □

Lemma 13.

$$\frac{A\,\mathtt{ue}>B}{A\,\mathtt{ue}>B'} \quad \textit{if } x \textit{ occurs in } B \textit{ but not } A, \textit{ if } x' \textit{ is in neither } A \textit{ nor } B \textit{ and if } B' \textit{ is the result of replacing every occurrence of } x \textit{ with } x' \textit{ in } B.$$

Proof. $A\,\mathtt{ue}>B$ implies $A\,\mathtt{u}>B$ hence, as we have just seen, $A\,\mathtt{u}>B'$ and it suffices to show $A\,\mathtt{e}>B'$. For that we have $A\,\mathtt{ue}>B$ and $B\,\mathtt{ue}>(x=x')$ from which we may infer $A\,\mathtt{ue}>B\wedge(x=x')$. Next use $B\wedge(x=x')\,\mathtt{ue}>B'$ to obtain $A\,\mathtt{ue}>B\wedge(x=x')\wedge B'$ hence $A\,\mathtt{ue}>B'$. □

Assuming the side condition on variables, we will show that $\mathtt{e}>$ when viewed as a binary relation on terms can be considered to be the transitive closure of $\mathtt{ue}>$. Note first, that if $A\,\mathtt{e}>B$ then there exists an "interpolating term" C such that $A\,\mathtt{ue}>C$ and $C\,\mathtt{ue}>B$ and every variable occurring in both A and B occurs in C: to wit, take C to be $M\wedge B$ where $A\,\mathtt{ue}>M\wedge B$. To show that $\mathtt{e}>$ is the transitive closure of $\mathtt{ue}>$ (subject to the side condition) it thus suffices to show

Lemma 14.

$$\frac{A\,\mathtt{ue}>B, B\,\mathtt{e}>C}{A\,\mathtt{e}>C} \quad \textit{if all variables common to } A, C \textit{ occur in } B.$$

Proof. Let M be such that $B\,\mathtt{ue}>M\wedge C$. If there are no variables common to M and A other than those already in B then we have $A\,\mathtt{ue}>B\wedge M\wedge C$ hence $A\,\mathtt{e}>C$. But if

there are variables in M that occur in A but not B we note first that any such variable cannot occur in C and we may use the last lemma to replace M with M' in which the variable in question has been replaced by an entirely fresh variable. By iterating this step, if necessary, we may reduce to the case that there are no variables common to M and A other than those already in B. □

Now for the remaining alpha-rule lemma.

Lemma 15.

$$\frac{A\,\mathtt{ue}>B}{A'\mathtt{ue}>B'}$$

if x occurs in A and x' occurs in neither A nor B and A' (resp. B') is the result of replacing every occurrence of x with x' in A (resp. B).

Proof. As above, $A\,\mathtt{ue}>B$ implies $A\,\mathtt{u}>B$ hence $A'\,\mathtt{u}>B'$ and it suffices to show $A'\,\mathtt{e}>B'$. For that, consider the sequence $A'\,\mathtt{ue}>A'\wedge(x'=x)\,\mathtt{ue}>A\wedge(x'=x)\,\mathtt{ue}>B\wedge(x'=x)\,\mathtt{ue}>B'$ which of course can be rewritten as $A'\,\mathtt{e}>A'\wedge(x'=x)\,\mathtt{e}>A\wedge(x'=x)\,\mathtt{e}>B\wedge(x'=x)\,\mathtt{e}>B'$. The side condition on variables can be easily checked to obtain $A'\,\mathtt{e}>B'$. □

Given a sequence of terms, $A,B,C,\ldots,Z$ and assertions $A\,\mathtt{e}>B, B\,\mathtt{e}>C\ldots$, the side conditions on variables needed to infer $A\,\mathtt{e}>Z$ can be reduced to those variables common to A and Z each of which must appear in every term. There may, of course, be other variables that appear only in the intermediate terms. But using the last lemma each of those variables may be replaced in a way to achieve the condition that all variables that appear in any two terms appear in all the terms between them. And that is exactly what is needed to insure the transitivity of $\mathtt{e}>$.

We may now establish the interchangeability of equivalent terms:

Lemma 16.

$$\frac{A\,\mathtt{ue}>B,\ B\equiv C}{A\,\mathtt{ue}>C}.$$

Proof. $B\equiv C$ gives $B\,\mathtt{e}>C$, hence $A\,\mathtt{e}>C$ and it suffices to show that $A\,\mathtt{u}>C$. But $A\wedge C\wedge C'\,\mathtt{a}>A\wedge B\wedge B'\,\mathtt{a}>y=y'$ (for relevant y, y'). □

Lemma 17.

$$\frac{A\equiv B,\ B\,\mathtt{ue}>C}{A\,\mathtt{ue}>C}.$$

Proof. As above, it suffices to prove $A\,\mathtt{u}>C$ and for that: $A\wedge C\wedge C'\,\mathtt{a}>B\wedge C\wedge C'\,\mathtt{a}>y=y'$. (This proof uses only that A and B have the same variables and $A\,\mathtt{a}>B$.) □

Thus, if $A \equiv B$ than A may be replaced with B in any assertion.

Two of the basic rules of inference with restrictive side conditions on the variables yield still-useful rules even when the side conditions fail. The proofs are left as exercises in the use of the alpha-rules.

Lemma 18 ($[> \wedge]$ *improved*).

$$\frac{A \,\mathtt{ue}> B,\ A \,\mathtt{ue}> C}{A \,\mathtt{ue}> B \wedge C'}$$

if C' is the result of replacing each variable in C that occurs in B but not A with a fresh variable.

Lemma 19 ($[\mathtt{ue}\gg]$: *improved*).

$$\frac{A \,\mathtt{ue}> B,\ B \,\mathtt{ue}> C}{A \,\mathtt{ue}> B \wedge C'}$$

if C' is the result of replacing each variable in C that occurs in A but not B with a fresh variable.

We will have need of:

Lemma 20.

$$A \,\mathtt{e}> A'$$

if A' is the result of replacing some of the occurrences of a variable x with a fresh variable x'.

Proof.

$A \,\mathtt{e}> A \wedge (x = x') \,\mathtt{e}> A'$. □

3. The free cartesian category: its construction

Given a cartesian theory the FREE CARTESIAN CATEGORY has the resulting derived predicates as objects. A DERIVED PREDICATE is named by a term together with linear ordering of its free variables. (Derived predicates have sort-types as do primitive predicates. Terms do not.) Given a derived predicate name, A, and a word of variables, w, of the same sort-type as A, then Aw denotes the term obtained by replacing each of the variables in the term with the corresponding variable from w. Two names, A and A', NAME THE SAME DERIVED PREDICATE if they have the same sort-type and if the terms are alpha-equivalent, that is if $Aw = A'w$ for all appropriate words (equivalently, if for any appropriate word without repeated variables). Note the equal sign. For A and A' to be equal as derived predicates, Aw and $A'w$ must be equal as terms. By convention, we understand every primitive predicate to name a derived predicate.

(One might wish to introduce here a lambda-style notation: $\lambda xy.x = y$ describes the derived predicate P such that $Pxy \equiv x = y$.)

A derived predicate, M, NAMES A MAP from A to B if:

$$Mxuy \,\mathtt{a}> Ax \wedge By$$

and

$$Ax \,\mathtt{ue}> Mxuy.$$

We are using a number of conventions here. First, x, u, and y are understood to denote words of variables of the appropriate sort-types. It is further understood that there are *no repetitions of variables among the words*. And it is finally understood that *different words have no variables in common*. These conventions will be used throughout.

If M' also names a map from A to B then M and M' NAME THE SAME MAP iff

$$Mxuy \,\mathtt{e}> M'xu'y.$$

(Note here the use of the convention just stated: the words u, u', x, y have no repetitions and no variables in common.) This definition is chosen for its brevity and its usefulness in proving certain lemmas. It is not chosen for its transparency: it is not even immediately reflexive, never mind symmetric. The transitivity of $\mathtt{e}>$ does make it obviously transitive. The reflexivity is obtainable by repeated application of the last lemma. For its symmetry consider the obviously symmetric definition:

$$Mxuy \wedge M'xu'y' \,\mathtt{a}> y = y'.$$

Matters are settled by:

Lemma 21. *When M and M' name maps from A to B then*

$$Mxuy \,\mathtt{e}> M'xu'y \quad \textit{iff} \quad Mxuy \wedge M'xu'y' \,\mathtt{a}> y = y'.$$

Proof. Starting with $Mxuy \,\mathtt{e}> M'xu'y$ consider $Mxuy \wedge M'xu'y' \,\mathtt{e}> Ax \wedge M'xu'y \wedge M'xuy' \,\mathtt{a}> u'y = uy' \,\mathtt{a}> y = y'$. Starting with $Mxuy \wedge M'xu'y' \,\mathtt{a}> y = y'$ consider $Mxuy \,\mathtt{e}> Mxuy \wedge Ax \,\mathtt{e}> Mxuy \wedge M'xu'y' \,\mathtt{e}> (y = y') \wedge M'xu'y' \,\mathtt{e}> M'xu'y$. □

N is a NICKNAME for a map from A to B if M names a map from A to B where $Mxuy \equiv Ax \wedge Nxuy \wedge By$. (For N to be a nickname it suffices that $Ax \,\mathtt{ue}> Nxuy \wedge By$. N' nicknames the same map iff $Ax \wedge Nxuy \wedge By \,\mathtt{e}> N'xu'y$.) When specifying nicknames we will occasionally use a more relaxed notion of derived predicate: an EXTENDED DERIVED PREDICATE is named by a term together with a finite set of "extending variables" and a linear ordering of the union of its free and extending variables. Given an extended derived predicate name, A, and a word of variables, w, of the same sort-type as A, then Aw denotes the term obtained by replacing each of the variables in the term with the corresponding variable in w. The variables that do not appear in the term used in the name disappear from Aw. It is thus possible, for example, to define N to be the extended derived predicate such that $Nxyz \equiv x = z$.

The identity map on A is nicknamed by I where $Ixx' \equiv x = x'$. Here it is understood that an equality sign between words of variables denotes the conjunction of the appropriate individual equalities.

If M names a map from A to B and N names a map from B to C then P names the composition of the maps where $Px(uyv)z \equiv Mxuy \wedge Nyvz$. (The parentheses appearing in the variable-string after P serve no mathematical role; they are there

only for the eye to separate the "mediating" variables from the "source" and "target" variables.)

It is no trouble to verify $Mxuy \wedge Nyvz \,\mathtt{a}> Ax \wedge Cz$. It is no trouble to verify $Ax\,\mathtt{e}> Mxuy \wedge Nyvz$ (because $Ax\,\mathtt{e}> Mxuy \wedge By\,\mathtt{e}> Mxuy \wedge Nyvz$). To verify $Ax\,\mathtt{u}> Mxuy \wedge Nyvz$ we must show $Ax \wedge Mxuy \wedge Nyvz \wedge Mxu'y' \wedge Nyv'z' \,\mathtt{a}> uyvz = u'y'v'z'$. For that: $Ax \wedge Mxuy \wedge Nyvz \wedge Mxu'y' \wedge Nyv'z' \,\mathtt{a}> (uy = u'y') \wedge By \wedge By' \wedge Nyvz \wedge Ny'v'z' \,\mathtt{a}> (uy = u'y') \wedge By \wedge Nyvz \wedge Nyv'z' \,\mathtt{a}> (uy = u'y') \wedge (vz = v'z')$.

Associativity of composition is an immediate consequence of the associativity of conjunction. That our recipe for identity maps works is an immediate consequence of the rules of equality. And our definition of equivalence of map-names was chosen to make it an immediate consequence of Lemma 20 that this definition of composition is well defined.

A few nice exercises: if a map from A to B is named by M then it is an isomorphism iff $By\,\mathtt{ue}> Mxuy$; monic iff $By\,\mathtt{u}> Mxuy$; split epi iff $By\,\mathtt{e}> Mxuy$. M names an isomorphism from A to B iff M also names a map from B to A. To be precise, the inverse of the map named by M is named by the derived predicate, M', obtained by rearranging the ordering of the variables, to wit, $M'yux \equiv Mxuy$. If $By\,\mathtt{e}> Mxuy$ let N be a derived predicate such that $By\,\mathtt{ue}> Nyvx \wedge Mxuy$. Then N may be used to name a map from B to A and it is a left-inverse for the map named by M.

The primitive predicate $\top$ is a terminator for this category and for any derived predicate, A, the unique map from A to $\top$ is named also by A (and nicknamed by $\top$). The uniqueness of such is an immediate consequence of the symmetric definition of equivalence of map names. (This last sentence works as well for any proposition, that is, nullary predicate.)

Given derived predicates, A and B, the product of the objects they describe is described by the derived predicate P such that $Pxy \equiv Ax \wedge By$. (Recall the convention: x and y are variable-words with no variables in common.) The projection map from P to A (respectively, B) is nicknamed by the extended derived predicate L where $Lxyx' \equiv x = x'$ (resp. R where $Rxyy' \equiv y = y'$). There is no great difficulty in seeing that this pair of maps is a jointly monic pair of maps.

There is no great difficulty in checking that if C is a derived predicate and if M names a map from C to A and N a map from C to B, then $\langle M,N\rangle$ may be constructed as the derived predicate such that $\langle M,N\rangle z(uv)(xy) \equiv Mzux \wedge Nzvy$. (Again the parentheses are only for the eye: (uv) comprise the "mediating" variables, (xy) the "target" variables.)

Given derived predicates, A and B, and maps named by M and M' from A to B, their equalizer is given by the derived predicate E such that $Exuu'y \equiv Mxuy \wedge M'xu'y$. The map from E to A is nicknamed by I where $I(xuu'y)x' \equiv x = x'$. There is no great difficulty in seeing that this map is monic. If C is a derived predicate and if N names a map from C to A such that the $Nzvx \wedge Mxuy\ yeNzv'x' \wedge M'x'u'y$ let Q be the derived predicate such that $Qzv(xuu'y) \equiv Nzvx \wedge Mxuy \wedge M'xu'y$. Then Q names a map from C to E that when composed with the map nicknamed by I from E to A is the map named by N.

The CANONICAL INTERPRETATION of the theory is obtained by setting $[\![x]\!]$ as the derived predicate $x=x$. Note that if x and x' are of the same sort-type then $[\![x]\!]=[\![x']\!]$. If α is the sort-type of x we may thus define $[\![\alpha]\!]=[\![x]\!]$. For each primitive predicate, P, we take its canonical interpretation, $[\![P]\!]$, to be P (recalling that every primitive predicate was to be considered a special case of derived predicate). For ω the type of P, the canonical interpretation of $p_i : P \to [\![\omega_i]\!]$ is nicknamed by I, where $Iwy \equiv w_i = y$ for w any variable-word of sort-word ω and y a variable not in ω of sort ω_i. (Note that the definition of the derived predicate I is independent of choice of w and the definition of p_i is independent of choice of y.)

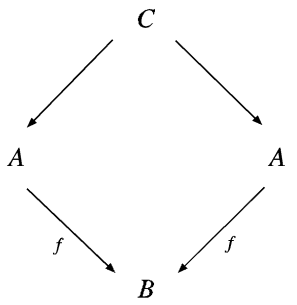
For each term, A, we take its canonical interpretation, $[\![A]\!]$, to be the derived predicate, P, such that $Pw \equiv A$ where w is a variable-word that comprises without repetition the variables in A. (Thus, the definition depends on a choice of ordering of the variables in A. We could, instead, take an ordering on the alphabet of all variables as given and insist that w be in ascending order.) For $x = w_i$ the canonical interpretation of $m_x : [\![A]\!] \to [\![x]\!]$ is $p_i : P \to [\![x]\!]$.

It is routine to verify that this interpretation does follow the rules for interpretations. For terms A and B the maps $\ell_{A,B} : [\![A \wedge B]\!] \to A$ and $r_{A,B} : [\![A \wedge]\!] \to B$ are easily nicknamed with equalities and easily checked to yield a pullback diagram.

4. The completeness theorem

The completeness of the rules for cartesian inference now follows from the observation that $\ell_{A,B}$ is an isomorphism in the free category (i.e., that $Ax'y' \,\mathtt{ue}\, (xy = x'y') \wedge Axy \wedge Byz$) iff $Axy \,\mathtt{ue}\, Byz$. The Cayley representation theorem can now be used to obtain the completeness theorem for the elemental semantics: if it is not possible to infer $Axy \,\mathtt{e}\, Byz$ from a given theory then the covariant representable functor from the free cartesian category to the category of sets represented by the object A yields an interpretation of the theory in which $\ell_{A,B}$ is not an epi; if it is not possible to infer $Axy \,\mathtt{u}\, Byz$ then the functor represented by C yields an interpretation of the theory in which $\ell_{A,B}$ is not monic where $Cxyzz' \equiv Axy \wedge Byz \wedge Byz'$.

(A direct argument easily shows that C does as advertised. No cleverness was needed to find C; it arises as the top of the pullback:



Let $g : A \to C$ be the common left inverse of the maps from C to A. In any cartesian category, f is monic iff g is an isomorphism. Since g is automatically right-invertible, f is monic iff g is left-invertible, that is, a split epi. For both cases, $\mathtt{e}>$ and $\mathtt{u}>$, we use the easily verified fact that the functor represented by an object reflects the existence of left-inverses of maps targeted at that object.)

5. The free cartesian category: its freeness

Given a theory and a model thereof in a cartesian category $\mathbb{A}$ there is a representation of cartesian categories from the free category $T : \mathbb{F} \to \mathbb{A}$ that carries the canonical model to the given model, and the representation is unique up to natural equivalence.

We will assume that an interpretation of each term has been chosen in A. Assume, further, that

Define T on objects by sending a derived predicate A to the interpretation, $[\![Aw]\!]$, where

The proof that this is so is a task worth doing. It is to be hoped that someone actually does it.

6. Every cartesian category is free

Given a cartesian category, $\mathbb{A}$, we seek a cartesian theory whose free cartesian category is equivalent to $\mathbb{A}$. The objects of $\mathbb{A}$ will become the sorts of the theory. Each map of $\mathbb{A}$ will become a predicate of the theory. To be precise: for every $f : A \to B$ in $\mathbb{A}$ there is a predicate P_f of type AB. We will write, when convenient, $P_f xy$ as $fx = y$.

The axioms:

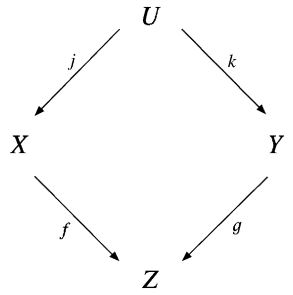
$x = x \,\mathtt{ue}> fx = y$ for every $f \in \mathbb{A}$;

$x = x \,\mathtt{a}> ex = x$ for every identity map e;

$(fx = y) \wedge (gy = z) \,\mathtt{a}> hx = z$ for every equation $fg = h$ from $\mathbb{A}$;

$\top \mathtt{ue}> x = x$ for x of the type of a terminator;

$(fx = z) \wedge (gy = z) \;\mathtt{ue}> (ju = x) \wedge (ku = y)$ for every pullback diagram:



This theory has a REGULAR MODEL in $\mathbb{A}$ obtained by taking for each object (that is, sort) A, $[\![A]\!]$ to be A, and for each map $f : A \to B$, $[\![P_f]\!]$ to be A, $p_1 : [\![P_f]\!] \to [\![A]\!]$ to be the identity map, and $p_2 : [\![P_f]\!] \to [\![B]\!]$ to be f. The axioms are easily checked.

Letting $\mathbb{F}$ denote the free category for this theory, we note that there is a functor $\mathbb{A} \to \mathbb{F}$ that carries A to $[\![A]\!]$ and $f : A \to B$ to the map named by P_f. This functor is a representation of cartesian categories and it carries the regular model in $\mathbb{A}$ to the canonical model in $\mathbb{F}$.

The cartesian functor from $\mathbb{F}$ to $\mathbb{A}$ that carries the canonical model to the regular model is easily seen to yield the identity functor: $\mathbb{A} \to \mathbb{F} \to \mathbb{A}$. The uniqueness condition for the freeness of $\mathbb{F}$ gives that $\mathbb{F} \to \mathbb{A} \to \mathbb{F}$ is naturally equivalent to the identity functor.

7. Exercise: a little more parsimony

The two rules for equality could be replaced by more parsimonious rules (in which it should be kept in mind that terms are nothing more than sequences of symbols): $[= >]$:

$$A \wedge (y = y')\,\mathtt{ue} > A' \qquad \begin{array}{l}\textit{if } A \textit{ is an atomic term and } A' \textit{ is the result}\\ \textit{of replacing the first occurrence of } y \textit{ in } A\\ \textit{with } y'.\end{array}$$

$[> =]$:

$$A\,\mathtt{ue} > y' = y \qquad \begin{array}{l}\textit{if } A \textit{ is an atomic term and } y \textit{ occurs in } A\\ \textit{and } y' \textit{ does not.}\end{array}$$

8. Example: the cartesian theory of categories

What is the shortest definition in cartesian logic for categories? One possibility: the one-sorted theory with a unary predicate E and a ternary predicate C. Axioms:

$$x = x \;\mathtt{ue} > Cxty \wedge Et \wedge Csxz \wedge Es$$

$$Cxyz <\mathtt{ue}> Cxex \wedge Ee \wedge Ceyy$$

$$Cxyu \wedge Cuzw <\mathtt{ue}> Cxvw \wedge Cyzv$$

$A <\mathtt{ue}> B$ denotes the pair of assertions $A\,\mathtt{ue} > B$ and $B\,\mathtt{ue} > A$. We will say that xy is defined if there is a z such that $Cxyz$. We will say that e is an identity map if Ee. The first axiom says for every x that there is a unique identity map t such that xt is defined and a unique identity map s such that sx is defined. We will denote the first as $x\square$ and the second as $\square x$ (using $\square$ for "quad"). The second axiom in the forward direction says that if xy is defined then $x\square = \square y$ moreover that $Cx(x\square)x$ and $C(\square y)yy$. But going back to the first axiom we have that for every x there is a t such that xt is defined

hence for all x we have $Cx(x\Box)x$ and similarly $C(\Box x)xx$. The second axiom in the back direction now says that if $x\Box = \Box y$ then xy is defined moreover; there is a unique z such that $Cxyz$. We may infer that whenever xy is defined then there is a unique z such that $Cxyz$ (if xy is defined then $x\Box = \Box y$, hence as just seen, z is unique). We will write $xy = z$ for $Cxyz$. Since $(\Box x)x$ is defined we have $(\Box x)\Box = \Box x$ and similarly $\Box(x\Box) = x\Box$. It follows that $\Box(\Box x) = \Box((\Box x)\Box) = (\Box x)\Box = \Box x$ and similarly $(x\Box)\Box = x\Box$. The third axiom says that if either $(xy)z$ or $x(yz)$ is defined then so is the other and $(xy)z = x(yz)$. If xy is defined then $x(y(y\Box))$ is defined and thus so is $(xy)(y\Box)$ hence $(xy)\Box = \Box(y\Box) = y\Box$ and similarly $\Box(xy) = \Box x$. Finally, we note that Ee implies $e = \Box e$ because we know that $(\Box e)e$ is defined and the first axiom says that there is a unique identity map such that $(\Box e)t$ is defined, hence Ee implies that e is that unique identity map and $e = (\Box e)\Box = \Box e$.

(If the first axiom had been $x = x\,\mathtt{ue}> Cxtx \wedge Et \wedge Csxx \wedge Es$ we would not have been able to fix the "identity maps". Consider, for example, the case that the category is a group. As long as it is the case that the identity element verifies E it would not matter what else verifies E. And we could not prove $Ee \wedge Cxey\,\mathtt{a}> x = y$.)

9. Exercise: the syntax of toleration

In *Categories*, *Allegories* Scedrov and I develop first-order logic based on an assertion called "toleration". The elemental interpretation: A tolerates B if whenever all the variables of A and B are instantiated so that A is validated then so is B.

In Cartesian Logic we let $A\,\mathtt{t}> B$ ("A TOLERATES B") denote

$$A \wedge (y = y)\,\mathtt{a}> B,$$

where y is a word of all the variables that appear in B.

$\mathtt{t}>$ is not a transitive relation on terms: consider the theory with one sort, one nullary predicate U and one axiom $x = x\,\mathtt{ue}> U$. We have $\top\,\mathtt{t}> x = x$ and $x = x\,\mathtt{t}> U$. But in the elemental interpretation where the one sort is interpreted as the empty set and U as the false predicate we certainly do not have $\top\,\mathtt{t}> U$. The side condition on variables needed for the transitivity of $\mathtt{t}>$ is the same as stated in *Cats and Alligators* (and as different as could be from that which holds for $\mathtt{e}>$):

Lemma 22.

$$\frac{A\ \mathtt{t}> B, B\ \mathtt{t}> C}{A\ \mathtt{t}> C} \quad \textit{if each variable in } B \textit{ occurs either in } A \textit{ or in } C.$$

Proof. Letting z be a word of all the variables that occur in C we wish to infer $A \wedge (z = z)\,\mathtt{a}> C$ from $A \wedge (y = y)\,\mathtt{a}> B$ and $B \wedge (z = z)\,\mathtt{a}> C$. The side condition says that every variable in B occurs either in A or $z = z$ hence $A \wedge (z = z)\,\mathtt{a}> B$. Clearly, then $A \wedge (z = z)\,\mathtt{a}> B \wedge (z = z)$ and we have the side condition needed for the transitivity of $\mathtt{a}>$. □

Since $\mathtt{t}>$ is defined using $\mathtt{a}>$ we have the most unrestricted of the alpha rules:

Lemma 23.

$$\frac{A\ \mathtt{t}>B}{A'\ \mathtt{t}>B'}$$ *if A' (respectively B') is the result of replacing every occurrence of x with x' in A (resp. B).*

This allows, just as it did in *Cats and Alligators*, an immediate improvement of the transitivity rule:

Lemma 24.

$$\frac{A\ \mathtt{t}>B, B\ \mathtt{t}>C}{A\ \mathtt{t}>C}$$ *if the sort of every variable in B occurs as the sort of a variable in either A or C.*

The best of the transitivity rules in *Cats and Alligators* relied an existential quantification. Its approximation in cartesian logic:

Lemma 25.

$$\frac{A\ \mathtt{t}>B, B\ \mathtt{t}>C}{A\ \mathtt{t}>C}$$ *if the sort of every variable in B occurs as the sort of a variable on the right of an assertion of the form $A\wedge C\,\mathtt{e}>x=x$ provable from the theory.*

10. Adjoining a generic subobject

The immediate motivation for this work was the need that arose in an analysis of logic programming for a workable description of the result of adjoining a generic subobject of an object in a category. In the case of a topos the construction is easy: if B is an object in a topos $\mathbb{A}$ then the slice category $\mathbb{A}/PB$ (where PB is the power-object of B) is the answer. But topoi are not the natural setting for logic programming. A construction could be obtained by going from a given cartesian category to the cartesian category for which it is free and then adding a unary predicate of the relevant sort, then going on to its free cartesian category. An analysis of this rather indirect construction revealed a much simpler construction.

Let $\mathbb{A}$ be a cartesian category and B an object therein. We wish to construct the result of adjoining a generic subobject B' to B which result will be denoted as the cartesian category $\mathbb{A}[B']$. Its objects are defined to be pairs $\langle A,\mathscr{F}\rangle$, where A is an object in the original $\mathbb{A}$ and $\mathscr{F}$ is a finite set of maps from A to B. A map from $\langle A,\mathscr{F}\rangle$ to $\langle A',\mathscr{F}'\rangle$ is a map $A\xrightarrow{g}A'$ in $\mathbb{A}$ such that the composition $A\xrightarrow{g}A'\xrightarrow{f'}B$ is in $\mathscr{F}$ for every f' in $\mathscr{F}'$. This is a cartesian category and the forgetful functor back to $\mathbb{A}$ is a representation of cartesian categories, which statement comes close to showing how finite limits are constructed: given a finite diagram apply the forgetful functor, take the limit in $\mathbb{A}$, then take the defining finite set of maps to B to be the minimal set of maps needed to allow all the maps to live in $\mathbb{A}[B']$. In particular, note that if

all defining finite sets for the diagram objects are empty then so is the defining finite set for the limit which is just what is needed to see that the inclusion of $\mathbb{A}$ into $\mathbb{A}[B']$ (obtained by attaching empty defining sets) is a representation of cartesian categories. The identity map on B yields a monomorphism from $\langle B, \{1\}\rangle$ to $\langle B, \emptyset\rangle$ and we take it as the name of the generic subobject. Given any cartesian representation $T : \mathbb{A} \to \mathbb{C}$ and subobject C' of TB, define $T : \mathbb{A}[B'] \to \mathbb{C}$ to be the functor that sends $\langle A, \mathscr{F}\rangle$ to

$$\bigcap_{f \in \mathscr{F}} (Tf)^{-1}(C')$$

(that is, the subobject of TA constructed as the intersection of all inverse images of C' with respect to the maps of the form Tf for f in $\mathscr{F}$). Then $T\langle A, \emptyset\rangle = TA$ and $T\langle B, \{1\}\rangle = C'$. It is easy to check that $T : \mathbb{A}[B'] \to \mathbb{C}$ is a representation of cartesian categories and, up to natural equivalence, the only representation that agrees with T on $\mathbb{A}$ and that carries the generic subobject of B to C'.

11. Alternating logic

The next step is the logic of "extensive" categories, that is, categories with disjoint coproducts that are preserved by pullbacks (in particular, in which the natural map from $(A \times B) + (A \times C)$ to $A \times (B + C)$ is an isomorphism).

The temptation to add a connective for exclusive disjunction must be postponed. Instead we add a new type of assertion. The terms remain the same. We understand

$$A \,\mathtt{ue}\!> B_1 \mid B_2 \mid \cdots \mid B_n$$

to mean – in the elemental semantics – that for every instantiation of the variables that validates A there is a unique index, i, such that the remaining variables of B_i can be instantiated to validate B_i and, further, that instantiation is unique.

The key definition for the categorical semantics: the assertion above is modeled by an interpretation of a language in an extensive category iff the collection of maps of the form

$$\ell_{A,B_i} : [\![A \wedge B_i]\!] \to [\![A]\!]$$

combine to give an isomorphism

$$[\![A \wedge B_1]\!] + [\![A \wedge B_2]\!] + \cdots + [\![A \wedge B_n]\!] \to [\![A]\!].$$

As an example, to obtain the theory of (decidable) fields adjoin to the cartesian theory of unital rings the axiom:

$$(x = x) \;\mathtt{ue}\!> (x = 0) \mid (xy = 1).$$

The free extensive category for an alternating theory has finite-indexed families of derived predicates as objects. A map from $\{A_i\}_I$ to $\{B_j\}_J$ is named by a finite indexed

family of derived predicates, $\{M_k\}_K$, together with functions $s:K\to I,\ t:K\to J$ such that for each $k\in K$

$$M_k xu\,y\ \mathtt{a}> A_{s(k)}x\wedge B_{t(k)}y$$

and for each $i\in I$

$$A_i x\ \mathtt{ue}> M_{n_1}xu_1\,y_1\mid M_{n_2}xu_2\,y_2\mid\cdots\mid M_{n_m}xu_m\,y_m,$$

where $\{n_1,n_2,\ldots,n_m\}=\{k\in K\mid s(k)=i\}$.

As with cartesian logic, the proof of completeness for alternating logic will use a categorical representation theorem. For any extensive category the set-valued representations collectively reflect monics and split epis. (If one is willing to use a lot of machinery: first represent an extensive category in its category of pre-canonical sheaves and use the representation theorem for positive pre-logoi.)

It would appear that the theory of algebraically closed fields is not an alternating theory: the existential condition is not a unique existential. But the theory of real closed fields is an alternating theory. First, obtain the theory of ordered fields, by adding to the cartesian theory of unital rings a unary predicate denoted $0<x$ and the alternating axiom:

$$(x=x)\,\mathtt{ue}>(0<x)\mid(0=x)\mid(0<-x).$$

and the cartesian axioms:

$$(0<x)\,\mathtt{ue}>(xy=1),$$

$$(0<x)\wedge(0<y)\,\mathtt{a}>(0<x+y)\wedge(0<xy).$$

Given a polynomial whose values at 0 and 1 have different signs we can guarantee uniqueness of the root between 0 and 1 by adding the condition that each of its derivatives at 0 and 1 does not change sign. That is, for each $n\geqslant 2$:

$$(1=a_nb_n)\wedge\left(0<-a_0\sum_{i=0}^{n}a_i\right)\wedge\bigwedge_{i=1}^{n-1}\left(b_i^2=a_i\sum_{j=i}^{n}\binom{j}{i}a_j\right)\ \mathtt{ue}>\ (0<x)\wedge(0<1-x)\wedge\left(0=\sum_{i=0}^{n}a_ix^i\right).$$

One may prove that this suffices for real closure.

The resulting free extensive category for this theory is of some interest. It is equivalent to the category whose objects are real semi-algebraic sets and whose maps are those functions whose graphs are semi-algebraic sets. Tarski's theorem on quantifier elimination implies that it is an "AC regular" category (every map factors as a split epi followed be a monic) consequently a boolean logos (all of first-order logic may be interpreted).