ACT 1: THE SEQUENT CALCULUS

In technique, thought confronts a dimension of necessity and exteriority immanent to its realization. This can happen either: (1) through the work of *materialization*, such as we encounter in the work of the artist or engineer; or (2) the work of *formalization*, as carried out by the logician or mathematician. Both confront an idea with *antagonisms* latent in its structure but obscured in its pre-technical immediacy. To resolve these, technique has two options: (a) to ameliorate antagonism by an extraneous regulatory device that curtails conflicting capacities (as when an engine prone to overheating is outfitted with a water-cooling mechanism, tenuously linked to the engine by a belt), or (b) to reforge the apparatus so that the previously conflicting capacities converge in a harmonious synthesis (as when the air-cooled engine is designed so as to cool itself by a process of thermal convection indissociable from its generation of heat). The first choice leads to what Gilbert SIMONDON calls an *abstract technical object*, while the second *concretizes* the apparatus.

I'm going to speak to you, today, about the concretization of logic, or, rather, about a single *concretizing trajectory* that logic has taken since 1934, and which is still in motion as we speak. It begins with Gentzen's invention of the sequent calculus, an invention affording a groundbreaking insight into role of *symmetry* in logic, which would propel the discipline on a path of technical concretization with *logical symmetry* as its centre of gravity. In light of this idea, it would become possible for the first time to discern certain structural antagonisms in logical technique and painstakingly carve out a path to their resolution—not through *mere compromise* between conflicting tendencies, but through a *convergence* of the discipline towards its essence. With the sequent calculus, Gentzen did not present a 'new logic', but a *new formalism* by which existing logics—classical and intuitionistic—could be reshaped. It was nevertheless sufficient to transform the core *concepts* of the discipline. The gestures by which the sequent calculus accomplished this are as follows:

First: The vantage point of the sequent calculus is a bit higher than its predecessors'. Where previous formalisms manipulated *propositions*, the sequent calculus concerns the manipulation of *relations of entailment between finite sets of propositions*, relations called "sequents". They are written in the form [Γ |– Δ], which means that *all the propositions on the* LEFT *(in* Γ*), together, entail one or more of the propositions on the* RIGHT *(in* Δ*).*

Second: GENTZEN'S rules are 'purer' than his predecessors'. Where the formalisms of RUSSELL, HILBERT, and LUKASIEWICZ deploy tangled skeins of axioms and definitions, knotting together the various *logical connectives*—'AND', 'OR', 'NOT', 'IF-THEN'—in various combinations, GENTZEN operates with *pairs of inference-rules unique to each connective*: one for introducing the connective on the RIGHT side of the sequent (the side of consequences), and one for introducing it on the LEFT (the side of premises). The technical advantage of this gesture was of the same order as that had by the Arabic numeral system over its Roman predecessor. It was an act of *differentiation* of the sort that Simondon has shown to be essential to concretization (SIMONDON 1989, 31-2). Without the careful work of differentiation, the diverse capacities at work in a machine converge only at the price of interference and confusion.

These gestures allow for an elegant configuration of proofs as '*trees*'. The 'root sequent' of the tree is the sequent to be proved. To prove it, we work backwards and up, *decomposing* each compound formula according to its rule of its main connective. The anabasis continues until we reach the '*leaves'* of the tree: primitive, *symmetrical* sequents of the form [A |– A]. We can stop here, thanks to the IDENTITY AXIOM, which gives us [A |– A] for any formula A.

A theorem's proof ***is*** *its decomposition into these basic symmetries.* But what is meant by 'symmetry', here? Logical symmetry, GENTZEN shows, is essentially bound up with *negation*. Negation, in the sequent calculus, is just the movement of a formula across the 'ENTAILS' sign. If we have [Γ,A |– Δ], we can derive [Γ |– ¬A, Δ], and vice versa. (If Γ and A entail some propositions in Δ, then Γ alone

must entail *either* the negation of A, *or* some propositions in **Δ**.) This *classical* symmetry makes negation INVOLUTIVE, or reversible: every proposition A is equivalent to its double negation, ¬¬A.

The decomposition of a theorem into its basic, negational symmetries is complicated by a second operation, called 'CUT'. CUT governs the use of *lemmas* in a proof. Given a set of premises, and a theorem to deduce, a *lemma* is a proposition we deduce from our premises to help us reach the *theorem*. Suppose we want to prove, from the PEANO AXIOMS OF ARITHMETIC, the equation [(14756 × 67) = (67 × 14756)]. The easiest thing to do would be to use the lemma 'MULTIPLICATION COMMUTES' ($a \times b = b \times a$, for all a, b), from which our *theorem* follows as a special case. We then show that this lemma follows from our *premises* (the PEANO AXIOMS). CUT says: once we have done this, we have shown that the theorem *itself* follows from the premises. Formally, CUT has the shape:

$$\frac{\Gamma, L \mid\!- A \qquad \Gamma \mid\!- L}{\Gamma \mid\!- A}\ [\text{CUT}]$$

Like the IDENTITY AXIOM, CUT plays on the symmetries of the logic, but it plays a different tune. IDENTITY waits for the symmetry between the LEFT and RIGHT of a *single* sequent to become *total*, and then seizes upon it to terminate a branch in the proof-tree. CUT looks for *traces* of symmetry between the LEFT and RIGHT *of different sequents* on the same stratum, and then collapses those sequents together—like a '*Mad Fold-in*'—to produce one where those traces of symmetry have vanished.

GENTZEN's greatest theorem—his *Hauptsatz*—is that there is a procedure by which *any proof employing* CUTS can be *turned into a* CUT-FREE PROOF, boiling it down to its bare, explicit kernel, a *proof without lemmas*, where nothing occurs without necessity. This is the CUT-ELIMINATION THEOREM.

ACT 2: TOWARDS LINEAR LOGIC

Here we encounter a problem: *Classical* CUT-ELIMINATION *is equivocal*. The procedure always terminates—each classical proof can reduce to an explicit, CUT-FREE kernel—but the result's not

uniquely determined. A proof, using CUT, may boil down to several *distinct* CUT-FREE kernels. In technical terms, the CUT-ELIMINATION PROCEDURE FOR CLASSICAL LOGIC **IS NON-CONFLUENT**.

Strangely, this is not the case for *intuitionistic logic*. Intuitionistic logic, more or less, is classical logic without the LAW OF THE EXCLUDED MIDDLE—the law: '**Either** A **or not-**A' for any proposition, A—and so without the principle of INVOLUTIVE NEGATION: A is not generally equivalent to NOT-NOT-A. This logic, too, has a sequent calculus. The restrictions on classical logic that render it 'intuitionistic' amount to a partial breaking of the *symmetry* of the sequent calculus: the RIGHT side of an *intuitionistic sequent* (the side of consequence) can contain *at most one formula* (by contrast, the LEFT side, the side of premises, can still hold as many formulae as it needs); nothing else needs to be altered. What's surprising is that THE CUT-ELIMINATION PROCEDURE FOR INTUITIONISTIC LOGIC, THIS LOGIC OF BROKEN SYMMETRIES, **TURNS OUT TO BE CONFLUENT**.

There is no contradiction here, no antinomy, but what SIMONDON might have called a ***structural antagonism***. CUT and CUT-ELIMINATION appear to be direct expressions of the *negational symmetry* that the classical calculus seems to captured most completely and perfectly, and yet there seems to be *something* in these symmetries that *derails* CUT-ELIMINATION. Intuitionistic logic executes CUT-ELIMINATION in a way that is CONFLUENT—its proofs always collapse to their *unique kernel*—but at the price of *breaking* the symmetry on which the procedure operates. It offers a solution to the antagonism, but one that *compromises* the structures behind it. The symmetries of intuitionistic logic are well-behaved, but intrinsically debilitated.

If technical progress, as SIMONDON insists, takes place by producing "a *convergence* of functions in a structural unity, rather than by seeking out *compromise* between conflicting demands" (SIMONDON 1989, 22), then there is no great progress in shifting from classical to intuitionistic logic. Intuitionistic logic is a *compromised form of classical logic* (classical logic 'with a hand tied behind its back'); it proposes to resolve the problem of non-confluence as it does the other faults of classical

logic, by *stymieing* the classical machinery. What would be necessary for there to be a true act of *technical concretization* in this domain, one that would effect a *real convergence* between the CONFLUENCE captured by intuitionism, and the full SYMMETRY expressed by its classical counterpart?

Let's examine the problem more closely. What is it about the sequent calculus that *opens* the problem of symmetry? It is that it lets us see a proof not only as an 'order of reasons', but as a *spatial deployment of demonstrative 'forces'*. It is the *geometrical structure* of the calculus that gives life to the problem of symmetry. The cause of our antagonism, therefore, should be sought in whatever formal features of the calculus let something like *logical space* slip through our fingers—anything blinding us to the *geometrical patterns* that formulae etch out as they thread their way through a proof. If our grasp of these patterns is too clumsy, or if our vision of them is obscured by inadequate rules, then the symmetries classical logic displays are unlikely to be exact, and the equivocal behaviour of its CUT-ELIMINATION is likely the expression of a deeper equivocity *within the symmetries themselves*. This idea works itself out through two, converging tasks: a ***symmetrization of intuitionistic logic***, and a ***polarization of classical logic***.

The ***symmetrization of intuitionistic logic*** begins with the question: What is so significant about the intuitionistic breaking of classical symmetry? What 'geometrical indefiniteness' does it correct? Here, we need to look at two rules of classical sequent calculus that I have not yet mentioned. These are the *structural rules*: WEAKENING and CONTRACTION. WEAKENING lets us add a new formula to either side of a sequent for free. In practice, WEAKENING is used to license repetitions in a proof. CONTRACTION lets us collapse two identical formulae, occurring on the same side of a given sequent, into one. **These are rules for repetition and forgetting**. They dissociate a formula's capacity to influence a proof from its position in the proof-tree. Since CUT and CUT-ELIMINATION work on the *locations* of formulae in a proof, it's not surprising that these rules might derail the classical CUT-ELIMINATION PROCEDURE, and be responsible for its NON-CONFLUENCE.

Now, in order for the structural rules to operate on a given side of a sequent, that side must be capable of holding *more than one formula*. By restricting the RIGHT side of its sequent so as to hold *at most one formula*, intuitionistic logic "serves to mark a place where weakening and contraction are forbidden" (GIRARD 1987, 4). Once this is recognized, a way of maintaining symmetry while preserving confluence becomes obvious: bind the power of formulae to their sites by a *complete elimination of the structural rules*.

The idea behind the ***polarization of classical logic*** is this: Classical logic fails to make certain, subtle distinctions regarding the spatial trajectories that formulae and their subformulae trace out as they thread their way through a proof. The classical rules for manipulating a conjunction, like [A∧B], for example, are indifferent to the relative 'genealogies' of A and B in the proof-tree. Classical logic establishes symmetries between terms—and sets them up for a CUT—irrespective of their 'logical histories'—the paths that trail behind them in the proof-tree. If CUT-ELIMINATION is non-confluent for classical logic, perhaps this is because it works on symmetries that are too loosely determined, symmetries which, from a perspective attuned not only to the *local shape* of formulae, but to *their subformulae's trajectories in the proof*, would appear equivocal. POLARIZATION is a device for assigning to each formula 'polarity' (POSITIVE or NEGATIVE) determined by the polarities of its constituents and its connective, giving each formula something like "a memory of its relative position" in the proof (TRONÇON 277). Full symmetries will still obtain, but *only between formulae of opposite polarity*. Only on these *polarized symmetries* will CUT play its tune. The CUT-ELIMINATION PROCEDURE on *polarized* classical proofs turns out to be CONFLUENT, while the symmetry of negation is not broken but *refined*.

These two mutations converge, and the result is what Jean-Yves GIRARD, in 1987, baptised "LINEAR LOGIC". In Linear Logic, the connectives familiar to us from classical and intuitionistic logic, like AND and OR, are split by polarization into *two distinct forms* of conjunction and disjunction.

What distinguishes the rules for the positive and negative connectives is their *behaviour* in a proof. A negative connective decomposes in a way that keeps all of its information intact—the rules for negative connectives are 'reversible' and 'deterministic' in practice. The rule for a positive connective fractures a sequent in a voluntaristic and irreversible way; information is lost and cannot be recovered. Now, the linear rules for each connective, if transposed into the classical calculus, would be made **equivalent** to both their polar opposites *and* their classical counterparts by the STRUCTURAL RULES, but in Linear Logic these rules are no longer operative. A linear proof neither scatters nor forgets what it inscribes in its *space of demonstration*. As a result, linear formulae behave less like idealized propositions than like *resources to be consumed in the work of demonstration*. This should not tempt us to read Linear Logic as just another 'special' logic, instrumentally adapted to the demands of an extra-logical domain—it is not a logic of "food and cigarettes", as Jean-Yves BEZIAU once joked. It emerges through an *adaptation of logic to itself*, an immanent overcoming of logic's internal antagonisms. Linear Logic nevertheless shows itself capable of formalizing something quite like a logic of *material process*, in way that its antecedents could not imagine. In addition, it recuperates the full force of its predecessors. Not only is it *both symmetrical and confluent*, but *any intuitionistic or classical proof can be translated into a linear demonstration*—so long as we augment the logic with two "opaque" modal operators (called "OF COURSE" and "WHY NOT") which, when applied to a formula, *locally subject it to the structural rules*, letting it be repeated or forgotten as required. These operators, in fact, suffice to transform the linear connectives back into their classical counterparts.

A few years after GIRARD invented Linear Logic, the logician Jean-Marc ANDREOLI discovered that the splitting of the connectives into POSITIVE and NEGATIVE varieties enables a subtle reshaping of proofs, through a procedure called *focalization*: Any series of NEGATIVE moves in a proof can be collapsed into a single step, expressed by a *composite rule* that yields a *compound connective* built up from any number of negative connectives. The same can be done for the POSITIVE rules. A

linear proof can now be grasped according to the strict, '*focalized*' rhythm of alternation between POSITIVE and NEGATIVE moves. The significance of this gesture will become clear when we turn to the next phase in logic's concretization.

ACT 3: LUDICS

The next phase begins with GIRARD's 2001 monograph, *Locus Solum*, where we encounter his *second* revolutionary act: to drive even further the idea of logical symmetry, and the bet that the essence of logical forms can be found *in the geometrical structure of a proof alone*. The result is the invention of LUDICS, whose idea we can split into three gestures: *locative, daimonic* and *interactionist*.

First, the *locative gesture*: focalized linear proof-trees are stripped of all 'logical' information—all connectives, etc. All that is retained are positional 'DESIGNS' or 'STRATEGIES', determined by (1) the *locations* where formulae would have been, and (2) the polarized rhythm of POSITIVE and NEGATIVE moves, by which these locations decompose in ramified series of *sub-locations*. In the sequent calculi, the logical formulae occupying each location *prescribe all their possible decompositions in advance*, but Ludics' locations decompose as if through a *fog of war*; at each step they know only the *immediate* sub-locations afforded to a '**potential opponent'**—to be explicated on the next step, of opposite polarity. Since no logical information regulates the decomposition of locations, their rhythmic explication may carry on indefinitely. Never do we strike 'atomic' formula, or non-decomposable cores.

It is only thanks to the *second* gesture that these processes can terminate. This happens through the intervention of a new operator, called the DAIMON. The DAIMON arrests the process of decomposition as if by *brute decision*, letting any locative sequent whatever become a stopping point of explication, the end of a branch. The DAIMON takes the place of the IDENTITY AXIOM in the sequent calculus, replacing appeal to an "irreducible intuition" of sameness with a "cessation of thought" (which Cavaillès saw as the truth of such appeals). The recourse STRATEGIES have to this

'para-logical' operator yields a world more vast and labyrinthine than the world of proofs, for we now have STRATEGIES expressing the geography not only of 'proofs', but of 'false' or 'unfinished' proofs as well. Ludics' is much more *symmetrical* universe, since to every proof-strategy a symmetrical pseudo-proof-strategy can be made to correspond, a strategy that can be seen as its *test*.

The nature of this symmetry, and the 'testing' it makes possible, is expressed by the third, *interactionist* gesture, which sets rules of engagement for strategies. Two strategies can 'INTERACT', or become '**opponents**', if, at their root, they share a location on alternate sides of the sequent: a strategy with a negative root [ξ |– **Γ**] 'interacts' with a positively-rooted strategy [|– ξ]. The negatively-rooted strategy can be seen as affording possibilities to its 'opponent' on POSITIVE moves, and detailing its own decisions with NEGATIVE moves, while the positively-rooted strategy does just the opposite. The process of interaction that ensues is analogous to CUT-ELIMINATION. When two strategies meet at the root, an array of polar symmetries, structured like CUT relations, appears—showing us where each strategy *anticipates* the other's tactics. These mutual anticipations orchestrate an intricate collapse of the two strategies, which either carries on indefinitely, and **diverges** in an endless dispute, or **converges** on a STRATEGY called **Dai**, in which a single *locative sequent* remains, marked by the DAIMON. When two strategies X and Y converge on **Dai**, they are said to be ORTHOGONAL (X ⊥ Y). We should see in ORTHOGONALITY *the* ***strategic Idea*** *of negative symmetry*. If we think of strategy X as deploying the geography of a proof for, say, P, then its orthogonal Y displays the geography of its 'TEST', the strategy of a pseudo-proof for ~P (or *vice versa*). Of two orthogonal strategies, the one expressing the 'pseudo-proof' will be the 'quitter', the one that invokes the DAIMON, while a strategy that is not forced (and does not decide) to invoke the DAIMON '*passes'* the TEST OF ORTHOGONAL INTERACTION.

Through the dynamics of orthogonality, anonymous rhythms of interacting strategies *immanently generate* PROPOSITIONS *and* CONNECTIVES. Through a peculiar torsion,

we no longer decorate proof trees with propositions, but compose propositions from the hollow trees of STRATEGIES. A PROPOSITION will now be given by what is called a 'BEHAVIOUR', a *set of strategies identical to their own* BIORTHOGONAL. To say that a set **B** is 'identical to its biorthogonal' means that if X belongs to the behaviour **B**, and Y is orthogonal to X, then anything orthogonal to Y *also* belongs to **B**. If, as the saying goes, *the enemy of my enemy is my friend,* then a BEHAVIOUR is just such a circle of friends. A LOGICAL CONNECTIVE emerges wherever BEHAVIOURS stitch together in a way that PRESERVES AN IDENTITY WITH ITS BIORTHOGONAL.

This discovery, which shows the apophantic crust of logic—the forms of proposition and connective—to be intelligible as expressions of the dynamics of logical symmetry, which Ludics captures with its notion of *orthogonal strategies*, transforms in retrospect the entire terrain of logical history. Only now can we grasp the symmetrical displays of Gentzen's sequent calculus as something more than an artifice of his formalism. Symmetry is the Idea, embedded in logical technique, by virtue of which logic, today, pursues its concretization.