L'éclat de la dialectique dans les courts-circuits de la syntaxe: Hegel, Miller, Petersen, Girard

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Abstract

Le sujet de cette intervention fait partie d'une recherche en cours autour des différents efforts dans la direction d'une formalisation de la logique dialectique. Je me concentrerai ici sur ce que Hegel considérait l'obstacle le plus fondamental pour un traitement mathématique d'une logique authentique (c'est-à-dire, dialectique) : la « fixité » et extériorité mutuelle des signes mathématiques. Je tenterai donc d'analyser cette notion de « fixité » , et de montrer dans quelle mesure celle-ci est réalisée à l'intérieur des mathématiques elles-mêmes, au lieu d'être une condition a priori de la logique. Je prendrai appui sur l'analyse d'une Remarque de Hegel, où il élabore une critique des lois traditionnelles de la pensée, trouvant à chaque fois le mouvement syntaxique qui articule chaque loi, pour présenter la loi avec son exception et sa relève. La question qui se pose alors est : qu'est-ce qui empêche la syntaxe logico-mathématique de se comporter de cette manière ? Après examiner certains aspects de cette question concernant le caractère antidialectique de la stratification du discours formel, à partir du débat que J.-A. Miller et A. Badiou entretiennent vers la fin des années soixante autour de la lecture de la philosophie de Frege, je me pencherai sur les oeuvres d'Uwe Petersen et de Jean-Yves Girard. Ces dernières oeuvres effectuent une « destratification » partielle de la logique à travers l'abolition de types à deux

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niveaux différents : prédication (Petersen) et preuves (Girard). A chaque fois, cette stratification s'accompagne d'une ouverture de la logique sur des phénomènes d'ambiguïté systémique et d'interférence syntaxique qui semblaient à l'oeuvre dans le texte de Hegel. A chaque fois aussi, il se produit une suspension du principe de fixité le plus élémentaire, que Boole appelait la « loi fondamentale de la pensée » . La clarté et la rigueur dont la dialectique est reflétée dans ces oeuvres ne sont pourtant pas achevées à travers un refus total de la stratification discursive, mais par l'influence de la structure stratifiée de la formalisation logique sur elle-même. Un formalisme stratifié apparaît alors comme le pivot non dialectique de la pensée dialectique, refusant et restituant à la fois sa propre force.[1]

Ṫhe central problem of a speculative philosophy is, even more so than in the case of Frege's enterprise of a logical foundation of the concept of number, a clarification of the question "How do the empty forms of logic come to disgorge so rich a content?" '

— Uwe Petersen, citing Gottlob Frege, *The Foundations of Arithmetic*, § 16: 22

Go back to An-fang, the Peace Square at An-Fang, the Beginning Place at An-Fang, where all things start (. . .) An-Fang was near a city, the only living city with a pre-atomic name (. . .) The headquarters of the People Programmer was at An-Fang, and there the mistake happened: A ruby trembled. Two tourmaline nets failed to rectify the laser beam. A diamond noted the error. Both the error and the correction went into the general computer.

— Cordwainer Smith, *The Dead Lady of Clown Town*, 1964. Epigraph to Jean-Yves Girard's *Locus Solum* (2001).

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Contents

1 The revolution and division of logic in the 19th century: speculative and mathematical logic **4**
- 1.1 The twin birth . . . 4
- 1.2 The idea of dialectical logic's formalization as a concentration of this division and as its 'excluded third' . . . 5

2 Hegel's judgment concerning the sterility of formal logic and its mathematization (techniques 'void of concept') **7**
- 2.1 Against formality . . . 8
- 2.2 Against fixity . . . 9
- 2.3 Boole's 'Fundamental Law of Thought' as a crystallization of formality and fixity . . . 10
- 2.4 The sparks of the dialectic in the short-circuiting of syntax: Hegel's destruction of 'the laws of thought' . . . 12
  - 2.4.1 The law of identity . . . 12
  - 2.4.2 The law of the excluded middle . . . 13
- 2.5 Suture and stratification: Miller's repetition of Hegel, and the repercussions on Hegel of Badiou's critique of Miller . . . 16

3 Uwe Petersen's dialectical logicism **23**
- 3.1 Antinomy and anti-dialectical stratifications: types and metalanguages . . . 23
- 3.2 Fixed points, diagonalization and systemic ambiguity: The non-cancellation of logical time as precondition for their study *in vitro* . 28
  - 3.2.1 Systemic ambiguity, and the *in vitro* cultivation of the dialectic's rational kernel . . . 29
  - 3.2.2 The sequent calculus . . . 31
- 3.3 A logic of unrestricted abstraction, without contraction: a sketch of Petersen's calculus . . . 34

4 Jean-Yves Girard's locative dialectics **38**
- 4.1 Transcendental syntax . . . 38
- 4.2 The return of type-stratification: propositions as types of proofs . . 44
- 4.3 Locativity, and the 'hidden necessity of adding to abstract identity the extra factor of that movement' inscribed in syntax . . . 46

5 Conclusion: The dialectical challenge to the 'classical image of thought' **52**

5.1 A war on two fronts: the stratification of types *versus* dialectics (are Petersen and Girard's projects compatible?) 52
5.2 The classical constellation 54
5.2.1 Formality 55
5.2.2 Fixity 55
5.2.3 Atomicity/Transparency 55

1 The revolution and division of logic in the 19th century: speculative and mathematical logic

1.1 The twin birth

In the nineteenth century, after two millennia of dormancy, which Kant took as the sign of the science's sure path, the study of logic underwent an explosive fission, which split the discipline in two and, for the first time in ages, set it back into motion. This motion took the form of two diverging trajectories: the path of *speculative logic*, born through Hegel, and that of *mathematical logic*, born through Boole. In terms of language, method, institutions and concepts, the split could not have been more dramatic.

Communication between the two sides, though rare, nevertheless persisted, and it is not impossible to locate a common kernel, an essential unrest that propels both movements (speculative and mathematical): an unprecedentedly radical *reflection* of the discipline back onto itself, but a refractory reflection, a reflection that never ceases to escape itself. The concept of this kernel is itself split in two, and all of my current research is staked on trying to grasp its unity in and through this division: *dialectics* and *formalization*. These are the two names given to a divided logic's common kernel, its essential and restless reflexivity. *Formalization* drives mathematical logic as *dialectics* drives its speculative twin. (The practice of formalization, consubstantial with the emergence of mathematical logic as a historical force, is, after all, nothing other than the effort to grasp

mathematical thought as *itself* a mathematical object, and makes possible the study of what happens when this thought escapes itself at the moment it seizes itself.) This is, at least, the vague intuition underlying my current research.

We're indebted to the French epistemological tradition – from Cavaillès and Lautman through to Desanti, Gonseth, Granger and the young Badiou – for bringing to light an essential aspect of the history of mathematics and logic that can be summed up with the watchword, '*the dialectics of formalization*'. The study and continuation of that tradition, I think, has been one of the main objectives of the seminar series we're taking part in, today ("*Formalisme(s) : la philosophie frança̧ise et les sciences formelles*"). I consider this tradition to be nothing short of magnificent. But it is its obscure counterpart that I wish to approach today: less the problem of the 'dialectics of formalization' than that of the 'formalization of dialectics'.

1.2 The idea of dialectical logic's formalization as a concentration of this division and as its 'excluded third'

There is no tradition under the banner of 'the formalization of dialectics', but it doesn't quite flap in the void. What we find there is a non-cohesive multiplicity of undertakings, historically excluded from the mainstream traditions of both mathematical and speculative logic. Neither Hegelian philosophers nor mathematical logicians have much to do with them, with rare exceptions. These attempts to do something like capture the mathematical structure of dialectical thought, produce a symbolic calculus that unfolds the dialectic's operations, or institutionalize in a formalized and technical apparatus the speculative work of the concept – these are the products of logicians working in almost complete isolation from one another, only rarely citing each other's work, and almost never engaging with one another substantively. There is no common 'paradigm', no agreement – and scarcely even debate – on what it would mean to produce a mathematical dialectics; on what forms must or may be extracted from their Hegelian environment (which took itself to be presuppositionless) in order to be

enshrined as the first principles, rules or axioms of a formal calculus; on what, in the classical image of logic, must be undone or completed in order for the structure of dialectical logic to come to the surface, or even on what logic *is* and what it *should be*. Without a tradition or a community of researchers to recognize them, the majority of these attempts have ended up as islands unto themselves, and have been, for the most part, forgotten. That they have a 'history' at all is almost accidental; at most they posses one as a sort of an environmental characteristic, owing only to the history of mathematics and logic out of whose orbits they've fallen.

I admit that there's a certain romanticism of the archivist in trying to assemble and, to some extent, revive these attempts, as I hope to do. There is nevertheless a complex and, I think, profound set of *philosophical* problems condensed in these attempts to formalize dialectical logic, *whether or not the project turns out to be possible*. Even if the idea of a fully formalized dialectical logic is, as Badiou asserts, a *phantasy* (Badiou 2009 (1982, 02-05-1977: 149),[2] it is a phantasy that must be traversed if we want to grasp the split concept of logic as clearly and concretely as possible. The common achievement of these formalizations, as I see it, is to finally let us read the conceptual and structural contours of this split, which goes to the root of a concept that might otherwise remain illegible in its transparency: *the concept of logic*.

This is quite a broad project, and one that is still in its early stages. Today, I'm going to restrict my attention to two points of intersection between dialectical and mathematical logic: the work of Uwe Petersen, whose project is to reforge dialectical logic *from scratch* in a strictly mathematical framework, and that of Jean-Yves Girard, whose investigation into the deepest strata of proof theory

[2]The context of this remark is as follows: 'Consider also the fascination that Marx and Engels feel for differential calculus and their somewhat naïve intent to seek therein the matrix of the 'laws of the dialectic'; or Marx's fallacious conviction, displayed in his numerous writings on mathematics, that he was a mathematician because he was a dialectician. These are all signs that the enigma of writing is tied to the **fantasy of a formalized dialectic**, with mathematics being its restricted speciality from which, upon close scrutiny, it would be possible all the same to extract the universal principle.' (Boldface added.)

has led him to stumble – somewhat reluctantly – across the figure of the dialectic.

2 Hegel's judgment concerning the sterility of formal logic and its mathematization (techniques 'void of concept')

It's a well-known fact that Hegel considered mathematical techniques to be essentially incompatible with a genuine science of logic, and considered the mathematization of logic to be an 'otiose and sterile' undertaking. These judgments were made public about 35 years before Boole's publication of *The Mathematical Analysis of Logic*, in 1812's *Science of Logic*.

It's nevertheless clear that Hegel saw logic's tendency towards mathematization as already inscribed in the *formal* understanding of logic of contemporary logicians. Efforts to mathematize logic, such as those early (and not yet successful) attempts by Leibniz or Euler, with which Hegel was familiar, Hegel took to ossify and arrest logical thought at 'the form which the understanding gives' to it 'and in which the determinations of the concept are taken as *abstract* formal determinations' (Hegel 2010 (1812-31, 605). Hegel stands in opposition both to logic's scholastic history *and* to its mathematical future. His two targets are the **formality** of logic, already enshrined by the scholastic tradition and philosophically cemented by Kant, and the **fixity** in which mathematization threatens to cast that formality.

It is worth noting that Hegel was not, and could not have been, addressing what we now call the *formalization* of logic, a movement that would push the mathematization of logic to its reflexive extreme: *extreme*, because formalization aims at a sort of exhaustive mathematical explication of the demonstrative process, one that leaves no gaps that an appeal to the intuitive understanding of space,

number, or whatever, would have to fill in; *reflexive*, because formalization has, in the end, no other meaning than making mathematical propositions and proofs *themselves* into mathematical *objects*. Formalization must be distinguished from *formality*, a notion that it radicalizes, completes and ultimately *unravels* – with force that is sufficient to shatter the image of thought that sustains the formal conception of logic that Hegel attacks. The motif that I'm playing on here could be summed up in this way: that *formalization* can be grasped as the *Aufhebung*, the *rélèvement* of the two aspects of logic that Hegel most fiercely opposed: the *formality* and the *fixity* of its determinations, both of which are pushed to their extreme and finally undone by logic's mathematization and formalization.

2.1 Against formality

Hegel's protests against the 'formal' conception of logic are numerous and well-known. Already in the Introduction to *The Science of Logic*, Hegel bemoans the inadequacy of the tradition whereby

> Whenever logic is taken as the science of thinking in general, it is thereby understood that this 'thinking' constitutes *the mere form* of a cognition; that logic abstracts from all *content*, and the so-called *constitutive piece* that belongs to the cognition, namely the *matter*, must be given from elsewhere; hence that logic, since this matter does not in the least depend on it, can give only the formal conditions of genuine knowledge, but does not itself contain real truth. (Hegel 2010 (1812-31, 24)

This vision of logic certainly survives to the present day, but the *progress* of mathematical logic owes nothing to it. Its name, today, is *semantics*, an approach to – or, rather, through – logic which is a no doubt useful, and indeed mathematically critical employment of logic as a noiseless channel through which diverse systems of mathematics communicate. But this clarity is bought at the price of

logic's transparency to itself. We will see that it is rather through a reflection upon its own *syntax*, rather than through its servile semantic employment, that logic begins to accede to the ideal that Hegel set for it: to be a '*formal* science, yet the science of *absolute form* [... which] has in it a content or reality of its own [...] and the content is only these determinations of the absolute form and nothing else' (Hegel 2010 (1812-31, 523). The clue for this shift in perspective, and one which is fastened onto firmly by Girard, is already furnished by Hegel in the Introduction: 'For, since thinking and the rules of thinking are supposed to be its subject matter, in these logic already has a content specifically its own; in them it has that second constituent of knowledge, namely a matter whose composition is its concern' (Hegel 2010 (1812-31, 24).

Perhaps against expectations, it will be nothing other than the *formalization* of logic that will deliver it the reflective apprehension of its own formal content, in a fashion that marks a rupture with all '*formality*'.

2.2 Against fixity

This eventuality is nevertheless invisible to Hegel himself. For when, in a series of Remarks in the Doctrine of the Concept, Hegel outlines his critique of the idea of 'mathematizing' logic, he places the blame *on mathematical symbolism as such*. 'It is characteristic' of the signs of mathematics, 'as contrasted with the determinations of the concept', he writes, 'that they are mutually *external*, that they have a *fixed* determination. **Now when concepts are made to conform to such signs, they cease to be concepts**' (Hegel 2010 (1812-31, 544). Moreover,

> Since the human being **has in language a means of designation that is appropriate to reason**, it is otiose to look for a less perfect means of representation to bother oneself with. It is essentially only spirit that can grasp concept as concept, for the latter is not just the property of spirit but its pure self. It is futile to want to fix it by means of

> spatial figures and algebraic signs for the sake of the *outer eye* and a *non-conceptual, mechanical manipulation*, such as a *calculus*. (Hegel 2010 (1812-31, 544)

There is something mysterious about this objection. After all, the 'raw material' of both mathematical syntax and philosophical writing appears to be the same: a stock of discrete signs, mutually external to one another, and susceptible to combination and repetition. If such signs behave differently in the two discourses – such that one is appropriate to reason and the other not – then this must be either because there is something in language which redeems the sign *from* fixity, or something in mathematics which condemns the sign *to* fixity – or, likely, both.

2.3 Boole's 'Fundamental Law of Thought' as a crystallization of formality and fixity

George Boole's great merit was not only to be the first to at last set logic on the unsure path of science, but to grasp the possibility of embedding logic in, while demarcating it from, the algebra of quantities by means of a single law – a law he called 'THE FUNDAMENTAL LAW OF THOUGHT' – which would serve to capture in a single but articulate crystalline structure the most essential obstacles to dialectical logic: the *formality* and the *fixity* of conceptual determinations.

What, in Boole's eyes, justifies the 'fundamental law of thought' – the idempotence of conjunction, or the identity of A *and* $A = A$ – is precisely the thought that when our signs have a 'fixed interpretation' – when they refer univocally to some *thing* outside thought, a thing whose unconditional exteriority of thought is already presupposed in Boole's putatively non-metaphysical claim that the laws of logic are 'laws of thought, and not laws of things' – then the repetition of the thought does not affect the identity of the thing thought. 'Suppose,' for instance

> that by a definite act of conception the attention has been fixed upon men, and that by another exercise of the same faculty we limit it to those of the race who are white. Then any further repetition of the latter mental act, by which the attention is limited to white objects, does not in any way modify the conception arrived at, viz., that of white men. (Boole 2005 (1853, 32).

Considerations of the same nature give Boole his justification for the law of *commutativity*, the identity $AB = BA$.

> Difference in the order of the qualities or attributes of an object, apart from all questions of causation, is a difference in conception merely. The law [of commutativity] expresses as a general truth, that the same thing may be conceived in different ways, and states the nature of that difference; and it does no more than this. (Boole 2005 (1853, 21)

In sum, we can say that, for Boole, logic's operations – like the freedoms of modern democracy – are licensed to the extent that *they leave everything in its place*. They express the radical exteriority of thought to its objects, and the pure 'formality' of logic. The freedoms they grant to language – both to logical reasoning and, Boole observes, to poetic diction, understood as pure ornament – is underwritten by the indifference of the thing to which discourse is tethered.

Boole's genius is that, even as he instates this law as an unyielding precondition of logical thought, he leaves perfectly legible the conditions under which it would have be challenged. What justifies the nullity of repetition (or commutation) in logic, Boole seems to argue, is the *stability* and *exteriority* of what thought thinks with respect to the thought that think it. So long as thought and its object remain mutually exterior, it does indeed appear unthinkable that the *course of reasoning* itself would change the constitution of its object. But a logic that would study the structural effects of thought thinking *itself*, and the rational relations between concepts capable of taking *themselves* as objects... What jurisdiction could Boole's law have there?

I want to keep this point in view. It will resurface again before long.

2.4 The sparks of the dialectic in the short-circuiting of syntax: Hegel's destruction of 'the laws of thought'

The problem I want to look at now is this: if mathematical and speculative logic begin with the same, raw syntactic material, how is it that, for Hegel, speculative reason absolves this material of its fixity and sets in it genuinely conceptual motion? This is a delicate question, and it is helpful to have a clear syntactic specimen in view as we try to work through it. Hegel, fortunately, provides the best we could ask for, in a series of Remarks occurring in the Doctrine of Essence, where he undertakes a destructive analysis of the scholastic 'laws of thought'.

2.4.1 The law of identity

Consider, for example, his Remark on the law of identity, $A = A$. Starting with the observation that statements of the form 'A plant is… a plant' seem to leave us where we started while promising, in virtue of their grammatical structure, a movement forward and increase of information, Hegel remarks that

> *More* is entailed, therefore, in the *form of the proposition* expressing identity than simple, abstract identity, entailed by it is this pure movement of reflection in the course of which there emerges the other, but only as reflective shine, as immediate disappearing: 'A is' is a beginning that envisages a something different before it to which the 'A is' would proceed; but the 'A is' never gets to it. 'A is … A': the difference is only a disappearing and the movement goes back to itself. **The propositional form can be regarded as the hidden necessity of adding to abstract identity the extra factor of that movement**. (Hegel

2010 (1812-31, 360)

The syntax of the law, here, is not what *arrests* conceptual movement, but what *insinuates* a movement of concepts that seems to contradict the law itself. What Hegel demands, here, is entirely contrary to his warning in the Doctrine of the Concept – the warning that when concepts are made to conform to such signs as those of formal syntax they cease to be concepts. Here, with a twist that is characteristic of Hegel's method, were are told that the concept should find its movement precisely by conforming to the signs that articulate it: '*The propositional form can be regarded as the hidden necessity of adding to abstract identity the extra factor of that movement*'. The *fixity* of the concept of identity seems, then, not to lie in syntax that articulates it, but in a certain collapse of syntactic difference at the level of the law's (standard) *interpretation*: the image of a static reality hovers *outside* of the syntax, as what Girard will call a 'quotienting' of syntactic movement, and it is this which is challenged by the Hegelian call to return to the immanent *form* of the syntax.

2.4.2 The law of the excluded middle

The most dramatic scene of Hegel's analysis is played out in his Remark on the law of the excluded middle.

> The principle of the excluded middle is further distinguished from the previously examined principles of identity or contradiction that said, 'Nothing is A and not-A at the same time'. What this implies is that ***there is* nothing which is not *either* A *or not-*A; that there is no third that would be indifferent to that opposition**. But in fact a third indifferent to it *is given* in the principle itself, for the A itself is there. This A is neither $+A$ nor $-A$ and just as much also $+A$ and $-A$. – The something which ought to have been either $+A$ or $-A$ is here attached to the $+A$ as well as the $-A$; and again, inasmuch as it is attached to the A, it ought *not* to be attached to the $-A$, just

> as it ought to be attached to the A inasmuch as it is to the not-A. The **something** itself is therefore the third which ought to have been excluded. Since in this something the opposite determinations are equally posited and sublated, that third which is here in the shape of an inert something, when taken more profoundly, is the unity of reflection into which the opposition returns as into ground. (Hegel 2010 (1812-31, 381)

Hegel's argument is somewhat clumsy and muddled in its exposition. The idea is nevertheless intriguing. If we paraphrase Hegel's reasoning in the more spacious syntax of modern predicate calculus, his tactics become clear:

1. Let 'A' be any property whatsoever. The LAW OF THE EXCLUDED MIDDLE states that any given x is either A or else *not-A*, with no third possibility. In modern notation, the law reads as follows:

$$\text{For any property } A, \quad (\forall x)(Ax \vee \neg Ax)$$

2. By contraposition, this law has an equivalent, negative form: 'There is nothing [no x] which is not *either A or not-A*; that there is no third that would be indifferent to that opposition.' Formally:

$$\text{For any property } A, \quad \neg(\exists x)\neg(Ax \vee \neg Ax)$$

3. We can now pinpoint that obscure element that Hegel is groping for, calling it sometimes 'something' and sometimes 'A' (in contrast to the gerrymandered '$+A, -A$' pair): it is the **bound variable** x – a structural *point* on which the law rests, but which remains invisible or virtual in Aristotelean idiom Hegel is working with – that Hegel is seeking to isolate as the *exception* to the LAW OF THE EXCLUDED MIDDLE (as well as to the LAW OF CONTRADICTION, for that matter).

4. Paraphrasing Hegel's remark, we now get:

> But in fact a third indifferent to it *is given* in the principle itself, for the x itself is there. This x is neither A nor $\neg A$ and just as much also A and $\neg A$. – The *variable*, the x which ought to have been either A or $\neg A$ is here attached to the A as well as the $\neg A$; and again, inasmuch as it is attached to the A, it ought *not* to be attached to the $\neg A$, just as it ought to be attached to the A inasmuch as it is to the $\neg A$. The **variable** x is therefore the third which ought to have been excluded. Since in this x the opposite determinations are equally posited and sublated, that third which is here in the shape of an inert something, when taken more profoundly, is the unity of reflection into which the opposition returns as into ground. (Hegel 2010 (1812-31, 381)

5. The x is dislodged from of its place in the syntax of the law, and exhibited as the law's own exception – as the impossible object that both supports and subverts the law from within. As Hegel interprets it, the 'x' does what the LAW OF CONTRADICTION says is impossible, attaching itself to both 'A' *and* '$\neg A$'. But this law *says* that such a such a situation is impossible, and insists that x's attachment to A *prevents* it from attaching to to $\neg A$, and *vice versa*. Since both attachments are barred to x, it now seems to float freely. But then the LAW OF THE EXCLUDED MIDDLE is violated, since it says that any object must be such that it is *either* A or $\neg A$, and so it seems to demand us to take seriously – *literally* – the attachments with which x variably flirts. x is, again, *both* A and $\neg A$, and so it goes, with x endlessly oscillating between a violation of one law and then the other. The two laws of CONTRADICTION and EXCLUDED MIDDLE, which classical logic took to be *equivalent*, now show themselves to ceaselessly *sublate* one another, as Hegel set out to demonstrate.[3]

[3] '[O]n closer examination, the *several propositions* that are set up as *absolute laws of thought* are *opposed to each other*: they contradict each other and mutually sublate each other. [...] The thoughtless examination of them enumerates them *one after the other*, so that they appear unconnected; it merely adverts to their reflectedness without paying attention to their other moment, to the *positedness*, or the *determinateness* as such which propels them on to transition or to their

6. Hegel's argument therefore seems to hinge upon a short-circuiting of the law's syntactic expression. The '*x*' (or the 'something'), to begin with, was never *meant* to be an object over which the law ranged, which is intended to be a law about objects and their properties, or about propositional contents, and not a law about the behaviour of the grammatical components of its own syntactic structure. So far as the law itself is concerned, '*x*' is not an 'object' at all.

What we find in these Remarks is this: unlike the challenges to the traditional laws of thought advanced by, say, Brouwer, Simondon, Von Neumann or Engels, Hegel at no point seeks to hold the laws up to a pre-given domain, in view of which they appear inadequate. What he does, instead, is locate an exception to each law which arises from the *syntactic structure* of the law itself. The law produces its own exception by stumbling over its own syntax, by an inadvertent short-circuiting of its own syntactic machinery.

Each law, moreover, can be seen to found itself precisely on a *repression* of its own syntactic conditions of actuality, an exclusion from its 'field of truth' of the syntax that sustains and haunts it.

2.5 Suture and stratification: Miller's repetition of Hegel, and the repercussions on Hegel of Badiou's critique of Miller

Hegel's strategy is thus remarkably similar to the one advanced by Jacques-Alain Miller in his 1966 article, 'Suture (éléments de la logique du signifiant)' (Miller 1966).

Miller, recall, sought to show that underneath every discourse – even and especially the most desubjectified and formalized – moves a quasi-dialectical 'logic of the signifier', that attaches to the manifest discourse by means of an operation called 'suture', which situates the subject in the signifying chain of the latter,

negation' (Hegel 2010 (1812-31, 356).

even and especially when that chain endeavours to leave no mark of the subject itself.

The specimen that Miller analyses in that text is Gottlob Frege's *Foundations of Arithmetic* (Frege 1960), in which Frege seeks to provide a purely logical foundation and reconstruction of arithmetic and it fundamental concepts of equality, succession and number. The crucial logical innovation Frege uses to do this is his second-order logic of relations, equipped with a principle that we may call the PRINCIPLE OF ABSTRACTION. This principle – grounded, essentially, on Frege's Fifth Basic Law[4] – allows us to treat a proposition with an unbound variable as a determinate *concept*, and to pass freely from that concept to its *extension*, the set of objects that satisfy that proposition (objects which the concept, conceived as a mathematical *function* will map to 'THE TRUE'). Frege aims to define numbers in relation to concepts formed in this way: *the number that belongs to a concept F* will be the extension of the concept 'equinumerous to *F*', which is to say, the set of concepts whose extensions can be placed into a one-to-one correspondence with the objects falling under *F*. In order to ground his construction of the series of natural numbers, Frege begins by constructing the *zero*:

> I define [zero] as follows:
>
> > 0 is the Number which belongs to the concept "not identical with itself".[5]

[4]The Fifth Basic Law (*Grundsatz V*): Two concepts f and g have the same extension if and only if everything that falls under f also falls under g and vice-versa.

$$\vdash (\acute{\varepsilon} f(\varepsilon) = \acute{\alpha} g(\alpha)) = (\smile_{\mathfrak{a}} f(\mathfrak{a}) = g(\mathfrak{a}))$$

In modern notation:

$$\lambda x(f(x)) = \lambda x(g(x)) \longleftrightarrow (\forall x)(f(x) \leftrightarrow g(x))$$

A consequence of this law is the ability to construct, from any propositional form the language can express, a *concept*, obtained by *abstracting* from the determinate (names of) objects or bound variables occurring in the proposition.

[5]In a remark whose significance will be clearer later on, Frege goes on to anticipate certain objections to his use of this concept:

> Some may find it shocking that I should speak of a concept in this connexion. They will

Observing that Frege defines identity as *substitutability without loss of truth*, Miller reasons that the 'not identical with itself' can only gesture to a breach in the symbolic order of logic: after all, if a sign could not be substituted for itself, where would that leave truth? 'Absolute is its subversion' (Miller 1966, 43).[6] But: '*Truth is*,' and so, '*Each thing is identical to itself.*' The extension of the concept "not identical with itself" must therefore be empty: to its Number the name of 'zero' will be given. The invocation of the non-identical, as Miller reads it, opens up a rift in the 'field of truth' and lets us glimpse an exception to logical discourse's conditions of possibility, invoked and annulled in a single stroke. This is what he calls the moment of **suture**.

Frege goes on to define each natural number $n + 1$ as the 'Number belonging to the series that ends with n', thereby demanding that zero be repeatedly 'counted

> object, very likely, that it contains a contradiction and is reminiscent of our old friends the square circle and wooden iron. Now I believe that these old friends are not so black as they are painted. [...] All that can be demanded of a concept from the point of view of logic and with an eye to rigour of proof is only that the limits to its application should be sharp, that we should be able to decide definitely about every object whether it falls under that concept or not. But this demand is completely satisfied by concepts which, like 'not identical with itself', contains a contradiction; for of every object we know that it does not fall under any such concept. (Frege 1960, § 75:87)

[6] As Miller writes,

> Those things which are identical are those which can be substituted for one another *salva veritate*, without loss of truth.
>
> You can no doubt sense the importance of what is accomplished in this statement: the emergence of the function of truth. But what it takes for granted is more important than what it expresses: self-identity. That a thing should not be substitutable with itself, where would that leave truth? Absolute is its subversion.
>
> If we follow Leibniz's argument, the failure of truth, the possibility of which is opened for an instant, its loss in the substitution of one thing for another, shall nevertheless be followed by its reestablishment in a new relation: truth is recovered in that the thing substituted, insofar as it is identical with itself, can be made the object of a judgment and enter into the order of discourse; identical-to-itself, it is articulable.
>
> But that a thing should not be identical with itself would subvert the field of truth, ruining and abolishing it. (Miller 1966, 43)

as one', as the first number, while annulling the extension of the seemingly magical concept 'not-identical-with-itself' in order to ensure that the 'field of truth' is unbreached. Miller therefore takes the moment of 'suture' to be the agency that animates the numerical chain, and names 'subject' – which is nothing but the flickering of lack in the suturing of the signifier – 'that which functions' in the function of succession. The non-object that falls under the concept 'not identical with itself' is thus judged to be the 'impossible object' correlative to the Lacanian subject: '*objet* (a)'.

Here, just as in Hegel's analysis of the LAW OF THE EXCLUDED MIDDLE, we are dealing with a condition of the possibility of a (classical) logical discourse which must be suppressed (not thematized as such within the discourse's 'field of truth', to use Miller's expression) in order for that discourse to operate. In each case, moreover, we are dealing with an element that is *not identical to itself*, and so strays in an essential but intolerable fashion from the field of truth it helps to articulate.

In each case, moreover, we are faced with an element that seems to fall loose from the fixed gears of syntax, while throwing those gears into motion. It is only when viewed from the vantage point of the subjectifying suture of its *objet* (a) that Frege's numerical series *appears* as a moving progression. And it is only when seen from the oscillations of the unthematized support of the '*something*' that the LAW OF THE EXCLUDED MIDDLE betrays its dialectical fluxions.[7]

However, it is through Badiou's critique of Miller in '*Marque et manque: à propos du zéro*' that the most important analogy between the Hegelian '*something*' and the Millerian '*objet* (a)' comes to light: both are engendered through the short-circuiting of a syntax that is, in itself, essentially *stratified*.

What Badiou argues is this: Miller is *wrong* to see anything resembling suture or

[7]We could add: it is by seizing on something which seems to fall outside the syntactic order it sustains, Miller and Hegel seem to have caught the tail of what Girard, in an altogether different context, calls the '*perfective*' stratum of logic, the stratum on which what are at stake are moments and actions, rather than truths and functions (the '*imperfective*' realm)...

lack in Frege's logical reconstruction of Number. Logic can only be constituted, Badiou argues, through a series of distinct syntactic *strata*. He distinguishes, in particular, three stratified 'mechanisms' that are operative in any logical calculus. *First*, there is a mechanism of CONCATENATION (M1), which assembles its raw materials – a stock of letters and marks – into arbitrary sequences, exhausting all possibilities of combination, heedless of grammatical correctness. *Second*, a mechanism of FORMATION (M2), which sorts the output of M1 into well-formed expressions (whatever can be read as 'propositions' of the system) and ill-formed expressions (ungrammatical strings of characters), according to iterable rules of construction. *Third*, a mechanism of DERIVATION, which sorts the output of M2 – the stock of grammatically correct, well-formed expressions – into those which can be proven and those which cannot, according to a set of axioms and rules of inference.

The entire drama of logic is staged between such stratified syntactic mechanisms as these. The operation of each stratum requires, as material, only what the strata 'beneath' it have produced. When Frege forms the concept 'non-identical to itself' $(x \neq x)$, and then judges its extension to be empty

$$\vdash \neg(\exists y)(y \in \lambda x(x \neq x))$$

he is not, even for a moment, invoking an exception to the self-identity of marks, which Badiou takes to be an indispensable condition of possibility of any scientific symbolism.[8] *That* law of self-identity, he argues, is a rule brought into force already on M1, and no system of notation whatsoever can stray from its reign. The 'absolute subversion of truth' that Miller takes to haunt logic with the invocation of $(x \neq x)$ is foreclosed long before the question of truth is even raised, and

[8] As a rule governing the products of M1, Badiou claims, 'the rule of self-identity allows of no exceptions and does not tolerate the evocation of that which is withheld from it; not even in the form of rejection. What is not substitutable-for-itself is something radically unthought, of which the logical mechanism *bears no trace*. It is impossible to turn it into an evanescence, a shimmering oscillation, as Frege does when he phantasmatically (ideologically) convokes then revokes the thing that is not self-identical in order to assign the zero. What is not substitutable-for-itself is foreclosed without appeal or mark.' (Badiou 1969, 157)

long before M2 determines '$(x \neq x)$' to be a well-formed formula. Indeed, the identity of x, its self-substitutability across all of its instances, is *presupposed* by the very writability of statements like $(x \neq x)$, in which 'x' occurs *twice* (Badiou 1969, 158).

The 'lack' with which $(x \neq x)$ is associated, Badiou observes, refers to nothing external to the system's stratified machinery. It is nothing but the absence **in M3** of any inscription of the form $a \neq a$, where a is a constant. But inscriptions of this sort, of the form $a \neq a$, are fully present in the output **of M2**, since they are unambiguously well-formed formulae, and violate none of the logic's *grammatical* rules.

The sign 'o', then, does not serve to suture a rupture in the fabric of logical discourse as such, but only to mark *the lack of a mark* in the output of M3, a mark that, in M2, is clearly and distinctly present. All of scientific discourse, Badiou argues, can be seen to follow this pattern. Science is a signifying order that is

> stratifié, tel qu'aucun manque n'y est marqué qu'on ne puisse découvrir marque lui-même dans l'ordre sous-jacent dont se différencie le premier, résulte l'exception. La science ne tombe pas sous le concept de la logique du signifiant. Au vrai, c'est de n'y pas tomber qui la constitue: la coupure épistémologique doit être pensée sous les espèces irreprésentables de la désuturation.
>
> En sorte qu'il n'y a pas de sujet de la science. Stratifiée à l'infini, réglant ses passages, la science est l'espace pur, sans envers ni marque ou place de ce qu'elle exclut.
>
> Forclusion, mais de rien, on la peut dire psychose d'aucun sujet. Donc de tous; universelle de plein droit, délire partagé, il suffit de s'y tenir pour n'être plus personne, anonymement dispersé dans la hiérarchie des ordres.

La science est le Dehors sans point de cécité.(Badiou 1969, 161-2)

The 'suturation' of scientific discourse, when it occurs, amounts to a *de-stratification* of the discourse, and its *relapse* into ideology's clutches, where the gears of *subjectivity* again begin to turn. It is in this way that *epistemological obstacles* are formed, the destruction of which – in the form of *epistemological breaks* – is the *essence* of scientific labour.

It is just such a *destratification* of formal logic's mechanisms that Hegel stages in his analysis of the 'laws of thought'. What Hegel attempts to do in his interpretation of the 'something', the logical variable x, in the LAW OF THE EXCLUDED MIDDLE, is to short-circuit the machinery of the law: a description of this syntactic element, and its instances in the law, is dislodged from its proper stratum and reinserted into the stratum of objects to which the law is taken to apply, so as to furnish the law with an apparent *counter-model*. In an interference between grammatical and semantic orders, the law malfunctions and spins on its head, producing a bizarre display of dialectical vacillations.

It is through this torsion of the syntax that the variable, the 'something', is cast as *objet* (a) in this particular scene of Hegel's dialectic. It is through its interference that he is able, at this moment, to wrest the syntax of the law the '*fixity*' that he claims to be incompatible with the movements of logic.

If this is the *only* way of freeing logic from its anti-dialectical fixity, Badiou's arguments can indeed be seen to corroborate Hegel's own thesis: it is 'natural language' alone which affords the means of expression appropriate to Reason, and this is so – Badiou's text may be taken to suggest – insofar as it refuses stratification,[9] and demands the suturation of discourse that, alone, can subjectify it.[10] If *objet* (a) is, indeed, an *essential* dialectical operator, such that no logic can be

[9]Indeed, Miller makes a direct connection between the *necessity of suture* and the *impossibility of a meta-language*, linking, in this way, two crucial Lacanian theses (see Miller 1966, 46, fn.1).

[10]In a letter to Hegel, commenting on the newly published *Science of Logic*, the mathematician Pfaff writes:

> C'est pourquoi je crois toujours que vous êtes obligé de vous **mouvoir dans un**

called dialectical that refuses it a place and if this operator can *only* be articulated by a suture, by an 'ideological' de-stratification of scientific discourse, then the prospects of formalizing dialectical logic seem dim indeed.

This is an incomplete, but seductive, argument. It pivots on the notion that stratification is an essentially *anti-dialectical* operation, that it serves to foreclose the sort of short-circuiting and suturation that animates dialectical reasoning. It is, to a certain extent, the extrapolation of a thesis I take from Mladen Dolar: that something structurally akin to Lacan's *objet* (a) rears its head at every turn of the dialectic, when a conceptual formation fails to measure up to itself, insofar as it engenders its own exception by short-circuiting (suturing) components that the understanding – or, for Badiou, scientific discourse – struggles to keep separate.

I nevertheless take this conclusion to be precipitous.

3 Uwe Petersen's dialectical logicism

3.1 Antinomy and anti-dialectical stratifications: types and metalanguages

In the concept that Frege used to define the zero – the concept '*not identical to itself*' – Miller thought he had found the point at which Fregean logic propped itself on an inarticulable lack, necessary for its development but accessible only to a

> **cercle**… On part d'un point, on pose par avance ces opérations, etc., qui cependent ne viennent qu'ensuite; si en conséquence tout doit être juste, on est obligé de revenir au point dont on était parti. Si dans votre cercle vous n'êtes pas obligé en ligne droit (comme les mathématiciens), non point selon une parabole comme les cométes, mais selon une courbe fermée comme les planétes, séjour des dieux, **c'est parce que vous avez besoin du langage, tandis que le mathématicien est muet**. (Hegel 1962, 359)

If we take the speaking subject to be – as Lacan and Miller would insist – the sutured subject, and the circle to be the refusal of stratification, we can take Pfaff's conclusion to chime with Badiou's.

(broadly dialectical) 'logic of the signifier'. Badiou then showed that this reading can be sustained only at the cost of obfuscating the essential, productive structure of scientific discourse, whose unyielding stratification into non-interfering and univocal mechanisms forecloses all lack, subject and suture. Frege's own justification of his use of the contradictory concept, '*not identical to itself*', thus seems vindicated: 'All that can be demanded of a concept,' Frege writes,

> is only that the limits to its application should be sharp, that we should be able to decide definitely about every object whether it falls under that concept or not. But this demand is completely satisfied by concepts which, like 'not identical with itself', contains a contradiction; for of every object we know that it does not fall under any such concept. (Frege 1960, § 75:87)

What shores up Frege's method, then, is a difference of strata separating the identity of marks, the formation of a concept of non-self-identity, and the determination of the emptiness of that concept's extension – a difference that should definitively bar confusion and prevent the question of suture from even arising.

Formally stratified and foreclosing all lack, Frege's logic would therefore seem safe from the dialectics of the signifier – except for a small problem, first observed by Bertrand Russell and communicated to Frege in a letter written on June 16, 1902. After praising Frege on the first volume of *The Basic Laws of Arithmetic*, where the programme sketched out in *The Foundations of Arithmetic* is executed with the aid of Frege's own *Begriffsschrift*, Russell remarked:

> There is just one point where I have encountered a difficulty. You state that a function, too, can act as the indeterminate element ⟦*i.e., as the bound variable in a quantified proposition*⟧. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? **From each answer its opposite follows.** Therefore we must conclude that w is not a

> predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [*Menge*] does not form a totality (Russell 1967, 124-5).

The situation with which this letter confronted Frege could not have been more serious: the free exercise of rational thought, proceeding only according to the strict laws of logic and manipulating only those concepts which it is able to form for itself, has led to an *antinomy*.

The fault, as Russell's sees it, lies in the assumption of the right to pass freely from a propositional form to it concept, and from a concept to its extension: the right of ABSTRACTION.

The responses with which the mathematical community – Russell himself included – greeted this antinomy were various, but overwhelmingly 'Kantian' in spirit: in every case it was a matter of *limiting* the freedom of logic to perform and apply abstractions.[11] In each case, moreover, it was a matter of reinforcing the *stratification* of logico-mathematical discourse.

This state of affairs was by no means limited to the Russell paradox.

True to Hegelian expectations, the same phenomenon erupted in the attempt to grasp the actual infinite mathematically. If we take S to be any set – even an actually infinite set – we can show that the assumption that we can map the set of S's *subsets* back onto its *elements* leads to a paradox formally identical to

[11] As Kurt Gödel observed in 1933,

> only one solution [to the paradoxes] has been found, although more than 30 years have elapsed since the discovery of the paradoxes. This solution consists in the theory of types. [...]
>
> It may seem as if another solution were afforded by the system of axioms for the theory of aggregates [i.e. set theory], as presented by Zermelo, Fraenkel and von Neumann; but it turns out that this system is nothing else but a natural generalization of the theory of types, or rather, it is what becomes of the theory of types if certain superfluous restrictions are removed.

For a detailed discussion of the problems that Gödel's (controversial) observation raises, see **?**, from which this quotation of Gödel is taken.

Russell's. To avoid disaster we are forced to concede that the set of subsets of an infinite set constitutes an infinite set that is *quantitatively larger* than the first. The infinite recoils before paradox, stratifying itself indefinitely into the hierarchy of alephs.

When Alfred Tarski encountered a structurally analogous antinomy attending the introduction of a 'truth predicate' into classical logic, his solution took the same general shape: in order to prevent a language from being capable – as natural languages always are – of formulating the 'Liar Paradox' ("*This sentence is not true!*"), we must enforce a *stratification* of discourse, separating syntaxes from their semantic interpretations and mediating between the two sides with a 'metalanguage', whose own semantics could, in turn, only be formulated by appealing to *its* metalanguage, and so on, to infinity (a paradigmatically 'bad' infinity, by Hegelian lights).

And again, if we turn our attention to the *lambda calculus*, we find a familiar situation. Developed in the 1930s in order to rigorously articulate the methodological concept of 'effectively calculable procedure', the lambda calculus is a pure theory of functions. Instead of simply identifying functions with their 'graphs', or with the set of ordered pairs that coordinates their inputs with their outputs, as is done in set theory, the lambda calculus grasps functions *intensionally* or algorithmically, as syntactic processes of *rewriting*. Such functions can be freely applied to one another in order to build functions of greater complexity. In the pure lambda calculus, functions roam wild. Each can be applied to any other. The result, however, may be monstrous – the process may diverge or 'crash' – and when interpreted logically, the results are often paradoxical: no sooner can we construct an implication operator in the calculus, than we can construct a term $T = T \rightarrow X$, from which X inevitably follows, *whatever X might be*.[12] Solutions to

[12] An informal example: Let T be something like 'If this statement is true, then God exists.' On being uttered, the statement is not *yet* true, and so the antecedent of its condition is false. This, however, makes the conditional *true*, like any conditional with a false antecedent. A second time around, 'this statement is true' is *indeed* true. But now we are faced with (1) a true antecedent, and (2) a true conditional. The consequent inevitably follows, giving us an ontological argument

this paradox, and similar troubling phenomena, varied, but virtually all involved hierarchizing – stratifying – the calculus by means of an apparatus of *types*. As with Russell, the typing of lambda terms served to restrict the domain of terms that they may take as arguments – barring reflexivity and 'preventing functional incest'. The resulting calculi behave well, but at the cost of sacrificing the freedom and reflexivity of the pure lambda calculus.

The phenomenon would appear, as well, in 'indirect' forms, which, though not antinomical, were no less startling or paradoxical. This takes place, for instance, in the proof of Gödel's famous incompleteness results. By encoding the syntax of arithmetic back into arithmetic itself, Gödel was able to construct a perfectly ordinary numerical expression F that is also an *indirect* fixed point for the numerical code for the propositional schema, 'F is not provable'. This yields a proposition that is *provable* if and only if its *negation* is also provable. This does not lead to a contradiction, but forces the conclusion that there exists, and must forever exist, a proposition which the system can formulate but neither prove nor refute: an 'undecidable' proposition, a symptom of incompleteness. The most prevalent reaction to Gödel's results has been to take the 'Gödel sentence' – which seems to say 'F is not provable' – to be self-evidently *true*, and conclude that a system of sufficient expressive power cannot syntactically prove all of its own 'truths'. To capture these truths in a systematic fashion, we must ascend through a stratified hierarchy of formal systems, each with its own 'formally undecidable truths'.

The common *dialectical kernel* of each of these results is the existence of a *fixed point phenomenon*, put to work in a technique called *diagonalization*. And in each case, the solution has been to escape paradox through a systematic stratification of discourse.

A minority position is nevertheless gaining ground, at least in the annals of 'philosophical logic': this is the position of 'dialetheic' or 'paraconsistent' logic,

for everything imaginable!

whose advocates frequently cite Hegel as their authority. Their strategy is to reform the underlying logic so as to admit what appear to be 'true contradictions', conjunctions of A and $\neg A$ which do not explode into triviality by entailing every proposition whatsoever (according to the old law, '*ex falso sequitur quodlibet*'). I will not say much about these approaches today, except to say that *nothing* could be further from 'Hegel's thesis' than that, as the Brazillian paraconsistentist Ayda Arruda put it, some contradictions are true 'and we do not need to surmount them!' (**?**, 4-5). The contradictions countenanced by the paraconsistent schools are neither classical *nor* dialectical: they are contradictions neutered and declawed, which do no harm and force no sublation.

3.2 Fixed points, diagonalization and systemic ambiguity: The non-cancellation of logical time as precondition for their study *in vitro*

Uwe Petersen's strategy is essentially different. Unlike the paraconsistent approach, which affirms contradictions as *actual* but in truth as little more than castrated subcontraieties, and unlike the classical – Kantian – approach, which forecloses the engine of contradiction through a stratification of discourse, Petersen preserves the engine of the antinomies *in its virtuality*, capturing it in action '*before*' it explodes into triviality. This demands a restoration of an immanently logical notion of *time* into the calculus, a notion of time that 'emerges from the contradictions that *would* be, if there weren't any time' (Petersen 2002, Vol. III: 1392).

Now, logical time is unthinkable within the bounds of the classical image of thought, where the speed of logic is infinite and the time of syntax is null. But it is not to be grasped by *adding* something *to* logic – a metric of temporal parameters, for instance – but by *taking something away*: 'only the oblivion of syntax can find it' (Girard 2012, 8) as Girard will say. What we are called upon to do is not

to *add time*, but to *subtract eternity*, or rather the eternalizing principles encoded – for instance – in what Boole called THE FUNDAMENTAL LAW OF THOUGHT, the identity of '*A*' with '*A and A*'. To excise this principle, however, we need sharper instruments than Boolean logic provides. These we will examine in a moment. Before doing so, however, I would first like to draw a clearer picture of the stakes of Petersen's project.

3.2.1 Systemic ambiguity, and the *in vitro* cultivation of the dialectic's rational kernel

The engine of the antinomies foreclosed by conceptual stratification – which Petersen takes to be nothing other than the 'rational kernel of the Hegelian dialectic' – is family of *fixed point phenomena* we surveyed just a moment ago: these restless reflections whereby syntax trips over its tail and confuses form and content, enegendering effects of 'systemic ambiguity' that prevent the classical notions of truth or falsity from taking root.

'Systemic ambiguity' is to be distinguished from the sort of ambiguity characteristic of ordinary language, and it should not be seen as an effort to accommodate the vagueness of the latter by 'blurring' the semantic boundaries of logic – such as is attempted in the host of 'fuzzy logics' and the like. What Petersen desires to study are the ambiguities of sense and reference, and use and mention, to which a rigidly constructed syntax is forced to give rise the moment it is capable of reflecting back upon itself. An example would be formalized arithmetic's capacity to 'inadvertently' generate propositions in which we cannot untangle what is being said of their objects – numbers and their relations – and what is being said of their own syntactic structure and logical relations. The infamous 'Gödel sentence', for instance, is just such a proposition: it is at one and the same time a perfectly banal statement about ordinary numbers, *and* a statement about its own logical status, owing simply to the science's susceptibility to stumble into the thicket of its self-encodings. Another example would be the

structure of the 'Russell Paradox', where the demonstrative oscillation between $\lambda x(x \notin x) \in \lambda x(x \notin x)$ and its negation attests to an *instability* in the sense of the statement, an oscillation animated by nothing external to logic, but by the demonstrative movement itself.

To study the phenomena of systemic ambiguity, Petersen abandons the terrain of natural language that Hegel took to be reason's proper medium. Natural language, holds Petersen, is too vague and equivocal, to messy, to let such delicate things as systemic ambiguity and dialectical contradictions be grasped clearly. The point, however, is not to expel ambiguity, contamination, confusion or chance from dialectical or mathematical logic, as if they were extraneous or inessential to the subject – a 'mere shine' that would have to be screened to see the matter clearly. 'The mess', as Petersen calls it, is something *essential* to logical thought, and dialectics is nothing if it does not pursue this mess to the bitter end. 'The point,' however, 'is to see the mess *clearly*' (Petersen 2002, Vol. III: 1513) and for this a measure of artifice is needed.

> The usual mess won't do; it is contingent; it can be explained away, one way or another; it is unreliable; it's a messy mess. I want a tidy mess and I am prepared to *create* one myself, a paradigmatic mess which can serve as an appropriate object of study. Recall that Galilei (is said to have) studied gravitational acceleration by causing balls to roll down an inclined plane. He did not bother himself with any old motion; he set up his own conditions. In this vein, *I want to set up a laboratory to study certain disturbances; the point is to construct a device, the mechanism of which I know, but which still produces disturbances of the kind that interest me* (Petersen 2002, Vol. III: 1390).

Seeking to cultivate the rational kernel of the dialectic *in vitro*, extracting it from its natural environment (but not inverted), Petersen constructs a *formal laboratory* where the classical principle expressed by Boole's law no longer holds. The resulting calculus, like the one Badiou reconstructs, 'is sutured to nothing but itself' – but this, it turns out, is already enough to yield the kind of dialectical

phenomena that Hegel and Miller sought to grasp in the idiomatic shadows of formal logic.

The logic itself is based on profoundly simple principles. Its most salient positive characteristic is its refusal to give way on THE PRINCIPLE OF ABSTRACTION, or regulate its power through an extrinsic apparatus of *types*. The *direct fixed point property* thus reigns freely over this logic of *the concept conceiving itself*.

The research programme that Petersen undertakes in this laboratory is a strange hybrid of Frege's and Hegel's, and the rubric of 'dialectical logicism' under which Petersen places it is well-chosen. From the empty forms of logic – and their capacity, through the fixed-point phenomena afforded by *unrestricted abstraction* – to seize themselves as *content*, Petersen seeks to unfold an entire series of speculative constructions. The Hegelian itinerary through BEING, ESSENCE and CONCEPT, however, is replaced by one that resembles Frege's own: the first great destination of Petersen's enterprise shall be NUMBER, though the path to reach it – which traverses far harsher terrain and which is far slower to travel than Hegel's text – will take us through the qualitative notions of INFINITY and MODALITY as well.

3.2.2 The sequent calculus

The basic instruments of Petersen's laboratory are furnished by the *sequent calculus*, a formalism invented by Gerhard Gentzen in the 1920s. A full introduction to this calculus is unnecessary for our purposes today. What follows here should be enough for us to grasp the main outlines of both Petersen and Girard's projects.

The sequent calculus differs from axiomatic formalizations of logic, such as Hilbert's, Frege's or Russell's, for instance, in its absence of 'axioms', in the strict sense. Or, rather, in its reduction of all axiomaticity to assertions of the form

$$A \vdash A$$

read 'A implies A'. What it offers in place of axioms is an elegant collection of *rules of inference*, a pair of which attends to each *connective* in the logic in question. (I say 'the logic in question' because the sequent calculus can, in fact, be used to formalize a great number of *different* logics, as we will see.)

A sequent calculus proof takes the form of a 'tree'. At its root is placed the 'sequent' that is to be proved, which has the form

$$\Gamma \vdash \Delta$$

where Γ and Δ are (possibly empty) series of formulas. The series on the left – Γ – contains the 'premises' of the sequent; the series on the right – Δ – its 'consequences'. The left series is read (informally) as the 'conjunction' of its formulas; the right aside, as the 'disjunction' of the formulas in Δ. This difference reflects an essential logical duality, or symmetry, between conjunction and disjunction, such as is expressed – in classical logic – by the De Morgan equivalences, for instance:

$$A \wedge B \leftrightarrow \neg(\neg A \vee \neg B)$$

The formulas in Γ or Δ may, of course, themselves contain connectives, and the purpose of the calculus' *rules of inference* is to show how these complex formula can be systematically decomposed into their simplest constituents, branching the sequent-tree accordingly, in order to arrive at its 'leaves': axiomatic assertions of identity, of the form $A \vdash A$.

The classical sequent calculus is also equipped with three, more fundamental, '*structural rules*', which determine the behaviour of formulas irrespective of the logical constants or connectives which they contain.

Structural Rules

$$\textsc{Left Weakening [LW]}\ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \textsc{Right Weakening [RW]}$$

$$\textsc{Left Contraction [LC]}\ \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \textsc{Right Contraction [RC]}$$

$$\textsc{Left Exchange [LE]}\ \frac{\Gamma \vdash \Delta}{\sigma\Gamma \vdash \Delta} \quad (\sigma\Phi \textit{ is a permutation of } \Phi) \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \sigma\Delta}\ \textsc{Right Exchange [LE]}$$

Weakening is the rule that lets us forget or discard 'unneeded' information, as we move upwards in the proof, or add potentially 'irrelevant' information as we move downwards.

Exchange is the rule that lets us disregard the *order* of information in a given series of formulas. It is equivalent to Boole's law of *commutativity*, the collapse of syntactic succession into simulated simultaneity.

Contraction is the most important of the structural rules for our purposes today. It transposes Boole's *fundamental law of thought* into the sequent calculus, and serves to 'idealize' formulas in the proof, so as to make them eternally reusable and available. Without contraction, as we will see, formula behave less like ideal propositions and more like exhaustible 'resources'.

Finally, we have cut.

$$\frac{\Gamma \vdash C, \Delta \qquad \Gamma', C \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \textsc{cut}$$

which is, essentially, *modus ponens*. It expresses our freedom to make use of *lemmas* in a proof, or to combine proofs together into more complex demonstrations.

Now, Gentzen's most famous and significant theorem – his *Hauptsatz* – is that cut is *eliminable* from the calculus. Every proof using cut can be transformed, in a systematic and algorithmic fashion, into one in which cut is absent: an 'explicit' proof, which makes no appeal to lemmas, and in which every subformula decorating the 'leaves' of the proof appears as well in the proven 'root' sequent. *Cut-elimination* is of crucial importance in logic for a number of reasons. For now, let's mention just one. *Inconsistency* in a sequent calculus formalization of a logic

is equivalent to the derivability of the empty sequent

$$\vdash$$

But the only possible way of deriving such a sequent is through CUT. If a system enjoys *cut-elimination*, then no sequent derivable without CUT is derivable *with* CUT, and the system is therefore consistent.

3.3 A logic of unrestricted abstraction, without contraction: a sketch of Petersen's calculus

This is not the place to unfold, in all its details, the intricate technical apparatus that Petersen has spent forty years constructing,[13] but a series of observations should serve to give at least an impression of how it works and what it does.

- ⋆ Petersen's dialectical calculus consists of two interacting levels: that of *concepts* and that of *propositions*. Its extraordinarily restricted economy employs only *two* primitive signs: an operator of *abstraction*, written '$\lambda x(\mathfrak{F}[x/a])$' (where $\mathfrak{F}[x/a]$ is a proposition in which the term or concept a occurs zero or more times, and x is the variable that will replace a under abstraction), which transforms a proposition into a concept, an a relational predicate of *subsumption* (or inclusion), written '$a \subseteq b$' (where a and b are concepts), which builds propositions out of concepts. [*Examples in natural language would be helpful.*] The idea underlying Frege's GRUNDSATZ V is embodied here in a set of rules which regulate the traffic between the two levels (see table below).

RULES FOR ABSTRACTION & SUBSUMPTION

[13] This is something which I will be doing in a series of logic seminars that I will be conducting at the Jan van Eyck Academie in Maastricht, and which will be free and open to the public. Please feel free to contact me if interested, at zlfraser@gmail.com.

$$[\mathrm{L}\in]\ \frac{\Im[s], \Gamma \vdash \Delta}{s \in \lambda \Im[x], \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Im[s], \Delta}{\Gamma \vdash s \in \lambda x \Im[x], \Delta}\ [\mathrm{R}\in]$$

$$[\mathrm{L}\subseteq]\ \frac{\Gamma \vdash r \in s \qquad r \in t, \Pi \vdash \Delta}{s \subseteq t, \Gamma, \Pi \vdash \Delta} \qquad \frac{a \in s, \Gamma \vdash a \in t}{\Gamma \vdash s \subseteq t}\ [\mathrm{R}\subseteq]$$

- ⋆ On the basis of these four rules (or two principles), Petersen is able to reconstruct *all* of the familiar logical connectives, but in the absence of CONTRACTION they find themselves unfamiliarly *split*: disjunction and conjunction are both severed into two distinct forms, an *idempotent* form and a *non-idempotent* form.
- ⋆ Implication in Petersen's calculus differs from classical implication insofar as it *consumes* its antecedent in delivering its consequent. As such, $A \to B$ can be read as '*A passes over into B*', or '*A transits into B*', recovering in this way a distinctly *Hegelian* mode of implication. A sort of *logical temporality* is thereby retained in implication, giving it a 'quasi-causal' structure.
- ⋆ The quasi-temporal, quasi-causal nature of dialectical implication gives to paradoxical biconditionals such as Russell's a genuinely oscillatory character: it is no longer an immediate consequence from statements of the form

  $$A \leftrightarrow \neg A$$

  (if A, then not-A, and if not-A, then A) that we have

  $$A \wedge \neg A$$

  (A and not-A). This indeed seems to come closer to the structure of the 'contradictions' Hegel examines in his *Logic*, which rarely, it seems to me, take the form, treasured by Priest and the dialetheists, of a *conjunction* connecting a statement and its negation.

⋆ Having taken on Frege's task of rebuilding the science of Number from the materials of logic alone, and having sacrificed contraction on the altar of abstraction, Petersen now faces a profound difficulty, for it has always been contraction, or its equivalents, that have given logic access to the infinite iterability of numerical thinking, furnishing the condition of possibility of inductive or recursive reasoning. Suppose, for instance, that we have in hand the proposition that, for any n, if n is a number, then so is $n+1$. This is a universal proposition, and so we can easily derive from it an instance. For example, we can take from it the proposition that if 0 is a number, then so is $0+1$. How, then, do we get to 2? Deprived of contraction, the 'fundamental' identity the Boole articulated between A and $A \wedge A$, we have burned our bridge in crossing it: dialectical implication exhausts its antecedent, and so we no longer have in hand the universal proposition 'for any n, if n is a number, then so is $n+1$'.

Petersen's strategy here is remarkable: to perform the work of contraction, he enlists the very principle for which it had been sacrificed, and exploits the power of the fixed point property in order to construct self-replicating, potentially infinitizing propositional forms.[14] The technicalities of these constructions are quite intricate, but they are built around a relatively simple mechanism: the operator[15]

$$\check{\gamma}[A] :\equiv \lambda x((x \in x) \,\square\, A) \in \lambda x((x \in x) \,\square\, A)$$

This operator works by replicating itself *ad libitum*, each time spitting out a new copy of A, where A may be any proposition we like. On its basis, Pe-

[14] Petersen states the problem and sketches its solution as follows:

> [I]f contraction is abandoned, then what is going to take its place? The basic line of my answer is simply: the fixed point property. That's what contraction has been given up for, and that because there was a conflict between the two, i.e., there must be some application of the fixed point property that is contraction-like. (Petersen 2002, Vol. III: 1609)

[15] The sign '$\square$', below, is the symbol for the non-idempotent form of conjunction.

tersen is able to craft the thought-determinations of Necessity and Number that he will need in order to see Frege's project through to completion, by means as much Hegelian as Fregean.

⋆ Recall Hegel's argument that the mathematical concept of *ratio* comes closer to approaching the true infinite than the notion of infinite progression (the bad infinite). But what is it that condenses the infinite in the ratio? Consider the ratio as a universally quantified proposition. Its infinity is that it condenses in itself, in its 'finitude', infinite instances. But what is the logical principle that lets this infinity be expressed from the finite? In classical logic, this is, precisely, the work of CONTRACTION, the 'fingernail of the infinite in logic', as Girard called it. But this principle is, as we have seen, profoundly anti-dialectical. Petersen's effort to rebuild – on the basis of $\check{\gamma}[A]$, a fixed point construction that stages a concept conceiving itself – the principle whereby a universal ratio accedes to its infinitization can thus be seen as a strictly mathematical reactivation of the Hegelian idea of the 'true infinite'.

⋆ It is difficult to overestimate the strangeness that results from allowing every function a fixed point. Consider, for example, the function of *complementation*. In classical logic, it is always possible to split the world in two: an arbitrarily selected part p and everything else, its 'complement': $\mathcal{C}(p)$. In the dialectical calculus where we now find ourselves, $\mathcal{C}$, like every other function, has a fixed point: there must be some object, call it a, which is its own complement: $\mathcal{C}(a) \equiv a$. It is impossible to define complementation in such a way as to escape this state of affairs. Every cut leaves a remainder, even if this is just the glint of the knife.

⋆ It should not be overlooked that this result alone – the existence of a fixed point for complementation – suffices to subvert the LAW OF THE EXCLUDED MIDDLE, at least in some of its forms, and to do so along lines which are quite close to Hegel's own, without any need to illegitimately 'suture' or short-circuit the logical machinery. Operating entirely according to its prin-

ciples, Petersen's dialectical calculus illustrates exactly *how* it is that the division of reality into A and not-A trips over its own machinery, and produces its own exception in the form of its reflection.

⋆ A final point to mention is the provability of the dialectical calculus' *consistency* – the underivability of the empty sequent from its principles. This is significant, in particular, in view of the *inconsistency* that erupts – through Russell's paradox – when unrestricted abstraction is appended to classical logic. Why then is Petersen's logic, which upholds the full rights of abstraction, demonstrably consistent? The reason is fairly simple: the proof of cut-elimination requires an *induction* on some formal aspect of the proofs that can be written in the system. Two candidates present themselves as potential bases for induction: (1) the length of formulas, and (2) the length of branches in the proof. The rules of abstraction fail to reduce the length of formulas – they even threaten to increase their length in several instances – and so unrestricted abstraction makes method (1) unviable. CONTRACTION, on the other hand, fails to shorten the length of proof-branches when it is chased up the proof-tree by cut-elimination, rendering method (2) unusable when it is operative. But when CONTRACTION is absent, method (2) works perfectly well, and cut-elimination can be demonstrated without needing to know anything about the internal structure of the cut formulas. It is able to operate, instead, on the global structure of the proof itself.

4 Jean-Yves Girard's locative dialectics

4.1 Transcendental syntax

After several decades of extraordinarily innovative work littered with hints to this end, Jean-Yves Girard has finally announced, explicitly, the *transcendental* ambition of his research programme in mathematical logic. His transcendentalism

differs markedly from that of Kant or Husserl – it is not a question, for Girard, of swaddling classical logic with the all-too-human conditions of its efficacy, while leaving its laws and forms enshrined as a court of last appeal. His task is not to produce a 'transcendental logic', but to unearth the transcendental conditions *of* logic – and to unearth them *from within* the structural 'blind spots' of logic itself (as suggested by the title of his 2006-7 Rome lecture series, *Le Point Aveugle* (Girard 2006, 2007)).[16]

Nothing so clearly separates the task of *transcendental syntax* from the transcendental logics of Husserl or Kant, and so clearly unites it with the Hegelian vision of logic, than its radical subversion of *the semantic conception of logic* – the conception that Hegel attacks under the heading of '*formality*'. The semantic conception of logic, in a nutshell, is this: that the logical structure of judgments is to be sought in their *truth conditions*, in the external evaluation of syntactic expressions in a 'model' or 'world' that the syntax would be taken to merely *represent*. For example, following Tarski (the hierophant of semantics), the meaning of *A and B* is simply the equivalence

$$A \textit{ and } B \text{ is true} \iff A \text{ is true } \textbf{and } B \text{ is true}$$

where the only difference between the first '*and*' and the second '**and**' is that first belongs to the 'object language' and the second belongs to the 'metalanugage'. If we then try to define the meta-**and**, we are referred to a meta-metalanguage. There is no end to this *transparent* flight into the 'bad infinity' of the Tarskian hierarchy of metalanguages. But this transparency is just another name for blindness,[17] and the black comedy of semantics, as Girard never tires of pointing out, is easily brought to light with a joke: suppose we want to define a new connec-

[16]See, for instance: 'My thesis is that the meaning of logical rules is to be found in the *well-hidden geometrical structure of the rules themselves*,' in (Girard 1998, 1).

[17]For an extended treatment of this concept of 'transparency', see Girard (2008).

tive, called '*broccoli*'. The way forward is simple:

$$A \textit{ broccoli } B \text{ is true} \iff A \text{ is true } \textbf{broccoli } B \text{ is true}$$

While semantics is an incontestably *useful* employment of logic, in opening channels of communication between various mathematical systems, its usefulness is purchased at the expense of making logic itself *disappear*, at the expense of logic's *transparency*. When 'the question at stake is the question of *meaning*, and [...] not a substitute like *use* or *interpretation*,' transcendental syntax advances the 'thesis that the meaning of logical rules is to be found in the *well-hidden geometrical structure of the rules themselves*' (Girard 1998, 1). The meaning of '*not A*' will not be taken to be '*A* is **not** true', but rather be sought in the symmetry that negation induces in logic: the symmetry that relates introduction and elimination rules in natural deduction, which organizes the structure of sequent-calculus proofs, and which guides the dynamics of cut-elimination, for instance. A symmetry that will, at last, appear in its fundamental role in *ludics*, the formalism we will soon visit. The excavation of such deep syntactical structures demands not a badly infinite *reflection* of syntactic forms, but rather their controlled and tempered *oblivion*.[18]

Girard takes the dominance of the semantic approach in logic to be the symptom of the ideological dominance of a form of 'objectivism' or 'realism', and takes the vacuity of the former to point to a philosophical weakness in the latter. This leads him to adopt, like Petersen, a form of Hegelian idealism. Just as what Petersen

[18] As Girard remarks,

> The idea of transcendental syntax is quite the opposite of what people do with Kripke models and similar constructions: they take the language as it is, call it reality, and state a completeness theorem. Here, the idea is to start with a good system and "forget" its syntax: the reason is that syntactical manipulations – although the only possible manipulations – are extremely involved. In particular, there is an entanglement between the actual information and the normative (deontic) apparatus which controls its use. Typically, what is the effect of writing twice the same formula at different locations? If there is a reason for doing so, *only the oblivion of syntax can find it*. (Girard 2012, 8)

is 'interested in is how the world can be left outside and *truth* can still prevail' (Petersen 2002, Vol. III: 1561), Girard will assert 'that logic is better off when there is no "reality" at all, and [...] restores the priority of syntax over anything else' (Girard 2012, 1).

Intimations of the task of transcendental syntax are already legible in the 1987 monograph, 'Linear Logic', which opens with an invocation of Audiberti's '*secrète noirceur du lait*' (Girard 1987, 2), and proceeds to uncover 'a logic behind logic' (ibid.), logic's 'perfective stratum' – something which we have already, in part, encountered in Petersen's work: the immanently temporalizing logic that appears in light of the structural rules' oblivion, and which begins – hesitatingly – to expose a logical dynamism in which we may glimpse the forms of classical logic taking shape *in the act*. The resistance to the Tarskian – or, more generally, Kantian – image of truth (derided by Hegel) is already evident in this logic

> *qui abandonnerait la sacro-sainte « réalité » , pour se concentrer sur sa propre structure [et] parvient ainsi à toucher ce point aveugle où l'essentialisme nous ment, ou du moin se refuse à toute justification autre que « c'est comme ça »* , (Girard 2006, 12)

and this is so to the extent that, deprived of the perennializing force of the structural rules, the formulae of the logic no longer resemble references to external states of affairs – taking the shape, instead, of immanently syntactical acts.[19]

The structure of linear logic is already somewhat familiar to us: it corresponds, precisely, to the structure of Petersen's dialectical calculus – though without use of unrestricted abstraction (in which Girard seems to have taken only a passing interest) and without the structural rule of WEAKENING, the rules which grants us the freedom to 'forget' about unneeded propositions in a proof (a rule which Petersen has not yet found a satisfactory way to eliminate).

[19]No one since Boole has had clearer insight than Girard into the 'essential' character of the structural rules in the determination of the classical image of thought, or has seen so clearly that in these principles of *fixity* lies the lynchpin of *formality* – the lynchpin of the semantic conception of logic that renders its object transparent, which is, precisely, the 'blind spot' of logic.

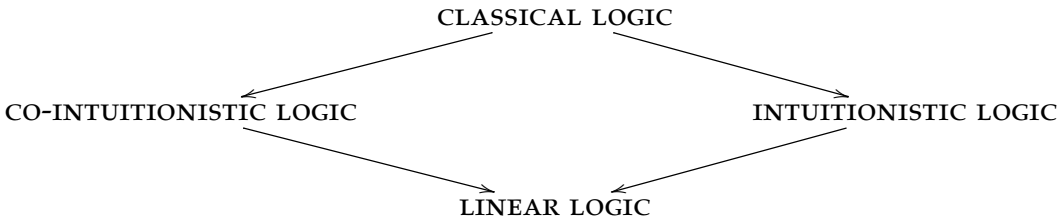
Girard tactically diverges from Petersen, however, in reintroducing the power of the structural rules, not by means of fixed point constructions, but by brute modal operators: 'of course!' ($!A$) and 'why not?' ($?A$). '*Il est intérressant*,' Girard remarks, '*que la logique linéaire réduise l'essence à une noyau modal opaque*' (Girard 2006, 12). The exponentials are indeed 'opaque': again and again, his investigations into the structure and potential of linear logic encounter them as stumbling blocks, as indigestible bones in the throat of logic. The opacity and contingency of these 'essentialist' principles has as much methodological significance for Girard as the inevitability of the infinitizing fixed-point constructions has for Petersen, for it signals the possibility of their free variation and transformation, the possibility of experimenting with new forms of infinitization undreamt of in classical logic.

The genesis of linear logic is extremely interesting, but it would take us too far afield to (adequately) recount it today. A few brief points will have to suffice.

The actual history of linear logic finds it emerging from an analysis of a topological interpretation of intuitionistic logic's proof theory. What this interpretation – the theory of *coherent spaces* – shows is that intuitionistic implication can be decomposed into two distinct, geometrically more primitive operations, which (for algebraic reasons) Girard terms 'linear implication' and 'exponentiation' (so that $A \to B$ breaks down into $!A \multimap B$). The syntax of linear logic was then reverse-engineered from this topological semantics, and the resulting system is the one we have just mentioned: effectively, Petersen's dialectical calculus without abstraction or subsumption, and without weakening.

It is also possible, however, to see linear logic as emerging through a sublation of the contradiction between intuitionistic logic (which rejects the law of the excluded middle) and the paraconsistent logic that is the 'dual' of the latter, 'co-intuitionistic logic' (which rejects the law of contradiction). Both of those logic depart from classical logic in symmetrical but opposed directions, and both enjoy a quality of which classical logic is incapable: cut-elimination in each case

in *confluent*, yielding a unique cut-free normal form for every proof. This was the intention of neither the paraconsistentists nor the intuitionists, but, by Girardian lights (though Girard does not mention this himself), constitutes their most important virtue – for cut-elimination can be seen as the procedural *core* of logic, its immanently computational dimension. Considered proof-theoretically, or*computationally*, and not in terms of truth-preservation, classical logic is 'inconsistent': the effort to normalize its proofs, to distill their explicit kernels, *diverges*, yielding unequal and unpredictable results. What Girard observes for intuitionistic logic is equally true of co-intuitionistic logic: their rejection of the 'laws of thought' (excluded middle or non-contradiction, accordingly) is, on a more fundamental level, a *partial restriction* of the 'FUNDAMENTAL LAWS OF THOUGHT', a partial restriction of CONTRACTION and WEAKENING. This is visible only in light of the sequent calculus formalizations of both logics, each of which differs from that of classical logic only in one respect: the imposition that we may only have *one* formula on a given side of the sequent (the side of consequences, for intuitionistic logic, and the side of premises for co-intuitionistic logic). If we radicalize these restrictions, and render them symmetrical, we obtain *linear logic* (or its 'perfective fragment': linear logic minus the exponentials). Like Petersen's calculus, linear logic splits conjunction and disjunction in two, into idempotent and non-idempotent forms. In doing so, it delivers *both* the law of the excluded middle and the law of non-contradiction, *and* their rejections: LEM and LNC are preserved in the realm of non-idempotent connectives, and lost in the realm of idempotent connectives – quite a surprising twist! This sublation, moreover, enables linear logic to retain *both the symmetry of classical logic* ***and*** *the convergent cut-elimination of the asymmetrical logics*. It is a textbook case of sublation through a 'negation' both double and determinate!



CLASSICAL LOGIC *preserves LEM and LNC*	$\vdash A \vee \neg A$ $\vdash \neg(A \wedge \neg A)$
INTUITIONISTIC LOGIC *preserves LNC, rejects LEM*	$\nvdash A \vee \neg A$ $\vdash \neg(A \wedge \neg A)$
CO-INTUITIONISTIC LOGIC *preserves LEM, rejects LNC*	$\vdash A \vee \neg A$ $\nvdash \neg(A \wedge \neg A)$
LINEAR LOGIC *all of the above,* *but in different forms*	$\vdash A \parr \neg A$ $\vdash \neg(A \otimes \neg A)$ $\nvdash A \oplus \neg A$ $\nvdash \neg(A \,\&\, \neg A)$

The most important dimension of Girard's shift into linear logic – and its requisite suspension of the structural rules – has to do with how it facilitates a *geometrical turn* in proof theory. Now that propositions can no longer wander freely from their site in a proof – the principles of errancy encoded by the structural rules being lost to us – their logical significance comes to be welded to the geometrical structure of the proof as a whole. This is an essential precondition for the next episode in Girard's research programme that we will examine: the plunge into the *locative*, and the challenge this presents to a yet-unseen form of (anti-dialectical) type-restriction in logic.

4.2 The return of type-stratification: propositions as types of proofs

I have already mentioned the appearance of fixed-point phenomena in the pure lambda calculus, and the stratification of programme-types that was introduced in order to prevent the cyclones of reflexivity from throwing logical implementations of the calculus into turmoil. It so happens that the type-systems imposed on the chaos of lambda terms are reflected back upon logic itself, illuminating a form of typing that we would otherwise have failed to notice. The agent of this illumination is the *Curry-Howard correspondence*, which maps proofs in the intuitionistic sequent calculus (or natural deduction) onto programmes/functions in the simply-typed lambda calculus. The execution of programmes – which is to say, the systematic rewriting of lambda terms – thereby corresponds to the process of cut-elimination between proofs.

This gives us a new way of looking at proofs, seeing the process by which they interact and combine with one another – as when a theorem is used as a *lemma* in the proof of a more complex theorem – as a dynamic procedure, structured like a computational programme or algorithm. It also hints at what we might be missing, or occluding, when we behold the interactions of proofs through cuts and their procedural eliminations – for the same correspondence that maps proofs onto programmes maps *propositions* onto *types* in the typed lambda calculi, types furnished by the same abstract apparatus that saddles the '*id*' of pure lambda terms with the '*superego*' or restrictions designed to 'prevent functional incest' [CITE? TRONCON, JOINET OR GIRARD?] – the external disciplining of calculative chaos.

The *type* of a programme or function circumscribes, in advance, the domain of arguments to which that function can be applied. That same *type*, modulo Curry-Howard, circumscribes the domain of *proofs* with which a given proof can interact through cut and cut-elimination. The form this circumscription takes – the correlate of *types* in proof theory – is *the name of the proposition being proved*. What chaos

then is occluded by the subordination of proofs – conceived as structures interacting through cut and cut-elimination – to predelineated 'propositions', propositions built up from *atoms* and a pre-given stock of *connectives*?

This is the question that must be answered if we are to plumb the depths of the proof-theoretic symmetries towards which the cut-elimination algorithm gestures, and it is a question answered by *ludics*, a formalism developed by Girard at the beginning of the current century.

4.3 Locativity, and the 'hidden necessity of adding to abstract identity the extra factor of that movement' inscribed in syntax

To Hegel's pregnant remark that '*propositional form can be regarded as the hidden necessity of adding to abstract identity the extra factor of that movement*' (Hegel 2010 (1812-31, 360), ludics plays the midwife.

It does this in several stages – first, by destroying the distance between the proposition as such and the geometrical structure of the proof in which it is inscribed. In the Preface to *The Phenomenology of Spirit*, Hegel complained of the 'defectiveness' of mathematical cognition, and took its defect to lie in the gap between a mathematical 'result' and the proof that determines it as true. In *ludics*, this complaint is redressed: Girard strips proofs (written, say, in the sequent calculus for linear logic) of all propositional information, leaving only the geometrical or 'locative' structure of the proof-tree intact – replacing formulas and subformulas with 'loci' and their 'sub-loci', respectively – thereby dissolving the slightest gap between demonstrative processes and their 'results'.

This reduction is facilitated by a prior reformatting of sequent-calculus proofs in linear logic, the possibility of which is arrived at through Andreoli's 'focalization' technique. What Andreoli saw was that the elimination of the structural rules served to split the existing logical connectives in not just one, but two dis-

tinct fashions: not only do we witness their division into idempotent and non-idempotent forms (i.e. *additive* and *multiplicative*), but into *positive* and *negative* varieties.

	Positive *(Irreversible)*	Negative *(Reversible)*
Multiplicative *(Non-idempotent)*	$\otimes$ *conjunction*	$\parr$ *disjunction*
Additive *(Idempotent)*	$\oplus$ *disjunction*	& *conjunction*

It turned out to be possible to restructure any given proof into one in which rules are regrouped into a strict, alternative sequence of *negative* and *positive* series, and to regard each group of negative rules – punctuated by positive rules – as being applied simultaneously. It is this rhythm alone that is retained in the locative purification of proof-trees: an alternating sequence of *positive* and *negative* turns, which can be interpreted as an alternation between a protagonist's actions and anticipations of the reactions of their antagonists.

The geometrical and rhythmic artefacts that survive this double process are called 'designs' or 'strategies' – the locative skeletons of proofs. Now, in the sequent calculus, a proof ordinarily terminates when the rule-bound decomposition of sequents arrives at what are called 'axioms': sequents of the form

$$A \vdash A$$

where A is atomic – not composed of any more basic subformulas. Ever since Plato, the word '*axiom*' has been nothing if not a name for an obstacle to dialectics. With ludics – in this respect, at least – the obstacle is structurally dissolved. Stripped of any propositional encumbrances, the locative ramification of strategies can progress indefinitely. *If* they are interrupted, then the interruption will be pure and without reason: by a gesture of distilled contingency called **dai-**

mon, which can at any time be invoked to abruptly terminate a branch in a design.

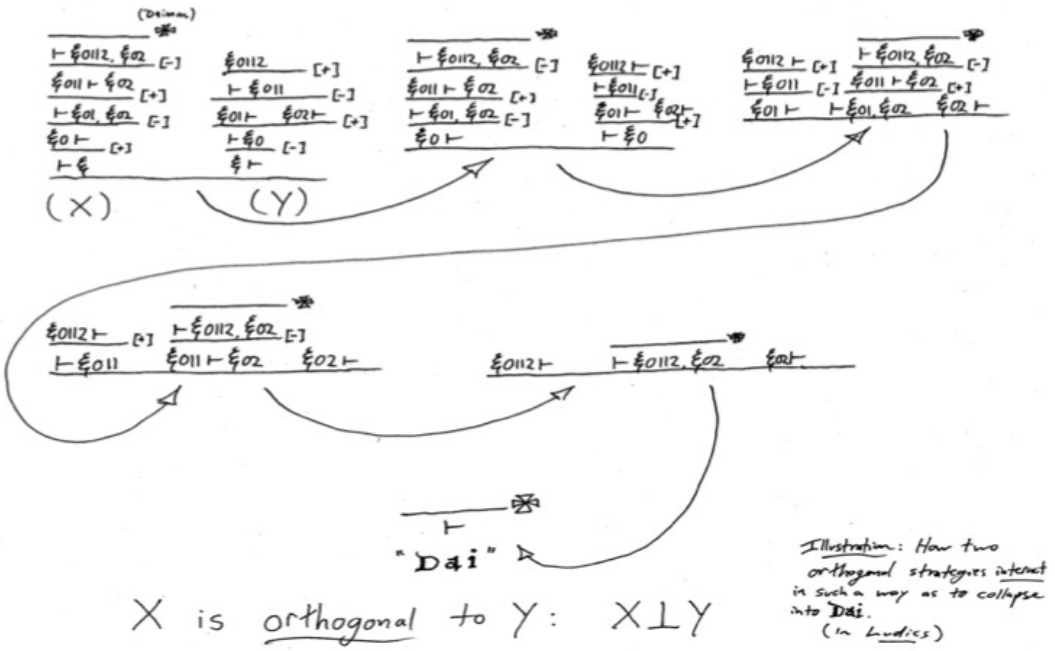
The access to pure contingency that the *daimon* affords yields a space enough for more subtle and complex forms of necessity to emerge, in an arena more vast and labyrinthine than the arena of proofs. The reason for this is that we now have recourse to 'false' or 'unfinished' proofs as well: the geometrical skeletons of paralogisms and sophisms, ruse and regress.[20] The need for such structures is simple: Girard's ambition is to reconstruct logic, as far as possible, on the basis of the dynamics of cut-elimination alone. In order to fully unfold the symmetry that cut and cut-elimination describe, we need to examine how proofs of $\vdash A$ interact – through cut – with their 'duals': proofs of $A \vdash$, or $\vdash \neg A$. And here is the problem: in traditional logics, the existence of a proof of $\vdash A$ *precludes* the existence of proofs of $\vdash \neg A$, on pain of paradox. With the intervention of the *daimon*, however, this restriction is lifted: for every design rooted in $\vdash \xi_n$ there exists a plethora of counter-designs rooted in $\xi_n \vdash$.

When two designs share a common locus at their root, there begins a process, which is essentially that of cut-elimination, but which Girard calls *interaction*. Two possible outcomes await every interaction: either the interaction *diverges* in an endless dispute (a phenomenon Girard names '$\mathfrak{Fid}$', for '*fidelité*'), or else it *converges* on a single, minimal design called '$\mathfrak{Dai}$'. $\mathfrak{Dai}$ is nothing but the brute positing of an empty sequent, by means of the *daimon* (✠):

$$\frac{}{\vdash}\ \maltese$$

When the interaction between two designs X and Y converges on $\mathfrak{Dai}$, X and Y are said to be 'orthogonal' (written: $X \perp Y$), as in the example below:

[20] See, for example, Quatrini (2008), where Quatrini shows how the sophistic or 'dialectical' strategems that Schopenhauer enumerates in *The Art of Controversy* can be formalized in ludics. Schopenhauer, of course, is using 'dialectical' in the sense Kant uses this term, to refer to fallacious reasoning.



When the interaction between two designs converges on $\mathfrak{Dai}$, a 'winner' can be declared: the winner is the design that persevered, and refused to invoke the *daimon*, while leading its opponent to do just this. In the diagram above, Y stands victorious.

In general, we can define the 'orthogonal' of X (written $X^{\perp}$) as the set of designs Y such that $X \perp Y$. On this basis it becomes possible to reconstruct the *propositional forms* as 'types' of designs. These types are not imposed on the arena in an exterior or*ad hoc* fashion, as is the case with the typing of the lambda calculus. They are defined *immanently*, in virtue of the symmetries that emerge between interacting designs. In particular, we can define a type – a proposition, or, in Girard's lexicon, a 'behaviour' – as a set of designs that is *equal to its biorthogonal*: X belongs to the type $\mathcal{X} = \mathcal{X}^{\perp\perp}$. *The strategy delivers its concept through its double negation.*

From this we can reforge a fully immanent concept of logical *truth*, which makes

no reference to any semantic interpretation whatsoever. A proposition (i.e., a behaviour) $\mathcal{Y}$ is 'true' when it contains a design that wins every match against its orthogonals, and 'false' when *its* orthogonal $\mathcal{Y}^{\perp}$ contains such a design.

This places *contradiction* – in the form of orthogonality – at the very basis of logic, in a fashion that Girard takes to capture the essential Hegelian precept of '*contradictory foundations*'. But note: it is not 'alethic' or semantic contradiction that is at issue here. It is not a contradiction between simultaneously asserted truth claims, but an effective mode of contradiction that is the *condition of possibility* of determining 'truth' as such. It is far 'beneath' the abstract determinations of truth or falsity that the negativity of ludics operates. Only there do we witness anything like the Hegelian drama of *determinate negation*, a struggle of concrete – locative – oppositions anterior to and generative of 'truth'. In a recent text, which began circulating only two weeks ago, Girard makes the leap only hinted at in *Locus Solum*, and directly connects ludic contradiction to the Hegelian dialectic: The breakthroughs of ludics, he writes

> vindicate one of the greatest philosophers, Hegel, and his *contradictory foundations*. From a fregean viewpoint, founding A on its negation is stupid, since A and $\neg A$ cannot coexist by consistency! The fact is that [...] ludics rests upon contradictory foundations without the slightest problem. The mistake about Hegel is to take negation in the *alethic* (i.e., -1) sense, for which there cannot be any honest contradictory foundations; there are dishonest ones, e.g., paraconsistent "logic", making Hegel a sibling of Bernard Madoff... a real insult for such a great philosopher! (Girard 2012, 6)

Now, as we come to study the conditions under which determinately structured designs and behaviours combine with one another – so as to rediscover the forms of logical connectivity on this plane – there is a chance of encountering phenomena of *interference* between designs. Interference takes place when two interacting designs share loci at sites other than their roots, a situation which causes tangles and snarls in their combination, in such a way that we cannot predict, on the ba-

sis of the 'truth values' of the designs alone, and without knowing *where* a design is inscribed, whether their combination will be 'winning' or 'losing'.[21]

I began this quick survey of ludics with a promise to show how it activates Hegel's insight that '*propositional form can be regarded as the hidden necessity of adding to abstract identity the extra factor of that movement*' (Hegel 2010 (1812-31, 360). How, then, does it do this?

What Hegel draws attention to, in the most basic sense, is that there is, at least, a purely *graphical* difference between the two A's incorporated into the statement $A = A$. This difference is nevertheless occluded by the understanding's *notion* of identity, which fails to even make a problem of it. The same can be said for the 'axiom' nodes of a proof tree where the sequent calculus considers its work to be finished: symmetrical nodes of the form $A \vdash A$, which are taken to stand in no need of proof. Since ludics strips designs of propositional nametags, such an occlusion of difference in identity is not an option for it.

What replaces identity in ludics is an abstract machine of *delocation*, a species of design called $\mathfrak{Fax}$. In order to produce an 'identical' (or rather, isomorphic) copy of a design D with base $\vdash \xi$, a copy that would have a new location $\vdash \xi'$ as its base, it is necessary to make D *interact* with a $\mathfrak{Fax}$ design of base $\xi \vdash \xi'$. The *result* of this process will be a new design, D', which is structurally isomorphic to D, but which has $\vdash \xi'$ and not $\vdash \xi$ as its base. It is $\mathfrak{Fax}$ which serves to mediate between the *locative* combinations of designs, the 'locative connectives', and what Girard calls the 'spiritual connectives'. Locative combinations of designs always risk being caught off-guard by interferences – which, recall, occur when the designs employ intersecting sets of locations, causing the syntax to trip over its own graphical trace. 'Spiritual' combinations avoid such complications by first *delocating* the designs that are to be combined, by submitting them beforehand to a

[21] This touches on the *philosophical* concept of locativity that orients Girard's research, through which 'the locative' is understood as *'l'hors-cadre, l'hors-champ'*: the locative as that aspect of '*lalangue*' that is immanent to logic as such, but which can be grasped without falling into the pitfalls prepared by Miller. See Girard (2009) for details.

Fa‡ing operation. It is only then that the traditional logical connectives – the conjunctions, disjunctions and implications we know so well – can be found.

The necessity of this detour is indeed, as Hegel observed, inscribed in the *propositional forms* themselves, but propositional forms that are united with the inner necessity of their demonstrations, stripped of all extrinsic control (all typing), and submitted to the daimonic contingency of sophistry that animates the strategic arena. In short, the necessity of adding to abstract identity the extra factor of its movement is inscribed in the form of the *design*, the primitive artefact of *ludics*.

5 Conclusion: The dialectical challenge to the 'classical image of thought'

5.1 A war on two fronts: the stratification of types *versus* dialectics (are Petersen and Girard's projects compatible?)

A recurring motif in this study has been the essential antagonism between typing apparatuses and what is dialectical in logic. This motif can already be heard in Hegel's confrontation with Kant. If conceptual thinking is such that its domain of validity is always circumscribed in advance – if they can, for instance, only legitimately be brought to bear on what is supplied by the manifold of intuition, and neither (anticipating Russell) to themselves nor (anticipating Zermelo) to the All, then dialectics is an engine of illusion. *Thought stratifies itself in types as a bulwark against the dialectic.*

There are many common threads between Girard and Petersen's projects, but the one in tune with this motif is their effort to strip logic of type restrictions. The war, however, is fought on two different fronts, and it is not yet clear whether their simultaneous victory is possible. Petersen's focus is on the undoing of

predicate types (of a Russellian variety) and Zermelo-like constraints on abstraction. Girard's is on proof or programme types – types inscribed, modulo the Curry-Howard isomorphism and its analogues, in the names of propositions and connectives that decorate a proof. The harmony between these two strategies consists in their mutual elimination of the structural rules. The motive for this elimination differs, however, and this difference will result in a dissonance that it may or may not be possible to reduce: Petersen needs to eliminate the structural rules – primarily CONTRACTION – so that it may be possible to grasp logically, in a consistent (and not paraconsistent) fashion, the motor of dialectical contradictions: *the fixed point phenomena* engendered by unrestricted abstraction.

In the presence of contraction, these phenomena collapse instantaneously into antinomies. In the absence of contraction, it becomes possible to prove their logical consistency. The reason for this is that in the absence of contraction, it is possible to prove cut-elimination for a logic equipped with unrestricted abstraction without performing an induction on the length of the cut formula. All of the information we need in order to inductively prove the eliminability of CUT can be read off the global structure of the proof, for it is on the length of the proof itself (or rather its branches) and not on that of its propositional decorations that the induction is performed.

This already gestures towards Girard's type-elimination programme, which, however, changes the focus of the problem significantly. The intersection of the two problematics lies in this change of focus that the elimination of structural rules makes possible: from propositions and their provability to the global structure of proofs themselves, taken as quasi-geometrical objects. The structural rules can be seen as encoding a sort of logical surplus of propositional structure over proof structure, of the local over the global. Their effect, after all, is to unbind the logical potential of a proposition from its locus in a proof, allowing it, to some extent, to float freely through the demonstration, multiplying or vanishing as needed.

The elimination of structural rules welds a proposition to its *place*, fusing it to

the global structure of the proof itself. The essential gesture of Girard's ludics – whose other name is 'dialectics' – is a radicalization of this operation. The liberation of proof structures from propositional surplus is at the same time their liberation from computational type restrictions.

The dissonance between these two strategies has entirely to do with the fixed-point effects of unrestricted abstraction at the propositional level, to the extent that they seem *to reintroduce a surplus of the local over the global*, of 'force' over 'place' to speak like the young Badiou. (They have logical effects which do not *seem* to be easily reducible to their calculative topography.) It is *not yet clear* whether or not this surplus can be reabsorbed into the geometry of proofs. *It is an open, mathematical problem whether or not the techniques of ludics, for instance, can apprehend Petersen's dialectical calculus in terms of its locative structure alone – or, to put it another way, whether we can have a fully type-free logic, a logic free of both predicative and proof types, exhibiting both locative interaction and unrestricted abstraction.* At the time being, the froth of reflexivity seems to float freely over the surface of the demonstrative ocean, after eternity is shipwrecked and its kernels softened and dissolved.

5.2 The classical constellation

A result of this study, so far, is that we can at least *negatively* characterise the dialectic in terms of its conflicts with 'the classical image of thought', of the 'way of the understanding', to speak like Hegel. It seems, today, that at least three stars in the classical constellation must be destroyed in order for dialectical logic to chart its course. And it should not be overlooked that the necessary destruction can be – and has been – carried out *within the trajectory of logic's formalization*:

5.2.1 Formality

5.2.2 Fixity

5.2.3 Atomicity/Transparency

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