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0.1 The problem of beginning and the dead-end of calculation

In 1812, the Nuremberg professor of mathematics, J.W.A. Pfaff, exchanged a series of letters with Hegel on the subject of the recently published *Wissenschaft der Logik*. From what we can gather from Pfaff's remaining letters – Hegel's replies being lost – the exchange hinged on a series of deadlocks in the relation between speculative and mathematical thought.

What Pfaff asks Hegel – repeatedly, it seems, and without satisfaction – is *how* one proceeds in speculative logic, *how* one step succeeds another, given that everything must proceed from an utterly empty beginning, rejecting every appeal to laws or axioms. As Pfaff sees it, however, the axiomaticity Hegel rejects at the outset takes its revenge by obscurely and inexplicitly infecting every step of the logic with arbitrariness: hidden postulates blindly absorbed from the medium of language in which the speculative logician swims (and which, if made explicit, can be justified only in a retroactive and circular fashion, as Pfaff observes).[1] 'There was some mathematical audacity on my part,' Pfaff writes, in a passage that's worth quoting at length,

> to assert that [in your *Logic*] I could not see any *proofs*, that in every case all I could see were *new postulates*, pretensions to have the new *spring* from what already exists. [...] This is why I went a bit further, and said that there was something *arbitrary* there (a lack of necessary connections between asserted propositions). [...]
>
> After an attentive reading of your letter, my persisting rebellion now has to do simply with the fact – of which I am now wholly convinced –

[1] In Pfaff's second letter, we read: 'It seems to me that you are always compelled to move in a circle... Starting from a given point, you posit, in advance, these operations, etc., that are only arrived at later on; and so, in order for it all to be justified, you are compelled to return to the point from which you started. If, in your circle, you are not compelled to move in a straight line (like a mathematician), nor even, like a comet, in a parabola, but in a closed curve like the planets, like the sojourn of the gods, *this is because you have need of language, while the mathematician is mute*' (**?**, 359).

> that not only do I fail to perceive the difference between philosophical and mathematical thought, but that, furthermore, I am unable to practice the former, because each of its acts appears to me as a new postulate. The enigma is always this: HOW does the thinking subject develop a new proposition? HOW does the new, which was not yet there *in the thought*, spring from the old? HOW is synthesis possible? HOW does thought progress? HOW does one unite freedom and necessity, creation and construction, invention and demonstration? [...] This is the only question that can truly interest a man who thinks... the 'how'...; as for the results, well, there they are, *collected* in mathematics (the collections of logarithmic tables, of integral formulas, etc.). Their formation, their necessary formation, *that* is what is truly of interest. (**?**, 360)

What's peculiar about Pfaff's criticism is the extent to which, at least on the surface, it echoes Hegel's own criticism of mathematical demonstration. In the Preface to *The Phenomenology of Spirit* and in several Remarks in *The Science of Logic*[†] he argues that it is the *mathematical* mode of demonstration that fails to establish a necessary connection between its several steps. 'We see that 7 and 5 make 12,' for example,

> by adding five more ones to the seven, on our finger tips or in some other way; and the result is then imprinted in memory *by rote*, FOR THERE IS NO INTERNAL CONSTRAINT TO THE PROCEDURE. Similarly, we know that $7 \times 5 = 35$ by counting off on finger tips, etc. (by adding to one seven another seven and by repeating the operation five times), and the result is equally memorized. The labor of this counting, the ascertaining of the sums or the products, is relieved by ready-made addition or multiplication tables which one has only to learn by heart.' (**?**, 172)

What breaks the symmetry between Pfaff and Hegel's reciprocal criticisms of method is the Hegelian distinction between *internal* and *external* necessity. If Dominque Dubarle is correct in detecting an indirect reply to Pfaff's '*how*'s in a passage inserted into the *Logic*'s second edition, then this is, indeed, how Hegel himself marks his conflict with Pfaff: 'The question of the *how*,' he writes,

> means: in which way and manner? in what relation? and so forth, and requires the application of a particular category; but there can be no question here of a 'way' or 'manner', of the categories of the understanding. The question of the *how* is itself one that belongs to the bad practices of reflection, which demands comprehensibility, but for that it presupposes its fixed categories and is thereby assured from the start to be forearmed against the answer to what it asks. (**?**, 72)

[†]See specimen texts # 5 and # 6.

Calculation is, for Hegel, an arbitrary sequencing of facts because, as he understands it, it relies on a separation of its necessitating principles – embalmed as axioms or 'fixed categories' – from the subject matter (whatever that might be) whose development those principles guide. This separation or abstraction leaves the subject matter of mathematics 'void of concept', an empty husk that can be adequately represented by a meaningless stock of letters manipulated by the calculator from without. This mode of operation, argues Hegel, is so utterly foreign to genuine, logical reasoning – which refuses any *a priori* separation of method and subject matter[2] – that any attempt to mathematise logic, genuine logic, is doomed to failure. 'It is characteristic' of the signs of mathematics, 'as contrasted with the determinations of the concept', he writes, 'that they are mutually *external*, that they have a *fixed* determination. Now when concepts are made to conform to such signs, they cease to be concepts', and so 'It is futile to want to fix [them] by means of spatial figures and algebraic signs for the sake of the *outer eye* and a *non-conceptual, mechanical manipulation*, such as a *calculus*' (**?**, 544).

The abstraction to which calculative thought subjects its material seems, at times, more than Hegel's dialectic can bear: when it appears in that process – in the Remarks interjected into his treatment of quantity, for instance – it occupies the peculiar and uncommon position of an *unsublatable* exteriority. As a *speculative* category, of course, 'Quantity' is not immune to sublation, but the calculative practice associated with it, Hegel freely admits, *cannot* be given a philosophical treatment, cannot be immanently and conceptually developed.[†] The concept can make its way past calculation only by sidestepping it, as an undialectical by-product of the dialectic.[‡]

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[2] As Hegel writes at the very beginning of his Introduction to *The Science of Logic*, 'In no science is the need to begin with the fact [*Sache*] itself, without preliminary reflections, felt more strongly than in the science of logic. In every other science, the matter that it treats, and the scientific method, are distinguished from each other; the content, moreover, does not make an absolute beginning but is dependent on other concepts and is connected on all sides with other material. It is therefore permitted to these sciences to speak of their ground and its context, as well of their method, in the form of lemmas; to apply presupposed forms of definitions and the like without further ado, as known and accepted; and to make use of customary ways of argumentation in order to establish their general concepts and fundamental determinations.

Logic, on the contrary, cannot presuppose any of these forms of reflection, these rules and laws of thinking, for they are part of its content and they first have to be established within it' (**?**, 23).

[†] See specimen text # 6.

[‡] Consider, for example, the following remark, reprinted in specimen text # 6: 'The step-by-step determination of the species of calculation that we have just given cannot be said to be a philosophical treatment of them, not an exposition of their inner meaning as it were, for it is not in fact an immanent development of the concept. However, philosophy must know how to distinguish what is by nature a self-external material; it must know that, so far as this material goes, the concept can make its way forward only externally, and its moments also can be only in the form peculiar to their externality, as here equality and inequality. To distinguish the spheres to which a specific form of the concept belongs, that is, in which the concept is present in concrete existence, is an essential requirement when philosophizing about real objects. This is to prevent ideas from interfering with the peculiar nature of externality and accidentality, and the ideas themselves, because of the disproportionateness of the material, from being distorted and reduced to a formalism. Here, however, the externality in

This is not, however, the first time the notion of unsublatable exteriority, devoid of concept and devoid of inner negativity, surfaces in Hegel's logic. What is surprising is that what appears here as a dead branch of the dialectic – pure exteriority without concept – is not treated in an even remotely similar fashion when it first makes an entrance in the *Logic*, in the scene of the *Logic*'s *beginning*, which, unsublated, 'is the ever present and self-preserving foundation of all subsequent developments, remaining everywhere immanent in its further determinations' (?, 49), as Hegel himself puts it. There is, indeed, no inner negativity animating the empty repetition of 'being, pure being' or 'nothing, the pure nothing' (?, 59), which is mediated by neither limitation, reflection or development. In the beginning, they are nothing but empty *loci*, names purged of all content, all 'determinations'. The dialectic nevertheless proceeds, and must proceed, from this not-yet negative moment. How then does the artificial separation of necessity from the bare letters of a calculus result in a dead end, something speculation can only side-step, rather than return to and retroactively *ground*? What is the *obstacle* that blocks a genuinely dialectical apprehension of calculation, an obstacle not present in the no less empty exteriority of the *beginning*?

The bar of formality. The bar that separates the dead letters of calculation from the empty beginning of Hegel's logic, the bar that reflects every mathematician's 'how' *away* from the dialectic rather than into it, and which makes the exteriority of the calculative a dead end rather than an unsublatable beginning for dialectical thought, can – it seems – be summed up in a single word: it is the *formality* of mathematical thought that prevents it from being able to apprehend that logic of *absolute form* that is the dialectic.

0.2 Formality and formalisation

Hegel does not waste any time distancing the project to be undertaken in the *Science of Logic* from the 'formal' conception of the discipline that we find in Kant's characterisation of what he calls 'general logic' – the logic of the Aristotelean and Stoic tradition, which he considers a finished and unshakable science – and which persists, in an attenuated and enriched fashion, in the Kantian conception of '*transcendental logic*'. This is the conception of logic which, Hegel writes,

> has hitherto rested on a separation, presupposed once and for all in ordinary consciousness, of the *content* of knowledge and its *form*, or of *truth* and *certainty*. Presupposed *from the start* is that the material of knowledge is present in and for itself as a ready-made world outside thinking; that thinking is by itself empty, that it comes to this material

which the moments of the concepts appear in this external matter, number, is the appropriate form; since these moments display the subject matter in the conceptual form of the understanding which is appropriate to it, and also, since they contain no demand for speculative thought and therefore have the semblance of being easy, they deserve to be employed in elementary textbooks' (?, 177).

> as a form from outside, fills itself with it, and only then gains a content, thereby becoming real knowledge.
>
> Further, these two component parts [...] are said to stand to each other in this order: the object is complete and finished all by itself and, for its actuality, can fully dispense with thought; thought, for its part, is something deficient and in need of a material in order to complete itself, and also, as a pliable indeterminate form, must adapt itself to its matter. Truth is the agreement of thought with the subject matter, and in order to produce this agreement – for it is not there on its own account – thought is expected to be subservient and responsive to the subject matter. (**?**, 24)[3]

Hegel's various criticisms of mathematics all pivot on the conviction that mathematical thought remains 'formal' in this sense, and it is for this reason that he takes a mathematisation of *logic* to be a way of detaining the latter in a tradition whose subversion is long overdue.[4]

This thesis, which, at the time Hegel was writing, certainly seemed reasonable enough, is nevertheless *false*. The protagonists of logic's early mathematisation, in the nineteenth century, indeed *meant* to codify a vision of logic that was, undisputably, *formal* in orientation. But the mathematical language in which they carried out this task was more truthful than their intentions, and the project of logic's mathematisation gave way to logico-mathematical *formalisation* – taking a direction very different from the steady path planned for it.

The formalisation of logic is not its submission to formality, but the process of formality's *sublation*. This is the thesis I want to defend here, today. It is *through logic's formalisation* that the science grasps a deeper and more concrete concept of logical form, one that opposes and outstrips the notion of formality dismissed in the first pages of Hegel's *Logic*, destroying the grip of this notion on logic while at the same time preserving and grounding it as the *use* of a λόγος that subtends the distinction of content and form. Since vague talk of sublation comes cheap, I'll try to make this more precise: logic's formalisation culminates in a vision of logical form that is so profoundly analogous to Hegel's own dialectic of *essence, ground*,

[3] As Hegel makes clear, this conception of logic is left unaffected by the 'Copernican hypothesis' by which Kant adjusts the programme of metaphysics. A methodological hiatus remains between Kant's transcendental theory of objects – according to which objects must conform to the *a priori* structures of subjectivity in order to be objects of experience in the first place – and the *formal* theory of logic, and even of 'transcendental' logic, which is just formal logic enriched with rules and axioms obtained through the transcendental study of the *a priori* conditions of experience. (See CPR A56-7/B80-1, e.g.)

[4] This is how the Preface to the first edition of *The Science of Logic* begins: 'The complete transformation that the ways of philosophical thought have undergone among us in the past twenty-five odd years, the higher standpoint in self-awareness that spirit has attained in this period of time, has so far had little influence on the shape of the *logic*' (**?**, 7).

form and matter that its products deserve to be called *formalisations of dialectical logic* – at least up to a certain point.[5]

The process of formalisation

Abstract algebra and axiomatic method

Note: if time is tight, skip to 0.2 on page 9.

In the classical era, sculpture was essentially an art of monument. The artwork joined real or mythical history at two points: through its shape, as a figurative representation of a historical figure or moment, and through the pedestal that connected it to the site that grounded its meaning. The movement of sculptural modernity, as Rosalind Krauss argues in 'Sculpture in the Expanded Field' (**?**), consisted in a gradual abstraction from these two bonds – absorbing the pedestal into the figure, absolving it of its site, and abstracting its figure from its historical or mythical referents, absolving it of representational duties. An analogous movement characterises mathematical modernity. Where geometry, in the classical era, served to represent and idealise supposedly given or intuited space, to whose evidence its axioms served to rivet it like a pedestal, with the dawn of mathematical modernity this would change dramatically. As with the emergence of abstract algebra, geometrical (and other mathematical) constructions were absolved of their representational function to become mathematics for mathematics' sake – if they retained representational usefulness, this was in some sense accidental to their validity as mathematical. And the axiom, like the modern pedestal, no longer served to attach these abstract constructions to an evidential *base* whose reality or truth could not be doubted; the positing of axioms, instead, served to mark a point of rupture with the lifeworld, their arbitrariness was made a virtue that usurped that of evidence, and they were fully integrated into the activity of mathematical construction as that which guaranteed the independence and freedom of the latter. This allowed, among other things, for the free proliferation of non-Euclidean geometries. On the algebraic front, the same attitude (often working with different methods) opened onto the study of *operations* satisfying certain formal properties (like the distributivity of one operation over another, invertibility, commutativity, associativity, etc.) without limiting *what* those operations are operating on to the narrow channels of number – or, indeed, to any pre-given domain.

These two gestures profoundly transformed the very idea of mathematics, which, when Hegel was writing, could still be described as *the science of magnitudes* without straining anyone's credulity. The epistemological break represented by abstract algebra and axiomatic method has made such an attitude impossible to maintain. The 'objects' of mathematics would, from the late nineteenth century onwards, be noth-

[5] We could go further and say that, today, logic is in the process of answering to the Hegelian demand to become a '*formal* science, yet the science of *absolute form* [… which] has in it a content or reality of its own […] and the content is only these determinations of the absolute form and nothing else' (**?**, 523).

ing but the correlates of an autonomous movement of decision and construction, constrained only by the modes of consistency to which it commits itself. [6]

Mathematisation of logic

I do not think that it is an accident that the first genuinely abstract algebras, the first mathematical practices to fully understand their freedom from any objective given, were those which struggled to apprehend rational thought *itself* as an object: these were the algebras of logic, that came with Boole and his successors. It is as if the objectification of the subject was a sort of threshold for the subject to cross in order to be freed of any *particular* object.

Boole's work presents a particularly interesting case, on the threshold of the practices of logical formalisation and mathematical logic in such a way that it condenses the historical contradiction between the epistemological break currently in progress and the epistemological obstacles that still hampered it. Boole's algebra of logic is the first non-numerical algebraic system, an algebra whose signs are not to be understood as referring to *quantities*. His work, however, precedes – by mere decades – the fully modern era of universal algebra, which gave the algebraist the right to construct abstract systems in complete autonomy from the domain of quantity.

Boole begins by observing that the algebraic laws that govern the addition and multiplication of quantities – the *commutativity* of the operations (that $a+b = b+a$ and $a \times b = b \times a$), and the *distributivity* of $\times$ over $+$ – also govern the *logical* operations of *disjunction* and *conjunction*, 'or' and 'and'. Logical deduction, he saw, can be mapped onto algebraic calculation, embedded in the algebra of quantities, with the variables being interpreted as ranging over classes (predicates and subjects of judgments) or propositions, rather than numbers – provided, that is, that we only interpret as logical values (classes or propositions) those values that satisfy an additional law, which Boole called *the fundamental law of thought*, which is simply

$$x \times x = x$$

or *idempotency under multiplication*, understood as the identity of 'x and x' with 'x', itself.

What, in Boole's eyes, justifies the 'fundamental law of thought' – the idempotence of conjunction, or the identity of A *and* A = A – is precisely the thought that when our signs have a 'fixed interpretation' – when they refer univocally to some *thing* outside thought. 'Suppose,' for instance

> that by a definite act of conception the attention has been fixed upon men, and that by another exercise of the same faculty we limit it to those of the race who are white. Then any further repetition of the latter

[6] To adopt another analogy, what took place in mathematics in the late nineteenth century is roughly comparable to what took place in political theory in the seventeenth, with Hobbes – who, by the way, had a few prescient insights into the future of mathematical formalisation.

> mental act, by which the attention is limited to white objects, does not in any way modify the conception arrived at, viz., that of white men. (?, 32).[7]

This is how Boole generally proceeds, guided by the notion that logic's operations are licensed to the extent that *they leave everything in its place*. His algebra encodes thought's exteriority to its objects, and logic's austere 'formality'. The freedoms his laws grant to language – opening the way for logical reasoning and, Boole observes, allowing poetic diction to be understood as pure ornament – is underwritten by the indifference of the thing to which discourse is tethered. 'The object,' to make Hegel speak through Boole, 'is complete and finished all by itself and, for its actuality, can fully dispense with thought' (?, 24). What emerges from this vision of logic is a set of algebraic laws that serve to neutralise anything we could call *syntactic time*, imposing explicitly the very structure of 'fixity' that Hegel took to be inherent in mathematical symbolism, a fixity in virtue of which 'concepts made to conform to such signs cease to be concepts'.[†] By making explicit the laws of fixity, and formulating them mathematically, Boole nevertheless succeeds in opening a gap between the condition of fixity and mathematical symbolism as such – the fundamental law of thought, after all, did not come for free, but had to be explicitly *imposed* on the movements of syntax.[8]

My reasons for emphasising this point, which for the time being seems tangential, will be clear in a moment, when we see the extent to which Girard's work hinges on a suspension of Boole's law.

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[7] Considerations of the same nature give Boole his justification for the law of *commutativity*, the identity $xy = yx$. As Boole puts it, 'Difference in the order of the qualities or attributes of an object, apart from all questions of causation, is a difference in conception merely. The law [of commutativity] expresses as a general truth, that the same thing may be conceived in different ways, and states the nature of that difference; and it does no more than this. (?, 21)

[†] See specimen text # 6.

[8] By the same token, Boole leaves perfectly legible certain conditions under which the fundamental might be challenged. What justifies the nullity of repetition (or commutation) in logic, Boole seems to argue, is the *stability* and *exteriority* of what thought thinks with respect to the thought that think it. So long as thought and its object remain external to one another, it does indeed appear unthinkable that the *course of reasoning* itself would change the constitution of its object. For this reason, though Boole's logic is indeed a mathematisation of mathematical thought, it is not yet the mathematics of *thought thinking itself*.

Boole's own interpretation of 'the fundamental law of thought' gives us every reason to anticipate the failure of every attempt to formalise dialectical logic that remains within its scope: the ultraboolean algebra proposed by Dominique Dubarle, where the relations between the universal, the particular, the singular and the void – which Dubarle sees as an essential moment of the Hegelian dialectic, misrecognised by Hegel himself – are mapped onto the algebraic relations obtaining in a $\mathbb{B}^2$ Boolean ring or lattice, and the more sophisticated lattice-theoretic formalisation of the dialectic proposed by Juan Grompone. This law serves to enforce the *fixity* of logical determinations by appealing to the *formality* of their manipulation: to the notion that they leave their object untouched, and that there is no such thing as *logical time*.

Formal languages and logico-mathematical axiomatisation

By the beginning of the twentieth century, the movement of mathematics' abstraction – or 'modernisation' – had come full circle. So long as axiomatisation continued to be carried out in the medium of natural language, its practitioners could not be assured of the integrity and independence of their abstractions – there is great difficulty in making it absolutely clear that hidden presuppositions and appeals to intuition did not leak into the axiomatic vessel so long as it was constructed from the dark and noisy elements of language. This is, of course, the threat that Pfaff took to unravel Hegel's claims to have made a *presuppositionless beginning* in his *Logic* – and there is some suggestion in one of Pfaff's letters that Hegel, himself, observed that even mathematics is not immune to this threat (see **?**, 361). In the element of language, even mathematics cannot be as 'mute' as Pfaff desired.[†]

The sophisticated mathematisation of logical quantification and relations that Frege and Russell developed around the turn of the century provided a means to solve the problem of *mute axiomatisation*. The extremely basic mathematical structures of *functions* and *sets* could now be expressed and specified without any explicit use of natural language (at least on the strata concerned, a complete departure from natural language being impossible, as the most serious formalists never failed to point out[9]). By this route, the axioms of a system could be stated *in a language that was itself formalised*. Inferences from the axioms to the theorems could, likewise, be encoded entirely in the formal language of the underlying logic. And the logic, itself, proved to be amenable to axiomatic treatment.

The result of this was that, under the seal of the axiomatic method, a complete and mutual embrace of mathematics and logic became possible.

The greatest protagonist of this new axiomatic method was undoubtedly David Hilbert, with the movement of *formalism* that he inaugurated, and the *metamathematical* research programme that he advanced. We have already learned something of this programme, and its dialectical significance, from John Bova and Emmanuel Barot's presentations, yesterday, so I won't spend too much time on it. For now, it can be summed up along three lines:

FIRST, Hilbertian formalism is a means for mathematising epistemological reflection on mathematics, transforming, for example, epistemological theses concerning the consistency and completeness of a mathematical system into mathematical conjectures *bearing mathematical, and not only philosophical, consequences*. The most important objective of 'the Hilbert Programme', as it came to be called,

[†]See specimen text # 7.

[9]As Haskell B. Curry, who we'll meet in a moment, remarks, 'The construction of a formal system has to be explained in a communicative language understood by both the speaker and the hearer. We call this language the *U-language* (the language being used). It is a language in the habitual sense of the word. It is well determined but not rigidly fixed; new locutions may be introduced in it by way of definition, old locutions may be made more precise, etc. *Everything we do depends on the U-language; we can never transcend it; whatever we study we study by means of it*. Of course, there is always vagueness inherent in the U-language; but we can, by skillful use, obtain any degree of precision by a process of successive approximation' (**?**, 25).

was to *prove* that the axioms of arithmetic and analysis *are not contradictory*, and to do so by means internal to arithmetic itself – means that are essentially *finitary*.

SECOND, the precondition for this reflexivity was, for Hilbert, a prior reduction of a mathematical system to 'a game of marks on paper', as some of the formalists polemically put it: a simple set of rules for rewriting formulas, beginning with a set of *axioms*, so as to transform them into *theorems*. In order to make mathematical reason an *object* for itself, the formalists saw, it is first necessary to identify it, as far as possible, with its mode of expression – seeking, in the play of inscriptions, both a materialisation of mathematical ideality and an idealisation of its material support. This gesture presses towards an increasing closure of the gap between the rational *necessity* of mathematics and the apparent *accidentality* of its expression, and it is only by seeing how and where this gulf is sealed that we can understand the legitimacy, for example, of incorporating calculations over the number of characters occurring in a given formula into demonstrations bearing on the mathematical structures those formula express – to indicate just one practice common in formal mathematics.

THIRD, formalism should thus be seen as an effort to achieve a *finite but nevertheless adequate apprehension of the infinite*. Any effective syntactic *technique*, any game of inscriptions that can actually by *played*, must ultimately be finite, even if indefinitely extendible. To be something that can actually be *written*, for instance, a proposition can only contain a finite number of symbols, and a proof only a finite number of steps. The *theme* of these demonstrations is nevertheless, in general, mathematical structures of infinite complexity and size: the multitude of sets, numbers, spaces, etc. whose *concepts* formalisation transubstantiates into techniques of inscription.

The logical formalism Hilbert used differed only slightly, and inessentially, from Russell's. Its propositional fragment contains only one rule, *modus ponens*. To this, the quantificational fragment adds the rule of universal generalisation: to infer from $\mathrm{A}[x]$, when x is free (i.e. left completely indeterminate), $(\forall x)\mathrm{A}[x]$. A formal sys-

$$\frac{\mathrm{A} \to \mathrm{B} \qquad \mathrm{A}}{\mathrm{B}}$$

Figure 1: Modus ponens

tem, in Hilbert's sense, would also contain a series of *axioms*: a set of logical axioms, being a more or less arbitrary selection of tautologies (true on any interpretation of their variables), and a set of axioms specific to the theory in question (specifying the properties of sets, groups, numbers, or whatever domain is being axiomatised).

The resulting logical syntax is a useful object for analysis, so far as metalogical proofs of the system's completeness (or incompleteness) and consistency (or inconsistency) were concerned, but a bit of a mess to read and painful to work with. Formalised proofs, with Hilbert, began to become mathematical objects in their own right, but through a glass darkly – only their most general features could be dis-

cerned (typically, whether or not a proof *exists* for certain mathematical formulas, such as a contradiction, or the set of statements true of some structure).[10]

The sequent calculus. Everything changes with Gerhard Gentzen's invention of the sequent calculus, constructed in the dusk of the 'Hilbert programme' and the wake of its death at the hands of Gödel's incompleteness theorems, in an effort to prove that, even if we cannot demonstrate the consistency of arithmetic by finite means, we may do so by *transfinite* means – which means proving the consistency of arithmetic by means of super-arithmetic. The tactic worked, but apart from giving the Hilbert Programme the funeral procession it deserved, it was of little independent interest. The tools he constructed to prove the result, however, were revolutionary.

The sequent calculus differs from axiomatic formalizations of logic, such as Hilbert's, Frege's or Russell's, for instance, in its absence of 'axioms', in the strict sense. Or, rather, in its reduction of all axiomaticity to assertions of the form

$$A \vdash A$$

read 'A implies A', taken as a primitive principle of identity. For the rest of the logic, what the sequent calculus offers in place of axioms is an elegant collection of *rules of inference*, a pair of which attends to each *connective* in the logic in question. (I say 'the logic in question' because the sequent calculus can, in fact, be used to formalize a great number of *different* logics, as we will see.)

A sequent calculus proof takes the form of a 'tree'. At its root is placed the 'sequent' that is to be proved, which has the form

$$\Gamma \vdash \Delta$$

where Γ and Δ are (possibly empty) series of formulas. The series on the left – Γ – contains the 'premises' of the sequent; the series on the right – Δ – its 'consequences'. The left series is read (informally) as the 'conjunction' of its formulas; the right aside, as the 'disjunction' of the formulas in Δ. This difference reflects an essential logical duality, or symmetry, between conjunction and disjunction, such as is expressed – in classical logic – by the De Morgan equivalences, for instance:

$$A \wedge B \leftrightarrow \neg(\neg A \vee \neg B)$$

[10] As Girard remarks, 'Before Gentzen, logic was formulated in 'Hilbert-style' formal systems; basically a lot of axioms and a couple of rules, essentially *Modus Ponens*: from A and $A \Rightarrow B$ deduce B, and *generalisation*: from $A[x]$, deduce $\forall x A[x]$. These systems have no good structural property, for instance, it is impossible to try the slightest automated deduction in the presence of *Modus Ponens*: indeed, in order to prove B (supposedly obtained by *Modus Ponens*), one must guess the premise A, which can be any formula, including B. The only setting in which Hilbert-style systems are justified is the narrow context of the incompleteness theorem: indeed, one does not seek positive properties of formalism, but rather its limits. In this very negative perspective, Hilbert-style systems do no worse than other systems and some of them have the advantage of being immediately understandable' (**?**, 41).

$$[\mathrm{L}\neg]\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, \vdash \neg A, \Delta}\ [\mathrm{R}\neg]$$

Figure 2: Negation in the sequent calculus

The formulas in Γ or Δ may, of course, themselves contain connectives, and the purpose of the calculus' *rules of inference* is to show how these complex formula can be systematically decomposed into their simplest constituents, branching the sequent-tree accordingly, in order to arrive at its 'leaves': axiomatic assertions of identity, of the form $A \vdash A$ (see figure 7 on page 35 for details). The tree can also be read from the top down, with each downward step representing a deductive inference.

As observed by the Belgian dialectician, Leo Apostel, Gentzen's sequent calculus (and systems of natural deduction)

> aim at the most complete solidarity possible between the definition of logical constants and the demonstrations in which those constants appear: to demonstrate a thesis is now equivalent to constructing it on the basis of the simplest theses. The proof thereby becomes a genesis, and it is no longer possible to assert that the operations are applied to concepts from the outside. (**?**, 113)

The complaint Hegel makes against mathematical demonstration in the Preface to the *Phenomenology of Spirit*[†] loses its grip on mathematical logic at this point, where its capacity to outstrip the old notion of formality is becoming increasingly apparent.

The convergence of 'form' and 'content', or the apprehension of logic's form *as* its content, is pressed further than this by Gentzen's calculus. The form and formalism of the system exposes certain deep geometrical properties underlying the 'pure play of symbols' articulated by the formalists.[11] In its syntactic structure, for instance, it *shows* how the fundamental concept of *implication* can be understood as an axis of symmetry, which, at bottom, is nothing other than logical *negativity* itself. This can be seen in the rules for the negation operator (figure 2). Moreover, as Girard points out, the very rules of the calculus 'are unchanged if one swaps left and right, provided we perform the permutations $\vee/\wedge$ and $\exists/\forall$. One knows that negation exchanges these connectives: one concludes that negation is nothing but the exchange left/right. [...] This is an obvious change of paradigm,' he goes on to argue, for 'instead of concentrating on the intended meaning of negation, one works on its geometry, which does not care about our intentions' (**?**, 47).

[†]See specimen text # 5.

[11]'The current formalist ideology says that mathematics is a pure play on symbols. [...] The mistake might lie in a subtle shift from pure play to 'meaningless': who told us that a play on symbols is meaningless, has not its own geometry? By the way, Gentzen – Hilbert's most conspicuous follower – disclosed a structure (sequent calculus) underlying the 'play on symbols' ... and sequent calculus has its own geometrical structure-this is precisely what ludics is about' (**?**, 132).

The deepest aperture that Gentzen's formalism opens onto the life of logical form is focused through his reflections of the *cut* rule of the calculus (figure 3). This

$$\frac{\Gamma \vdash C, \Delta \qquad \Gamma', C \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ CUT}$$

Figure 3: The cut rule

rule, essentially, is *modus ponens*. It expresses our freedom to make use of *lemmas* in a proof, or to combine proofs together into more complex demonstrations.

Now, Gentzen's most famous and significant theorem – his *Hauptsatz* – is that CUT is *eliminable* from the calculus. Every proof using CUT can be transformed, algorithmically, into one where CUT is absent: an 'explicit' proof, which makes no appeal to lemmas, and in which every subformula decorating the 'leaves' of the proof appears as well in the proven 'root' sequent. *Cut-elimination* is of crucial importance in logic for a number of reasons. Consider, for instance, the light it throws on the problem of consistency. *Inconsistency* in a sequent calculus formalization of a logic is equivalent to the derivability of the empty sequent

$$\vdash$$

But the only possible way of deriving such a sequent is through CUT. If a system enjoys *cut-elimination*, then no sequent derivable without CUT is derivable *with* CUT, and the system is therefore consistent.

As we will see in a moment, moreover, the algorithm of cut-elimination is even more revealing than its results: its study opens onto an unsuspectedly rich and complex field of investigation, where the *dynamics* of logical form come to light.

The convergence of 'form' and 'content' brought on by the sequent calculus, or the progressive sublation of formality, which Apostel notes, encounters a revealing obstacle in what are called the *structural rules* of the calculus. These concern the global structure of proofs, the format of the logical horizon that is in place before a single logical constant is written, and determine the very nature of what it is to be a logical *proposition*. Collectively, they endow the formula with a certain *ideality*, and open a gap between the type of logical information it contains and its particular occurrences in the syntactic structure of the proof. In effect, the structural rules can be seen as a sophisticated refinement of Boole's fundamental insights as to what constitutes the logical. The names of the rules are WEAKENING, EXCHANGE and CONTRACTION.

EXCHANGE is the rule that lets us disregard the *order* of information in a given series of formulas. It is equivalent to Boole's law of *commutativity*, the collapse of syntactic succession into simulated simultaneity.

WEAKENING is the rule that lets us forget or discard 'unneeded' information, as we move upwards in the proof, or add potentially 'irrelevant' information as we move downwards.

CONTRACTION, together with weakening, transposes Boole's *fundamental law of thought* into the calculus, and works to 'idealise' formulas in the proof, so as to make them eternally reusable and available. Without contraction formula behave less like ideal propositions and more like exhaustible 'resources'.

The collective effect of these rules is to reopen a gap between the punctual location of formulas and the global geometry of the proof: it lets logical information 'wander' through the proof tree, unbinding, in a sense, logical content from the syntax that inscribes it. By and large, these rules are fundamentally antagonistic to the smooth operation of cut-elimination: so long as contraction reigns free, in particular, cut-elimination does not converge towards a unique result, a unique cut-free kernel, and retains a degree of 'algorithmic inconsistency', which we will look at more closely in a moment.

Formal systems

The investigation into the *dynamics* of proofs, a manner of studying logical forms that owes nothing to the form-content dichotomy, was made possible when another of Hilbert's greatest students, and perhaps the most radical of the formalists, Haskell B. Curry, pushed the movement of formalisation to its limits, and developed the concept of a *completely formalised system*.

A formalised system, in Curry's sense, is specified by (1) an initial stock of primitive entities – generally represented by letters, and (2) rules for constructing new entities by combining the primitives – new entities that are essentially *identical* to the process of their construction (**?**, 15). It is in this last respect that the concept of a formal system differs from that of an abstract algebra, where, typically, 'the same element may be obtained by the operations in many different ways' (**?**, 17), or, to put it another way, where the *equality* of various constructions is treated as a more or less *immediate fact*. In a formal system, a proposition expressing an equality is *itself* an entity that must be constructed by a determinate process, on the basis of syntactical differences that it does not simply neutralise.[12]

A *completely* formalised system is one that is utterly univocal, one that admits no pre-given division of its objects into *types*. Types, in logic and computation theory, are classifications imposed on terms so as to restrict their scope of application, usually for the sake of avoiding paradox, incoherence, or the 'crashing' of computational processes. It was by stratifying propositional functions through a hierarchy of types, for instance, that Russell avoided the antinomy he discovered at the heart of Frege's logic. In an untyped, or univocal system, by contrast, every entity operates on every other, and 'any construct formed from the primitive entities by means of the allowed operations must be significant in the sense that it is admissible as an entity' (**?**, 5).

[12]The spirit of a formal system is, in this way, suprisingly close to that of Hegel's own reflections on quasi-algebraic identities – as when he says, for example, that the identity expressed by $A = A$ conceals a difference and negative movement inscribed in the *propositional form* itself (see **?**, 360).

For a system to be considered 'formal', it must, at least, traffic in schematic expressions whose terms can be freely substituted for other terms – whose terms play the role of *variables*, or empty slots. So long as the work of substitution is guided by an independent *substitution rule*, then the formality of the system is in some sense 'abstract', or ungrounded by the milieu in which it operates. Now, the implementation of a substitution rule, as Curry has shown, generally presupposes that a *typing system* has been put in place, usually in an 'intuitive' or implicit fashion (**?**, 33). The typing of entities is thus a precondition for grasping the expressions of a formal system as *formal*, in the sense of being *schemata* that are essentially indifferent to their content, which can be interchanged at will – provided that the right *type* of content is inserted into their empty slots. It is this notion of formality – a notion close to, and in any case necessary for, the one that Hegel rejects at the outset of his *Logic* – that collapses under the force of *complete formalisation*, which demands a univocity of logic incompatible with the presupposition of types.

Apostel sums the situation up well in his Appendix to *Logique et dialectique*, where he recounts how

> After mathematical logic's period of construction, which culminates with Russell, followed a period of formalisation, which transformed the systems of the preceding epoch into systems speaking of signs and combinations of signs. But, in a characteristically dialectical transformation, with the extreme formalist Haskell Curry, those systems, which in Carnap speak of signs, are transformed into systems speaking of indeterminate [*quelconques*] structures, of objects in the most general sense of the word. After having sought to exhaust the richest content, the development of logic is restrained to the narrowest form, only to once again overflow into the most complete generality.

He then goes on to articulate a suggestion that I will try to develop in some detail in the last section of this presentation, writing that

> Nothing more clearly illustrates the dialectic of form and ground that Hegel recognised as essential than this process, which he gave short shrift in criticising, at certain places in his 'Logic', the diminution to which the pure concept is subjected when the Euler circles reduce it to merely sensory. It is the inevitable and constant transmutation that he tells us of in his dialectic of essence, a dialectic at work before his very eyes, and which he willfully, without reason, discounts.
>
> We conclude,

he writes,

> that if we examine the profound affinity between form and matter in the Hegelian system, then formal and dialectical logic should be two faces of the same unity. (**?**, 117)

To unfold this suggestion, we must look at things more closely.

The completely formalised systems at the heart of Curry's research programme (in the mid-twentieth century) were what he called *combinatory logic* and the related system (which, strictly speaking, is still 'less' formalised than combinatory logic, but still essentially untyped) called the *lambda calculus*, as developed by Alonzo Church. Without going into too much detail, we can say that the main purpose of these systems is to analyse the idea of an *effectively calculable function*, in a struggle to give exact meaning to the notion of determinate, finite procedure at the heart of the formalist enterprise. The concept that emerged was, in effect, that of a *computer programme* in its most basic and abstract form (Alan Turing's universal abstract machine was, in fact, a contemporary, parallel and provably equivalent prong in this campaign).

With W.A. Howard, Curry went on to discover an isomorphism that would link together formalisations two rebellions against the formal: the lambda calculus developed by Curry and Church and the sequent calculus developed by Gentzen. The Curry-Howard isomorphism, as it came to be called, is a strict correspondence between *proofs* in the sequent calculus (or natural deduction) of intuitionistic logic and functions, or *programmes* in the lambda calculus. Through its lens, the procedure of CUT-ELIMINATION – which justifies the combination of proofs and the use of theorems as *lemmas* in proofs of greater complexity, a practice indispensable to mathematical practice – fully reveals its dynamic and computational significance: Each CUT, or combination of proofs – as lemmas in a more complex proof – becomes a combination of programmes whose *execution* is the process of cut-elimination itself.

This correspondence, however, demands a prior bridling of the lambda calculus with an apparatus of *types* – it is, strictly speaking, an isomorphism of logic with the *typed* lambda calculi – and so still remains at the *threshold* of logic's sublation of its own formality, a threshold which it, here, outstrips only in part. Types, here, are imposed to prevent certain paradoxical and self-referential formations from arising in the calculus, so as to ensures that *every programme terminates, or yields a result rather than 'crashes'*. The Curry-Howard isomorphism, which correlates logical *proofs* with *programmes*, also correlates programme *types* with *propositions*. A proposition, from this point of view, is perceived as a set of proofs, or, more precisely, as a *classification* determining which proofs can be combined, or CUT, with which. The proposition, on this view, finds its meaning not by referring to a model, or pre-given state of affairs, but according to the formal dynamics obtaining between the ways it can be proven and the ways those proofs can be used.

Syntax and semantics, and the relapse into formality

Even if these developments herald a radical and thoroughgoing sublation of the notion of logic's formality through its formalisation, and a conception of the discipline in terms of the action of form on form, an action that develops its own content immanently and dynamically, without any need of extraneous hypostases, the image

of formality would nevertheless continue to detain logic's historical development through the notion of *semantics*, or through a certain use of that notion, which, in *The Concept of Model*, Badiou has diagnosed as ideological (**?**).

The great bastion of abstract formality in modern mathematical logic is what Alfred Tarski called the *semantic conception of truth*.[13] In non-technical terms, this is the conception of truth that proceeds according to definitions of the form

'snow is white' is true **if and only if** snow is white.

When it comes to defining the truth conditions of formal, compound propositions, this conception operates by equivalences such as

$$\begin{aligned}
\mathrm{A} \wedge \mathrm{B} \texttt{ is true} &\Leftrightarrow \mathrm{A} \texttt{ is true } \textbf{and } \mathrm{B} \texttt{ is true} \\
\mathrm{A} \vee \mathrm{B} \texttt{ is true} &\Leftrightarrow \mathrm{A} \texttt{ is true } \textbf{or } \mathrm{B} \texttt{ is true} \\
\neg \mathrm{A} \texttt{ is true} &\Leftrightarrow \mathrm{A} \texttt{ is } \textbf{not } \texttt{true} \\
(\forall x)\mathrm{A}[x] \texttt{ is true} &\Leftrightarrow \textbf{for all } \texttt{objects } \mathfrak{x} \texttt{ in any model, } \mathrm{A}[\mathfrak{x}] \texttt{ is true}
\end{aligned}$$

and so on. These are not *quite* the empty tautologies that they appear to be at first sight, since there *is* a difference between the left and right sides of the equivalences.[14] The expressions which are *named* and whose truth is asserted on the left are expressions in the 'object language', the syntax under investigation. But as Tarski and, before him, Gödel, showed, a non-contradictory syntax *cannot* express its own truth predicate. If it could, it would be exposed to the famous 'Liar Paradox' – through a simple act of diagonalisation it could construct statements that amount to saying '$\mathrm{A} \Leftrightarrow \neg(\mathrm{A} \texttt{ is true})$' – statements which are at once true and false (or which, if Boole's fundamental law is suspended, *oscillate* between truth and falsehood). The truth predicate for a syntax $\mathscr{S}$ must therefore be expressed in *another syntax*, which would be a *metalanguage* with respect to $\mathscr{S}$ (call it $\mathscr{S}'$). The connectives on the right hand side of the equivalences just mentioned must therefore be connectives in the *metalanguage*. When we define the truth of $\mathrm{A} \wedge \mathrm{B}$ in $\mathscr{S}'$, we *use* the metalinguistic '**and**', but only mention or name the object-linguistic *and* (denoted by the wedge

[13] For a clear overview of Tarski's theory of truth, see **?**.

[14] Regarding the 'snow is white' example, Tarski offers the following clarification: 'It might seem at first sight that [this example], when regarded as a definition, exhibits an essential flaw widely discussed in traditional logic as a vicious circle. The reason is that certain words, for example 'snow', occur in both the definiens and the definiendum. Actually, however, these occurrences have an entirely different character. The word 'snow' is a syntactical, or organic, part of the definiens; in fact the definiens is a sentence, and the word 'snow' is its subject. The definiendum is also a sentence; it expresses the fact that the definiens is a true sentence. Its subject is a name of the definiens formed by putting the definiens in quotes. [...] Hence the word 'snow', which undoubtedly occurs in the definiendum as a part, does not occur there as a syntactical part. [...] However, words which are not syntactical parts of the definiendum cannot create a vicious circle, and the danger of a vicious circle vanishes' (**?**, 64). To further dispel fears of a vicious circle, Tarski points out that the definition, ' ''snow is white' is true if and only if snow is white" ' could just as well have been written: "The string of three words, the first of which is the string of the letters Es, En, O and Double-U, the second is the string of letters I and Es, and the third is the string of the letters Double-U, Aitch, I, Te, and E, is a true sentence if and only if snow is white" (ibid.).

symbol, $\wedge$). Of course, the metalanguage, so long as it is non-contradictory, is subject to the same limitations. *It* must call on yet another metalanguage to express *its* truth predicate, and so on, in a badly infinite progression that, as Hegel would say, *expresses* the contradiction at the heart of the semantic conception of truth by the very flight that avoids it *without resolving it*.[15]

Equivalences of this sort, shuttling across the badly infinite hierarchy of metalanguages, are neither false nor useless. They perform the invaluable service of mediating between various syntactic constructions, between explicit formalisms and informal, historically given mathematical structures through which the formalisms can be 'interpreted': they open the channels of communication on which the theory of *models* depends. What *is* both false and useless is the notion that such a procedure in any way serves to explicate the logical forms themselves.[†] Semantics is a good way of *using* logic, of *applying it* to material that comes from elsewhere (even if it is simply furnished by a neighbouring domain of logic itself). But its efficiency is bought at the price of logic's *transparency*, and to attempt to *explain* a syntax through its semantics is to get lost in a hall of mirrors.

The way in which the syntax/semantics opposition is often handled bears further traces of the formal image of logic, which Hegel attributes to Kant, in framing it as a relation between a *subjective technique of representation*, on the one hand – finitary syntax – and a *represented object*, on the other – a (usually infinite) structure, interpreted as the semantic content or model of the syntax. This notion of *syntax as subject* is one which neither Hegel nor Girard will reject entirely. The question for both, rather, will be how to radicalise it so as to release it from the image of formality, over-awed with semantics. What syntax will retain of the subject will be, moreover, precisely what is occluded by the formal image of logic that charges it with little more than representing the specular image of its own semantics: its temporal, procedural aspect (the dimension of syntactic time), its autonomy or self-legislation, and its 'existential' negativity.

0.3 The idea of transcendental syntax

The same *modus operandi*, and the same formal image of logic found in the semantic conception of truth animates the canonical versions of *transcendental logic*, which tout themselves as an inquiry into logic's conditions of possibility or grounds.

Hegel's double criticism of the Kantian programme of transcendental logic is well-known: his transcendental logic preserves unquestioned the *form* of logic received from the scholastic tradition, and goes on to bar any explication of those forms *in and for themselves* by keeping transcendental logic an essentially *formal* inquiry – abiding by the prejudice that the forms of thought are incapable of determining their own truth and internal content:

[15] 'The process to infinity is in general the expression of contradiction [...] *not the resolution of it*' (?, 191).

[†] This is the butt of Girard's 'Broccoli logic' joke. See specimen text # 15.

> Just as the Kantian philosophy did not consider the categories in and for themselves, but declared them to be finite determinations unfit to hold the truth, on the only inappropriate ground that they are subjective forms of self-consciousness, still less did it subject to criticism the forms of the concepts that make up the content of ordinary logic. What it did, rather, is to pick a portion of them, namely the functions of judgments, for the determination of categories, and simply accepted them as valid presuppositions. Even if there were nothing more to the forms of logic than these formal functions of judgment, for that reason alone they would already be worthwhile investigating to see how far, by themselves, they correspond to the *truth*. A logic that does not perform this task can at most claim the value of a natural description of the phenomena of thought as they simply occur. (?, 525)

For Hegel, it will be through a reflective subversion of logic's *formality* that the *forms*, themselves, can be scientifically grasped – which is to say, rationally transformed and not merely repeated. On this point, Girard's approach is infinitely closer to Hegel's than to Kant's, as we will see.[16]

Husserl is no better, where the question of actually *grounding* the syntactical structures of logic are concerned. We can consider, for example, his attempt to explicate the *transcendental* grounds of the law of non-contradiction

$$\vdash \neg(A \wedge \neg A)$$

by pointing out that

> whoever thinks of two judgments as judged by someone, and, on making them distinct, recognises that one *contradicts* the other, *cannot do otherwise* than *deny* the *conjunctive* judgment formed out of them. (?, § 75, 190)

It is clear that this game of mirrors leads nowhere, other than on a badly infinite flight away from the syntax itself, whose blind repetition in the (meta-) language of subjective consciousness does nothing to explicate any grounds whatsoever. It's true that the quest for logic's grounds never ceases to concern Husserl, but the effect of this quest on his work is consistently centrifugal. Where the forms of logic should be explicated *in and for themselves* remains the empty centre of his entire corpus, with every attempt to penetrate logic's depths winds up being flung outwards into the living context of logic's *use*. This centrifuge has been quite productive for transcendental phenomenology in general, but it must be admitted that nothing like a transcendental *logic* has actually been carried out in those quarters.

It is from this point of view that we must recognise Girard's project of *transcendental syntax* as a radical departure from the Kantian and Husserlian traditions, traditions with which, I think, he is too quick to associate himself. The project was

[16] See also Hegel's remarks on Kant's transcendental logic at ?, 40-42.

first articulated under that name in 2011 (in ? and again in ?), but retrospectively embraces the whole of his research since the mid-1980s.

Girard's own methodological categories help to explicate the distance he takes from the Kantian and Husserlian methods of transcendental logic. In 'Syntaxe transcendentale, manifeste', he delineates four strata – or 'infernos' – of logical investigation.

-1 THE ALETHIC INFERNO: this is the realm of 'truth', semantically conceived. Within limited contexts – those where the infinite or the numerical is not at stake, and where indirect modes of self-reference cannot take shape – it serves as a baggy garment for *provability*. Propositions are alethically equal when they are equi-demonstrable; semantic 'truth', where it works best, is effectively the *quotient* we obtain from the syntax when we identify all provable statements with *truth*, in the abstract, and all refutable statements with the equally abstract *false*. What disappears in this collapse of *proofs* into *provabilities* is the *action of proof* itself. Alethic or semantic interpretations cannot discern between two syntactically different proofs of the same theorem, and so quite a bit falls into the blind spots of this particular perspective. What is retained is a bleak and impoverished *repetition* of the initial syntax – or the quotient obtained when one identifies all equi-provable formula.[17] This approach to logic is best suited to its CLASSICAL variant: the traditional 'laws of thought': the law of the excluded middle, non-contradiciton, Boole's fundamental law of thought, etc. all find a fairly straightforward justification from this point of view. Kantian and Husserlian transcendental logic plays itself out entirely on this plane, hence their permanent indenture to the 'formal' image of logic, which they might supplement with additional content (gleaned from various critical or phenomenological reflections on the *a priori* conditions of experience), but whose horizon they perpetually fail to outstrip.[18]

-2 THE FUNCTIONAL INFERNO: this inferno goes somewhat deeper than the preceding by attending to the *action* of proving, interpreted in terms of the

[17] As in a Lindenbaum algebra, which is a lattice structure in which we take the equivalence classes of each formula as the elements of the lattice, defined as $|\alpha| = \{x : x \Leftrightarrow \alpha\}$, and the ordering relation $|\alpha| \leq |\beta|$ to hold whenever $(\exists x \in |\alpha|)(\exists y \in |\beta|)(x \Rightarrow y)$. A Boolean lattice can be seen as the Lindenbaum algebra of classical propositional logic.

[18] Things only get worse for classical logic, from the point of view of the commited proof theorist, with Gödel's incompleteness theorems. There have traditionally been two kinds of reaction to the incompleteness results: (1) the transcendent, semantics-oriented reaction, which, when seeing that there are 'unprovable truths', cries, 'So much the worse for provability! Truth flies free!' (2) the immanent, syntax-oriented reaction, which responds, 'So much the worse for truth! It's too clumsy to capture even *provability* without stumbling and overstepping its bounds!' From the latter point of view, the existence of undecidable statements makes a plea against one of classical logic's emblematic laws: the law of the excluded middle, since it does not survive the shift from semantic truth to syntactic provability in any sufficiently powerful system, where '$\vdash$ A **or** $\vdash \neg$A' is refutable. For the intuitionist, who *identifies* A with the set of A's proofs, A $\vee \neg$A is not a valid law.

functional transits linking one proof to another. Rather than interpret A → B as **If** A **is true then** B **is true**, for instance, it interprets it as a *function* or map sending a proof of A to a proof of B. Category theory and the lambda calculi, mediated by the Curry-Howard isomorphism, make invaluable contributions to the exploration of this inferno, whose native logic tends to be INTUITIONISTIC (i.e. rejects the law of the excluded middle, and favours a somewhat asymmetric relation between premises and conclusions).[19]

-3 THE INTERACTIVE INFERNO: the dynamics of *cut-elimination* are what fuel the third inferno. Gentzen and Curry's advances furnish crucial prerequisites for its exploration, but since cut-elimination is essentially a phenomenon of *logical symmetry* – the symmetry of negation, of ¬¬A with A – its adequate investigation is quite limited by the intuitionistic framework, which falls short of this symmetry. It's no use retreating to classical logic at this point, since classical logic maintains the symmetry of negation at the price of a cut-elimination process that is algorithmically incoherent (non-convergent) – it is not in general possible to reduce a given classical proof to a unique cut-free kernel. Though alethically non-contradictory, classical logic has for this reason been said to be *computationally inconsistent*. What Girard showed in 1987 was that a far deeper and more systematic critique of classical logic than intuitionism offers is needed in order to grasp the phenomenon of cut-elimination adequately. In particular, he showed that where classical logic fails, where it 'leaks', is in its collapse of the subtle differences of logical time – a collapse enshrined in Boole's fundamental law, which identifies A with its repetitions. [20] In order to grasp cut-elimination in a way that is both computationally sound – traces a convergent dynamic process – *and* fully symmetrical, it is necessary to *fully suspend the structural rules of the sequent calculus* – the structural rules which, in embryonic form, in Boole, were the very buttresses of the *formality* and *fixity* of logical determinations. This leads to *linear logic*, a logic where attention is paid to the *occurrences* of formulas in a proof rather than the mere 'types' that occur, where something like logical

[19] I.e., in the sequent calculus, intuitionistic logic is distinguished from classical logic only in this: one may have at most *one* formula on the right side of the sequent, while the left may contain any finite number of formula. This is enough to prevent the law of the excluded middle, or the identity of A with ¬¬A from being provable in the intuitionistic calculus, since proving either of these propositions requires the use of *contraction* on the right side of the sequent (see 8 on page 35 for an illustration of the proof of the law of the excluded middle in the classical sequent calculus, and notice that this proof could not be written without placing more than one formula on the right hand side).

[20] The reason intuitionistic logic has a convergent cut-elimination procedure (an algorithmically coherent dynamics of proofs) is that it effectively prohibits the use of contraction and weakening on *one* side of the sequent. This, in fact, amounts to an asymmetrical notion of logical time, as Kripke models of intuitionistic logic help to illustrate: whereas in classical logic, propositions are always either *eternally true* or *eternally false*, in intuitionistic logic they are either *eternally proven* or *currently unproven*. Linear logic symmetrises logical time, by distinguishing statements that are *currently active* in a proof from those which are *currently inactive*. I discuss the intuitionistic image of logical time to some extent in ?.

time once again becomes visible, and where negation is no longer an operation that inverts 'truth values' (swaps the true for the false, and vice versa), but which expresses something like the relation between an action and *its* reaction – its 'determinate negation' as a Hegelian reading might have it. Implication in linear logic, moreover, is no longer – as in classical or intuitionistic logic – a relation expressing the entailment of one timeless proposition by another, but the action of one logical determination *passing into* another. If we combine A and A ⊸ B, for example, we obtain B, but in doing so, we *lose* A – A is 'consumed' in the process of producing B, so to speak.

Negation in linear logic lends itself well to an interpretation in terms of the alternation of agency between two players in a 'game', or two adversaries in an argument or struggle. For this reason, the linear-logical study of the dynamics of cut-elimination lends itself quite naturally to certain varieties of *game semantics*.

-4 THE DEONTIC INFERNO: The concrete problem that compels the descent to the fourth inferno, in a nutshell, is this: I've mentioned that the second, 'functional' inferno surpasses the explanatory power of the first, by opening up the study of *proofs*, rather than just the *provability* of propositions, by interpreting the connectives (and, or, etc.) as functional or algorithmic modes of combination of proofs. The deepest aperture into this field is provided by the dynamics of *cut-elimination*, but cut-elimination seems to pivot on a logical symmetry – the symmetry of logical negativity – that the *intuitionistic* structure of the second inferno stymies. A deeper grasp on logical symmetry is provided by shifting to the third inferno, the interactive, where *linear logic* prevails. But something about negation still escapes this perspective. According to the functional interpretation, the negation of A is understood as A → ⊥, as a function sending a proof of A to the proof of a contradiction or falsehood. But what could the 'proof' of a contradiction be, *how* could it be? The notion seems to be void of constructive content, and it was for this reason that the radical intuitionist – and, in fact, the proponent of an idealism that would attempt to synthesise Bergson and Hegel (?) – G.F.C. Griss made a motion for the complete rejection of negation from logic (?). But as the moderate intuitionist Arend Heyting pointed out, this would not only destroy logic's *formality*, but forbid any means of reconstituting it.[21] The no-

[21] 'According to Brouwer, and also in this point Griss agrees with him, logic is not presupposed in mathematics, but yet in mathematical reasonings we can detect certain regularities the study of which belongs to logic. One might suppose that the logic of negationless mathematics is obtained by omitting from the intuitionistic logic all those rules which contain a negation, but this simple method does not work because it is necessary to impose a restriction upon the conjunction of two propositions. "*a* is a square" is a proposition; "*a* is a circle" as well, but "*a* is a square and *a* is a circle" is not a proposition. Thus the conjunction of two propositions *p* and *q* can only be formed if an instance is known which makes this conjunction true. In other words, the notion of a well-formed formula in this logic will depend upon the interpretation; a purely formal logic is impossible' (?, 94-5).

> tion of a 'proof of the contradictory' remains, in Girard's words, the 'empty pivot' of the constructive understanding of logic (**?**, 131,145).
>
> To grasp the concrete, procedural content of logical negativity, we must allow proofs to interact with something like 'false proofs' – but false proofs cannot be produced if we forearm the activity of 'proving' against contradiction in advance, by an apparatus of rules and types. Nor is it enough to tamper with those rules so as to allow 'true contradictions', as Priest and the other paraconsistentists try to do, since this would be, as Emmanuel has already pointed out, to neuter the element of negativity whose deep logical structure we are trying to encounter. The grail of the fourth inferno, therefore, would be a form of contradiction or opposition that is *prior* to, and, indeed, constitutive of logical rules – prior, in fact, to the very idea of *truth* or *provability*, and hinging on a notion of negation that would not mean *refuation* but *recusal* or *challenge*.
>
> This is what leads us to the fourth inferno, which Girard calls 'deontic', and it is on level that we begin to sense the intention of his downward flight. The study of 'interaction' in the third inferno is indeed a study of form acting on form, form disgorging its content as the dynamics of that action. It is nevertheless constrained by a normative apparatus of *types* – fixed determinations arming the logician in advance against the question of how form, left to itself, will perform its operations. The 'how' the logician asks the third inferno is still, to some extent, the 'how' which, in Hegel's words, 'means: in which way and manner? in what relation? and so forth, and requires the application of a particular category,' and so 'presupposes its fixed categories and is thereby assured from the start to be forearmed against the answer to what it asks' (**?**, 72). But here, in the fourth inferno, whose exploration is animated by a desire to see what is *behind* the 'fixed categories' of the typing apparatus constraining logic,'there can be no question here of a 'way' or 'manner', of the categories of the understanding' (**?**, 72).

The motive of Girard's descent is the motive of Platonic dialectic: to seek the unconditioned conditions for the possibility of λόγος, through a meticulous destruction of its presuppositions, a careful 'oblivion of syntax' as he calls it (in **?**). ('My dream,' he told me when I met him in February, 'would be to have no axioms at all'.) This is, ultimately, I think, an endless task, but one which can nevertheless serve to orient logical research. The apprehension of what is at stake in this task, for transcendental syntax, thus demands a return to the question of *beginning*.

To strip logic of presupposed *types* means to move, as close as possible, to the dimension of bare literality that formalisation progressively lays bare – to the grounds of symbolic manipulation, no longer made subject to what Hegel called the 'fixed' and 'formal' determinations of the understanding. It will be from this perspective, to the extent to which is is actually obtainable in mathematical practice, that transcendental syntax will seek to reconstitute logical formality, to show how it emerges, immanently, from the empty exteriority of the beginning it sights.

0.4 An-Fang: introduction to ludics

It's a well-known custom in the Platonic dialogues to fall back on myth and allegory to guide thought's movement towards the the obscure foundations of λόγος. A similar practice is found in Girard's writings, executed in a somewhat more fragmentary fashion, drawing on snatches of poetry or fiction with little explanation before proceeding with (often daunting) technical elaborations.[22] The text of 2001's 'Locus Solum: From the rules of logic to the logic of rules' – the first, and, for all its labyrinthine complexity, most elementary text to broach the 'deontic inferno' – is exemplary of this practice. There is, first of all, of course, the title's allusion to Roussel's novel, *Locus Solus*, or 'Solitary Place', from which Girard draws the pun of *Locus Solum*, 'Solely the Place', or 'only the place matters'. Now, neither Girard nor I have actually gotten around to reading Roussel's book, so I won't spend much time reconstructing this particular allusion.

More to the point is the text's epigraph, which repeats the first lines of Cordwainer Smith's short story, 'The Dead Lady of Clown Town'. It is an extraordinarily strange text to find at the head of an essay in mathematical logic, an essay to which the journal *Mathematical Structures in Computer Science* devoted an entire issue. It is also what marked my first impression of Girard's work, when I encountered it entirely by accident about five years ago, before I could understand a word of it.

This is it:

> Go back to An-fang, the Peace Square at An-Fang, the Beginning Place at An-Fang, where all things start (…) An-Fang was near a city, the only living city with a pre-atomic name (…) The headquarters of the People Programmer was at An-Fang, and there the mistake happened: A ruby trembled. Two tourmaline nets failed to rectify the laser beam. A diamond noted the error. Both the error and the correction went into the general computer. (Epigraph to **?**)[†]

It's all here, the entire ludic formalism, in allusion, pun and code, and I can't think of a better way to introduce it than to go through this allegory line by line.

Go back to An-Fang, the Peace Square at An-Fang, the Beginning Place at An-Fang, where all things start… When Goethe's Faust translated the Gospel of John, he chose to subvert the Lutheran translation, '*Im Anfang war das Wort*' with the subversively romantic '*Im Anfang war die Tat*', in the Beginning was the Act. If there is a contradiction between the Gospels of Faust and John, it is effectively sublated in the formalist Hilbert's famous pronouncement, that

[22] For example, before introducing *linear logic*, in 1987, as 'a logic behind logic […] a continuation of the constructivization that began with intuitionistic logic', we are reminded of the surrealist Audibert's invocation of '*La secrète noirceur du lait*' (**?**, 2).

[†] The excerpt is reproduced as specimen text # 16.

> The solid philosophical attitude that I think is required for the grounding of pure mathematics – as well as for all scientific thought, understanding, and communication – is this: *In the beginning was the sign* [*Am Anfang ist das Zeichen*]. (?, 202)[23]

or the word as act, as technique. What we have here is a bit different: in the beginning is *the beginning*, or rather the *signs*, "the beginning" – but these are not the type of signs that, in formalism, are preserved from ambiguity and ensure certainty. They are the material of a multi-lingual pun! *Im Anfang ist "An-Fang"* ': broken Chinese for 'Peace Square'! ('*An*' = 'peace', '*Fang*' = 'square', but in the sense of an equilateral rectangle rather than a plaza or place).[24] What's the meaning of this slip of logic into *lalangue*, at the very moment when its *beginning* is at stake?

Ludics approaches the Hilbertian sign from the angle of its *locativity*, present but scarcely visible in the formalist framework. 'Locativity' designates the quasi-material trace of the sign's occurrence, subtracted from the type of sign that occurs (from its codified uses). It concerns the place or *locus* where the sign is inscribed – the *letter* rather than the *signifier*, to use slightly similar Lacanian distinction. Insofar as their can be ambiguity in syntax, or interference between syntactic expressions – interference normally avoided, for instance, by renaming variables before combining two syntactic expressions in a third, so that there is no confusion as to what binds them – this is due to the *locative* aspects of the sign. The condition of possibility of a pun – and a bilingual pun above all – is precisely the *locative* aspect of signifiers: the accidental but always possible circumstance that the phonemes the German language uses to express *beginning*, for example, might interfere with those used to designate 'Peace Square' in mock-Chinese. Ordinarily, the use of formal languages abstracts from this aspect of the sign, according to techniques that Girard calls 'spiritual', as opposed to 'locative' – techniques concerned with *structure* alone, whereby everything is identified with its isomorphic images. Ludics takes the locative aspects of syntax seriously, it *begins* with the locative, and seeks to study *how* what has traditionally been taken as accidental dross can contribute to the very structuration of logic's 'spiritual', trans-locative dimension. This, by the way, is the meaning of the monograph's title: nothing takes place but the place…

An-Fang was near a city, the only living city with a pre-atomic name… Since Girard makes free use of ellipses in this epigraph, we should take seriously every line he makes a point of including, even if this means stretching the pun at some points. The stretch is worth making here, exploiting this reference to pre-atomic history to anticipate ludics's subversion of logical *atomicity*.

Logical *atoms* are unanalysable propositions, normally indicated by naked letters, without any connective symbols or grammatical structure. In a sequent cal-

[23] Michael Detlefsen suggests that, as an allusion to Goethe's 'In the beginning was the act', Hilbert's remark might have been meant as a jab at Brouwer's (Faustian?) maxim that 'Mathematics is created by a free *action* independent of experience'. See Detlefsen's helpful lecture notes on formalism at `http://philosophy.nd.edu/people/all/profiles/detlefsen-michael/documents/InSearchofFormalism.L3f.HO_000.pdf`.

[24] Thanks to Tzuchien Tho for pointing this out to me.

culus proof, one works by decomposing the sequent to be proved – following the logical rules assigned to each connective – until one reaches atomic propositions, and the proof or analysis terminates. If, at this point, every 'leaf' of the proof-tree contains a sequent of the form

$$A \vdash A$$

then the sequent at the root is *proved*, it is a theorem. The notion that every meaningful proposition can ultimately be decomposed into *atoms* has a long history. It is one of the fundamental theses of Leibniz, one with which Hegel broke emphatically, and one reasserted against Hegel by Russell, who took the practical need for atoms in mathematical logic to show that Hegel was mistaken in rejecting semantic and logical atomism. If we do not presuppose the existence of *atomic propositions*, Russell argued, the acquisition of knowledge has *nowhere to begin*.[25]

Of atoms, however, ludics has none. At bottom, it does not even operate on *propositions*, and retains only the *loci* of propositions as its base. The basic entities of ludics are what Girard calls *designs*, or strategies. They resemble proof-trees in the sequent calculus, except that while a proof-tree has a sequent at its root, such as

$$\Gamma \vdash \Delta$$

where Γ and Δ are sequences of propositions, the root of a design is a sequent or '*pitchfork*', of the form

$$\xi_n \vdash \xi_{\mathrm{I}=\{m_1 \ldots m_i\}}$$

which is a relation between a *locus* (which we can think of as a certain *action* or gesture) and collection of several other *loci* (which we can think of as anticipations of certain *re*-actions).

In a sequent proof, we would proceed by isolating the main connectives on each side of the sequent – they might be conjunction, disjunction, or whatever – and determining the structure of the immediately upper nodes of the proof tree in accordance with the rules assigned to those connectives. A design has no set of rules to follow. Instead, it is allowed to branch out in any fashion we like (so long as a certain rhythm between 'positive' and 'negative' moves is preserved – but this technicality isn't altogether crucial here). Instead of having each proposition branch out into

[25] As A. Ushenko explains, 'the question at issue,' in much of Russell's dispute with Hegel, 'is the principle of atomicity at the basis of analysis. According to the principle of atomicity, a single statement can be a complete unit of meaning and therefore completely true: if we know the logical structure, or syntax, of the statement together with the dictionary meanings of its connotative constituents, we have the necessary and sufficient equipment for the understanding of the statement as a whole without the aid of a wider context [...]. If our concern is knowledge, or true propositions alone, the principle of atomicity is summed up by the dictum that a single true statement is completely true even though it can never be the complete truth. Another way to state the dictum is to observe that the oath which binds a witness to tell 'the whole truth' forces him into perjury, although he can, if he wants to, tell the court 'nothing but the truth'. Russell proceeds to argue that Hegel and the enemies of analysis cannot account for the *beginning* of knowledge, since knowledge must begin with a single sentence, and continue through a sequence of single sentences. (?, 108-9)

sub-propositions – for example, having

$$\Gamma, \Delta \vdash A \wedge B$$

branch into

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}$$

– we have each locus branch into its *subloci*. But since we began with bare loci, without any propositional structure imposed on them in advance, there is nothing written at the root of a design that preordains *how* the design will decompose or ramify. The process of ramification may indeed carry on indefinitely, in an infinite analysis of the initial loci – actions and anticipations – that never encounters a single *atom*.

The headquarters of the People Programmer[26] *was at An-Fang and there the mistake happened. A ruby trembled…* Since there are no atoms to be found in designs, there are no *axioms* either. A kernel of what we could call *axiomaticity* nevertheless persists. It persists in the form of a pure, abstract arbitrariness, as pure contingency, under the name of the *daimon*.

The *daimon* is the rule of ludics that allows any locus whatsoever to be inscribed without justification, without being subject to further inquiries from the 'opponent'. It is the only action by which the branching of a design may be brought to a halt. Formally, it is written

$$\frac{}{\vdash \xi_n}\ \maltese$$

It can be seen as the bare form of the axiom, purified of all content and every appeal to evidence. It is an interruption of thought, or a brute decision.

Allowing designs to have recourse to the *daimon* opens the field of logic considerably. Its intervention populates the arena not only with the locative skeletons of *proofs*, but also with the bones of 'paraproofs' – unfinished proofs and sophisms. These are what occupy the 'pivot' left vacant by functional proof theory (see **?**, 131, 139-145), and open logic to a complete duality between proofs and tests (every design has a counter-design). To give some illustration of the expressive capacities this affords the formalism, we could turn to Myriam Quatrini recent work, where she shows that most of the sophistical stratagems Schopenhauer catalogues in *The Art of Always Being Right* – stratagems which this arch-enemy of Hegel liked to call *dialectical* – can be transcribed into ludics with the *daimon*'s aid.

Two tourmaline nets failed to rectify the laser beam. A diamond noted the error. Both the error and the correction went into the general computer… The 'error' in question, of course, is the *daimon*, which we have just invoked. The 'general computer',

[26] The reference to 'the People Programmer' isn't at all irrelevant here – it hints, in fact, at a certain concern Girard has with the transcendental conditions for, and 'formatting' of, logical subjectivity – but it would take me too far afield to go down this path today, so we'll cut straight to the trembling ruby and get on with it.

on the same allegorical reading, is nothing other than ludics itself, or 'dialectics', as Girard suggests the formalism could have also been called. What's won from this deal with the devil is a complete arena in which designs can be 'tested' against one another, eliminating the practical need to shuttle between syntactic proofs, on the one hand, and semantic counter-models on the other. With the return of the sophist to the arena of logic, the logician no longer needs to hypostatise a transcendent world – an external model – against which to test syntactical operations.

Testing, here, is what Girard calls the *interaction* between designs. Interaction is a process analogous to *cut-elimination* in the sequent calculus. 'Cut' does not exist as a *rule* in ludics, but as a phenomenon built into its very geometry. There is a 'cut' between two designs when they share the same locus at their roots, on opposite sides of the pitchfork, as in figure 0.4. *Interaction* proceeds by a progressive elimination

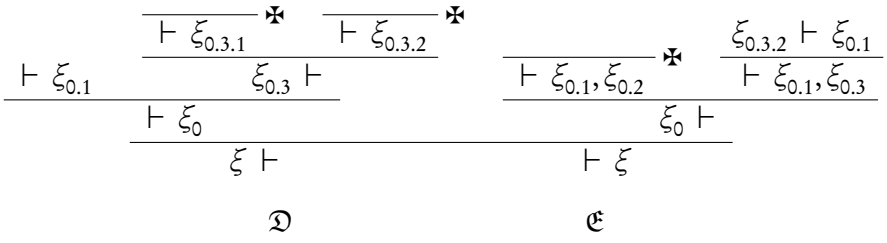


Figure 4: Poised for combat

of 'cuts' – the mutual deletion of shared loci on opposite sides of the pitchfork. The interaction poised to take place in figure0.4 results in the arrangement seen in figure0.4 after one round.

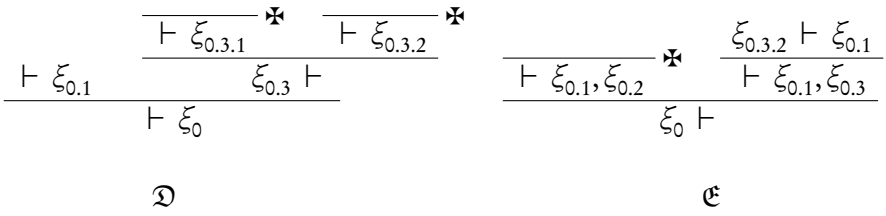


Figure 5: After the first round

The process continues so long as shared loci, oppositely positioned, link the two designs. The first design to invoke the *daimon* 'loses' – this is the price of axiomaticity in ludics. The victory, however, is not necessarily eternal, since the winning strategy must engage with further 'tests', against which it too would be forced to summon the daimon and admit defeat – the judges must also be judged. In a moment, we will see how this provides for a completely syntactical notion of falsity, and therefore truth, to be defined in ludics.

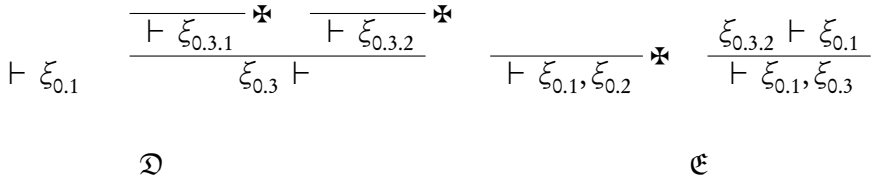


Figure 6: After the second round: 𝔈 invokes the *daimon*, and 𝔇 wins

0.5 Ludics and the dialectic of form

It is with 'Locus Solum', and the ludic formalism it advances, that Girard first makes explicit his project's proximity to dialectics – though with some hesitance. On the first page of the text, right after the epigraph, Girard directs the reader's attention to a lengthy (> 100 pp!) 'dictionary' appended to the monograph, which he calls 'a sort of final introduction, since one can only be introduced to known material. For instance,' he adds, 'if you go to DIALECTICS you will learn about the word ***ludics***' (**?**, 303). The entry itself reads as follows:

> DIALECTICS: Illustrated by Hegel, Marx, not to mention comrade Stalin…, this is a word with a history. For this reason, even if ludics has a lot to do with (the early) dialectics, I decided to create a new expression rather than having to assume the (mostly negative) consequences of the reuse of such a notorious word. By the way, this proximity of ludics to dialectics, which it would be unfair to deny, is not the result of, say, an attempt in the style of Engels' *Dialektik der Natur*, but only the result of a long familiarity with logic, and the contemplation of the symmetries of proof-theory. (**?**, 418)

Girard's hesitancy has two motives: (1) a distrust of the philosophical tradition of dialectical thought, and (2) an even greater distrust of what is called, today, *philosophical logic* – a field he routinely caricatures for its *ad hoc* and superficial attempts to subordinate and mould mathematical logic to the shape of a externally supplied subject matter, usually coming from metaphysics or common sense, inventing 'non-classical logics' tailored to fit the customer. For Girard, what demands formalisation is not something external to mathematical logic, but the leaks and blind spots already immanent to historically existing mathematical logic. It isn't difficult to sense an effectively Hegelian attitude at work in this, or to see how this attitude leads to an antipathy towards many attempts to 'formalise the dialectic' – such as those of Priest or Dubarle, for instance – an antipathy whose motives are perfectly congruent with those of the strictest Hegelian: a hatred of the *ad hoc*.

Nevertheless, the next ten years would dissolve Girard's resistance towards inscribing his research programme, and ludics in particular, in the space of the PFD, without giving way on the principles of his initial reluctance. In 2012's 'Tran-

scendental Syntax 2.0', Girard, without backpedalling, asserts that ludics is indeed enough to

> vindicate one of the greatest philosophers, Hegel, and his *contradictory foundations*. From a fregean viewpoint, founding A on its negation is stupid, since A and ¬A cannot coexist by consistency! The fact is that [...] ludics rests upon contradictory foundations without the slightest problem. The mistake about Hegel [*a mistake Girard takes Priest and Da Costa to have made*] is to take negation in the *alethic* (i.e., -1) sense, for which there cannot be any honest contradictory foundations; there are dishonest ones, e.g., paraconsistent "logic", making Hegel a sibling of Bernard Madoff... a real insult for such a great philosopher! (?, 6)

The argument remains aphoristic, and ends here. What I want to do now is to show that it can be developed in detail, that there is indeed, in ludics, a concept of the (sub-alethic) contradictory foundations, and of the contradictory foundations of formality, in particular. This concept of formality founded through contradiction – and not imposed from above, in order to avoid contradiction – is, in its contours, essentially Hegelian, even as it is developed without any explicit effort to mimic Hegel's thought (with which Girard seems only passingly familiar).

Though the motif of contradictory foundations runs throughout Hegel's speculative and dialectical thought, its most explicit thematisation is to be found in Book II of *The Science of Logic*, under the heading of *The Doctrine of Essence*. There, the concept of contradiction serves as a bridge between the study of the determinations of reflection (or 'essentialities'), and that of the category of *ground*, in which the concept of *form*, itself, finds its place. An analysis of ludics will show that this concept of form can indeed be related back to the notion of form which, earlier, I said is sublated through the historical process of *formalisation* – a notion of form that the radical formalist Haskell Curry showed is contingent on a confinement of the calculative by means of *types*. Recall that the *a priori* imposition of typing apparatus on the calculative represents the traditionally *Kantian* response to the discovery of mathematical antinomies, as alethic contradictions. What we will find in ludics is an essentially dialectical approach to the immanent genesis of types – or *forms* – through a free reflection on *non*-alethic contradiction.

*
* *

The semantic, or, more precisely, *alethic* vision of logic submits fully to the image of formality that Hegel decries from the outset of the *Logic* onwards – even if, as we have seen, it never ceases to provide the material required for a sublation of this image[27] The ideological arrangement of this image in terms of a syntactical 'appearance' finding purchase in a semantic 'reality', may, however, lead us to expect the alethic vision to be susceptible to the same dialectic of appearance and reality

[27] As Badiou, for example, shows in *The Concept of Model* (?)

that drives the movement of the Hegelian Doctrine of Essence – unveiling its deeper rational structure not in the badly infinite ascent of metalanguages piled on metalanguages in pursuit of a fugitive phantom of truth, but in a purely negative game of mirrors that immanently unfolds on a single and univocal plane.

This is what takes place in Girard's descent to the 'fourth inferno' of transcendental syntax – an inferno that, in fact, is to be found in the 'well-hidden geometry' of the syntactic surface itself (?, 1). A proof – grasped, at first, as the tree-like object that the sequent calculus formalisations of logic make of it – is stripped of all imaginary decoration, all the impenetrable atoms that the realist (Russell, for example) takes to promise a metaphysically atomistic interpretation in the real, and the entire apparatus of pre-formulated rules that refer the question of *how* we must proceed to the authority of presupposed forms. What results is what Girard calls a *design*, or a strategy, determined in a purely locative fashion. The strategy is not tested against a *model*, a (typically infinite) construction brought into a more or less external relation with the syntactic proof-structure and susceptible to being (ideologically) understood as the *given* reality against which the proof, or a collection of proofs, is measured, but against *other strategies*, without any their correctness or erroneousness being determined, beforehand, in any fashion. The basic unit of these strategies is not, any longer, a formula or proposition, but a positive or negative *locus*, an empty position answering to – or making a demand upon – its polar counterpart.

The relation between a positive locus and its negative counterparts (or vice versa) is not a static or extrinsic relation, but, immediately, a process of interaction that proceeds like a generalised form of the *cut-elimination algorithm* discovered by Gentzen. This process is an effective and concrete process of *contradiction*, but a form of contradiction that is deeper than and prior to any determination of truth-values – anterior to any fixed support for alethic reference to reality. When this takes place, one of three things may happen:

1. The conflict between the strategies goes on endlessly, in the sort of badly infinite progression that Hegel took to be the symptom of an unresolved contradiction. In this case, Hegel's diagnosis is apt: the situations in which interaction *diverges* are those in which the two designs are locked in a form of abstract 'dissensus'. It is a form of antagonism on which no propositions can be grounded, a logical phenomenon that has no counterpart on the alethic plane. The name that Girard gives to this outcome, or lack of an outcome, is $\mathfrak{Fid}$ or *fidelity*.

2. The conflict *converges* on a particular design, usually different from the two that entered the interaction. Interaction can therefore be used as a way of transforming one design into another, in a concrete and effective fashion. The most methodologically important of these transformations is a procedure called *faxing*, which serves to delocate or transpose a design onto a different locative base – performing the material labour underlying abstract or

'spiritual' conceptions of proof structures, mediated by a particular class of designs called $\mathfrak{Fax}$. If, for example, we want to transpose a design $\mathfrak{D}$ with base $\vdash \xi$ onto base $\vdash \rho$, we have it interact with a $\mathfrak{Fax}$ design of base $\xi \vdash \rho$, a process which converges on a design isomorphic (structurally identical) to $\mathfrak{D}$ but rooted in $\vdash \rho$.

3. The third possible outcome is that, step by step, the opposed loci cancel each other out entirely, collapsing the two trees into a design called $\mathfrak{Dai}$

$$\frac{}{\vdash}\ \maltese$$

The empty ground into which the contradiction founders is, here, a pure void, the empty pitchfork (sequent) summoned by the *daimon*. When this takes place, the two strategies – call them $\mathfrak{X}$ and $\mathfrak{Y}$ – are said to be *orthogonal* to one another (written $\mathfrak{X} \perp \mathfrak{Y}$).

Understood as structural invariance, form owes its existence to the mobility of $\mathfrak{Fax}$.[28] But it is from the third sort of outcome – the foundering of contradiction into the void, $\mathfrak{Dai}$ – that *form*, as typing, is reconstituted. To get this, we first define the *orthogonal* of a set $\mathscr{S}$ of designs as

$$\mathscr{S}^{\perp} := \{\mathfrak{D} \mid \exists \mathfrak{X} \in \mathscr{S} \,.\, \mathfrak{D} \perp \mathfrak{X}\}$$

A form, as type, proposition, or, in the terminology of ludics, as 'behaviour', is a set of designs that is *equal to its biorthogonal*:

$$\mathscr{S} \text{ is a type, iff } \mathscr{S} = (\mathscr{S}^{\perp})^{\perp}$$

If the reflective play of negativity between designs describes a logic of reflection, or essence, then by taking the pivot of form's constitution as $\mathfrak{Dai}$, we are taking as the *ground* of form the moment in which, in Hegel's words, 'the positive and the negative constitute the essential determination in which essence is lost in its negation,' such that 'these self-subsisting determinations of reflection' – the designs – 'sublate themselves,' cancel each other out through a process of interaction or elimination, 'and the determination that has foundered to the ground is the true determination of essence' (?, 386). In contrast to $\mathfrak{Fid}$, the bad infinity of unresolved contradiction, $\mathfrak{Dai}$ is the 'resolved contradiction' into which finite determinations – designs orthogonal to one another, and therefore punctuated by the mark of finitude, which is the *daimon* – 'founder', or fall to ground, to use Hegel's own pun. $\mathfrak{Dai}$ is, relative to essence, an 'absolute' which *is*, 'because the finite is the immanently self-contradictory opposition, because it *is not*' – it receives its logical determination only in the case of its orthogonality with what destroys it.

[28] We could parrot Hegel here to describe this process as one of 'adding to abstract identity the extra factor of that movement' whose 'hidden necessity' is inscribed in the form of the proposition (?, 360), to recall Hegel's reflections on the law of identity in the Doctrine of Essence.

Formality, to put it another way, now appears as the interaction of bare negativities – the opposition of polarised designs, which stripped of all logical decoration, presuppose no distinction between form and content – under the quotient of their collapse into ground ($\mathfrak{Dai}$) or as invariant under bi-orthogonality.

This is where we find the buttresses of the alethic inferno in the deontic, the supports on which *propositional forms*, capable of bearing *truth*, will rest. A behaviour, or interactive type, is, in effect, the ludic embodiment of the logical *proposition*. Its equality with its biorthogonal captures a notion of completeness that owes nothing to an external or semantic reference to a model. To understand this, we can think of a *complete* set of proofs $\mathscr{D}$ as one that is turns out to be valid when measured against all potential counter-examples, or counter-models. Ludics fulfills this office with the set of counter-proofs or 'tests', $\mathscr{D}^{\perp}$.[29] But $\mathscr{D}$ measures up to these tests if it is orthogonal to them, if it meets them head-on, and hence if it is equal to the orthogonal of its orthogonal, $\mathscr{D} = \mathscr{D}^{\perp\perp}$. A propositional form, type or behaviour is 'true' when it contains a design that 'wins' against all challengers – all of its orthogonal counter-designs. The technical conditions for truth, in ludics, are somewhat subtle, but broadly speaking, they boil down to two essential properties: to be victorious, a design must be (1) 'exact and parsimonious' – these are delicate conditions, but effectively they mean that the victorious design must move in a way that closely anticipates its opponents, without too many unnecessary moves, and (2) 'stubborn'; the winning design must not give up, must not invoke the *daimon*, which is nevertheless the quilting point of all logical determinacy. If neither party in a dispute plays the *daimon*, no logical form will crystalise: the cut-net will diverge, and be lost to $\mathfrak{Fid}$. Truth owes its existence to the errors of its opponents, without which it is nothing more than faith.

*
* *

Contradiction, on the deontic plane – the plane of recusals rather than refutations – is orthogonality, and the foundering of the contradictory to ground is the convergence of orthogonal designs on $\mathfrak{Dai}$. This is, in any case, how I am attempting to map the dialectic of essence onto the dynamics of ludics, in a way that works out and satisfies, I think, suggestions which, in Girard's writings, remain on the level of aphorism. $\mathfrak{Dai}$ is the ground into which essence founders, the *result* of essential (deontic and locative) contradiction. *Logical form*, or *types*, emerge in ludics on the basis of the symmetry that this empty ground induces between designs: it is identity in biorthogonality, and nothing more. In dramatic contrast to the way in

[29] A technical aside: if we have a purported proof of a sequent $\vdash \Delta$ in the sequent calculus, then its counter-model, if one exists, can be obtained by attempting to prove $\Delta \vdash$ or, equivalently, $\vdash \neg\Delta = \{\neg A : A \in \Delta\}$, and then building a model out of the subformula unfolded in that 'counter-proof'. Now, the locative infrastructure of these two structures – the attempted proofs of $\vdash \Delta$ and $\Delta \vdash$ – will be orthogonal, in the sense described above. The pairing of a design with its orthogonal design is thus a way of bringing the duality between syntactic proofs and semantic counterexampels onto a single, homogeneous plane.

which types are *applied*, transcendently, to the pure lambda calculus – 'such that,' as Jean-Baptiste Joinet writes, 'typing would appear as an apparatus of control, a disciplinary action taken on an otherwise vagabond dynamics' (**?**, 44), here, in ludics, typing answers to Hegel's own description of *form* as

> absolute negativity itself or the negative absolute self-identity by virtue of which essence is indeed not being but essence. [...] One cannot therefore ask, *how form comes to essence*, for form is only the internal reflective shining of essence, its own reflection inhabiting it. (**?**, 391)[30]

[30]Jean-Baptiste Joinet sums the situation up well when he recounts how 'The vision, which could be called 'cybernetic', of logic as external, *transcendent* control, imposing particular properties on a pre-existing interaction (*anterior* to that control), that was methodologically useful insofar as it came to disengage (in virtue of having 'enlarged, displaced and refined' the point of view on the dynamics, and in virtue of having recovered symmetries, isomorphisms and quotients in the space of proofs) a fruitful and general frameowrk for taking up those processes *in abstracto*, later found itself replaced or at least ompleted by a point of view that may be contrasted with it by being qualified as *immanentist*, a point of view according to which the regulative norm of the dynamics, *logic*, emerges directly from the interaction as a solution to the problem of *internal completeness*' (**?**, 50).

AXIOM AND CUT

$$[\text{AXIOM}]\ \frac{}{A \vdash A} \qquad \frac{\Gamma_1 \vdash C, \Delta_1 \qquad \Gamma_2, C \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ [\text{CUT}]$$

OPERATIONAL RULES

$$[\text{L}\neg]\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, \vdash \neg A, \Delta}\ [\text{R}\neg]$$

$$[\text{L}\wedge]\ \frac{\Gamma, A_{i\in(1,2)} \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta} \qquad \frac{\Gamma_1 \vdash A_1, \Delta \qquad \Gamma_2 \vdash A_2, \Delta}{\Gamma_1, \Gamma_2 \vdash A_1 \wedge A_2, \Delta}\ [\text{R}\wedge]$$

$$[\text{L}\vee]\ \frac{\Gamma_1, A_1 \vdash \Delta \qquad \Gamma_2, A_2 \vdash \Delta}{\Gamma_1, \Gamma_2, A_1 \vee A_2 \vdash \Delta} \qquad \frac{\Gamma \vdash A_{i\in(1,2)}}{\Gamma \vdash A_1 \vee A_2, \Delta}\ [\text{R}\vee]$$

$$[\text{L}\rightarrow]\ \frac{\Gamma_1 \vdash A_1, \Delta \qquad \Gamma_2, A_2 \vdash \Delta}{\Gamma_1, \Gamma_2, A_1 \rightarrow A_2 \vdash \Delta} \qquad \frac{\Gamma, A_1 \vdash A_2, \Delta}{\Gamma \vdash A_1 \rightarrow A_2}\ [\text{R}\rightarrow]$$

$$[\text{L}\forall]\ \frac{\Gamma, A[x/t] \vdash \Delta}{\Gamma, \forall x A[x] \vdash \Delta} \qquad (y \text{ not free in } \Gamma, \Delta)\ \frac{\Gamma \vdash A[x/y], \Delta}{\Gamma \vdash \forall x A[x], \Delta}\ [\text{R}\forall]$$

$$[\text{L}\exists]\ \frac{\Gamma, A[x/y] \vdash \Delta}{\Gamma, \exists x A[x] \vdash \Delta}\ (y \text{ not free in } \Gamma, \Delta) \qquad \frac{\Gamma \vdash A[x/t], \Delta}{\Gamma \vdash \exists x A[x], \Delta}\ [\text{R}\exists]$$

STRUCTURAL RULES

$$\text{LEFT WEAKENING [LW]}\ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \text{RIGHT WEAKENING [RW]}$$

$$\text{LEFT CONTRACTION [LC]}\ \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \text{RIGHT CONTRACTION [RC]}$$

$$\text{LEFT EXCHANGE [LE]}\ \frac{\Gamma \vdash \Delta}{\sigma\Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \sigma\Delta}\ \text{RIGHT EXCHANGE [LE]}$$

$\sigma\Phi$ is a permutation of Φ

Figure 7: The rules of the classical sequent calculus

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A}\,[\text{Identity}]}{A \vdash A \vee \neg A}\,[\text{R}\vee]}{\vdash (A \vee \neg A)\neg A}\,[\text{R}\neg]}{\vdash (A \vee \neg A)(A \vee \neg A)}\,[\text{R}\vee]}{\vdash A \vee \neg A}\,[\text{Right Contraction}]$$

Figure 8: Classical sequent proof of the law of the excluded middle