U N I V E R S A L C O A L G E B R A S

by

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Abstract

We investigate the category $\underline{C}$ of coalgebra objects in a given locally-presentable closed monoidal category $\underline{X}$. We show that $\underline{C}$ is the cartesian closure of $\underline{X}$ and that all other categories propable over $\underline{X}$ admit natural enriched HOM functors into $\underline{C}$. These HOM functors are studied and applications are given to the theories of algebraic deformations and extensions. Included are results, necessary to our work, concerning pure subobjects in $\underline{X}$.

Résumé

Cet ouvrage traite la catégorie $\underline{C}$ des objects co-algébriques dans une catégorie donnée $\underline{X}$, localement-présentable, monoidale et close. Nous montrons que la catégorie $\underline{C}$ est la clôture cartésienne de la catégorie $\underline{X}$ et que toute autre catégorie propable sur $\underline{X}$ admet un foncteur HOM naturel et enrichi dans $\underline{C}$. Nous étudions ces foncteurs HOM et en donnons des applications aux théories des extensions et déformations algébriques. Inclus dans notre travail sont ces résultats concernant les objets purs dans la catégorie $\underline{X}$ qui sont necessaires à nos recherches.

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Introduction

Coalgebras over a field k occur frequently in the study of algebraic topology and homological algebra, and have been studied in great detail by many authors. However, most of this effort has gone into the structure theory for coalgebras and Hopf-algebras, while the category of coalgebras per se has been neglected. This in itself would be of little note, except that the category $\underline{C}$ of k-coalgebras is the cartesian closure of Mod-k. Moreover, $\underline{C}$ turns out to be a category over which all categories of "universal algebras" are enriched. This may be gleaned from the work of Sweedler by generalizing his notion of a "measuring", [19]. We show that these results also hold when considering categories defined over a given monoidal base category, e.g. a category of modules or banach spaces, and we give applications to cohomology and deformation theory.

In chapter I we discuss the standard notion of a monoidal category $\underline{X}$ and introduce "monoidally-defined" categories over $\underline{X}$, an extension of Mac Lane's propable categories which include the usual categories of $\underline{X}$-algebras, bialgebras, modules, etc. We define the category $\underline{C}$ of $\underline{X}$-coalgebras and show $\underline{C}$ is the cartesianization of $\underline{X}$.

In chapter II we assume $\underline{X}$ is closed and locally-presentable and show that every subobject in $\underline{X}$ can be embedded in a not-too-much-bigger pure subobject. This technical result allows to prove that every coalgebraic-type category over $\underline{X}$ (including $\underline{C}$) is again locally-presentable. The adjoint functor theorems then imply that $\underline{C}$ is cartesian closed and that cofree coalgebras exist.

Given a category of algebras $\underline{A}$ over a monoidal base category $\underline{X}$, e.g. k-algebras over Mod-k, $\underline{A}$ usually does not admit an $\underline{X}$-valued hom functor, even though $\underline{X}$ may be closed. In chapter III we show that there is a $\underline{C}$-valued HOM functor from any monoidally-defined category over $\underline{X}$. Since $\underline{C}$ is cartesian closed it forms very natural "universal" base category, and the basic properties of the usual $\underline{Set}$-valued hom lift to the richer $\underline{C}$-valued HOM, e.g. most important functors and adjointness relations may be defined in terms of HOM-coalgebras rather than hom-sets.

Given an element X in $\underline{X}$ and a category of "algebras" $\underline{A}$ over $\underline{X}$, one may construct a coalgebra whose "points" exactly correspond to $\underline{A}$-structures on X. In chapter IV we show how this observation and cotriple cohomology may be used in clarifying and generalizing the basic results of Gerstenhaber on the theory of deformations, [11]. Basically, the deformations

can be realized as special elements in the constructed coalgebra. The connection of this with (non-singular) extension theory is also discussed.

In order to increase the readability of this thesis certain easy but gruesome details have been banished to the appendix. There, will be found the proof of each numbered result given without proof in the text. Also in the appendix are notes on chapters III and IV.

About notation - maps are composed from left to right, 1 is used to denote identity maps, and we often omit mention of forgetful functors.

Most of the results and definitions in chapters I-III are well known in the case when $\underline{X}$ is the category of vector spaces over a field. The general definition of an evaluation is due to Jon Beck and the author. Lemmas 1.1 and 2.5 are well known and 3.6 first appeared in [16]. With these exceptions the results here stated are original research.

Acknowledgement

I wish to thank Jon Beck for pointing out the categorical nature of coalgebra theory and suggesting that there should be applications to deformation theory; this work is the direct result of his stimulating presence at McGill. I thank Rick Ganong, Joachim Lambek, and Radu Diaconescu for commenting on an early draft, and Rachel Boller for deciphering and typing the final manuscript. Most of all, I thank Michael Barr for his good-natured and very patient guidance.

Chapter 1: Preliminaries

By a monoidal category we mean a category $\underline{X}$ equipped with a bifunctor, tensor, $\otimes : \underline{X}\ \underline{X} \longrightarrow \underline{X}$. It is assumed that $\otimes$ is coherently symmetric, associative, and unitary, with unit object I. If X is an object in $\underline{X}$, X^n denotes the n-fold tensor of X with itself and $X^0 = I$. If $\otimes$ is the product in $\underline{X}$, we say $\underline{X}$ is cartesian.

If $\underline{X}$ and $\underline{Y}$ are monoidal categories a functor $F : \underline{X} \longrightarrow \underline{Y}$ is said to be monoidal if the tensor product is preserved up to coherent natural isomorphism. F is a strict-monoidal functor if all the monoidal structure is preserved exactly — $\otimes$, I, and the symmetry, associative, and unit natural isomorphisms. Let $\underline{\mathrm{MON}}$ denote the category of monoidal categories and strict monoidal functors.

The trivial prop is the free monoidal category (i.e. $\underline{\mathrm{MON}}$-object) over the category $\underline{1}$; it is denoted by $\underline{\mathbb{P}}$. Its objects are those of the free non-associative monoid generated by a single element. As there is a bijection between the skeleton of $\underline{\mathbb{P}}$ and $\mathbb{N}$, the natural numbers, we write the tensor in $\mathbb{P}$ as $+$, use 0 to denote the unit object, and use natural numbers to denote the objects of $\underline{\mathbb{P}}$ (whenever it is unambiguous to do so). Note that this is not the usual definition, as in [18].

A prop $\underline{P}$ is a small monoidal category with a strict monoidal functor $\underline{\mathbb{P}} \longrightarrow \underline{P}$ which is a bijection on objects. $\underline{P}$ is said to be algebraic (resp. coalgebraic) if its set of maps is generated by a set of maps having codomain 1 (resp. domain 1). If σ is a map in $\underline{P}$, we let σd and σr denote its domain and range respectively (as elements of $\mathbb{N}$).

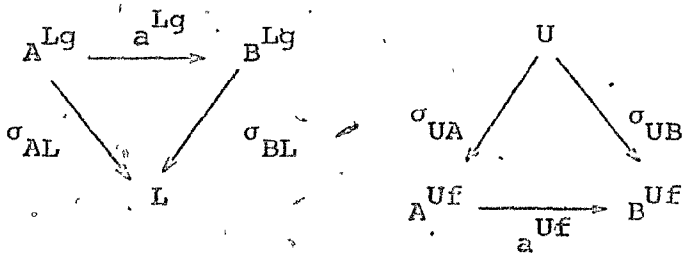
If $\underline{P}$ is a prop and $\underline{X}$ a monoidal category, let $\underline{X}^{\underline{P}}$ denote the class $\underline{MON}(\underline{P},\underline{X})$. $\underline{X}^{\underline{P}}$ is a category in a natural way and comes equipped with the usual forgetful functor to $\underline{X}$, sending A in $\underline{X}^{\underline{P}}$ to the object $(1)A$ in $\underline{X}$. For A in $\underline{X}^{\underline{P}}$, we also denote the object $(1)A$ by A. If σ is a map in $\underline{P}$, we often write σ in place of σA. A map $A \xrightarrow{a} B$ in $\underline{X}^{\underline{P}}$ is an $\underline{X}$-map such that for all maps σ in $\underline{P}$ the following diagram commutes:

(1)
$$\begin{array}{ccc} A^{\sigma d} & \xrightarrow{a^{\sigma d}} & B^{\sigma d} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ A^{\sigma r} & \xrightarrow{a^{\sigma r}} & B^{\sigma r} \end{array}$$

A category of the form $\underline{X}^{\underline{P}}$ is said to be propable over $\underline{X}$. It is called algebraic or coalgebraic if $\underline{P}$ is. If $\underline{A} = \underline{X}^{\underline{P}^{\circ}}$ and $A \in \underline{A}$, $\{\sigma A : \sigma \text{ is a map in } \underline{P}\}$ is the set of $\underline{A}$-structure maps on A in $\underline{X}$.

We wish to extend these notions to include comma-categories and their relatives. A monoidally-defined category $\underline{B}$ over $\underline{X}$ is given by a five-tuple $(\underline{A},\underline{U},\underline{L},f,g)$ where $\underline{A}$ is a propable category over $\underline{X}$, $\underline{U}$ and $\underline{L}$ are sets of objects in $\underline{X}$, and $\underline{U} \xrightarrow{f} \mathbb{N}$ and $\underline{L} \xrightarrow{g} \mathbb{N}$ are functions. $\underline{B}$ is then the subcategory of $\underline{A}$ consisting of objects A equipped with additional structure maps $(U \xrightarrow{\sigma_{UA}} A^{Uf})_{U \in \underline{U}}$ and $(A^{Lg} \xrightarrow{\sigma_{AL}} L)_{L \in \underline{L}}$. An $\underline{A}$-map ?

$A \xrightarrow{a} B$ between objects in $\underline{B}$ is a $\underline{B}$-map if the following diagrams commute for all $L \in \underline{L}$ and $U \in \underline{U}$:



Finally, by a decent category over $\underline{X}$ we mean a full subcategory of a monoidally-defined category over $\underline{X}$.

To reduce the number of special cases we must consider, for each element X of $\underline{X}$ denote by $(\ )^X$ the functor $\underline{X} \to \underline{X}$ with constant value X. If $\underline{B}$ is monoidally-defined over $\underline{X}$ and $A^n \xrightarrow{\sigma} L$ (resp. $U \xrightarrow{\sigma} A^m$) is a $\underline{B}$ structure map, define $\sigma d = n$ and $\sigma r = L$ (resp. $\sigma d = U$ and $\sigma r = m$). Then a $\underline{B}$ map $A \xrightarrow{a} B$ is an $\underline{X}$-map such that for all structure maps σ diagram (1) above commutes.

Example 1: Let $\underline{X} = \underline{Sets}$ with $\otimes$ the cartesian product and I the one element set. Propable categories over $\underline{X}$ include all of Lawvere's finitary algebraic categories-monoids, groups, rings, etc. If $\underline{A}$ is algebraic and $A \in \underline{A}$, then the comma-categories $(A,\underline{A})$ and $(\underline{A},A)$ are decent categories.

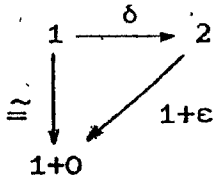
Example 2: Let $\underline{X} = \text{Mod-}R$ for some commutative ring R. Of course $\otimes = \otimes_R$ and $I = R$. The categories of not necessarily associative R-algebras (Alg-R), R-bialgebras, R-coalgebras,

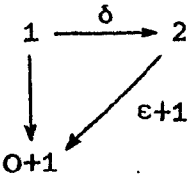
associative R-algebras, etc., are obvious propable categories over $\underline{X}$. R-lie algebras are a decent full subcategory of Alg-R . If $\underline{P}$ is the free prop generated by a single map $2 \to 0$, $\underline{A} = (\text{Mod-R})^{\underline{P}}$ is just the category of R-modules equipped with a bilinear map to R . Inner-product spaces then form a decent full subcategory of $\underline{A}$.

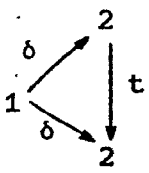
Example 3: Let $\underline{X}$ be the category of banach spaces with norm decreasing maps, $\underline{\text{Ban}}_1$, and the projective tensor product. Banach algebras form a propable category over $\underline{X}$, while spaces with real- or complex-valued inner products are decent.

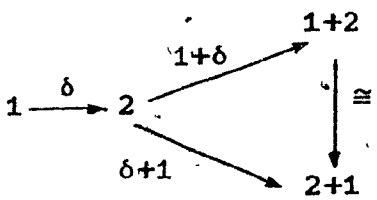
Of course there are many other examples (see [4] or [7]). Of particular interest to us will be the following two propable categories over an arbitrary monoidal $\underline{X}$.

Example-Definition: Let $\underline{P}$ be the prop generated by maps $\delta: 1 \to 2$ and $\varepsilon : 1 \to 0$, and the diagrams below, where t is the "twist" isomorphism in $\underline{P}$ and the indicated isomorphisms are the obvious ones.









Then $\underline{X}^P$ is the category of (counitary, coabelian, coassociative) coalgebras in $\underline{X}$, denoted $\underline{C}(\underline{X})$.

Let $\underline{H}(\underline{X})$ be the full subcategory of $\underline{C}(\underline{X})$ whose elements H have addition structure maps $H^2 \xrightarrow{m} H$, $H \xrightarrow{s} H$, and $I \xrightarrow{u} H$ satisfying $\varepsilon u = \delta(1\otimes s)\cdot m = \delta(s\otimes 1)\cdot m : H \to H$, as well as diagrams (equations) making m an associative, commutative, multiplication with unit u. $\underline{H}(\underline{X})$ is the category of Hopf objects in $\underline{X}$.

In the case where $\underline{X}$ is the category of vector spaces over a field the above categories have been studied in great detail (see Sweedler [19] for example). $\underline{C}(\underline{X})$ is then equivalent to the category of formal-schemes over $\underline{X}$, [5]. We drop the argument $\underline{X}$ whenever it is unambiguous to do so.

1. Lemma If C and D are elements of $\underline{C}$, then $C \otimes D$ has a natural $\underline{C}$-structure on it, making it the product of C and D in $\underline{C}$.

Note that if C is in $\underline{C}$, $C \xrightarrow{\delta} C^2$ is a coalgebra map, (the diagonal $C \to C \times C$ in $\underline{C}$), and I with its trivial coalgebra structure is the terminal object in $\underline{C}$. Thus $\underline{C}$ is a cartesian category and the objects of $\underline{H}$ are exactly the abelian group objects of $\underline{C}$. From [8] we have:

2. Proposition: Let CART be the category of cartesian categories and (strictly) product preserving functors. Then $\underline{C}(\ ) : \underline{MON} \to \underline{CART}$ is the right adjoint for the inclusion $\underline{CART} \to \underline{MON}$.

In chapters II and III we show that if $\underline{X}$ is locally-presentable and closed, then so is $\underline{C}(\underline{X})$. Thus, with the restriction of local-presentability, the above result holds for monoidal-closed and cartesian-closed categories, although one cannot require the functors involved to preserve the internal homs.

Chapter 2: Locally-presentable Categories and Coalgebraic Props

A diagram $\underline{D}$ is α-filtered, α a regular cardinal, if:

i) for every family $\{D_i\}_{i\in I}$ of objects in $\underline{D}$ such that $|I| < \alpha$, there is an object D in $\underline{D}$ and $\underline{D}$-maps $\{D_i \to D\}_{i\in I}$.

ii) for every family of maps $\{D \xrightarrow{d_i} D'\}_{i\in I}$ in $\underline{D}$ such that $|I| < \alpha$, there exists a map $D' \xrightarrow{d} D''$ in $\underline{D}$ such that $d_i d = d_j d$ for all i and j in I.

Given a functor $\underline{X} \to \underline{Y}$, if there exists a regular cardinal α such that F preserves α-filtered colimits, we say that F has rank, and rank $F \leq \alpha$. For an object X in $\underline{X}$, we write $\operatorname{rank}_{\underline{X}} X$ (or rank X) instead of rank $\underline{X}(X,-)$. A category $\underline{X}$ is α-<u>locally-presentable</u> if it is cocomplete and has a generating set $\underline{G}$ such that rank $G \leq \alpha$ for all G in $\underline{G}$. An object X in $\underline{X}$ is said to be β-<u>generated</u>, where β is a regular cardinal, if there is a proper epimorphism $\bigoplus_{i\in I} G_i \to X$, where $G_i \in \underline{G}$ and $|I| < \beta$.

Locally presentable categories have been studied extensively by Gabriel and Ulmer, [10], and by Barr, [1]. They include the categories <u>Sets</u>, <u>Mod-R</u>, $\underline{\text{Ban}}_1$, categories of sheaves, and many others. If $\underline{X}$ is α-locally-presentable, it is known that $\underline{X}$ is complete, that every element of $\underline{X}$ has rank, and that α-filtered colimits commute with α-limits, i.e. kernel pairs and products

with fewer than α factors. Also we may assume that every element of $\underline{X}$ is the colimit of an α-filtered diagram in $\underline{G}$.

Henceforth, let $\underline{X}$ be an α-locally-presentable monoidal category. We assume that $\underline{X}$ has images and that $X\otimes -$ and $-\otimes X$ preserve all colimits, for each $X \in \underline{X}$ (though see page 13).

1. Lemma: We may assume that $\underline{G}$ is closed under finite applications of the tensor.

Recall that a subobject $i : X \longrightarrow Y$ in $\underline{X}$ is pure if $i\otimes 1 : X\otimes Z \longrightarrow Y\otimes Z$ is a monomorphism for every Z in $\underline{X}$. We shall show that every "small" subobject is contained in a pure subobject that is not too much bigger. Using Th.II.3.2 in [1], we may choose a generating set $\underline{G}$ and a regular cardinal γ such that:

1) $\gamma > \alpha + |\underline{G}|$
2) Every object in $\underline{X}$ is the direct colimit of its γ-generated subobject
3) X in $\underline{X}$ is γ-generated if and only if $\underline{X}(X,-)$ preserves monomorphic γ-filtered colimits (see [10]).
4) X in $\underline{X}$ is γ-generated if and only if $\left|\bigcup_{G\in\underline{G}} \underline{X}(G,X)\right| < \gamma$.
5) Whenever X and Y are γ-generated, so are $X \times Y$ and $X\otimes Y$.

2. Lemma: Let X be a γ-generated subobject of Y in $\underline{X}$. For each $G \in \underline{G}$, there exists a 2^{γ}-generated subobject, $X(G)$, of Y containing X and such that the kernel pair of $i\otimes 1 : X \otimes G \longrightarrow Y \otimes G$ is a kernel pair for $i\otimes 1 : X \otimes G \longrightarrow X(G) \otimes G$, where i is the inclusion.

Proof: Let $\underline{U}_X$ (or just $\underline{U}$) denote the set of all subobjects of Y of the form $X \cup X'$ where X' is γ-generated. By the remarks preceeding the lemma we have $\varinjlim \underline{U} = Y$. Now let $K_G \rightrightarrows X \otimes G \xrightarrow{i\otimes 1} Y \otimes G$ be a kernel pair diagram, and let $K_U \rightrightarrows X \otimes G \xrightarrow{1\otimes 1} U \otimes G$ be a kernel pair for each $U \in \underline{U}$. Let $\underline{K}$ denote the set of all K_U, $U \in \underline{U}$. Since $\underline{U}$ is closed under unions of less than α elements (since $\alpha < \gamma$), $\underline{U}$ is an α-directed set and so is $\underline{K}$. Since $Y \otimes G = \varinjlim \underline{U} \otimes G$ and kernel pairs are α-limits, the canonical map $K_G \to \varinjlim \underline{K}$ is an isomorphism. By the choice of γ, both $X \otimes G$ and $(X \otimes G) \times (X \otimes G)$ are γ-generated. Since subobjects are determined by sets of maps from the generating set, again by the choice of γ there are fewer than 2^γ isomorphism classes of subobjects of $(X \otimes G) \times (X \otimes G)$, and hence fewer than 2^γ distinct members of $\underline{K}$. For each distinct $K \in \underline{K}$, choose an object $U \in \underline{U}$ such that $K \rightrightarrows X \otimes G \longrightarrow U \otimes G$ is a kernel pair and let $\underline{U}?$ denote the set of all chosen objects. Let $\underline{U}!$ be the set of all unions of fewer than α elements from $\underline{U}?$. Since $|\underline{U}?| < 2^\gamma > \alpha$, $\underline{U}?$ has fewer than 2^γ subsets of cardinality α. Thus $|\underline{U}!| < 2^\gamma$ and $\underline{U}!$ is α-directed. Defining $X(G) = \varinjlim \underline{U}!$, $X(G)$ is a subobject of Y and obviously contains X. Since $X(G)$ is the union of fewer than 2^γ γ-generated objects, it is 2^γ-generated. Finally, because $\underline{U}!$ is α-directed:

$$\ker(X \otimes G \to X(G) \otimes G) = \ker(X \otimes G \to \varinjlim \underline{U}! \otimes G) \cong$$

$$\varinjlim \ker(X \otimes G \to \underline{U}! \otimes G) = \varinjlim \underline{K} \cong K_G .$$

Iterating the above construction would yield a subobject of Y that is pure "for" the generator G, and subsequent diagonalization would yield a pure subobject. We perform both operations in the proof of:

3. Propositions: There exists a regular cardinal γ' such that each γ-generated subobject in $\underline{X}$ is contained in a γ'-generated pure subobject.

Proof: Let X be a γ-generated subobject of Y. Let $\underline{E}$ denote the set of (ordered) families of elements of $\underline{G}$ with fewer than α terms. Define $X(E)$ for $E \in \underline{E}$ by ordered application of the process $(-)(G)$ for $G \in E$.

More precisely, let $\underline{E} = \bigcup_{|\lambda|<\alpha} \underline{G}^{\lambda}$, the union taken over all ordinals λ whose cardinality is less than α. If $E \in \underline{G}^{\lambda}$, let $E|\mu$ denote the function E restricted to $\mu < \lambda$, and define $X(E)$ as follows:

$$X(E) = \begin{cases} (X(E|\lambda-1))(\lambda-1)E & \text{if } \lambda \text{ is a non-limit ordinal} \\ \bigcup_{\mu<\lambda} X(E|\mu) & \text{if } \lambda \text{ is a limit ordinal} \end{cases}$$

Let $X(\underline{E})$ denote $\{X(E) : E \in \underline{E}\}$. $X(\underline{E})$ is α-directed by lemma 4 below, so defining $X' = \varinjlim X(\underline{E})$ yields a subobject of Y containing X. Let γ' be such that X' is γ'-generated. Note that by lemma 2 above, γ' does not depend on X, or on X', but only on γ.

We show that X' is pure in Y. Given $G \in \underline{G}$ let $K' \rightrightarrows X' \otimes G \to Y \otimes G$ be a kernel pair diagram. $X' \otimes G = \varinjlim(X(\underline{E}) \otimes G)$

and since $X(\underline{E})$ is α-directed, $K' = \varinjlim K(\underline{E})$, where $K(E) \rightrightarrows X(E) \otimes G \to Y \otimes G$ is a kernel pair for each $E \in \underline{E}$. Let $X(EG) = (X(E))G$, obviously in $\underline{E}$. We have a diagram

$$\begin{array}{ccccc} K' & \rightrightarrows & X' \otimes G & & \\ \uparrow & & \uparrow & \searrow & \\ K(EG) & \rightrightarrows & X(EG) \otimes G & \longrightarrow & Y \otimes G \\ \uparrow & & \uparrow & \nearrow & \\ K(E) & \rightrightarrows & X(E) \otimes G & & \end{array}$$

The diagram $K(E) \rightrightarrows X(E) \otimes G \longrightarrow X(EG) \otimes G$ commutes by lemma 2 above, so $K(E) \longrightarrow K' \rightrightarrows X' \otimes G$ commutes. Since $K' = \varinjlim(K(\underline{E})$, $K' \rightrightarrows X' \otimes G$ commutes, and $X' \otimes G \to Y \otimes G$ is a monomorphism for every $G \in \underline{G}$.

Given $Z \in \underline{X}$, $Z = \varinjlim_{I} G_i$ for some α-filtered family $\{G_i\}_{i \in I} \subset \underline{G}$. Thus $X' \otimes Z \to Y \otimes Z$ is a monomorphism, being $\varinjlim_{I}(X' \otimes G_i \to Y \otimes G_i)$.

4. Lemma $X(\underline{E})$ is α-directed.

Proof: Let $\{E_i\}_{i \in I}$ be a subset of $\underline{E}$ with $|I| < \alpha$. If $E_i \in \underline{G}^{\lambda_i}$, let λ be a concatenate of the λ_i for all $i \in I$ and define $E \in \underline{G}^{\lambda}$ by $E|\lambda_i = E_i$. Clearly $XE_i \subset XE$, and $|\lambda| = \text{sum } |\lambda_i| < \alpha$ because α is regular.

Let $\underline{A}$ be a coalgebraic propable category over $\underline{X}$. Each object A in $\underline{A}$ has a generating set of structure maps of the form $A \xrightarrow{\sigma A} A^{\sigma r}$; let $[A]$ denote the set of these structure maps. As in the ease $\underline{A} = \underline{C}(\underline{X})$, $A \otimes B$ has a natural $\underline{A}$-structure defined by:

$$A \otimes B \xrightarrow{\sigma A \otimes \sigma B} A^{\sigma r} \otimes B^{\sigma r} \cong (A \otimes B)^{\sigma r}.$$

5. Lemma: The underlying functor $\underline{A} \to \underline{X}$ creates colimits. Hence $\underline{A}$ is cocomplete.

6. Lemma: Let A be in $\underline{A}$ and let $i: X \to A$ be a pure $\underline{X}$-subobject of A. Then X is an $\underline{A}$-subobject if and only if $i\sigma$ factors through $i^{\sigma r}$ for each $\sigma \in [A]$.

By choosing γ large enough, we may assume $\gamma > |[A]|$ (for any A in $\underline{A}$).

7. Lemma: Let A be in $\underline{A}$ and let $i: X \to A$ be a γ-generated $\underline{X}$-subobject of A. Then there exists a γ'-generated pure subobject of A, $\overline{X} \xrightarrow{j} A$, containing X and such that $i\sigma$ factors through $\overline{X}^{\sigma r} \xrightarrow{j^{\sigma r}} A^{\sigma r}$ for all $\sigma \in [A]$.

Proof: Let $\underline{U}$ be as in the proof of lemma 2. Then $\varinjlim \underline{U} = A$, so for any $n \in \mathbb{N}$, $A^n = \bigotimes_n \varinjlim \underline{U} = \varinjlim_{U_i \in \underline{U}} (U_1 \otimes \ldots \otimes U_n) = \varinjlim_{U \in \underline{U}} U^n$ since the diagonal is cofinal. Now for each $\sigma \in [A]$, $A^{\sigma r} = \varinjlim \mathrm{Im}(U^{\sigma r} \to A^{\sigma r})$ the latter being a monomorphic γ-filtered colimit. Because X is γ-generated, the map $X \xrightarrow{i\sigma} A^{\sigma r}$ must factor through $\mathrm{Im}(U_\sigma^{\sigma r} \to A^{\sigma r})$ for some $U_\sigma \in \underline{U}$. Letting $Y = \bigcup_{\sigma \in [A]} U_\sigma$ and $\overline{X} = Y'$, we have $\mathrm{Im}(i\sigma) \subseteq \mathrm{Im}(U^{\sigma r} \to A^{\sigma r}) \subseteq \mathrm{Im}(Y^{\sigma r} \to A^{\sigma r}) \subseteq \mathrm{Im}(\overline{X}^{\sigma r} \to A^{\sigma r}) = \overline{X}^{\sigma r}$.

Iterating the construction in lemma 7 it is now easy to prove the main result of this section.

8. Theorem: There exist a regular cardinal $\gamma(\underline{A})$ such that each γ'-generated $\underline{X}$-subobject of $A \in \underline{A}$ is contained in a $\gamma(A)$-generated pure $\underline{A}$-subobject of A. Hence $\underline{A}$ has a set of generators.

9. Corollary: $\underline{A}$ is locally-presentable.

10. Corollary: There exists a right adjoint to the forgetful functor $\underline{A} \to \underline{X}$.

The right adjoint is the cofree $\underline{A}$-functor. The cofree coalgebra functor, i.e. when $\underline{A} = \underline{C}(\underline{X})$, is denoted by K. An explicit construction for K may be found in 19 for the case $\underline{X} = \underline{\text{Mod-k}}$, k a field.

Note that 9 implies $\underline{A}$ is complete and cowellpowered. By modifying the argument for vector spaces in [5] or for R-modules in [2] we have:

11. Proposition: $\underline{A}$ is cotripleable over $\underline{X}$.

The condition that $\otimes$ preserve colimits can be done away with when considering strictly coalgebraic propable categories, i.e. those whose props are quotients of free coalgebraic props by a set of diagrams each having domain 1. For example, this holds for $\underline{A} = \underline{C}(\underline{X})$. In such a case one need only assume that for all G in $\underline{G}$ the functors $G \otimes -$ and $- \otimes G$ have rank $\leq \alpha$. The stronger condition is only needed to prove 2.5 in the general case.

Chapter 3: The Coalgebra Valued HOM

Let $\underline{X}$ be monoidal locally-presentable category. Further, assume that the tensor product preserves all colimits. Then by the special adjoint functor theorem (SAFT), the tensor has a right adjoint, i.e. $\underline{X}$ is monoidal closed. Then by 1.1, 2.5, and 2.8, $\underline{C}(\underline{X})$ is a cartesian-closed category. Given (C,δ,ε) in $\underline{C}$, define $\delta_0 = \varepsilon$, $\delta_1 = 1$, $\delta_2 = \delta$, $\delta_n = (1\otimes\delta_{n-1})\cdot\delta : C \to C^n$ for all $n > 0$, and $\delta_X = \varepsilon$ for all X in $\underline{X}$.

Now let $\underline{A}$ be a monoidally-defined category over $\underline{X}$. Given A and B in $\underline{A}$, and C in $\underline{C}$, an $\underline{X}$-map $A \otimes C \xrightarrow{e} B$ is called an evaluation, and we say C maps A to B, if the following diagram commutes for each structure map σ:

$$(1)\qquad \begin{array}{ccccc} A^{\sigma d}\otimes C & \xrightarrow{1\otimes\delta_{\sigma d}} & (A\otimes C)^{\sigma d} & \xrightarrow{e^{\sigma d}} & B^{\sigma d} \\ \downarrow{\scriptstyle \sigma\otimes 1} & & & & \downarrow{\scriptstyle \sigma} \\ A^{\sigma r}\otimes C & \xrightarrow{1\otimes\delta_{\sigma r}} & (A\otimes C)^{\sigma r} & \xrightarrow{e^{\sigma r}} & B^{\sigma r} \end{array}$$

If $A\otimes C \xrightarrow{e} B$ is an evaluation, every $\underline{X}$-map $I \xrightarrow{f} C$ induces (or represents) an $\underline{X}$-map from A to B, $A \xrightarrow{\sim} A\otimes I \xrightarrow{1\otimes f} A\otimes C \xrightarrow{e} B$.

1. Lemma: If f is a coalgebra map, the induced map is an $\underline{A}$-map.

Denote the functor $\underline{C}(I,-)$ by P. For each C in $\underline{C}$, $(C)P$ is the set of points of C. Given A and B in $\underline{A}$

it is easy to construct a coalgebra whose points exactly represent the $\underline{A}$-maps from A to B ; define $C = \bigoplus_{\underline{A}(A,B)} I$ in $\underline{C}$. The only coalgebra maps $I \to C$ are the injections, so (C) $P \cong \underline{A}(A,B)$, and we have an evaluation $A \otimes C \xrightarrow{e} B$ defined by $(1_A \otimes i_f)e = f$ for the f^{TH} injection i_f (cf. [19] p. 141). We wish to find a largest coalgebra evaluating A to B , i.e. a coalgebra (A,B) and an evaluation $A \otimes (A,B) \xrightarrow{e} B$ such that given any other evaluation $A \otimes C \to B$, there is a coalgebra map $C \to (A,B)$ preserving the evaluations. Keeping A and B fixed, define the functor $\underline{C} \xrightarrow{E} \underline{Sets}$ by $CE = \{e : A \otimes C \xrightarrow{e} B$ is an evaluation$\}$. Showing E preserves colimits, applying 2.9 and the SAFT we have:

2. Proposition: The functor E is representable.

Let (A,B) be the representing coalgebra thus constructed. It comes equipped with a canonical evaluation $A \otimes (A,B) \xrightarrow{e} B$ as desired, and $(A,B)P \cong \underline{A}(A,B)$, for an $\underline{A}$-map $A \to B$ is just an evaluation $A \otimes I \to B$ inducing a unique point $I \to (A,B)$. If we allow A and B to vary , $(-,-)$ defines a functor $HOM_{\underline{A}} : \underline{A}^{op} \times \underline{A} \to \underline{C}$, the coalgebra valued HOM (we often drop the subscript $\underline{A}$). Note that $HOM_{\underline{A}} \cdot P \cong Hom_{\underline{A}} : \underline{A}^{op}\ \underline{A} \longrightarrow \underline{Sets}$, while composition and the identity have obvious (associative and unitary) liftings to $\underline{C}$ (see page 19). Thus $HOM_{\underline{A}}$ gives an enrichment of $\underline{A}$ as a $\underline{C}$-category (see 7). Since, for any full subcategory $\underline{A}'$ of $\underline{A}$, it makes perfect sense to define $HOM_{\underline{A}'}$ to be $HOM_{\underline{A}}$ restricted to $\underline{A}'$, we have shown,

3. Theorem: Every decent category over $\underline{X}$ has an enrichment over $\underline{C}(\underline{X})$.

We shall consider those properties of the Set-valued hom that lift to HOM, for example hom-set isomorphisms defining adjoint functors, set based powers and copowers, etc. In particular we show most natural functors are definable as $\underline{C}$-functors.

Let $\underline{A}$ be enriched over $\underline{C}$ and let A be an object in $\underline{A}$. A left adjoint for the functor $(A,-) : \underline{A} \to \underline{C}$ is called the tensor at A, and is denoted by $(-) \otimes A$. Similarly, a left adjoint for the functor $(-,A) : \underline{A}^{op} \to \underline{C}$ is called the cotensor at A, and is denoted by $(-) \pitchfork A$. If the tensor at A (resp. the cotensor at A) exists for all A in $\underline{A}$, we say $\underline{A}$ has tensors (resp. has cotensors). In that case we have natural isomorphisms $\underline{C}(C,(A,B)) \cong \underline{A}(C \otimes A,B)$ and $\underline{C}(C,(A,B)) \cong \underline{A}^{op}(C \pitchfork B,A) = \underline{A}(A,C \pitchfork B)$ for all $C \in \underline{C}$ and $A,B \in \underline{A}$.

4. Proposition: If $\underline{A}$ is coalgebraic over $\underline{X}$ then $\underline{A}$ has tensors and cotensors.

5. Proposition: Let $\underline{A}$ be algebraic over $\underline{X}$. Then $\underline{A}$ has cotensors. If all maps in $\underline{A}$ admit regular-epi/mono factori-

zations, then $\underline{A}$ has tensors.

Let A be in $\underline{A}$ and C in $\underline{C}$. If $\underline{A}$ is coalgebraic, $C \otimes A$ is just $C \otimes A$ with appropriate structure maps, while the existence of $C \rightarrow A$ follows from 2.9 and the SAFT. If $\underline{A}$ is algebraic, $C \rightarrow A$ is the hom object $[C,A]$ in $\underline{X}$ with its structure as a convolution algebra; $C \otimes A$ is shown to exist using the existence of free $\underline{A}$-objects and the adjoint functor theorem. See Barr [2] for the details in the case $\underline{X} = \text{Mod-R}$. Note that the factorization condition in Prop. 4 holds in all our major examples. In any case, we are usually concerned with cotensors in algebraic categories, as in [7].

Sweedler has studied all these functors (in labourious detail) when $\underline{X}$ is the category of k-vector spaces over a field k and $\underline{A} = k\text{-Alg}$, [19]. There (A,B) is the "universal measuring coalgebra", $C \rightarrow k$ is the dual algebra to C in $\underline{C}$, and an explicitly construct (A,k) is denoted A^o.

If $\underline{A}$ and $\underline{B}$ are enriched over $\underline{C}$, a strong functor $(F,\hat{F}) : \underline{A} \longrightarrow \underline{B}$ is a functor $F : \underline{A} \longrightarrow \underline{B}$ equipped with a natural transformation $\hat{F} : (A,A') \longrightarrow (AF,A'F)$ in $\underline{C}$, compatible with the (lifted) composition and identity and such that $\hat{F}P = F$; see [7]. When $\hat{F}$ is understood, we abuse notation and refer to F as a strong functor. If $\underline{A}$ and $\underline{B}$ both have tensors (cotensors),

$F : \underline{A} \longrightarrow \underline{B}$ is strong, and the induced map $C \underline{\otimes} AF \longrightarrow (C \underline{\otimes} A)F$ (resp. $(C \pitchfork A)F \longrightarrow (C \pitchfork AF)$) is an isomorphism, we say that F preserves tensors (resp. cotensors). If the functor F has a right adjoint G and there is a natural isomorphism $(AF,B) \longrightarrow (A,BG)$, we say that F and G are strong adjoints, and we write $F \rightleftharpoons G$. It is easy to see that the tensors and cotensors constructed above, in fact, yield strong adjoints to the functor HOM. Note that if $F \dashv G$ and F is strong, there are natural maps $(A,BG) \to (AF,BGF) \to (AF,G)$.

6. Lemma: Let $\underline{A} \underset{G}{\overset{F}{\rightleftarrows}} \underline{B}$ be functors between two categories enriched over $\underline{C}$ with $F \dashv G$. If G preserves cotensors or F preserves tensors, then $F \rightleftharpoons G$.

Proof: If F preserves tensors we have natural isomorphisms $\underline{C}(C,(A,BG)) \xrightarrow{\sim} \underline{A}(C \underline{\otimes} A,BG) \xrightarrow{\sim} \underline{B}((C \underline{\otimes} A)F,B) \xrightarrow{\sim} \underline{B}(C \underline{\otimes} AF,B) \xrightarrow{\sim} \underline{C}(C,(AF,B))$ for all $C \in \underline{C}$, $A \in \underline{A}$, $B \in \underline{B}$, so $(A,BG) \cong (AF,B)$ as desired. A similar argument holds if G preserves cotensors; see [16].

If $\underline{P}_1$ and $\underline{P}_2$ are props, a tensor preserving functor $\underline{P}_2 \xrightarrow{F} \underline{P}_1$ yields a functor $\underline{X}^F : \underline{X}^{\underline{P}_1} \longrightarrow \underline{X}^{\underline{P}_2}$ between the induced categories of algebras. A functor of the form $\underline{X}^F$ is called propable.

7. Proposition: Every propable functor is strong.

Proof. Every propable functor $\underline{A} \xrightarrow{F} \underline{B}$ has a degree, i.e. an integer k such that $AF = A^k$ for all $A \in \underline{A}$ (see [17]). Given an evaluation $A \otimes C \xrightarrow{e} A'$, where A and A' are

in $\underline{A}$, the map $AF \otimes C \xrightarrow{\bar{e}} A'F$ defined by $\bar{e} = (AF \otimes \delta_k)e^k$ is also an evaluation. Thus the canonical evaluation $A \otimes (A,A') \to A'$ induces the natural coalgebra map $(A,A') \xrightarrow{\hat{F}} (AF,A'F)$.

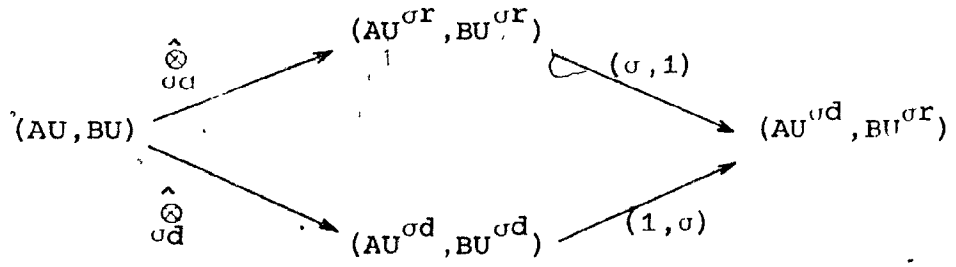
Let P_1 and P_2 be props, and let $U : \underline{P}_1 \to \underline{P}_2$ be a monoidal functor of degree 1 which is an injection on hom sets. Then the induced functor $\underline{X}^{\underline{P}_2} \to \underline{X}^{\underline{P}_1}$ is called a forgetful functor; the main example being the $\underline{X}$ valued forgetful functor induced by an injection from the trivial prop.

8. Proposition: If U is a forgetful functor between algebraic (resp. coalgebraic) categories over $\underline{X}$ having a left (resp. right) adjoint F, then U and F are strong adjoints.

Proof: U is strong by 7 and preserves cotensors (resp. tensors) by their constructions. Lemma 6 implies the result.

If $\underline{A}$ is enriched over $\underline{C}$, the strength of HOM is equivalent to the existence of the composition map, induced by the evaluation $A \otimes (A,B) \otimes (B,C) \xrightarrow{e \otimes 1} B \otimes (B,C) \xrightarrow{e} C$ for all A,B, and C in $\underline{A}$. Similarly, the bifunctor $\otimes$ on $\underline{X}$ is strong; for each W,X,Y, and Z in $\underline{X}$ the evaluation $W \otimes Y \otimes (W,X) \otimes (Y,Z) \xrightarrow{\sim} W \otimes (W,X) \otimes Y \otimes (Y,Z) \xrightarrow{e \otimes e} (X \otimes Z)$ yields the map $(W,X) \otimes (Y,Z) \xrightarrow{\hat{\otimes}} (W \otimes Y, X \otimes Z)$. For each $n \in \mathbb{N}$ define $\hat{\otimes}_n : (X,Y) \to (X^n,Y^n)$ to be δ_{n-1} followed by n-1 applications of $\hat{\otimes}$, the coassociativity of the tensor in $\underline{C}$ making this well defined. This may be used to give a more constructive de-

finition of HOM, as follows: Let [,] denote the $\underline{X}$ internal hom functor. The natural isomorphisms $\underline{C}(C,(X,Y)) \xrightarrow{\sim} \underline{X}(C \otimes X, Y) \xrightarrow{\sim} \underline{X}(C,[X,Y])$ show $(X,Y) \cong [X,Y]K$, so that we could have started by defining $(X,Y) = [X,Y]K$, the evaluation being adjoint to the canonical $[X,Y]K \to [X,Y]$. Exactly as above; this yields maps $\hat{\otimes}$ and $\hat{\otimes}_n$. Now let $\underline{A}$ be monoidally-defined over $\underline{X}$ with forgetful functor U. If A and B are in $\underline{A}$, we may redefine (A,B) in $\underline{C}$ to be the joint equalizer of all pairs of maps below, where σ runs over all $\underline{A}$-structure maps:



In terms of modules, $\hat{\otimes}$ is just the linearization of the very non-linear bifunctor $\otimes$ on $\underline{X}$, and its existence is what allows enrichments over $\underline{C}$, rather than over $\underline{X}$. Similarly, the non-linearity of certain conditions in $\underline{X}$ disappears when "lifted" to $\underline{C}$. For example, the map $(X,Y) \xrightarrow{\hat{K}} (XK,YK)$ using adjointness yields a map $XK \otimes YK \xrightarrow{\alpha} (X \otimes Y)K$, analogous to the map in $\underline{Sets}$ $XU \times YU \to (X \otimes Y)U$ (U the underlying functor). Now the non-linear condition $(X,X)m = 0$ for a lie-algebra multiplication $X \otimes X \xrightarrow{m} X$ can be expressed as $\alpha \cdot mK = \underline{0}: XK \otimes XK \to XK$, where $\underline{0}$ is the coalgebra map collapsing all points to the unique point in XK representing 0 in X.

Example: Let $\underline{X}$ = Mod-R for a commutative ring R, and let C be in $\underline{CX}$. Then a point of C is a coalgebra map $R \xrightarrow{p} C$ where R has its trivial coalgebra structure. Letting u denote the unit in R, we see from the commutivity of

$$\begin{array}{ccc} R & \xrightarrow{p} & C \\ \downarrow & & \downarrow \delta \\ R^2 & \xrightarrow{p^2} & C^2 \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{p} & C \\ & \searrow \quad \swarrow \varepsilon & \\ & R & \end{array}$$

that the element up in C is such that $(up)\delta = up \otimes up$ and $(up)\varepsilon = u$. Hence pt(C) is in one-to-one correspondence with

$$\{c \in C : c\delta = c \otimes c \ , \ c\varepsilon = u\}$$

Now let $\underline{A}$ = $\underline{UAlg\text{-}R}$, the category of unitary R-algebras, an eminent propable category over Mod-R.

Each A in $\underline{A}$ has structure maps $R \xrightarrow{u_A} A$ (the unit) and $A^2 \xrightarrow{m_A} A$ (multiplication, written as between its arguments). Let A and B be in $\underline{A}$ and suppose $A \otimes C \xrightarrow{e} B$ is an evaluation, i.e. (from (1) above) the following diagrams commute:

(3)
$$\begin{array}{ccccc} A^2 \otimes C & \xrightarrow{1 \otimes \delta} & (A \otimes C)^2 & \xrightarrow{e^2} & B^2 \\ \downarrow{\scriptstyle m_A \otimes 1} & & & & \downarrow{\scriptstyle m_B} \\ A \otimes C & & \xrightarrow{\qquad e \qquad} & & B \end{array} \qquad \begin{array}{ccc} A \otimes C & \xrightarrow{e} & B \\ \uparrow{\scriptstyle u_A \otimes 1} & & \uparrow{\scriptstyle u_B} \\ R \otimes C & \xrightarrow{1 \otimes \varepsilon} & R \end{array}$$

For a A and f in C we write af for $(a \otimes f)e$. If f is a point of C, the above diagrams yield for all a and a' in A:

$$(a \otimes a' \otimes f)(m_A \otimes 1)e = (a \otimes a' \otimes f)(1 \otimes \delta)e^2 m_B$$

$$((a \cdot a') \otimes f)e = (a \otimes f \otimes a' \otimes f)e^2 m_B$$

$$(a \cdot a')f = (af) \cdot (a'f)$$

and $u_A f = u_B$ since $f\varepsilon = u$ in R.

Suppose there is an element d in C such that $d\delta = d\otimes f + f\otimes d$ and $d\varepsilon = 0$. Then chasing around (3) again we find $(a\cdot a')d = ad\cdot a'f + af\cdot a'd$ and $u_A d = 0$, i.e. d represents an algebra derivation along f, from A to B.

For $A \in \underline{A}$ consider the coalgebra (A,A). Let $1 \in (A,A)P$ be the point representing the identity map, and let $\mathrm{Diff}(A) = \{d \in (A,A): d\delta = d\otimes 1 + 1\otimes d, d\varepsilon = 0\}$, where δ and ε are the coalgebra structure maps on (A,A). Then as above, the elements of $\mathrm{Diff}(A)$ represent algebra derivations from A into itself (along the identity map). To see that $\mathrm{Diff}(A)$ contains one representative for each derivation, consider the module $R(x)/(x^2)$ with coalgebra structure given by $u\delta = u\otimes u$ and $x\delta = x\otimes u + u\otimes x$ (u is the unit in R). A derivation $A \xrightarrow{d} A$ induces an evaluation $A\otimes R(x)/(x^2) \to A$ defined by $au = a$ and $ax = ad$ for all $a \in A$, thus yielding a unique coalgebra map $R(x)/(x^2) \to (A,A)$. The image of u is 1 in (A,A), while the image of x represents d. Similarly, it can be shown that (A,A) has representatives for all higher derivations as well.

Given A and B in $\underline{A}$, we already have shown that $(A,B)P \cong \underline{A}(A,B)$, and as above it is easily shown that for each $f \in (A,B)P$, the submodule $\{d \in (A,B): d\delta = d\otimes f + f\otimes d, d\varepsilon = 0\}$ <u>exactly</u> represents the module of f-derivations from

A to B . Denote this submodule of (A,B) by $D(A,B)_f$ the module of f-differentials. Given an R-module X , let XF be the tensor algebra over X ; this defines the left adjoint to the forgetful functor $\underline{A} \xrightarrow{U} \underline{X}$. By proposition 8 , these functors are strong adjoint, i.e. we have a natural isomorphism in $\underline{C}$: $(X,AU) \cong (XF,A)$. Any coalgebra map preserves points; the above isomorphism carries $(X,AU)P$, representing module maps $X \to AU$, into $(XF,A)P$, their unique liftings to algebra maps. The coalgebra maps also preserve "differentials" in the following sense: If $(X,AU) \xrightarrow{c} (XF,A)$ is a coalgebra map $f \in (X,AU)$ and $d \in D(X,AU)_f$ then $dc \in D(XF,A)_{fc}$. Now the elements of $D(X,AU)_f$ represent f-derivation from X to AU , but X and AU are only modules, i.e. algebras with zero multiplication, so any linear map $X \to AU$ is an f-derivation. The above isomorphism shows why not only do module maps $X \xrightarrow{f} AU$ lift to algebra maps $XF \xrightarrow{\bar{f}} A$, but also that any module map $X \xrightarrow{d} AU$ lifts to a unique algebra derivation (along f) $XF \xrightarrow{\bar{d}} A$.

Clearly this example may be greatly generalized, which we do in the following section.

Chapter IV: Deformations and Cohomology

Let A be a subcoalgebra of B. We say the inclusion $i : A \to B$ is singular if the following two conditions hold:

i) i has a splitting in $\underline{X}$

ii) $\delta: B \to B \otimes B$ factors through the inclusion $A \otimes B \cup B \otimes A \xrightarrow{j} B \otimes B$.

Condition ii) is equivalent to B being the pullback of δ and the inclusion j (see [19]§ 9). It is easy to see that the condition "$A \to B$ is singular" is just dual to the statement "the algebra A is a quotient of the algebra B by a square-zero ideal, the quotient map having an $\underline{X}$-splitting," (see Note in appendix). Thus a coalgebra C is said to be smooth if the hom functor $\underline{C}(-,C): \underline{C}^{op} \to \underline{Sets}$ carries singular inclusions to epimorphisms, i.e. if for every singular $i: A \to B$ and every coalgebra map $a: A \to C$ there exists a coalgebra map $b: B \to C$ such that $ib = a$. It is obvious that every cofree coalgebra is smooth. In fact the cofree coalgebras are $\underline{X}$-injective.

Now assume that $\underline{X}$ has a zero object and that the $\underline{X}$-image of a coalgebra under a coalgebra map is a subcoalgebra of the codomain. For the latter it is sufficient to assume that images and coimages in $\underline{X}$ coincide.

1. Lemma: If $A \to B$ satisfies ii) above and $B \xrightarrow{m} C$ is a coalgebra map, then $Im(A) \to Im(B)$ also satisfies ii).

2. Proposition: A coalgebra C is smooth if and only if every singular map $C \to E$ has a coalgebra splitting.

Thus a coalgebra C is smooth if and only if $H^2(N,C) = 0$ for every C-comodule N, where the cohomology groups are those determined by the cofree resolution as in [3]. (See [20] or [21] for the definition of a comodule).

Now assume that $\underline{X}$ is an additive category, i.e. that $X \otimes K$ is a Hopf object for all X in $\underline{X}$. Define a sequence of coalgebras $\{D_n\}_{n<\omega}$ as follows: In $\underline{X}$, $D_n = \bigoplus_n I$; let d_i be the $i^{\underline{TH}}$ injection and define $\delta: D_n \to D_n^2$ and $\varepsilon: D_n \to I$ by the equations below:

$$(1) \qquad d_i\delta = \sum_{j+k=i} d_j \otimes d_k \qquad d_i\varepsilon = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Clearly $D_0 = I$ and D_1 is a generalization of the ring of dual numbers from the last example. The injections $D_n \to D_{n+1}$ are coalgebra maps, and we define $D_\omega = \varinjlim D_n$. When $\underline{X} = \text{Mod-}R$, D_ω and $\{D_n\}_{n<\omega}$ are duals respectively of the ring of formal power series and its n^{th} degree truncations.

For C in $\underline{C}$, let $D_n(C)$ denote the set $\underline{C}(D_n, C)$. Clearly an element d of $D_n(C)$ uniquely determines a sequence $(d_i)_{i \le n}$ of elements of C, i.e. $\underline{X}$-maps $I \to C$, satisfying equations (1) above. In particular, d_0 is a point of C. An

element d of $D_\infty(C)$ is called <u>terminal</u> if for some $n \in \mathbb{N}$, $d_i = 0$ for all $i > n$, whence $\sum_{i<n} d_i$ is a point in C. Hence an element d of $D_n(C)$ is said to be an <u>n-deformation</u> of the point d_o (towards another point). The injections $D_n \to D_{n+1}$ are clearly singular, so if C is smooth every element of $D_n(C)$ lifts to an element of $D_{n+1}(C)$.

We often refer to an element d of $D_\infty(C)$ as a <u>formal point</u> in C (gotten by deforming the point d_o). It determines a formal sum $\sum_{i=o}^{\infty} d_i$, and given a notion of convergent series in C, e.g. when $\underline{X}$ is the category of Banach spaces, the sum above will (if it converges) determine an honest point in C; see Example 2 below. Note that if $\underline{X} = \text{Mod-}R$, R a commutative ring, each formal point d in C yields a whole family of formal points $\{d^r : r \in R, d_i^r = d_i r^1\}$.

Given a point p in C, we would like to consider the elements d of $D_n(C)$ having $d_o = p$. This may be done in a functorial way as follows: Consider the objects $I \xrightarrow{p} C$ and $I \xrightarrow{d_o} D_n$ in $\underline{C}^* = (I, \underline{C})$, the comma-category of pointed coalgebras. Then $\underline{C}^*(d_o, p)$ clearly represents the set of n-deformations of p, denoted $D_n(C_p)$.

Let $\underline{A}$ be enriched over $\underline{C}$ and let A and B be objects of $\underline{A}$. The elements of $D_\infty(A,B)$ can be thought of as <u>formal maps</u> from A to B, since a terminal or convergent formal map yields a point in (A,B) representing a real $\underline{A}$-map. If $C \in \underline{A}$, $a \in D_\infty(A,B)$, and $b \in D_\infty(B,C)$, the composition of a and b,

as formal maps, is given by

$$(2) \qquad D_\infty \xrightarrow{\delta} D_\infty \otimes D_\infty \xrightarrow{a\otimes b} (A,B)\otimes(B,C) \xrightarrow{comp} (A,C)$$

Denoting this by $a.b$, we have in terms of sequences:

$$(2') \qquad (a.b)_i = \sum_{k+j=i} a_k \cdot b_j \ .$$

Example 1: Let $\underline{AAlg}$ be the category of associative algebras over a ring R. Consider A and B in $\underline{AAlg}$, and a point p in the coalgebra (A,B). Then in chapter III it was shown that the elements of $D_n((A,B)_p)$ correspond to n-fold derivations along p. Now let A be in $\underline{AAlg}$ and let $\underline{A} = (\underline{AAlg},A)$, the abelian group objects of which are just A-modules. Let $B \xrightarrow{b} A$ be in $\underline{A}$ and consider a point p in the coalgebra $C = (A \xrightarrow{1} A,\ B \xrightarrow{b} A)$. As usual, p represents a splitting of b. The evaluation $A \otimes C \xrightarrow{e} B$ makes the diagram

$$\begin{array}{ccc} A\otimes C & \xrightarrow{e} & B \\ {\scriptstyle 1\otimes c}\downarrow & \swarrow{\scriptstyle b} & \\ A\otimes R \cong A & & \end{array}$$

commute. Thus for my element d of $D_1(C_p)$, d_1 must represent a map into the kernel of b, since $d_1\varepsilon = 0$. This kernel is an A module via p, and we recover the notion of module valued derivations.

Let A be in $\underline{X}$ and let $\underline{P}$ be a prop over $\underline{X}$. A $\underline{P}$-structure on A is clearly represented by a point of

$\times_{\sigma\in[\underline{P}]} (A^{\sigma d}, A^{\sigma r})$. Now $\underline{P}$ is generated by the quotient of a free prop by a set of diagrams D . Each element of D gives rise to a pair of maps in $\underline{C}$ of the form $\times_{\sigma\in[\underline{P}]} (A^{\sigma d}, A^{\sigma r}) \rightrightarrows (A^m, A^n)$. The intersection of the equalizers of all such pairs yields a coalgebra S whose points exactly correspond to $\underline{P}$-structures on A . Given $p \in (S)P$, i.e. a $\underline{P}$-algebra $A(p)$, an element of $D_n(S_p)$ may be considered as an n-fold deformation of $A(p)$ towards a new $\underline{P}$-algebra structure on A .

Example 2: Look at the situation in which $\underline{P}$ defines the category of associative algebra objects in $\underline{X}$. A $\underline{P}$-structure on an element A of $\underline{X}$ is given by a point contained in the equalizer of the pair of maps below:

$$(3)\qquad (A^2,A) \xrightarrow{\delta} (A^2,A)\otimes(A^2,A) \underset{(A\otimes -)\hat{}\otimes 1}{\overset{(-\otimes A)\hat{}\otimes 1}{\rightrightarrows}} (A^3,A^2)\otimes(A^2,A) \xrightarrow{\text{comp}} (A^3,A)$$

Let S denote the (coalgebra) equalizer of the above maps. If $d \in D_\infty(S)$ we find the sequence $(d_n)_{n\geq 0}$ satisfying the equations:

$$(3')\qquad \sum_{i+j=n} (d_i\otimes A)d_j - (A\otimes d_j)d_i = 0$$

$$(d_o\otimes A)d_o = (A\otimes d_o)d_o$$

Now let $\underline{X}$ be the category of vector spaces over a field k , and let $k((x))$ be the power series field in one variable. Then any sequence of k-linear maps $(d_n)_{n\geq 0} : A^2 \to A$ determines

a multiplication m on $A \otimes_k k((x))$, the linear extension of $(a \otimes b)m = \sum_{i=o}^{\infty} (a \otimes b) d_i x^i$ for $a,b \in A$. In the analytic theory, when $k = R$ or C, one may now substitute values from k for x and ask when the above sum converges, for all $a,b \in A$. In such a case m determines a new multiplication on A. Note that if A is finite dimensional this depends on the finite number of conditions that the sum converge for a and b in the basis. The multiplications (on $A \otimes_k k((x))$ and A) will be associative if the sequence of maps satisfy the equations (3') above. These notions may be generalized to arbitrary fields using the theory of formal convergence and places; see [11] and [12] for details.

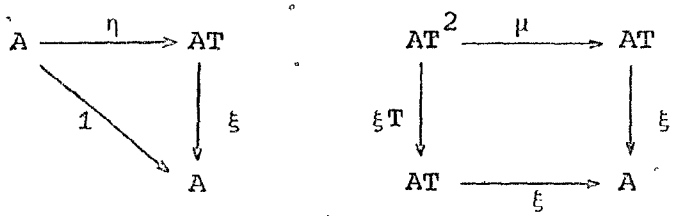
Return to an arbitrary base category $\underline{X}$ and look at the case where $\underline{P}$ is algebraic, yielding a triple T and a category of T-algebras $\underline{X}^T$ over $\underline{X}$. Let $T^2 \xrightarrow{\mu} T$ and $1 \xrightarrow{\eta} T$ denote the associative and unitary natural transformations with which T is equipped. Let A be an element of $\underline{X}$, and let S be the largest subcoalgebra of (AT,A) such that the following two diagrams commute:

$$(4) \qquad S \hookrightarrow (AT,A) \underset{\Gamma}{\overset{\eta \cdot (\ )}{\rightrightarrows}} (A,A)$$

$$S \hookrightarrow (AT,A) \underset{\mu \cdot (\ )}{\overset{(\ )T \cdot (\ )}{\rightrightarrows}} (AT^2,A)$$

Here $\Gamma = (AT,A) \to (AT,A) \otimes I \xrightarrow{\subset \otimes 1_A} I \otimes (A,A) \cong (A,A)$ and $(\ )T \cdot (\ ) = (AT,A) \xrightarrow{\delta} (AT,A)^2 \xrightarrow{T \otimes 1} (AT^2,AT) \otimes (AT,A) \xrightarrow{\text{comp.}} (AT^2,A)$, using the fact that T is strong. Then each ξ in SP defines

a map (also denoted ξ) $AT \to A$ such that the following two diagrams commute:



Thus the points of S_ξ correspond to T-algebra structures on A. Fixing such a ξ, we shall examine the elements of $D_\infty(S_\xi)$. As above, a terminal element of $D_\infty(S_\xi)$ would yield another point in S_ξ, i.e. a new T-algebra structure on A gotten by deforming ξ. Thus we may consider $D_\infty(S_\xi)$ as formal T-algebra structures on A.

(A,A) has a natural bialgebra structure given by composition, yielding the composition monoid structure on $D_n(A,A)$. The unit in $D_n(A,A)$ is given by the point 1 representing the identity map on A. Now the monoid $D_n(A,A)_1$ is a group; $(d_i^{-1})_{i\le n}$ is computable from the sequence of equations $d_o^{-1} = d_o = 1$ and $d_i^{-1} = \sum_{k+\ell=i,\ k,\ell>0} a_k \cdot a_\ell^{-1}$, the result of solving $d \cdot d^{-1} = 1$ using the composition (2). Denote this group by $Aut_n(A)$. The elements of $Aut_\infty(A)$ are the <u>formal automorphisms</u> of A.

Two n-deformations of an algebra structure ξ, i.e. two elements α and β in $D_n(S_\xi)$, are said to be <u>equivalent</u> if there exists an element a of $Aut_n(A)$ such that in $D_n(AT,A)$ we have

(5) $$aT \cdot \alpha = \beta \cdot a$$

If $n = \infty$, a is an isomorphism of the two formal algebra structures on A, α and β. Using (2'), the above equation yields:

$$(5') \qquad \sum_{j+k=i} a_j T\cdot\alpha_k = \sum_{j+k=i} \beta_j\cdot a_k \quad \text{for all } i \quad n$$

An element of $D_n(S_\xi)$ is called <u>trivial</u> if it is equivalent to the zero sequence over ξ, i.e. to $\hat{\xi} \in D_n(S_\xi)$, defined by $\hat{\xi}_o = \xi$, $\hat{\xi}_i = 0$ for $i > 0$.

Recall from Beck's thesis, [3], that the algebra structure ξ gives rise to a "non-homogeneous" cohomology complex of abelian groups, yielding cohomology groups denoted $H^n(\xi,\xi)$ of ξ with coefficients in ξ. The $n^{\underline{TH}}$ coboundary operator $\partial_n : \underline{X}^T(AT^n,A) \to \underline{X}^T(AT^{n+1},A)$ is given by

$$\partial_n = T^n\cdot(\ ) - \sum_{i=o}^{n-1} (-1)^i T^i\mu T^{n-i}\cdot(\ ) - (-1)^n(\ )T\cdot\xi$$

Now let d be an element of $D_n(S_\xi)$ and let d_j be the first non-zero term of the sequence $(d_i)_{i>o}$. Then from (1) we have $d_j\delta = d_j\otimes\xi + \xi\otimes d_j$ and from (4) $\mu\cdot d_j = d_jT\cdot\xi + \xi T\cdot d_j$, i.e. d_j is a one-cocycle. Analogous to [11] Proposition I.1, we have:

<u>3. Proposition</u>: Each element of $D_n(S_\xi)$ is equivalent to an element whose first non-zero term is not a coboundary.

<u>4. Corollary</u>: If $H^1(\xi,\xi) = 0$ then every element of $D_n(S_\xi)$ is trivial.

Let d be an element of $D_1(S_\xi)$. We shall investigate, in the spirit of [11], when such an element may be lifted to an element of $D_\infty(S_\xi)$. More generally, let d be in $D_{n-1}(S_\xi)$ and suppose there exists an element d_n extending d to an element of $D_n(S_\xi)$. Then from (1) and (4) we have:

$$\mu \cdot d_n = \sum_{i+j=n} d_i T \cdot d_j \quad . \tag{6}$$

Recalling $d_o = \xi$ this may be written as

$$\mu \cdot d_n - \xi T \cdot d_n - d_n T \cdot \xi = \sum_{\substack{i+j=n \\ i \neq o \neq j}} d_i T \cdot d_j \quad . \tag{6'}$$

Denote this last sum by d^*.

Note that any cofree coalgebra contains many deformations; every set of maps $\bigoplus_n I \to X$ in $\underline{X}$ lifts to an element of $D_n(XK)$. In particular, given X and Y in $\underline{X}$ and a sequence of maps $(x_i)_{i \leq n} : X \to Y$, there is an element $\hat{x}$ in $D_n((X,Y))$ such that $\hat{x}_i$ is a representative of x_i. Now recognizing the left side of (6') as the coboundary of d_n we see,

5. Lemma: An element d of $D_{n-1}(S_\xi)$ lifts to an element of $D_n(S_\xi)$ if and only if there exists a 1-cochain d_n such that $d_n \partial_1 = d^*$.

Although the following is a generalization of [11] Proposition I.3, the proof is now much easier, using only the strength of T and the naturality of μ.

6. Proposition: If d is an element of $D_{n-1}(S_\xi)$ then d^* is a two cocycle.

7. Corollary: If $H^2(\xi,\xi) = 0$ then every element of $D_n(S_\xi)$ lifts to an element of $D_\infty(S_\xi)$.

Let k be a commutative ring. In developing the theory of k-split singular extensions of a k-algebra A by an A-module M , one may proceed as follows: First, find one extension E (which is easy), then produce other extensions by "twisting" (or deforming) the algebra structure on E while preserving the vital properties of an extension. In the classical theory this deforming is done by the addition of cocycles in the multiplication on E . One then proves that all extensions of A by M may be gotten in this way from E . This connection between the extension theory and deformation theory may be generalized and made precise.

First a diversion. Let p be a point in an arbitrary coalgebra C , and denote the set $\{d \in C: d\varepsilon = 0, d\delta = d\otimes p + d\otimes d + p\otimes d\}$ by $\mathrm{Cord}(C_p)$; this is the set of cords at p and is obviously naturally isomorphic to $(C)P$, since the difference between p and another point is a cord at p . Now in certain cases there will be a natural correspondence between $\mathrm{Cord}(C_p)$ and $D_1(C_p)$. For example, let $\underline{A}$ be the category of algebra objects over a fixed algebra A as in example 1 above.

Let $B \xrightarrow{\beta} A$ and $G \xrightarrow{\gamma} A$ be in $\underline{A}$, and suppose $\ker(\gamma)$ has square zero. Denote by U the underlying functor $\underline{A} \to \underline{X}$, and recall that (β,γ) can be viewed as the largest subcoalgebra of $(\beta U, \gamma U)$ that maps β to γ. If $p \in (\beta,\gamma)P$ and $d : \beta U \to \gamma U$ is an $\underline{X}$-map, in $(\beta U, \gamma U)$ there are elements d_1 and d_2 each representing d, but such that $\varepsilon d_1 = 0 = \varepsilon d_2$, $d_1\delta = d_1 \otimes p + p \otimes d_1$, and $d_2\delta = d_2 \otimes p + d_2 \otimes d_2 + p \otimes d_2$. The first condition ensures that both d_1 and d_2 map into the kernel of γ, and d_1 will satisfy the evaluation condition if and only if d_2 will since the image of $d_2 \otimes d_2$ in $G \otimes G$ is annihilated by the multiplication.

Let $\underline{A}$ be the category of associative algebra objects in $\underline{X}$, and let A be in $\underline{A}$ with multiplication $a : A^2 \to A$. If $M \in \underline{X}$, an extension of A by M is an A-structure on $A + M$ with the projection $p : A + M \to A$ is an algebra map. Here $+$ denotes the biproduct in the additive category $\underline{X}$. Let $E = A + M$ and let S be the largest subcoalgebra of (E^2, E) that equalizes the pair of maps describing associativity, as in diagram (3), and such that every element $x \in S$ satisfies

$$x\varepsilon \cdot (p \otimes p) \cdot a = x \cdot p \quad . \tag{7}$$

If $s \in SP$, this means that s defines an extension of A by M, and the maximality of S ensures that every extension will be represented by a point in S. Note that (7) is equivalent to the commutativity of

$$\begin{array}{ccccc} S & \longrightarrow & (E^2,E) & \xrightarrow{\varepsilon} & I \\ & & \downarrow{\scriptstyle \int} & & \downarrow{\scriptstyle (p\otimes p)\cdot a} \\ & & (E^2,A)\otimes(E^2,M) & \xrightarrow[1\otimes\varepsilon]{} & (E^2,A) \end{array}$$

which is just a lifting to $\underline{C}$ of the condition that the projection be an algebra map.

Now assume that we are given a point s in S. For any two coalgebras B and C, $D_n(B\otimes C) \cong D_n(B) \times D_n(C)$, and in particular $(B\otimes C)P \cong BP \times CP$; since $(E^2,E) \cong (E^2,A) \otimes (E^2,M)$, s decomposes into $s = b \times e$ where $b \in (E^2,A)P$ and $e \in (E^2,M)P$. Further, $E^2 \cong A^2 + A\otimes M + M\otimes A + A^2$ so $e \in (E^2,M)$ has a decomposition into $e = c \times l \times r \times m$ where $c \in (A^2,M)P$, $d \in (A\otimes M,M)P$, $r \in (M\otimes A)P$, and $m \in (M^2,M)P$. Likewise corresponding to the obvious expansion of (E^2,A), b has a four-point decomposition. However, applying condition (7) it is clear that b acts non-trivially only as a point in (A^2,A), i.e. as a map $A^2 \to A$, and there $b = a$. Investigating what condition the associativity of s imposes we obtain the well known

8. Proposition: The point s in (E^2,E) is in S, and defines an extension, if and only if the following conditions hold:

i) $m : M^2 \to M$ is associative

ii) $(l\otimes M)\cdot m = (A\otimes m)\cdot l : A\otimes M^2 \to M$

iii) $(M\otimes r)\cdot m = (m\otimes A)\cdot r : M^2\otimes A \to M$

iv) $(l\otimes A)\cdot r = (A\otimes r)\cdot l : A\otimes M\otimes A \to M$

v) $(r\otimes M)\cdot m = (M\otimes l)\cdot m : M\otimes A\otimes M \to M$

vi) $(a\otimes M)\cdot l = (A\otimes l)\cdot l + (c\otimes M)\cdot m : A^2\otimes M \to M$

vii) $(r\otimes A)\cdot r = (M\otimes a)\cdot r + (M\otimes c)\cdot m : M\otimes A^2 \to M$

viii) $(a\otimes A)\cdot c + (c\otimes A)\cdot r = (A\otimes a)\cdot c + (A\otimes c)\cdot l : A^3 \to M$

All of the above equations have classical interpretations, for which we refer the reader to [14]. In particular, viii) says c must be a Hochschild 2-cocycle. If one requires that l and r define an A-bimodule structure on M, vi) and vii) would demand that c map into the (two sided) annahilator of M, and we would get the well known result that $H^2(A,\text{ann } M)$, this being Hochschild cohomology, classifies extensions with given (fixed) structure maps l, r, and m. Of course in the singular theory conditions i), ii), and iii) disappear.

Now let us search for deformations of s in S. Any n-deformations of s in (E^2,E) decomposes into $b' \times e'$ where $b' \in D_n(E^2,A)_b$ and $e' \in D_n(E^2,M)_e$. However, (7) applied to $b' \times e'$ demands that b' be the zero deformation of b, so in looking for deformations of s we need only look for deformations e' of $e = c \times l \times r \times m$. As above e' decomposes into $c' \times l' \times r' \times m'$, where $c' \in D_n(A^2,M)$, $l' \in D_n(A\otimes M,M)$, etc. In all the classical theories the actions of A upon M and the multiplication in M are presumed to be given and <u>fixed</u>, and here this corresponds to demanding that

$1'$, r', and m' be zero deformations; one only looks at deformations of the multiplication s on $A + M$ gotten by deforming c. We shall first look at this restricted theory and then at the more general case when $1'$, r', and m' are allowed to be non-zero.

We have seen that a 1-deformation s' is given by a 1-deformation c' of c in (A^2, M). This defines a linear map which also has a representation as a cord c'' at c. This defines a cord s'' at s, and it is easily seen that s' is in S if and only if s'' is in S. But s'' defines an actual new point in S, i.e. a new extension of A by M. Thus in finding those $\underline{X}$-maps $A^2 \to M$ that generate cords (or 1-deformations) within S, we find the set of all desired extensions. The associativity condition on S gives us the following: Assuming 1, r, and m are fixed, an $\underline{X}$-map $f : A^2 \to M$ yields a cord in S (at s) if and only if it satisfies the following equations:

$$(8) \qquad \begin{aligned} (f \otimes M)\cdot m &= 0 = (M \otimes f)\cdot m \\ (a \otimes A)\cdot f + (f \otimes A)\cdot r &= (A \otimes a)\cdot f + (A \otimes f)\cdot 1 \end{aligned}$$

Thus we have an isomorphism of $Z^2(A, \text{ann } M)$ and $D_1(S_s)$, the addition in the cocycle group being given simply by the additive group structure on the set of 1-deformations. This yields the well-known result that if SP is non-empty, i.e. if there

exists one extension of A by M, then the group of extensions is classified by $H^2(A,\operatorname{ann} M)$. Of course here one extends the original definition of Hochschild cohomology to include coefficients in the algebra M equipped with A actions l and r.

Returning to the general case of a deformation of s, let $c' = (c_n)_{n\geq 0} \in D_\infty(A^2,M)$, $l' = (l_n)_{n\geq 0}$, $r' = (r_n)_{n\geq 0}$, and $m' = \widetilde{(m_n)}_{n\geq 0}$. As we have seen these define a formal deformation of s lying in (E^2,E). Analogous to Proposition 8 we have

9. Proposition: The deformation of s defined by c', l', r', and m' lies in S if and only if the following equations hold for all $n \geq 0$, where the sums are taken over indices i and j such that $i + j = n$:

i) $\Sigma(M\otimes m_i)\cdot m_j = \Sigma(m_i\otimes M)\cdot m_j$

ii) $\Sigma(l_i\otimes M)\cdot m_j = \Sigma(A\otimes m_i)\cdot l_j$

iii) $\Sigma(M\otimes r_i)\cdot m_j = \Sigma(m_j\otimes A)\cdot r_j$

iv) $\Sigma(l_i\otimes A)\cdot r_j = \Sigma(A\otimes r_i)\otimes l_j$

v) $\Sigma(r_i\otimes M)\cdot m_j = \Sigma(M\otimes l_i)\cdot m_j$

vi) $\Sigma(A\otimes l_i)\cdot l_j = \Sigma(a_i\otimes M)\cdot l_j + (c_i\otimes M)\cdot m_j$

vii) $\Sigma(r_i\otimes A)\cdot r_j = \Sigma(M\otimes a_i)\cdot r_j + (M\otimes c_i)\cdot m_j$

viii) $\Sigma(a_i\otimes A)\cdot c_j + (c_i\otimes A)\cdot r_j = \Sigma(A\otimes a_i)\cdot c_j + (A\otimes c_i)\cdot l_j$.

Specializing this to $n = 0$ we recover Proposition 8 and assuming $r_i = l_i = m_i = 0$ for $i > 0$ yields (8). By considering the cases when various combinations of c^0, l^1, r^1, and m^0 are zero, the reader can easily find other conditions for deformations with certain fixed features. For example if everything but the left action l is assumed fixed, one finds that the sequence $(l_n)_{n\geq 0}$ must satisfy

$(l_n \otimes M)\cdot m = (A \otimes m)\cdot l_n$, $(l_n \otimes A)\cdot r = (A \otimes r)\cdot l_n$, $(M \otimes l_n)\cdot m = 0$,

$\sum_{i+j=n} (A \otimes l_i)\cdot l_j = (a \otimes M)\cdot l_n$, and $(A \otimes c)\cdot l_n = 0$.

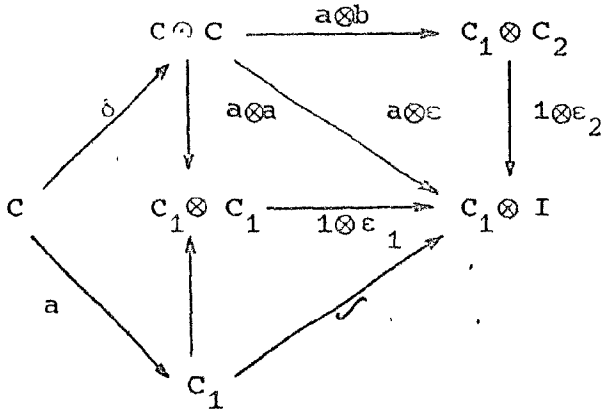
Obviously all of the above extension theory goes through when we replace the category of associative algebras by an algebraic category whose triple is additive. In order apply this to a non-additive triple T , one must assume that T at least partially respects the addition, but this is far out of the scope of this work.

Note that we have completely ignored the geometric interpretation of the deformation theory. Thinking of a coalgebra C as the linear envelope of its "space" of points, $D_1(C)$ is just the tangent bundle of the point-space. Given a point p in C $D_1(C_p)$ is the tangent space at the point p , and smooth coalgebras define smooth point-spaces in the usual sense. For the purely geometric version of deformation theory see [15] and the references there cited.

Appendix

Notes on chapter I

Proof of 1.1: We must first show $\underline{C}$ is cartesian. If $(C_1,\delta_1,\varepsilon_1)$ and $(C_2,\delta_2,\varepsilon_2)$ are in $\underline{C}$, their tensor product in $\underline{C}$ is defined to be $(C_1\otimes C_2,\ (\delta_1\otimes\delta_2)\cdot 1\otimes t\otimes 1\ ,\ \varepsilon_1\otimes\varepsilon_2)$. To show this is the product in $\underline{C}$, let (C,δ,ε) be in $\underline{C}$ and suppose we have $\underline{C}$-maps $a : C \to C_1$ and $b : C \to C_2$. The commutivity of the diagram below shows that $1\otimes\varepsilon_2 : C_1\otimes C_2 \to C_1$ is the product projection:



Now to finish the proof, we show the adjunction in MON is the forgetful functor $U : \underline{C}(\underline{X}) \to \underline{X}$. Let $\underline{Y}$ be a cartesian category and let $F : \underline{Y} \to \underline{X}$ be a MON-map. For Y in $\underline{Y}$ let $\Delta : Y \to Y \times Y$ be the diagonal and let ε be the composition $Y \xrightarrow{\sim} Y \times I \xrightarrow{\pi} I$. Define $\bar{F} : \underline{Y} \to \underline{C}(\underline{X})$ by $Y\bar{F} = (YF,\Delta F,\varepsilon F)$. $\bar{F}$ preserves product since it preserves the projection maps.

Notes on chapter II

Proof of 2.1: Let $\underline{G}'$ be the subset of $\underline{X}$ consisting of all finite tensor products of elements from $\underline{G}$, and let α' be a regular cardinal such that rank $G' < \alpha'$ for all G' in $\underline{G}'$. Now $\underline{G}'$ is closed under finite applications of the tensor, because $\underline{G} \subset \underline{G}'$ G' generates $\underline{X}$, and $\underline{X}$ is α'-locally presentable.

Proof of 2.5: This is well known. If $\underline{D}$ is a diagram in $\underline{A}$ with colimit D^* in $\underline{X}$, the structure maps $\upsilon^*: D^* \to D^{*\upsilon r}$ are given by the canonical arrow completing the square below, where i is the injection:

$$\begin{array}{ccc} D^* & & D^{*\upsilon r} \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i^{\upsilon r}} \\ D & \xrightarrow{\upsilon} & D^{\upsilon r} \end{array}$$

Now if the prop defining $\underline{A}$ is free we are done. If it is strictly coalgebraic (see p. 13) the maps υ^* must satisfy the commutivity of diagrams of the form $D^* \rightrightarrows D^{*n}$ (where the maps involved are built up out of the υ^* and various coherent natural isomorphisms). This pair of maps imbeds into a commuting diagram of the form

$$\begin{array}{ccc} D^* & \rightrightarrows & D^{*n} \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i^n} \\ D & \rightrightarrows & D^n \end{array}$$

The lower pair of maps are equal since $D \in \underline{A}$, and since the injections i are collectively epi the upper pair are equal and

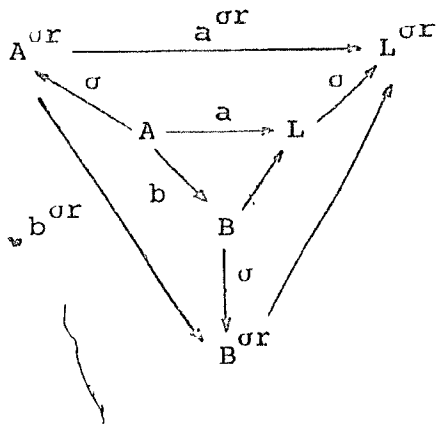
$D^* \in \underline{A}$. If $\underline{A}$ is not strictly coalgebraic, the maps required to be equal will be of the form $D^{*m} \to D^{*n}$. Since $\otimes$ preserves colimits, the maps $i^m : D^m \to D^{*m}$ are collectively epi and the argument goes as before.

Proof of 2.6: The "only if" statement is trivial. For the converse, the factorizations $i\sigma = \sigma' i^{\sigma r}$ define structure maps $\sigma' : X \to X^{\sigma r}$ on X . Now argue as in the proof of 2.5, above, the purity of X guarenteeing that the maps $X^n \xrightarrow{i^n} A^n$ are monomorphisms.

Proof of 2.8: Let $X_1 = \overline{X}$ and define X_β by $X_\beta = \overline{X}_{\beta-1}$ if β is a non-limit cardinal and $X_\beta = \bigcup_{\delta<\beta} X_\delta$ if β is a limit cardinal. Note that $\mathrm{Im}(X_\beta \to A \xrightarrow{\sigma} A^{\sigma r}) \subseteq \mathrm{Im}(X_\beta^{\sigma r} \to A^{\sigma r}) \subseteq \mathrm{Im}(X_{\beta+1}^{\sigma r} \to A^{\sigma r}) = X_{\beta+1}^{\sigma r}$ since $X_{\beta+1}$ is pure in $\underline{A}$. Choose a regular limit cardinal $\alpha' > \alpha$ and define $X^* = \varinjlim_{\beta<\alpha'} X_\beta$. Since α' is regular, $\{X_\beta\}_{\beta<\alpha'}$ is α-directed, so that $X^* = \bigcup_{\beta<\alpha'} X_\beta$. X^* is clearly pure in A and so $\mathrm{Im}(X^* \to A \xrightarrow{\sigma} A^{\sigma r}) = \mathrm{Im}(\bigcup_{\beta<\alpha'} X_\beta \to A \to A^{\sigma r}) = \bigcup_{\beta<\alpha'} \mathrm{Im}(X_\beta \to A \to A^{\sigma r}) \subseteq \bigcup_{\beta<\alpha'} [\mathrm{Im}(X_{\beta+1} \to A)]^{\sigma r} = \varinjlim_{\beta<\alpha} X_{\beta+1}^{\sigma r} = (X^*)^{\sigma r}$. Thus X^* is an $\underline{A}$-subobject of A by lemma 3 above. Finally choose $\gamma(\underline{A})$ such that X^* is $\gamma(\underline{A})$-generated. $\gamma(\underline{A})$ doesn't depend on X by proposition 4 above and 2.3 .

Now each element A of $\underline{A}$ may be written as $\varinjlim G_i$, where each G_i is a γ-generated $\underline{X}$-subobject of A . Thus $A = \varinjlim G_i^*$ in $\underline{A}$ and the set $\{G^* : G$ is γ-generated in $\underline{X}\}$ forms a generating set for $\underline{A}$.

Proof of 2.9: We show that if $\text{rank}_{\underline{X}} A < \beta$ for any $\beta > \gamma$, then $\text{rank}_{\underline{A}} A < \beta$. Thus the elements of the generating set constructed for $\underline{A}$ all have rank in $\underline{A}$. Suppose $\underline{D}$ is a β-filtered diagram in $\underline{A}$ having colimit L. Each $\underline{A}$-map $a : A \to L$ factors through a cofinal subdiagram $\underline{B}$ of $\underline{D}$ via $\underline{X}$-maps $b : A \to B$ for all $B \in \underline{B}$. Consider the diagram below:



Since $L^{\sigma r} = \varinjlim \underline{B}^{\sigma r}$ we have $\varinjlim b\sigma = a\sigma = \sigma a^{\sigma r} = \varinjlim \sigma b^{\sigma r}$ and for some $B_\sigma \in \underline{B}$ and $b_\sigma : A \to B_\sigma$ we have $b_\sigma \cdot \sigma = \sigma \cdot b_\sigma^{\sigma r}$. There exists some $B_* \in \underline{B}$ into which B_σ map, for all $\sigma \in [\underline{A}]$, since $\gamma > |[\underline{A}]|$ and $\underline{B}$ is γ-filtered. Clearly the factorization through B_* is an $\underline{A}$-map.

Proof of 2.10: Trivial by 2.6 and the SAFT.

Proof of 2.11: Let $B \rightrightarrows C$ be a pair of maps in $\underline{A}$ and suppose $X \longrightarrow B \rightrightarrows C$ is a split-equalizer diagram in $\underline{X}$. Let $\sigma : 1 \to \sigma r$ be a map in the prop defining $\underline{A}$. By the universality of split-equalizers, they are preserved by the functor $(\ )^{\sigma r}$, and $X^{\sigma r} \to B^{\sigma r} \rightrightarrows C^{\sigma r}$ is also a (split) equalizer. We have a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & B & \rightrightarrows & C \\ & & \sigma\downarrow & & \downarrow\sigma \\ X^{\sigma r} & \longrightarrow & B^{\sigma r} & \rightrightarrows & C^{\sigma r} \end{array}$$

defining a canonical map $X \to X^{\sigma r}$. Note that X is pure in B because the injection splits, and the result follows from 2.7, since the underlying functor $\underline{A} \to \underline{X}$ clearly reflects isomorphisms.

Notes on Chapter III: One can easily define a broader class of categories enriched over $\underline{C}$. If Φ is a set, a Φ-multiprop $\underline{M}$ is a monoidal category whose objects are those of the free-monoidal category on Φ generators in such a way that the full monoidal subcategory (FMS) generated by each element of Φ is a prop. Suppose $1 \in \Phi$ and let $\underline{M}'$ be the FMS of $\underline{M}$ generated by $\Phi - 1$. If $\underline{X}$ is our (monoidal) base category and $R: \underline{M}' \to \underline{X}$ is a MON-map, the category of R-widgets, denoted R-widg, has as its class of objects all MON-maps $\underline{M} \to \underline{X}$ that equal R when restricted to $\underline{M}'$.

Identify the elements of the FMS generated by 1 with $\mathbb{N}$ and assume $\underline{M}$ has a genrating set of maps $[\underline{M}]$ of the form $\sigma : \sigma U \otimes \sigma d \to \sigma L \otimes \sigma r$ where σU, $\sigma L \in \underline{M}'$ and σd, $\sigma r \in \mathbb{N}$. (The more general case is not hard, only messy.) As usual identify objects A, σU, and σL in $\underline{X}$ with $(1)A$, $(\sigma U)R$ and $(\sigma L)R$ respectively, where $A \in$ R-widg. Thus an object A in R-widg

is an object in $\underline{X}$ equipped with structure maps $(\sigma A\colon \sigma U \otimes A^{\sigma d} \to \sigma L \otimes A^{\sigma r})_{\sigma \in [\underline{M}]}$. An R-widg map $a : A \to B$ is an $\underline{X}$-map such that $(1 \otimes a^{\sigma d}) \cdot \sigma B = \sigma A \cdot (1 \otimes a^{\sigma r}) : U_\sigma \otimes A^{\sigma d} \to L_\sigma \otimes B^{\sigma r}$ for all $\sigma \in [\underline{M}]$.

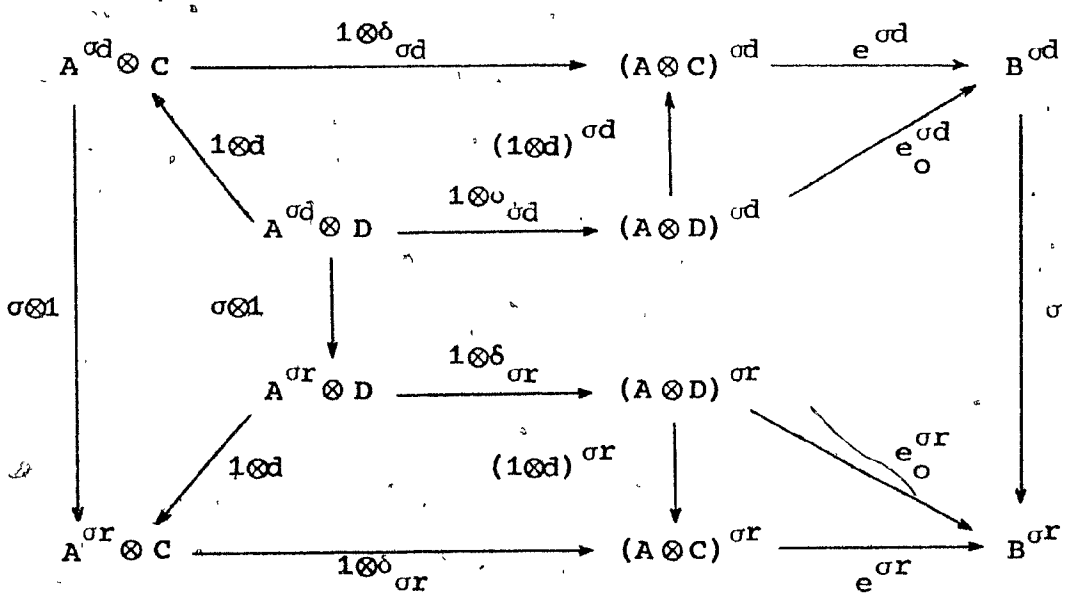
Let $\underline{A}$ be a category of widgets (over $\underline{X}$). Given $A,B \in \underline{A}$ and $C \in \underline{C}$, an evaluation may be defined to be an $\underline{X}$-map $e : A \otimes C \to B$ such that $(1 \otimes o_{\sigma d}) \cdot (1 \otimes e^{\sigma d}) \cdot \sigma B = (\sigma A \otimes 1) \cdot (1 \otimes \delta_{\sigma r}) \cdot (1 \otimes e^{\sigma r})$: $\sigma U \otimes A^{\sigma d} \otimes C \to \sigma L \otimes B^{\sigma r}$ for all $\sigma \in [\underline{M}]$. Just as for decent categories over $\underline{X}$, all widgets become enriched over $\underline{C}$. A widget is called algebraic (resp. coalgebraic) if $\sigma r = 1$ (resp. $\sigma d = 1$) for all $\sigma \in [\underline{M}]$. Maps between multiprops sending $\mathbb{N}$ to $\mathbb{N}$ yield <u>widgetable</u> functors. Now all the results of chapter III hold. In particular, all widgetable functors are strong.

<u>Proof of 3.1</u>: Let $\tilde{f}$ be the $\underline{X}$-map induced by f. For any n, $f^n = f\delta_n$ because f is a coalgebra map. Hence the condition $\tilde{f}^{\sigma d}\sigma = \sigma \tilde{f}^{\sigma r}$ becomes the commutivity of

$$\begin{array}{ccccccc}
A^{\sigma d} \cong A^{\sigma d} \otimes I & \xrightarrow{1 \otimes f} & A^{\sigma d} \otimes C & \xrightarrow{(1 \otimes \delta_{\sigma d}) e^{\sigma d}} & B^{\sigma d} \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma \otimes 1} & & \downarrow{\scriptstyle \sigma} \\
A^{\sigma r} \cong A^{\sigma r} \otimes I & \xrightarrow{1 \otimes f} & A^{\sigma r} \otimes C & \xrightarrow{(1 \otimes \delta_{\sigma r}) e^{\sigma r}} & B^{\sigma r}
\end{array}$$

The left rectangle commutes trivially, the right because e is an evaluation.

Proof of 3.2: Let $\underline{D}$ be a diagram in $\underline{C}$ with colimit C, and let $\{A \otimes D \xrightarrow{e_o} B\}_{D \in \underline{D}}$ be a compatible family of evaluations. Then $A \otimes C$ is the $\underline{X}$-colimit of $A \otimes \underline{D}$ and the evaluations e_o induce an $\underline{X}$-map $A \otimes C \xrightarrow{e} B$. To see that e is an evaluation consider the diagram below, where $D \xrightarrow{d} C$ is the colimit induced injection for $D \in \underline{D}$:



All the inner polygons commutes, and since the d's are collectively epi, the outer rectangle commutes.

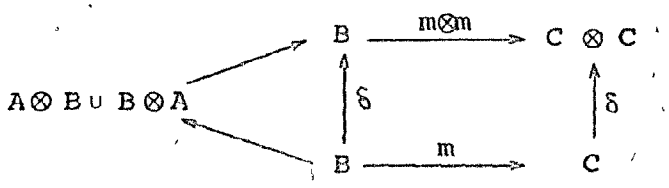
Proof of 3.4: Suppose $\underline{A}$ is coalgebraic over $\underline{X}$. For $A \in \underline{A}$ and $C \in \underline{C}$, define $C \underline{\otimes} A$ to be the object $C \otimes A$ equipped with structure maps equal to the composition $(\delta_\sigma \otimes \cdot) : C \otimes A \to C^{\sigma r} \otimes A^{\sigma r} \cong (C \otimes A)^{\sigma r}$. Since the associative and symmetric isomorphisms are coherent and δ commutes with them (i.e. C is coabelian and coassociative), the structure maps on $C \underline{\otimes} A$ satisfy of any equations the structure maps on A satisfy. Varification of the adjointness property is trivial. To see that $\underline{A}$ has cotensors using 2.9 and the SAFT, we must show that $(\ ,A): \underline{A}^{op} \to \underline{C}$ preserves limits, which follows trivially from the assumption that $\otimes$ preserves colimits and 2.5.

Proof of 3.5: The existence of cotensors is exactly dual to the first part of the proof of 3.4. Now it is trivial to check that for each $A \in \underline{A}$ the functor $(A,-): \underline{A} \to \underline{C}$ preserves limits, and as $\underline{A}$ is well-known to be complete and well-powered (and in fact co-wellpowered), we need only find a solution set in $\underline{A}$ for each given C in $\underline{C}$. Consider the free functor $F : \underline{X} \to \underline{A}$ and the object $(A \otimes C)F$ in $\underline{A}$. Let $\underline{Q}$ be the set of quotients Q of $(A \otimes C)F$ such that $A \otimes C \to (A \otimes C)F \to Q$ is an evaluation. Given a $B \in \underline{A}$ and a $\underline{C}$-map $C \to (A,B)$, this is equivalent to being given an evaluation $e : A \otimes C \to B$. e is a map in $\underline{X}$ and thus lifts to a map $f : (A \otimes C)F \to B$. Let $(A \otimes C)F \twoheadrightarrow Q \hookrightarrow B$ be a factorization of f. Since $Q \to B$ is a monomorphism and e is an evaluation, $A \otimes C \to (A \otimes C)F \to Q$ is an evaluation and $Q \in \underline{Q}$. Now the injection $Q \to B$ yields the obvious map

$(A,Q) \to (A,B)$ completing the diagram $(A,B) \leftarrow C \rightarrow (A,Q)$, and the set $\underline{Q}$ forms a solution set for C.

Notes on chapter IV: To understand the meaning of "singular" as we use it, note that the notion dual to that of union actually is intersection if one restricts one's attention to $\underline{X}$-split subobjects. If $i : A \to B$ is singular, let K be the cokernel of i in $\underline{X}$. Then the $\underline{X}$-spliting of i implies that $A \otimes B \cup B \otimes A$ is the kernel of $B \otimes B \to K \otimes K$. Thus if we wished to work with cokernels in $\underline{X}$, we could redefine singular to mean $B \to B \otimes B \to K \otimes K$ is zero. This is obviously dual to the notion of a square-zero kernel. One introduces kernels into the study of algebras precisely because quotients (cokernels) are so unwieldy. In the dual situation of coalgebras, there is no such need, and one may deal directly with the objects involved.

Proof of 4.1: We may assume m is an epimorphism in $\underline{C}$ by the assumption on factorizations preceeding the lemma. The following diagram commutes:



Thus we have $C\delta \subseteq (A \otimes B \cup B \otimes A)m \otimes m$

$= (A \otimes B)m \otimes m \cup (B \otimes A)m \otimes m \subseteq Am \otimes Bm \cup Bm \otimes Am = Am \otimes C \cup C \otimes Am$.

Proof of 4.2: One direction is trivial. Let $A \to B$ be singular and let $m : A \to C$ be a coalgebra map. Form the pushout diagram $\begin{array}{ccc} B & \longrightarrow & E \\ \uparrow & & \uparrow \\ A & \xrightarrow{m} & C \end{array}$ in $\underline{C}$. Since $A \to B$ splits in $\underline{X}$, so must $C \to E$. In particular $C \to E$ is a mono. Using 1 above it is obvious that $C \to E$ is singular. By assumption there exists a $\underline{C}$-map $E \to C$ splitting $C \to E$, and $B \to E \to C$ shows C is smooth.

Proof of 4.3: (cf. [10] p. 65) Let $d \in D_n(S_\xi)$ and let d_j be the first non-zero term of $(d_i)_{i>0}$. If d_j is a coboundary, let $a_* \in (A,A)$ be such that $a_*\partial_o = \xi \cdot a_* - a_*T \cdot \xi = d_j$, and let $a \in D_\infty(A,A)$, be defined by $a_o = 1$, $a_j = a_*$, $a_i = 0$ for $i > j$ and $1 < i < j$. Finally let $d' = aT \cdot d \cdot a^{-1}$. Using the composition formula (2), $d'_n = \sum_{i+k+1=n} a_i T \cdot d_k \cdot a_1^{-1}$. Since $d_k = 0$ for all $i < k < j$, we have $d'_n = 0$ for all $1 < n < j$. Now $a_j^{-1} = -a_*$, so $d'_j = d_j + a_*T \cdot \xi - \xi \cdot a_* = 0$ and the first non-zero term of $(d'_i)_{i>0}$ has subscript greater than j. The result follows by induction.

Proof of 4.5: The only if statement follows from the remarks preceeding the lemma. The converse follows from the remarks about cofree coalgebras.

Proof of 4.6: Let $d = (d_i)_{i<n}$ be in $D_{n-1}(S_\xi)$. Then using the composition as defined by diagram (2) we have

$$d^*\partial_2 = \sum_{\substack{i+j=n \\ i\neq o\neq j}} d_o T^2 \cdot d_i T \cdot d_j + T\mu \cdot d_i T \cdot d_j - \mu T \cdot d_i T \cdot d_j - (d_i T \cdot d_j) T \cdot d_o$$

Since T is strong $(d_iT\cdot d_j)T = d_iT^2\cdot d_jT$ and because μ is natural $T\mu\cdot d_iT = d_iT^2\cdot\mu$. Now for each $j < n$, d_j is an element of S_ξ yielding $\mu\cdot d_j = \sum_{k+l=j} d_kT\cdot d_l$ and similarly (again using the strength of T) $\mu T\cdot d_iT = \sum_{k+l=i} d_kT^2\cdot d_lT$.

Substituting into the equation for $d^*\partial_2$ above yields

$$d^*\partial_2 = \sum_{\substack{i+j=n\\ i\neq o\neq j}} (d_oT^2\cdot d_iT\cdot d_j + d_iT^2 \cdot \sum_{k+l=j} d_kT\cdot d_l - (\sum_{k+l=i} d_kT^2\cdot d_lT)d_j$$

$$- d_iT^2\cdot d_jT\cdot d_o)$$

$$= \sum_{\substack{i+j=n\\ i\neq o\neq j}} (\sum_{\substack{k+l=j\\ k\neq o\neq l}} d_iT^2\cdot d_kT\cdot d_l - \sum_{\substack{k+l=i\\ k\neq o\neq l}} d_kT^2\cdot d_lT\cdot d_j)$$

$$= \sum_{\substack{i+k+l=n\\ i\neq o\\ k\neq o\\ l\neq o}} d_iT^2\cdot d_kT\cdot d_l - d_iT^2\cdot d_kT\cdot d_l = 0$$

Proof of 4.8: The element s must satisfy the equation $(s\otimes E).s = (E\otimes s).s : E^3 \to E$ by the definition of S . $E^3 = (A+M)^3$ has an eight term expansion, and the eight equations of proposition 8 express the restriction on the injections made by the above restriction on s (cf. [14]).

Proof of 4.9: By definition a deformation (s_n) of s must satisfy the equation

$\sum_{i+j=n} (s_i\otimes E)\cdot s_j = \sum_{i+j=n} (E\otimes s_i)\cdot s_j$. As in the proof above,

this puts restrictions on the eight injections into E^3, the restrictions expressed by the eight points of the proposition.

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