

Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

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To cite this article: Thomas Fox (1976) Coalgebras and cartesian categories, Communications in Algebra, 4:7, 665-667, DOI: 10.1080/00927877608822127

To link to this article: http://dx.doi.org/10.1080/00927877608822127

Published online: 27 Jun 2007.

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COMMUNICATIONS IN ALGEBRA, 4(7), 665-667 (1976)

COALGEBRAS AND CARTESIAN CATEGORIES

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Let $\underline{X}$ be a monoidal category. We show that the best approximation to $\underline{X}$ by a cartesian category is the category of coalgebra objects in $\underline{X}$.

Recall that a monoidal category is a 7-tuple $(\underline{X}, \otimes, I, a, s, \ell, r)$ where $\underline{X}$ is a category equipped with a bifunctor, "tensor", $\otimes: \underline{X} \times \underline{X} \to \underline{X}$, an object I and coherent natural isomorphisms $a: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $s: X \otimes Y \to Y \otimes X$, $\ell: I \otimes X \to X$, and $r: X \otimes I \to X$, for all X, Y, Z in $\underline{X}$. We write $\underline{X}$ in place of $(\underline{X}, \otimes \ldots)$ whenever it is unambiguous. $\underline{X}$ is said to be cartesian if $\otimes$ is the categorical product. In that case I is terminal, and we may assume ℓ and r are projections.

Let $\underline{MON}$ be the category of monoidal categories and strict functors between them (see [M]). Maps in $\underline{MON}$ preserve all of the monoidal structure precisely, including the natural isomorphisms a, s, ℓ, and r. Similarly, let $\underline{CART}$ denote the

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category of all cartesian categories. Using the fact that MON and CART are both algebraic over CAT (the category of categories, see [L]), it is easy to show that the inclusion CART → MON has a left adjoint. We show that it has a right adjoint.

Given a monoidal category $\underline{X}$, a (coassociative, coabelian, counitary) $\underline{X}$-coalgebra is a triple (C, δ, ϵ), where C is an object in $\underline{X}$ equipped with maps $C \xrightarrow{\delta} C \otimes C$ and $C \xrightarrow{\epsilon} I$ such that $\delta(\delta \otimes C)a = \delta(C \otimes \delta)$, $\delta s = \delta$, $\delta(C \otimes \epsilon)r = C$, and $\delta(\epsilon \otimes C)\ell = C$. If $(C, \delta_C, \epsilon_C)$ and $(B, \delta_B, \epsilon_B)$ are coalgebras, an $\underline{X}$-map $C \xrightarrow{\alpha} B$ is a coalgebra map if $\delta_C(\alpha \otimes \alpha) = \alpha\delta_B$ and $\epsilon_C = \alpha\epsilon_B$. The category of $\underline{X}$-coalgebras is denoted $(\underline{X})\underline{C}$. Let $U: (\underline{X})\underline{C} \to \underline{X}$ be the natural forgetful functor.

Given $(C, \delta_C, \epsilon_C)$ and $(B, \delta_B, \epsilon_B)$ in $(\underline{X})\underline{C}$, define $\delta: C \otimes B \to (C \otimes B) \otimes (C \otimes B)$ and $\epsilon: C \otimes B \to I$ by $\delta = (\delta_C \otimes \delta_B)(C \otimes s \otimes B)$ and $\epsilon = (\epsilon_C \otimes \epsilon_B)r$. Then just as in the case $\underline{X} =$ Mod-R (see [S] or [B]), $(C \otimes B, \delta, \epsilon)$ is a coalgebra and is the product of $(C, \delta_C, \epsilon_C)$ and $(B, \delta_B, \epsilon_B)$ in $(\underline{X})\underline{C}$. The projections are $(C \otimes \epsilon_B)r$ and $(\epsilon_C \otimes B)\ell$. Since I has an obvious trivial coalgebra structure, we see that $(\underline{X})\underline{C}$ is a cartesian category in a way that U is a map in MON.

Proposition: $(-)\underline{C}$ defines a functor MON → CART right adjoint to the inclusion.

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Proof: We show that the forgetful functor $U:(\underline{X})\underline{C} \to \underline{X}$ is the adjunction in $\underline{MON}$. Given a cartesian category $\underline{Y}$, and a map $F:\underline{Y} \to \underline{X}$ in $\underline{MON}$, we define a map $\overline{F}: \underline{Y} \to (\underline{X})\underline{C}$ as follows: For each Y in $\underline{Y}$, let $\Delta: Y \to Y \times Y$ be the diagonal, and let $\epsilon: Y \to I$ be the isomorphism $Y \xrightarrow{r} Y \times I$ followed by the second projection. Let $Y\overline{F} = (YF, \Delta F, \epsilon F)$ in $(\underline{X})\underline{C}$. It is trivial to check that $\overline{F}$ preserves the necessary structure and is the unique lifting of F.

Note, if one requires that monoidal categories be cocomplete or that tensors preserve colimits, our results still hold; the forgetful functor U always creates colimits. If $\underline{X}$ is locally-presentable (hence closed), then $(\underline{X})\underline{C}$ is cartesian closed, [F].

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Received: August 1975

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