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Effect Algebras and Unsharp Quantum Logics

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The effects in a quantum-mechanical system form a partial algebra and a partially ordered set which is the prototypical example of the effect algebras discussed in this paper. The relationships among effect algebras and such structures as orthoalgebras and orthomodular posets are investigated, as are morphisms and group-valued measures (or charges) on effect algebras. It is proved that there is a universal group for every effect algebra, as well as a universal vector space over an arbitrary field.

1. BOOLEAN ALGEBRAS

In George Boole's seminal works *Mathematical Analysis of Logic* (1847) and *Laws of Thought* (1854), which led to the branch of algebra that now bears his name, the sum $a + b$ (now called the union) was defined only for *disjoint* classes a and b. In Boole's own words (Ref. 6, pp. 32–33), "In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words "and" "or" are analogous with the sign + in algebra, and their laws are identical." The more general notion of an inclusive sum or union of possibly overlapping classes was introduced later by Jevons, Peirce, and Schröder.

In contemporary mathematical terms, a Boolean algebra is usually defined either as a complemented distributive lattice or as an idempotent ring with unit. However, an equivalent definition can be formulated in the spirit of Boole using only a *partially defined sum*. Indeed, with the alternative notation $\oplus$ for this sum, we have the following theorem:

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1.1. Theorem. Let $(L, \oplus, 0, 1)$ be a system consisting of a set L with two special elements $0, 1 \in L$ and equipped with a partially defined binary operation $\oplus$ satisfying the following conditions for all $p, q, r \in L$:

(i) [*Commutative Law*] If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

(ii) [*Associative Law*] If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(iii) [*Orthosupplementation Law*] For every $p \in L$ there exists a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = 1$.

(iv) [*Zero-One Law*] If $1 \oplus p$ is defined, then $p = 0$.

(v) [*Coherence Law*] If $p \oplus q$, $p \oplus r$, and $q \oplus r$ are defined, then $(p \oplus q) \oplus r$ is defined.

(vi) [*Law of Compatibility*] For all $p, q \in L$, there exist $a, b, c \in L$ such that $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, $p = a \oplus c$ and $q = b \oplus c$.

Then L can be organized into a Boolean algebra in one and only one way so that $0 \leqslant p \leqslant 1$ for all $p \in L$, $p \oplus q$ is defined iff $p \wedge q = 0$, and $p \wedge q = 0 \Rightarrow p \oplus q = p \vee q$. Conversely, if L is a Boolean algebra, and if the partial binary operation $\oplus$ is defined by $p \oplus q = p \vee q$ iff $p \wedge q = 0$, then $(L, \oplus, 0, 1)$ satisfies conditions (i) through (vi).

Later in this paper, we indicate how Theorem 1.1 is proved.

2. EFFECT ALGEBRAS

If a quantum-mechanical system $\mathscr{S}$ is represented in the usual way by a Hilbert space $\mathscr{H}$, then a self-adjoint operator A on $\mathscr{H}$ such that $\mathbb{0} \leqslant A \leqslant \mathbb{1}$ corresponds to an *effect* for $\mathscr{S}$.[21, 22] Effects are of significance in representing *unsharp* measurements or observations on the system $\mathscr{S}$,[8] and effect-valued measures play an important role in stochastic quantum mechanics.[2, 29, 30] As a consequence, there have been a number of recent efforts to establish appropriate axioms for logics, algebras, or posets suggested by or based on effects.[10, 15, 19]

The set L of all effects for a Hilbert space can be organized into a system $(L, \oplus, \mathbb{0}, \mathbb{1})$ that satisfies Conditions (i)–(iv) in Theorem 1.1 by defining

$$A \oplus B = A + B \qquad \text{iff} \quad A + B \leqslant \mathbb{1}$$

This leads us to the following definition:

2.1. Definition. A system $(L, \oplus, 0, 1)$ that satisfies Conditions (i), (ii), (iii), and (iv) in Theorem 1.1 is called an *effect algebra.*

Unless confusion threatens, we say that L is an effect algebra when what we really mean is that $(L, \oplus, 0, 1)$ is an effect algebra. Also, when we write an equation such as $p \oplus r = q$ in an effect algebra, we are asserting both that $p \oplus r$ is defined and that $p \oplus r = q$.

For the remainder of this paper, we assume that L is an effect algebra.

2.2. Definition. Let $p, q \in L$.

(i) We say that p is *orthogonal* to q and write $p \perp q$ iff $p \oplus q$ is defined. If $0 \neq p$ and $p \perp p$, then p is called an *isotropic* element of L.

(ii) We say that p is *less than or equal to* q and write $p \leqslant q$ iff there exists an element $r \in L$ such that $p \perp r$ and $p \oplus r = q$.

(iii) The unique element q such that $p \perp q$ and $p \oplus q = 1$ is written as $p' := q$ and called the *orthosupplement* of p. (We use the notation $:=$ to mean "*equals by definition.*")

2.3. Lemma. Let $p, q \in L$.

(i) $p \perp q \Rightarrow q \perp p$,

(ii) $p'' = p$,

(iii) $1' = 0$ and $0' = 1$,

(iv) $p \oplus p' = 1$,

(v) $p \perp 0$ and $p \oplus 0 = p$,

(vi) $p \perp 1 \Leftrightarrow p = 0$,

(vii) $p \oplus q = 0 \Rightarrow p = q = 0$.

Proof. (i), (ii), (iii), and (iv) are obvious. (v): $1 = 1 \oplus 1' = (p' \oplus p) \oplus 0 = p' \oplus (p \oplus 0)$; hence, $p \oplus 0 = p'' = p$. (vi): $p \perp 1 \Rightarrow p = 0$ is the zero-one law and the converse follows from Part (v). (vii): If $p \oplus q = 0$, then $1 = 1 \oplus 0 = 1 \oplus (p \oplus q) = (1 \oplus p) \oplus q$, so $1 \perp p$, and it follows that $p = 0$. By symmetry, $q = 0$. □

2.4. Theorem. Let $p, q \in L$.

(i) $p \perp q \Rightarrow p \perp (p \oplus q)'$ and $p \oplus (p \oplus q)' = q'$,

(ii) $p \perp q \Leftrightarrow p \leqslant q'$,

(iii) $p \leqslant q \Rightarrow q' \leqslant p'$,

(iv) $p \leqslant q \Rightarrow p \perp (p \oplus q')'$ and $p \oplus (p \oplus q')' = q$.

Proof. (i): Let $r := (p \oplus q)'$. Then, $1 = (p \oplus q) \oplus r = q \oplus (p \oplus r)$, so $q' = p \oplus r$. (ii): If $p \perp q$, then $p \leqslant q'$ by Part (i). Conversely, if $p \leqslant q'$, there exists $r \in L$ with $p \oplus r = q'$; hence, $1 = (p \oplus r) \oplus q = r \oplus (p \oplus q)$, and it follows that $p \perp q$. (iii): By Part (ii), we have $p \leqslant q \Rightarrow p \leqslant (q')' \Rightarrow p \perp q' \Rightarrow q' \perp p \Rightarrow q' \leqslant p'$. (iv) follows from replacing q by q' in Part (i) and using Part (ii). □

Part (iv) of Theorem 2.4 is called the *orthomodular identity*.

2.5. Theorem (*Cancellation Laws*). Let $p, q, r \in L$ with $p, q \perp r$. Then:

(i) $p \oplus r = q \oplus r \Rightarrow p = q$,

(ii) $p \oplus r \leqslant q \oplus r \Rightarrow p \leqslant q$.

Proof. (i): Assume that $p \oplus r = q \oplus r$ and let $s := (p \oplus r)' = (q \oplus r)'$. Then $(p \oplus r) \oplus s = 1 = (q \oplus r) \oplus s$, so $p \oplus (r \oplus s) = q \oplus (r \oplus s) = 1$, and it follows that $p = (r \oplus s)' = q$. (ii): Assume that $p \oplus r \leqslant q \oplus r$. Then, there exists $s \in L$ with $(p \oplus s) \oplus r = q \oplus r$; hence, $p \oplus s = q$ by Part (i), so $p \leqslant q$. □

2.6. Definition. A subset $A \subseteq L$ is called a *sub-effect algebra* of L iff $0, 1 \in A$, A is closed under $p \mapsto p'$, and, for all $p, q \in A$, $p \perp q \Rightarrow p \oplus q \in A$.

If it is understood that we are working with effect algebras, we often refer to a sub-effect algebra simply as a *subalgebra*. Clearly, a subalgebra A of L is an effect algebra in its own right. As such, if A is a Boolean algebra, we call it a *Boolean subalgebra*.

3. THE ORDER STRUCTURE OF AN EFFECT ALGEBRA

3.1. Theorem. An effect algebra L is partially ordered by $\leqslant$ and $0 \leqslant p \leqslant 1$ holds for all $p \in L$.

Proof. All is clear except possibly for the antisymmetry of $\leqslant$. Thus, suppose that $a, b \in L$ with $a \leqslant b$ and $b \leqslant a$. Then there are elements $p, q \in L$ with $a \oplus p = b$ and $b \oplus q = a$. Hence, $a \oplus 0 = a = b \oplus q = (a \oplus p) \oplus q = a \oplus (p \oplus q)$, so $p \oplus q = 0$ by the cancellation law. Therefore, $p = q = 0$, by Part (iv) of Lemma 2.3, and it follows that $a = b$. □

If $(P, \leqslant)$ is a partially ordered set (poset) and $a, b \in P$ with $a \leqslant b$, we use the notation $P[a, b]$, or simply $[a, b]$ if P is understood, for the *interval* $P[a, b] := \{p \in P \mid a \leqslant p \leqslant b\}$. If P has a smallest element 0, we define an *atom* in P, as usual, to be a nonzero element $a \in P$ such that $P[0, a] = \{0, a\}$, and we say that P is *atomic* iff, for every nonzero element $p \in P$,

there is an atom $a \in P$ with $a \leqslant p$. We also use the standard notation $p \wedge q$ and $p \vee q$ for the meet (greatest lower bound) and join (least upper bound) of p and q *when they exist in* P. If P has a smallest element 0, then two elements $p, q \in P$ are said to be *disjoint* iff $p \wedge q = 0$, that is, iff $r \in P$ with $r \leqslant p, q \Rightarrow r = 0$.

A poset P is said to satisfy the *ascending* (respectively, *descending*) *chain condition* iff every nonempty subset of P has a maximal (respectively, minimal) element with respect to the partial order $\leqslant$. We say that P *satisfies the chain conditions* iff it satisfies both the ascending and descending chain conditions.

If we ascribe an order-theoretic property to the effect algebra L, we mean that the poset $(L, \leqslant)$ has that property. For instance, L is said to be *lattice ordered* iff $(L, \leqslant)$ is a lattice and it is said to be *linearly* (or *totally*, or *fully*) ordered iff at least one of $p \leqslant q$ or $q \leqslant p$ holds for all $p, q \in L$. Owing to the fact that the orthosupplement mapping $p \mapsto p'$ is an order-reversing involution on an effect algebra L, it follows that L satisfies the ascending chain condition iff it satisfies the descending chain condition.

3.2. Definition. A linearly ordered effect algebra is called a *scale algebra.*

It is important to realize that the order structure on an effect algebra is *derived* from its algebraic structure, and not vice versa. Indeed, there are bounded posets that can be organized into an effect algebra in more than one way. (See the example in Section 8.) Furthermore, there are bounded posets that cannot be organized into an effect algebra. (For instance, as a poset, an effect algebra is isomorphic to its own dual.)

If G is an additively written abelian group, then a subset C of G is said to be a *cone* in G if it satisfies $C + C \subseteq C$ and $C \cap -C = \{0\}$. A cone C in G induces a partial order $\leqslant$ on G such that $x \leqslant y$ iff $\exists z \in C$, $x + z = y$ and the induced partial order is *translation invariant* in the sense that $x \leqslant y \Rightarrow x + a \leqslant y + a$. Conversely, a translation invariant partial order $\leqslant$ on G determines a cone $C := \{x \in G \mid 0 \leqslant x\}$ in G which, in turn, induces the partial order $\leqslant$. A *partially ordered abelian group*[5, 14] is an abelian group G equipped with a translation invariant partial order $\leqslant$. If G is a partially ordered abelian group, then the set $\{x \in G \mid 0 \leqslant x\}$, called the *positive cone* in G, is often denoted by G^+. For instance, we use the notation $\mathbb{Z}^+$ for the positive cone in the additive group $\mathbb{Z}$ of integers with the usual ordering. We omit the simple verification of the following:

3.3. Theorem. Let G be a partially ordered abelian group, let $0 \neq u \in G^+$, and let $L := G^+[0, u] = \{g \in G \mid 0 \leqslant g \leqslant u\}$. Then L can be

organized into an effect algebra $(L, 0, u, \oplus)$ by defining $p \oplus q$ iff $p + q \in L$, in which case $p \oplus q = p + q$. In the effect algebra L, we have $p' = u - p$ and the effect partial order on L coincides with the restriction to L of the partial order on G.

3.4. Definition. An effect algebra of the form $G^+[0, u]$ as in Theorem 3.3 (or isomorphic to an effect algebra of this form) is called an *interval effect algebra*, or, for short, *an interval algebra.*

If G is any totally ordered abelian group with positive cone G^+ and $0 \neq u \in G^+$, then the interval effect algebra $G^+[0, u]$ is a scale algebra as in Definition 3.2. In particular, a finite chain consisting of $n + 1$ elements can be realized in this way as $\mathbb{Z}^+[0, n]$. It can be shown that every scale algebra is an interval algebra in a totally ordered abelian group.

3.5. Definition. If $\mathbb{R}$ is regarded as an additive totally ordered abelian group with the usual positive cone $\mathbb{R}^+$, the interval algebra $\mathbb{R}^+[0, 1]$ is called the *standard scale algebra.*

4. SUMS, MULTIPLES, AND DIFFERENCES IN AN EFFECT ALGEBRA L

We can extend the notion of orthogonal sums in an effect algebra L to finite sequences as follows:

4.1. Definition. Let $a_1, a_2, a_3, \ldots, a_n$ be a finite sequence in L. If $n = 1$, we say that $\bigoplus (a_1)$ exists in L and define $\bigoplus (a_1) := a_1$. By recursion, we say that the *orthogonal sum* $\bigoplus (a_1, a_2, a_3, \ldots, a_n)$ exists in L iff $\bigoplus (a_1, a_2, a_3, \ldots, a_{n-1})$ and $\bigoplus (a_1, a_2, a_3, \ldots, a_{n-1}) \oplus a_n$ exist in L, in which case we define

$$\bigoplus (a_1, a_2, a_3, \ldots, a_n) := \bigoplus (a_1, a_2, a_3, \ldots, a_{n-1}) \oplus a_n$$

If the orthogonal sum $\bigoplus (a_1, a_2, a_3, \ldots, a_n)$ exists in L, we say that $a_1, a_2, a_3, \ldots, a_n$ is an *orthogonal sequence* in L. Note that a, b is an orthogonal sequence in L iff $a \perp b$, in which case $\bigoplus (a, b) = a \oplus b$. More generally, for an orthogonal sequence $a_1, a_2, a_3, \ldots, a_n$ in L, we define

$$a_1 \oplus a_2 \oplus a_3 \oplus \cdots \oplus a_n := \bigoplus (a_1, a_2, a_3, \ldots, a_n)$$

Alternative notation for the orthogonal sum is $\bigoplus_{i=1}^{n} a_i$, or simply $\bigoplus_i a_i$ if n is understood.

The following theorem is proved by straightforward arguments using mathematical induction.

4.2. Theorem. Suppose that $a_1, a_2, a_3, \ldots, a_n$ is an orthogonal sequence in L. Then:

(i) If σ is a permutation of $\{1, 2, 3, \ldots, n\}$, then $a_{\sigma 1}, a_{\sigma 2}, a_{\sigma 3}, \ldots, a_{\sigma n}$ is an orthogonal sequence in L and $\bigoplus (a_1, a_2, a_3, \ldots, a_n) = \bigoplus (a_{\sigma 1}, a_{\sigma 2}, a_{\sigma 3}, \ldots, a_{\sigma n})$.

(ii) If $1 \leqslant k \leqslant n-1$, then $a_1, \ldots, a_k$ and $a_{k+1}, \ldots, a_n$ are orthogonal sequences in L and $\bigoplus (a_1, a_2, a_3, \ldots, a_n) = \bigoplus (a_1, \ldots, a_k) \oplus \bigoplus (a_{k+1}, \ldots, a_n)$.

(iii) If $b_1, b_2, b_3, \ldots, b_m$ is an orthogonal sequence in L, then $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m$ is an orthogonal sequence iff $\bigoplus (a_1, a_2, a_3, \ldots, a_n) \perp \bigoplus (b_1, b_2, b_3, \ldots, b_m)$.

(iv) If $b_k \in L$ with $b_k \leqslant a_k$ for $k = 1, 2, \ldots, n$, then $b_1, b_2, b_3, \ldots, b_n$ is an orthogonal sequence in L and $\bigoplus (b_1, b_2, b_3, \ldots, b_n) \leqslant \bigoplus (a_1, a_2, a_3, \ldots, a_n)$.

4.3. Definition. Let n be a nonnegative integer and let $a \in L$. If $n = 0$, we define $na := 0$, and if $n = 1$, we define $na := a$. If $n > 1$, let $a_i := a$ for $i = 1, 2, \ldots, n$. We say that na is defined iff $a_1, a_2, a_3, \ldots, a_n$ is an orthogonal sequence, in which case we define $na := \bigoplus (a_1, a_2, a_3, \ldots, a_n)$.

4.4. Lemma. Let $a \in L$, let n, h, k be nonnegative integers, and suppose that na is defined. Then:

(i) ka is defined for all $k \leqslant n$.

(ii) If $h + k \leqslant n$, then $(h + k)a = ha + ka$.

(iii) If $n > 0$, then $na = 0 \Leftrightarrow a = 0$.

(iv) If ha and ka are defined and $a \neq 0$, then $ha \leqslant ka \Leftrightarrow h \leqslant k$.

Proof. Use Definition 4.3, Part (ii) of Theorem 4.2, induction, the cancellation law, and Part (vii) of Lemma 2.3. □

If $0 \neq p \in L$, then p is isotropic iff $2p$ is defined in L.

If $a_1, a_2, \ldots, a_k \in L$, $n_1, n_2, \ldots, n_k \in \mathbb{Z}^+$, and $p \in L$, we write

$$p = \bigoplus_{i=1}^{k} n_i a_i = n_1 a_1 \oplus n_2 a_2 \oplus \cdots \oplus n_k a_k$$

to mean that $n_1 a_1, n_2 a_2, \ldots, n_k a_k$ are defined in L, $n_1 a_1, n_2 a_2, \ldots, n_k a_k$ is an orthogonal sequence in L, and $p = \bigoplus_i n_i a_i$. If there are integers

$n_1, n_2, ..., n_k \in \mathbb{Z}^+$ such that $p = \bigoplus_i n_i a_i$, we say that p is a (*finite*) *orthocombination* of the elements $a_1, a_2, a_3, ..., a_k$ with *coefficients* $n_1, n_2, ..., n_k$.

4.5. Definition. A subset X of L is said to *generate* L iff every element of L is a finite orthocombination of elements in X.

If $p, q \in L$ with $p \leqslant q$, there exists $r \in L$ with $p \perp r$ and $p \oplus r = q$. By the cancellation law, r is uniquely determined, and we can formulate the following definition:

4.6. Definition. If $p, q \in L$ with $p \leqslant q$, we define the *difference* $q - p$ to be the unique element in L that satisfies $p \oplus (q - p) = q$.

Proofs of the next three lemmas are omitted since they follow directly from Definition 4.6 and previously developed facts about effect algebras.

4.7. Lemma. If $p \in L$, then $p = p - 0$, $0 = p - p$, and $p' = 1 - p$.

4.8. Lemma. If $p, q \in L$ with $p \leqslant q$, then:

(i) $p = q \Leftrightarrow q - p = 0$,

(ii) $p = 0 \Leftrightarrow q - p = q$,

(iii) $q - p \leqslant q$ and $p = q - (q - p)$,

(iv) $r \leqslant q - p \Rightarrow p \leqslant q - r$ and $(q - p) - r = (q - r) - p$,

(v) $q - p = (p \oplus q')'$,

(vi) $p \oplus q' = (q - p)'$.

4.9. Lemma. If $p, q, r \in L$ and $p \leqslant q \leqslant r$, then $r = p \oplus (q - p) \oplus (r - q)$.

4.10. Remark. As a consequence of Lemma 4.9, a number of identities follow from the condition $p \leqslant q \leqslant r$ in an effect algebra. Among these are

(i) $r - p = (q - p) \oplus (r - q)$,

(ii) $(r - p) - (r - q) = q - p$,

(iii) $r - (q - p) = p \oplus (r - q)$, and so on.

4.11. Theorem. If L satisfies the chain conditions, then L is generated by the set of all atoms in L.

Proof. Because of the descending chain condition, L is atomic. Suppose $0 \neq p \in L$ and p cannot be written as a finite linear combination of atoms in L. Choose an atom $a_1 \leqslant p$, noting that $p - a_1 \neq 0$. Choose an

atom $a_2 \leqslant p - a_1$, noting that $(p - a_1) - a_2 \neq 0$. Continuing in this way, we produce an infinite sequence of (not necessarily distinct) atoms $a_1, a_2, a_3, \ldots$ in L such that $(a_1, a_2, \ldots, a_n)$ is an orthogonal sequence for every $n \geqslant 1$. Let $p_n := \bigoplus (a_1, a_2, \ldots, a_n)$ for $n = 1, 2, 3, \ldots$. Then $p_1 < p_2 < \cdots < p_n < \cdots$ contradicts the ascending chain condition. □

Kôpka and Chovanec[19] have introduced the following definition:

4.12. Definition. A *difference poset* is a partially ordered set $(D, \leqslant, 1)$ with a maximum element 1 and with a partially defined binary operation $(p, q) \mapsto q - p$, defined iff $p \leqslant q$, such that, for all $p, q, r \in D$ with $p \leqslant q \leqslant r$:

(i) $q - p \leqslant q$,

(ii) $q - (q - p) = p$,

(iii) $r - q \leqslant r - p$,

(iv) $(r - p) - (r - q) = q - p$.

In view of Part (iii) of Lemma 4.8 and Part (ii) of Remark 4.10, every effect algebra is a difference poset. Conversely, suppose D is a difference poset and define $0 := 1 - 1$. Also, for $p, q \in D$, define $p \oplus q \in D$ iff $p \leqslant 1 - q$, in which case

$$p \oplus q := 1 - ((1 - q) - p)$$

In Ref. 11, Dvurečenskij proves that $\oplus$ satisfies Conditions (i) and (ii) of Theorem 1.1 and that, for $p \leqslant 1 - q$, $p \oplus q$ is the unique element $r \in L$ such that

$$q \leqslant r \qquad \text{and} \qquad p = r - q$$

Furthermore, using the results in Ref. 19, it can be seen that $\oplus$ also satisfies Conditions (iii) and (iv) of Theorem 1.1; hence, L is an effect algebra. Consequently, *difference posets and effect algebras are the same thing*.

5. ORTHOALGEBRAS, ORTHOMODULAR POSETS, ORTHOMODULAR LATTICES, AND BOOLEAN ALGEBRAS

An *orthoalgebra*[12, 13, 16, 20] is an effect algebra in which the zero-one law (Part (iv) of Theorem 1.1) is replaced by the stronger

$$[\textit{Consistency law}]\ p \perp p \Rightarrow p = 0$$

Therefore, *every orthoalgebra is an effect algebra.* Evidently, an effect algebra is an orthoalgebra iff it contains no isotropic elements. In the standard scale algebra $\mathbb{R}^+[0, 1]$, every nonzero element in the interval $\mathbb{R}^+[0, 1/2]$ is isotropic, so $\mathbb{R}^+[0, 1]$ is an effect algebra, but not an orthoalgebra.

The weak orthoalgebras introduced by Giuntini and Greuling[15] can be shown to be equivalent to effect algebras. A connection between difference posets and orthoalgebras was originally discovered by Navarra and Pták.[25] In terms of effect algebras, this connection is as follows:

5.1. Theorem. For an effect algebra L, the following conditions are mutually equivalent:

(i) L is an orthoalgebra,

(ii) $p, q \in L$ with $p \perp q \Rightarrow p \oplus q$ is a minimal upper bound for p and q in L,

(iii) $p \in L \Rightarrow p \wedge p' = 0$,

(iv) L is orthocomplemented by the mapping $p \mapsto p'$.

Proof. (i) $\Rightarrow$ (ii) by Ref. 12, Theorem 2.10. (ii) $\Rightarrow$ (iii): Assume (ii) and suppose $r \in L$ with $r \leqslant p, p'$. Then $p, p' \leqslant r' \leqslant 1 = p \oplus p'$, so $r' = 1$, $r = 0$. (iii) $\Rightarrow$ (iv): By Parts (ii) and (iii) of Theorem 2.4, $p \mapsto p'$ is an order-inverting involution on the bounded poset L; hence, the deMorgan law is effective. Therefore, since $p \wedge p' = 0$ for all $p \in L$, it follows that $p \vee p' = 0' = 1$ for all $p \in L$. (iv) $\Rightarrow$ (i): Suppose L is orthocomplemented by $p \mapsto p'$ and let $q \in L$ with $q \perp q$. Then $q \leqslant q'$, so $0 = q \wedge q' = q$. □

5.2. Lemma. The only scale algebra that is also an orthoalgebra is the Boolean algebra $2 := \{0, 1\}$.

Proof. Suppose L is both a scale algebra and an orthoalgebra and let $a \in L$. Then either $a \leqslant a'$, in which case $a \perp a$ so $a = 0$, or else $a' \leqslant a$, in which case $a' \perp a'$, so $a' = 0$ and $a = 1$. □

Recall that an *orthomodular poset* (OMP) may be regarded as an orthoalgebra L that satisfies the *coherence law* for $p, q, r \in L$: $p \perp q$, $p \perp r$, $q \perp r \Rightarrow p, q, r$ is an orthogonal sequence; that is, Condition (v) in Theorem 1.1 holds. (See Ref. 12.)

5.3. Theorem. An effect algebra L is an OMP iff it satisfies the coherence law.

Proof. We have to prove that if L satisfies the coherence law, then L is an orthoalgebra. Assume the coherence law, and suppose $q \in L$ with

$q \perp q$. Then, with $p := q$ and $r := q'$, we have $p \perp q$, $p \perp r$, $q \perp r$, and it follows that (q, q, q') is an orthogonal sequence. Thus, $q \perp (q \oplus q') = 1$, so $q = 0$. □

An *orthomodular lattice* (OML)[17] may be defined as a lattice-ordered orthoalgebra. With the lattice of projection operators on a Hilbert space as exemplar, orthomodular lattices (or orthomodular posets) are often viewed as nonclassical *proposition systems* or *quantum logics*,[1, 3, 28] and it has been argued that the more general orthoalgebras can play the same role.[12] With this in mind, an effect algebra can be regarded as a (possibly) *unsharp* quantum logic.

An orthoalgebra is called a *centerium* iff it satisfies condition (v).[27] Thus, in view of Theorem 5.3, Theorem 1.1 says that a Boolean algebra is the same thing as an OMP that is also a centerium. This, however, is well known.[17, 27]

6. MORPHISMS, MEASURES, AND REPRESENTATIONS

6.1. Definition. Let L and P be effect algebras. A mapping $\phi: L \to P$ is said to be

(i) a *morphism* iff $\phi(1) = 1$ and $p, q \in L$ with $p \perp q \Rightarrow \phi(p) \perp \phi(q)$ and $\phi(p \oplus q) = \phi(p) \oplus \phi(q)$;

(ii) a *positive* morphism iff ϕ is a morphism and $p \in L$ with $\phi(p) = 0 \Rightarrow p = 0$;

(iii) a *homomorphism* iff ϕ is a morphism and $p, q \in L$ with $p \wedge q = 0 \Rightarrow \phi(p) \wedge \phi(q) = 0$;

(iv) a *monomorphism* iff ϕ is a morphism and $p, q \in L$ with $\phi(p) \leqslant \phi(q) \Rightarrow p \leqslant q$; and

(v) an *isomorphism* iff ϕ is a surjective monomorphism.

6.2. Definition. A *probability measure* for L is a morphism $\mu: L \to \mathbb{R}^+[0, 1]$ of L into the standard scale algebra.

6.3. Definition. Let P, Q, and L be effect algebras. A mapping $\beta: P \times Q \to L$ is called a *bimorphism* iff, for each $p \in P$ and each $q \in Q$, $\beta(p, \cdot): Q \to L$ and $\beta(\cdot, q): P \to L$ are morphisms and $\beta(1, 1) = 1$.

If L and P are effect algebras and $\phi: L \to P$ is a morphism, it is clear that $\phi(0) = 0$ and that, for $p, q \in L$, $p \leqslant q \Rightarrow \phi(p) \leqslant \phi(q)$ with $\phi(q - p) = \phi(q) - \phi(p)$. In particular, $\phi(p') = \phi(p)'$. By Part (ii) of Lemma 2.4, a

morphism $\phi: L \to P$ is a monomorphism iff $\phi(p) \perp \phi(q) \Rightarrow p \perp q$ for all $p, q \in L$. Also, if $\phi: L \to P$ is an isomorphism, then ϕ is a bijection and $\phi^{-1}: P \to L$ is an isomorphism.

When we speak of a *family* of morphisms on L, we do not necessarily mean to imply that all morphisms in the family map into the same effect algebra. The conditions given in the following definition are suggested by analogous conditions that have been studied for families of states on quantum logics.

6.4. Definition. Let Δ be a family of morphisms on L.

(i) Δ is *positive* iff, for every $p \in L$ with $p \neq 0$, $\exists \phi \in \Delta$, $\phi(p) \neq 0$.

(ii) Δ is *separating* iff, for $p, q \in L$ with $p \neq q$, $\exists \phi \in \Delta$, $\phi(p) \neq \phi(q)$.

(iii) Δ is *unital* iff, for $p \in L$ with $p \neq 0$, $\exists \phi \in \Delta$, $\phi(p) = 1$.

(iv) Δ is *order-determining* iff, for $p, q \in L$, the condition $\phi(p) \leqslant \phi(q)$ for all $\phi \in \Delta$ implies that $p \leqslant q$.

(v) Δ is *full* iff, for $p, q \in L$ with $p \not\leqslant q$, $\exists \phi \in \Delta$, $\phi(p) = 1$, $\phi(q) \neq 1$.

If L and P are orthomodular lattices, then a *homomorphism* $\phi: L \to P$ is usually defined to be a *lattice* homomorphism that preserves the orthocomplementation. In the following lemma and theorem, we show that this is consistent with our definition of a homomorphism of effect algebras in Part (iii) of Definition 6.1.

6.5. Lemma. Let L and P be orthomodular posets and let $\phi: L \to P$ be a morphism of L and P regarded as effect algebras. Suppose p and q are compatible (i.e., commuting) elements of L. Then $\phi(p)$ and $\phi(q)$ are compatible elements of P and $\phi(p \wedge q) = \phi(p) \wedge \phi(q)$.

Proof. Since p and q are compatible, there is an orthogonal sequence (a, b, c) in L such that $p = a \oplus c$ and $q = b \oplus c$. Because L is an orthomodular poset, $c = p \wedge q$. Thus, $(\phi(a), \phi(b), \phi(c))$ is an orthogonal sequence in P, $\phi(p) = \phi(a) \oplus \phi(c)$, and $\phi(q) = \phi(b) \oplus \phi(c)$, so $\phi(p)$ is compatible with $\phi(q)$ and $\phi(p) \wedge \phi(q) = \phi(c) = \phi(p \wedge q)$. □

6.6. Theorem. Let L and P be orthomodular lattices and let $\phi: L \to P$. Then ϕ is an effect-algebra homomorphism iff ϕ is an orthomodular-lattice homomorphism.

Proof. If ϕ is a homomorphism of orthomodular lattices, it is clear that it is an effect-algebra homomorphism. Conversely, suppose ϕ is an effect-algebra homomorphism. By the de Morgan law, it will be sufficient to prove that $\phi(p \wedge q) = \phi(p) \wedge \phi(q)$ for all $p, q \in L$. Obviously, $\phi(p \wedge q) \leqslant$

$\phi(p), \phi(q)$, so we have $\phi(p \wedge q) \leqslant \phi(p) \wedge \phi(q)$, and it will be sufficient to prove that $\phi(p) \wedge \phi(q) \wedge (\phi(p \wedge q))' = 0$. Now q and $(p \wedge q)'$ are compatible in L, so, by Lemma 6.5, and the fact that $p \wedge (q \wedge (p \wedge q)') = 0$, $\phi(p) \wedge \phi(q) \wedge (\phi(p \wedge q))' = \phi(p) \wedge \phi(q) \wedge \phi((p \wedge q)') = \phi(p) \wedge \phi(q \wedge (p \wedge q)') = 0$. □

Group-valued measures, which are closely related to morphisms, play an important role in the theory of effect algebras. Indeed, if $G^+[0, u]$ is an interval algebra and, $\phi: L \to G^+[0, u]$ is a morphism, then $\phi: L \to G$ is a G-valued measure in the sense of the following definition.

6.7. Definition. Let G be an additive abelian group.

(i) A *G-valued measure* (or *charge*) on L is a mapping $\phi: L \to G$ such that, $p, q \in L$ with $p \perp q \Rightarrow \phi(p \oplus q) = \phi(p) + \phi(q)$. Such a measure ϕ is said to be *strict* iff $\phi(p) = 0 \Rightarrow p = 0$ for all $p \in L$, and it is said to be *nondegenerate* iff $\phi(1) \neq 0$.

(ii) If V is a vector space over a field F, then a V-valued measure on L is called a *vector measure* over F.

6.8. Definition. Let Δ be a family of group-valued measures on L.

(i) Δ is *strict* iff, for every $p \in L$ with $p \neq 0$, $\exists \phi \in \Delta$, $\phi(p) \neq 0$.

(ii) Δ is *separating* iff, for $p, q \in L$ with $p \neq q$, $\exists \phi \in \Delta$, $\phi(p) \neq \phi(q)$.

6.9. Definition. Let P and Q be effect algebras and let G be an additive abelian group. A mapping $\xi: P \times Q \to G$ is called a *G-valued bimeasure* iff, for each $p \in P$ and each $q \in Q$, $\xi(p, \cdot): Q \to L$ and $\xi(\cdot, q)$: $P \to L$ are G-valued measures.

6.10. Definition. Let G be a partially ordered abelian group with positive cone G^+.

(i) A G-valued measure $\phi: L \to G$ such that $\phi(L) \subseteq G^+$ is called a *G-representation* of L. A representation $\phi: L \to G^+$ such that $\phi(p) \leqslant \phi(q) \Rightarrow p \leqslant q$ for all $p, q \in L$ is said to be *faithful*.

(ii) If V^+ is the positive cone in a partially ordered vector space over $\mathbb{R}$, then a representation $\phi: L \to V^+$ is called a *vector representation* of L.

6.11. Definition. A family Δ of representations of L is said to be *order-determining* iff, for $p, q \in L$, the condition $\phi(p) \leqslant \phi(q)$ for all $\phi \in \Delta$ implies that $p \leqslant q$.

7. PRODUCTS AND SUMS OF EFFECT ALGEBRAS

Cartesian products of effect algebras are defined as expected, and we can safely omit the proof of the following theorem.

7.1. Theorem. Let $(L_\alpha)_{\alpha\in I}$ be a family of effect algebras and let $L := \mathsf{X}_{\alpha\in I} L_\alpha$ be the cartesian product of the family. Then L can be organized into an effect algebra in such a way that, for $p, q \in L$, $p \oplus q$ is defined in L iff $p_\alpha \oplus q_\alpha$ is defined in L_α for all $\alpha \in I$, in which case $(p \oplus q)_\alpha = p_\alpha \oplus q_\alpha$ for all $\alpha \in I$.

The effect algebra $L = \mathsf{X}_{\alpha\in I} L_\alpha$ in Theorem 7.1 is called the *cartesian product* of the effect algebras in the family $(L_\alpha)_{\alpha\in I}$. As a special case, if X is a nonempty set and L is an effect algebra, then the set L^X of all functions $\gamma\colon X \to L$ is again an effect algebra under pointwise operations.

The next theorem is also easily proved.

7.2. Theorem. Let $(L_\alpha)_{\alpha\in I}$ be a family of effect algebras, so labeled that, for $\alpha, \beta \in I$ with $\alpha \neq \beta$, $L_\alpha \cap L_\beta = \{0, 1\}$, and let L be the set-theoretic union of the family. Then L can be organized into an effect algebra in such a way that, for $p, q \in L$, $p \oplus q$ is defined iff there exists $\alpha \in I$ such that $p, q \in L_\alpha$, $p \perp q$, in which case $p \oplus q$ is defined as in L_α.

The effect algebra L in Theorem 7.2 is called the *horizontal sum* of the effect algebras in the family $(L_\alpha)_{\alpha\in I}$.

The following theorem, proved by Dvurečenskij in Ref. 11, generalizes the main theorem of Ref. 4.

7.3. Theorem. If P and Q are effect algebras and if there exists a bimorphism mapping $P \times Q$ into an effect algebra, then there exists an effect algebra $P \otimes Q$ and a bimorphism $\otimes\colon P \times Q \to P \otimes Q$ such that:

(i) If $\beta\colon P \times Q \to L$ is a bimorphism, there exists a morphism $\phi\colon P \otimes Q \to L$ such that $\beta = \phi \circ \otimes$.

(ii) Elements of the form $\otimes(p, q)$ with $p \in P$, $q \in Q$ generate $P \otimes Q$.

The effect algebra $P \otimes Q$ in Theorem 7.3, which is obviously unique up to an isomorphism, is called the *tensor product* of the effect algebras P and Q. In Ref. 11, it is shown that a sufficient condition for the existence of $P \otimes Q$ is that both P and Q admit positive sets of probability measures.

8. EXAMPLES

The real vector space V of all self-adjoint operators on a Hilbert space $\mathscr{H}$, partially ordered by the usual positive cone

$$V^+ = \{T \in V \mid \langle Tx, x\rangle \geqslant 0 \text{ for all } x \in \mathscr{H}\}$$

gives rise to the archetypal effect algebra $V^+[\mathbb{0}, \mathbb{1}]$ which motivates much of our work. Such an effect algebra, or one isomorphic to it, is called a *standard effect algebra*. The only scale algebra that is also a standard effect algebra is the standard scale $\mathbb{R}^+[0, 1]$.

Difference posets (D-posets),[19] weak orthoalgebras,[15] and Brouwer–Zadeh posets (BZ-posets)[8] are also motivated by the structure $V^+[\mathbb{0}, \mathbb{1}]$. If $A \in V^+[\mathbb{0}, \mathbb{1}]$, then A is isotropic iff $A \leqslant \frac{1}{2}\mathbb{1}$, so the standard effect algebras satisfy the condition in the following definition (one of the axioms for a BZ-poset):

8.1. Definition. An effect algebra L is *BZ-regular* iff $a \perp b$ holds for all isotropic elements $a, b \in L$.

All orthoalgebras are obviously BZ-regular. If $\mathbb{Z}^+$ is the usual positive cone in the additive group of integers $\mathbb{Z}$, then the horizontal sum H of two copies of $\mathbb{Z}^+[0, 2]$ provides a simple example of an effect algebra that is not BZ-regular. Note that the lattice $(H, \leqslant)$ is isomorphic to the lattice of the Boolean algebra 2×2, whereas, as effect algebras, H and 2×2 are not isomorphic. Thus, *the structure of an effect algebra L determines, but is not determined by the structure of the poset $(L, \leqslant)$.*

Another interesting example of an effect algebra is the algebra L of perspectivity classes in a (possibly reducible) continuous geometry (Ref. 23, Hilfsatz 4.1). In Ref. 26, Part I, Chapter VI, it is shown that the perspectivity classes in an irreducible continuous geometry form a scale algebra L satisfying the condition that, if $p \in L$ and np is defined for every positive integer n, then $p = 0$.

Intervals $G^+[0, u]$ for $0 \neq u \in G^+$ in partially ordered abelian groups G provide abundant examples of effect algebras. Even very simple groups G yield interesting interval effect algebras. For instance, if the additive group of integers $\mathbb{Z}$ is partially ordered by the positive cone $C := \{3m + 8n \mid m, n \in \mathbb{Z}^+\}$, then the interval $C[0, 14] = \{0, 3, 6, 8, 11, 14\}$ forms a Heyting algebra. Another simple example is afforded by the interval $C[(0, 0), (2, 0)] = \{(0, 0), (1, 0), (1, 1), (2, 0)\}$ in the direct product $\mathbb{Z} \times \mathbb{Z}_2$ of $\mathbb{Z}$ and the integers modulo 2, partially ordered by the positive cone $C := \{(0, 0)\} \cup (\mathbb{Z}^+ \setminus \{0\}) \times \mathbb{Z}_2$. Note that the interval

algebra $C[(0,0),(2,0)]$ is isomorphic to the horizontal sum of two copies of $\mathbb{Z}^+[0,2]$.

Every Boolean algebra B is an interval effect algebra. In fact, if X is the Stone space of B, then the set of all functions $\phi: X \to \mathbb{Z}$ that are continuous when $\mathbb{Z}$ is given the discrete topology forms a partially ordered abelian group G under pointwise operations with positive cone G^+ consisting of all functions taking on values in $\mathbb{Z}^+$, and B is isomorphic to the interval $G^+[0,1]$, where 1 denotes the constant function $x \mapsto 1$.

9. THE UNIVERSAL GROUP FOR AN EFFECT ALGEBRA L

9.1. Definition. By a *universal group* for L, we mean a pair $(\mathcal{G}, \gamma)$ consisting of an additive abelian group $\mathcal{G}$ and a $\mathcal{G}$-valued measure $\gamma: L \to \mathcal{G}$ such that the following conditions hold: (i) $\gamma(L)$ generates $\mathcal{G}$. (ii) If G is an additive abelian group and $\phi: L \to G$ is a G-valued measure, then there is a group homomorphism $\phi^*: \mathcal{G} \to G$ such that $\phi = \phi^* \circ \gamma$.

Suppose X is a set, G is an additive abelian group, and $\psi: X \to G$. We define the *support* of ψ by $\operatorname{supp}(\psi) := \{x \in X \mid \psi(x) \neq 0\}$ and we define $G^{[X]}$ to be the set of all functions $\psi: X \to G$ such that $\operatorname{supp}(\psi)$ is a finite subset of X. Under pointwise operations $G^{[X]}$ forms an additive abelian group.

9.2. Theorem. If L is an effect algebra, then there is a universal group $(\mathcal{G}, \gamma)$ for L.

Proof. If $p \in L$, define $\chi_p \in \mathbb{Z}^{[L]}$ by $\chi_p(q) := 1$ if $q = p$ and $\chi_p(q) := 0$ if $q \neq p$ for all $q \in L$. Let H be the subgroup of $\mathbb{Z}^{[X]}$ generated by all elements of the form $\chi_p + \chi_q - \chi_r$ such that $p, q, r \in L$, $p \perp q$, and $p \oplus q = r$, let $\mathcal{G} = \mathbb{Z}^{[L]}/H$, and let $\eta: \mathbb{Z}^{[L]} \to \mathcal{G}$ be the canonical group epimorphism with kernel H. Define $\gamma: L \to \mathcal{G}$ by $\gamma(p) := \eta(\chi_p)$ for all $p \in L$. Then, for $p, q, r \in L$ with $p \perp q$ and $p \oplus q = r$, we have $\eta(\chi_p + \chi_q - \chi_r) = 0$, so that

$$\gamma(p) + \gamma(q) - \gamma(r) = \eta(\chi_p) + \eta(\chi_q) - \eta(\chi_r) = 0$$

and it follows that $\gamma: L \to \mathcal{G}$ is a $\mathcal{G}$-valued measure on L. The elements χ_p for $p \in L$ generate $\mathbb{Z}^{[L]}$ and $\eta: \mathbb{Z}^{[L]} \to \mathcal{G}$ is surjective, so $\gamma(L)$ generates $\mathcal{G}$.

Now let G be an additive abelian group and let $\phi: L \to G$ be a G-valued measure. Define the group homomorphism $\phi': \mathbb{Z}^{[L]} \to G$ by

$$\phi'(\psi) := \sum_{p \in \operatorname{supp}(\psi)} \psi(p)\,\phi(p) \qquad \text{for all} \quad \psi \in \mathbb{Z}^{[L]}$$

noting that $\phi'(\chi_p) = \phi(p)$ for all $p \in L$. If $p, q, r \in L$ with $p \perp q$ and $p \oplus q = r$, then

$$\phi'(\chi_p + \chi_q - \chi_r) = \phi(p) + \phi(q) - \phi(r) = 0$$

so $H = \ker(\eta) \subseteq \ker(\phi')$, and it follows that there is a group homomorphism $\phi^*: \mathscr{G} \to G$ such that $\phi' = \phi^* \circ \eta$. Thus, for $p \in L$,

$$\phi^*(\gamma(p)) = \phi^*(\eta(\chi_p)) = \phi'(\chi_p) = \phi(p)$$

so $\phi^* \circ \gamma = \phi$. □

If both $(\mathscr{G}, \gamma)$ and $(\mathscr{J}, \theta)$ are universal groups for L, it is clear that there exists a group isomorphism $\theta^*: \mathscr{G} \to \mathscr{J}$ such that $\theta = \theta^* \circ \gamma$ and, in this sense, a universal group for L is unique up to isomorphism. With this in mind, we refer to a universal group $(\mathscr{G}, \gamma)$ for an effect algebra L as *the* universal group for L.

Evidently, an effect algebra L admits no nonzero group-valued measure iff its universal group $\mathscr{G}$ is the trivial group $\mathscr{G} = \{0\}$. For instance, the orthomodular lattice constructed by Navara in Ref. 24 has $\mathscr{G} = \{0\}$ as its universal group.

Theorem 9.2 provides an intimate connection between effect algebras and additive abelian groups. For instance, by the following theorems, both effect-algebra morphisms and bimeasures are transformed into group homomorphisms:

9.3. Theorem. If P and Q are effect algebras with universal groups $(\mathscr{G}, \alpha)$ and $(\mathscr{H}, \beta)$, respectively, and if $\phi: P \to Q$ is a morphism, there is a unique group homomorphism $\phi': \mathscr{G} \to \mathscr{H}$ such that $\phi' \circ \alpha = \beta \circ \phi$.

Proof. $\beta \circ \phi: P \to \mathscr{H}$ is an $\mathscr{H}$-valued measure, so there is a group homomorphism $\phi': \mathscr{G} \to \mathscr{H}$ such that $\phi' \circ \alpha = \beta \circ \phi$. The uniqueness of ϕ' is a consequence of the fact that $\alpha(P)$ generates $\mathscr{G}$. □

9.4. Theorem. Let P, Q be effect algebras, let G be an additive abelian group, and let $\xi: P \times Q \to G$ be a G-valued bimeasure. Let $(\mathscr{P}, \alpha)$ and $(\mathscr{Q}, \beta)$ be universal groups for P and Q, respectively and let $\mathscr{P} \otimes \mathscr{Q}$ be the tensor product of the abelian groups $\mathscr{P}$ and $\mathscr{Q}$. Then there exists a unique group homomorphism $\xi^*: \mathscr{P} \otimes \mathscr{Q} \to G$ such that, for all $p \in P$, $q \in Q$, $\xi(p, q) = \xi^*(\alpha(p) \otimes \beta(q))$.

Proof. Let H be the subgroup of $\mathbb{Z}^{[P]}$ generated by all elements of the form $\chi_a + \chi_b - \chi_c$ with $a, b, c \in P$, $a \oplus b = c$ as in the proof of Theorem 9.2, and let K be the corresponding subgroup of $\mathbb{Z}^{[Q]}$. We

identify $\mathscr{P}$ with $\mathbb{Z}^{[P]}/H$ and $\mathscr{Q}$ with $\mathbb{Z}^{[Q]}/K$, and we define $\eta\colon \mathbb{Z}^{[P]} \to \mathscr{P}$ and $\kappa\colon \mathbb{Z}^{[Q]} \to \mathscr{Q}$ to be the canonical group homomorphisms with kernels H and K, respectively. Then $\alpha(p) = \eta(\chi_p)$ and $\beta(q) = \kappa(\chi_q)$ for all $p \in P$, $q \in Q$.

Define the group bihomomorphism $\xi'\colon \mathbb{Z}^{[P]} \times \mathbb{Z}^{[Q]} \to G$ by

$$\xi'(\phi, \psi) := \sum_{p \in \operatorname{supp}(\phi)} \sum_{q \in \operatorname{supp}(\psi)} \phi(p)\, \psi(q)\, \xi(p, q)$$

for $\phi \in \mathbb{Z}^{[P]}$, $\psi \in \mathbb{Z}^{[Q]}$, and let $\xi''\colon \mathbb{Z}^{[P]} \otimes \mathbb{Z}^{[Q]} \to G$ be the corresponding group homomorphism, so that $\xi'(\phi, \psi) = \xi''(\phi \otimes \psi)$. Let J be the subgroup of $\mathbb{Z}^{[P]} \otimes \mathbb{Z}^{[Q]}$ generated by all elements of the form $\phi \otimes \psi$ such that $\phi \in H$ or $\psi \in K$ (or both), and let $\theta\colon \mathbb{Z}^{[P]} \otimes \mathbb{Z}^{[Q]} \to (\mathbb{Z}^{[P]} \otimes \mathbb{Z}^{[Q]})/J$ be the canonical group homomorphism with kernel J.

If $a, b, c \in P$ with $a \oplus b = c$, then, owing to the fact that ξ is a bimeasure, $\xi''((\chi_a + \chi_b - \chi_c) \otimes \psi) = 0$ for all $\psi \in \mathbb{Z}^{[Q]}$, and it follows that $\xi''(\phi \otimes \psi) = 0$ for all $\phi \in H$, $\psi \in \mathbb{Z}^{[Q]}$. Likewise, $\xi''(\phi \otimes \psi) = 0$ for all $\phi \in \mathbb{Z}^{[P]}$, $\psi \in K$, and it follows that $J \subseteq \ker(\xi'')$. Therefore, there is a group homomorphism

$$\xi^{\#}\colon (\mathbb{Z}^{[P]} \otimes \mathbb{Z}^{[Q]})/J \to G$$

such that $\xi'' = \xi^{\#} \circ \theta$. By a standard theorem in multilinear algebra, there is a group isomorphism

$$\iota\colon (\mathbb{Z}^{[P]}/H) \otimes (\mathbb{Z}^{[Q]}/K) \to (\mathbb{Z}^{[P]} \otimes \mathbb{Z}^{[Q]})/J$$

such that $\iota(\eta(\phi) \otimes \kappa(\psi)) = \theta(\phi \otimes \psi)$ for all $\phi \in \mathbb{Z}^{[P]}$, $\psi \in \mathbb{Z}^{[Q]}$. Let $\xi^* := \xi^{\#} \circ \iota$. Then, for $p \in P$, $q \in Q$, we have

$$\begin{aligned} \xi^*(\alpha(p) \otimes \beta(q)) &= \xi^{\#}(\iota(\eta(\chi_p) \otimes \kappa(\chi_q)) = \xi^{\#}(\theta(\chi_p \otimes \chi_q)) \\ &= \xi''(\chi_p \otimes \chi_q) = \xi'(\chi_p, \chi_q) = \xi(p, q) \end{aligned}$$

The uniqueness of ξ^* follows immediately from the fact that elements of the form $\alpha(p) \otimes \beta(q)$ with $p \in P$, $q \in Q$ generate $\mathscr{P} \otimes \mathscr{Q}$. □

9.5. Corollary. Suppose that the effect algebras P and Q have an effect-algebra tensor product $P \otimes Q$. Let $(\mathscr{P}, \alpha)$, $(\mathscr{Q}, \beta)$, and $(\mathscr{T}, \gamma)$ be the universal groups for P, Q, and $P \otimes Q$, respectively. Then there is a unique group epimorphism $\tau\colon \mathscr{P} \otimes \mathscr{Q} \to \mathscr{T}$ such that $\tau(\alpha(p) \otimes \beta(q)) = \gamma(p \otimes q)$ for all $p \in P$, $q \in Q$.

Proof. In Theorem 9.4, let $G := \mathscr{T}$, let $\xi\colon P \times Q \to G$ be given by $\xi(p, q) = \gamma(p \otimes q)$, and let $\tau := \xi^*$. □

9.6. Theorem. Let $(\mathscr{G}, \gamma)$ be the universal group for the effect algebra L.

(i) γ is nondegenerate iff L admits a nondegenerate group-valued measure.

(ii) γ is strict iff L admits a strict family of group-valued measures.

(iii) γ is injective iff L admits a separating family of group-valued measures.

(iv) $\mathscr{G}$ can be organized into a partially ordered group in such a way that γ is a strict representation of L iff L admits a strict family of representations.

(v) $\mathscr{G}$ can be organized into a partially ordered group in such a way that γ is a faithful representation of L iff L admits an order-determining family of representations.

Proof. (i), (ii), and (iii) are obvious.

(iv): If $\mathscr{G}$ can be partially ordered in such a way that $\gamma(L) \subseteq \mathscr{G}^+$ and γ is strict, then $\Delta := \{\gamma\}$ is a strict family of representations of L. Conversely, suppose that Δ is a strict family of representations of L and define P to be the subset of $\mathscr{G}$ consisting of all finite linear combinations of elements of $\gamma(L)$ with nonnegative integer coefficients. Evidently, $P + P \subseteq P$ and $0 \in P$. Let $g \in P \cap -P$. Then there is a finite nonempty subset A of $L \setminus \{0\}$ and there are functions $\rho, \sigma: A \to \mathbb{Z}^+$ such that $g = \sum_{a \in A} \rho(a)\, \gamma(a)$ and $-g = \sum_{a \in A} \sigma(a)\, \gamma(a)$, so $0 = \sum_{a \in A} (\rho(a) + \sigma(a))\, \gamma(a)$. If $\phi \in \Delta$, then $\phi: L \to G^+$ for a partially ordered abelian group G and there exists a group homomorphism $\phi^*: \mathscr{G} \to G$ such that $\phi = \phi^* \circ \gamma$. Thus,

$$0 = \sum_{a \in A} (\rho(a) + \sigma(a))\, \phi^*(\gamma(a)) = \sum_{a \in A} (\rho(a) + \sigma(a))\, \phi(a) \in G^+$$

and, since each $\phi(a) \in G^+$ and $\rho(a) + \sigma(a)$ is a nonnegative integer, it follows that $0 = (\rho(a) + \sigma(a))\, \phi(a) \in G^+$ for all $a \in A$. For each $a \in A \subseteq L \setminus \{0\}$, we can choose $\phi \in \Delta$ with $\phi(a) \neq 0$, and it follows that $\rho(a) + \sigma(a) = 0$ in $\mathbb{Z}^+$; hence, that $\rho(a) = 0$. Consequently, $g = 0$, and so $P \cap -P = \{0\}$. Thus $\mathscr{G}$ can be organized into an ordered group with positive cone $\mathscr{G}^+ := P$. Evidently, $\gamma(L) \subseteq \mathscr{G}^+$, so γ is a representation of L. That γ is strict follows from Part (ii). (v) is now a corollary of (iv). □

As a consequence of Theorem 9.6, if L admits an order-determining family of probability measures (or states), then $\mathscr{G}$ can be organized into a partially ordered group in such a way that γ is a faithful representation of L.

10. THE UNIVERSAL VECTOR SPACE FOR AN EFFECT ALGEBRA L

10.1. Definition. Let F be a field. By a *universal vector space over* F for the effect algebra L, we mean a pair $(\mathcal{U}, v)$ consisting of a vector space $\mathcal{U}$ over F and a vector measure $v: L \to \mathcal{U}$ such that the following conditions hold: (i) $v(L)$ spans $\mathcal{U}$. (ii) If U is a vector space over F and $\phi: L \to U$ is a vector measure, there is a linear transformation $\phi^{\#}: \mathcal{U} \to U$ such that $\phi = \phi^{\#} \circ v$.

10.2. Theorem. If F is a field, there exists a universal vector space over F for the effect algebra L.

Proof. Let $(\mathcal{G}, \gamma)$ be the universal group for L and, regarding F as an additive abelian group, let $\mathcal{U} := F \otimes \mathcal{G}$. Define the $\mathcal{U}$-valued measure $v: L \to \mathcal{U}$ by $v(p) = 1 \otimes \gamma(p)$ for all $p \in L$. The additive abelian group $\mathcal{U}$ can be organized into a vector space over F in such a way that, for $\alpha, \beta \in F$ and $g \in \mathcal{G}$, $\alpha(\beta \otimes g) = (\alpha\beta) \otimes g$. Since $\gamma(L)$ generates $\mathcal{G}$, it is clear that $v(L)$ spans the vector space $\mathcal{U}$ over F. Let U be a vector space over F, let $\phi: L \to U$ be a measure, and let $\phi^*: \mathcal{G} \to U$ be the corresponding homomorphism into the additive abelian group U such that $\phi = \phi^* \circ \gamma$. Define the group bihomomorphism $\phi': F \times \mathcal{G} \to U$ by $\phi'(\beta, g) := \beta\phi^*(g)$ for all $\beta \in F$, $g \in \mathcal{G}$, and let $\phi^{\#}: \mathcal{U} \to U$ be the corresponding group homomorphism such that $\phi'(\beta, g) = \phi^{\#}(\beta \otimes g) = \beta\phi^*(g)$. Evidently, $\phi^{\#}$ is a linear transformation and, for $p \in L$, we have $\phi^{\#}(v(p)) = \phi^{\#}(1 \otimes \gamma(p)) = \phi^*(\gamma(p)) = \phi(p)$. □

10.3. Definition. If F is a field, we denote by $V(L, F)$ the set of all F-valued measures $\mu: L \to F$.

Under pointwise operations, it is clear that $V(L, F)$ is a vector space over F. We omit the straightforward proof of the following theorem.

10.4. Theorem. If F is a field, $(\mathcal{U}, v)$ is the universal vector space over F for the effect algebra L, and $\mathcal{U}'$ is the algebraic dual space of $\mathcal{U}$ over F, then the mapping $^{\#}: V(L, F) \to \mathcal{U}'$ given by $\mu \mapsto \mu^{\#}$ is a vector-space isomorphism of $V(L, F)$ onto $\mathcal{U}'$.

If $V(L, F)$ is finite dimensional (which happens, for instance, when L is finite), then Theorem 10.4 implies that $\mathcal{U}$ is finite dimensional and that $\mathcal{U}$ can be identified with the dual space $V(L, F)'$. Under this identification, the vector measure v is identified with the mapping $p \mapsto f_p$, where f_p is the linear functional on $V(L, F)$ obtained by evaluation: $f_p(\mu) := \mu(p)$.

Analogues of Theorems 9.3 and 9.4 can be established for universal vector spaces. In particular, the vector analogue of Theorem 9.4 generalizes the main result of Ref. 18. Numerous variations on the themes of Sections 9 and 10 suggest themselves. For instance, it is clear how to define a universal Banach space for bounded Banach-space-valued measures on L. The theorem of Bunce and Maitland Wright[7] shows that, if A is a von Neumann algebra with no type I_2 direct summand and L is the orthomodular lattice of self-adjoint idempotents (projections) in A, then the universal Banach space for L is the Banach space A itself.

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