Lettera Matematica (2018) 6:45–55
https://doi.org/10.1007/s40329-018-0208-6

For an epistemology of mathematical contents: Albert Lautman

Mario Castellana[1]

Published online: 5 February 2018
© Centro P.RI.ST.EM, Università Commerciale Luigi Bocconi 2018

Abstract
Albert Lautman was an attentive observer of modern mathematics from Riemann to the Bourbakist movement in the 1930s and developed his thought within that unique thread of philosophy of science that was the *philosophie mathématique* in France during the early twentieth century. He gave a coherent form to the philosophical concerns of mathematicians such as Hilbert, Weyl and Gödel, highlighting the unity of mathematics and their dynamic and creative character. He developed a philosophy of mathematics that differed from standard 1930s philosophy of science, and focussed on the analysis of the "contents" of mathematical thought, not considered only as a formal language. Thanks to this historical-epistemological choice, as claimed by Jean Dieudonné who knew him personally, Lautman was able to anticipate the themes and problems of the subsequent mathematical thought, and to provide some tools that allow us to better understand the complex figure of Grothendieck. His "epistemology of mathematical contents" also represents an important moment in French philosophy of the sciences, which even today in France does not receive adequate critical attention.

Keywords Albert Lautman · Philosophie mathématique · Philosophy of mathematics · Philosophy of physical mathematics · Riemann · Weyl · Grothendieck

> Mathematical discoveries, like springtime violets in the woods, have their season which no human can hasten or retard.
>
> Carl Friedrich Gauss [7], p. 263.
>
> If there is one thing in mathematics that fascinates me more than anything else ... it is neither "number" nor "size", but always form. And among the thousand and more faces that form chooses to reveal itself to us, the one that fascinates me more than any other and continues to fascinate me, is the structure hidden in mathematical things.
>
> Alexander Grothendieck [28].
>
> It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative.
>
> Francis W. Lawvere [39], p. 16.

1 Introduction

To best situate Albert Lautman (1908–1944) (Fig. 1; see Appendix), an original and almost unique figure of philosopher of mathematics, within the epistemological landscape of the twentieth century, little known and in recent years the focus of increasing interest,[1] it is very helpful to refer to some statements of two eminent mathematicians who knew and appreciated him already in the late 1930s: Jean Dieudonné and Paul Bernays. They identified some of his fundamental facets, such as the ability to identify a number of problematic issues in mathematical thought which would emerge only later.

Quoting Dieudonné on the occasion of the publication in 1977 of the first reprint of almost all of Lautman's works, written in the late 1930s, in parallel with the formation of

✉ Mario Castellana
mario.castellana@unisalento.it

[1] Dipartimento di Storia, Società e Studi sull'Uomo, Università del Salento, Complesso Studium 2000, 73100 Lecce, Italy

[1] See [35] and the translations in Spanish [36], in English [37] and in Italian (*La pensée mathématique* in [15] and three minor writings in [38]). About Lautman, see [44], [1], [6], [53], [14–18].



Fig. 1 Albert Lautman (graphical elaboration from Lautman's identity card photo, in: A. Lautman, *Ensayos sobre la dialectica, estructura y unidad de la matematicas modernas*, Univ. de Colombia, Bogotá 2011) (courtesy of Fernando Zalamea)

Bourbaki movement, Lautman was one of those rare philosophers who try to "get an idea of the major trends of mathematics of their time"; at the same time.

> he had acquired, about the mathematics of the 1920s-1930 s, views that were far more extensive and precise than most mathematicians of his generation, often locked in their specialist research; I can testify for it personally. One cannot but be amazed, forty years later, of the prophetic ability of the ideas presented in his two theses.[2]

Paul Bernays, on the other hand, reviewing these works in 1937, was one of the first to highlight the new approach, which was due to an uncommon critical attention to the work of Hilbert, Weyl, Gödel and Gentzen. He expressed an appreciation for this approach, concluding that.

> it is to be said in favor of Lautman's method that it is more suited than foundational discussions to give a philosopher an impression of the content and nature of modern mathematics. Indeed it is worth while to emphasize that foundational problems by no means constitute the only philosophically important aspect of mathematics [9], p. 22.

[2] The quote is from [21, p. 35]. The two theses referred to are [31] and [33].

But to better understand this path, which to a first, superficial reading may seem like an attempt to frame in an a priori way mathematical thought within certain philosophical viewpoints, thanks to an interest in dialectical philosophy, Platonism and some parts of Heidegger's thinking, it is necessary to keep in mind the vast, non-homogeneous body of historical-epistemological literature that developed in France from the second half of the nineteenth century regarding the nature of mathematics following the emergence of non-Euclidean geometries and on the nature of the changes within physics. Further, to better understand Lautman's point of view, at the same time it is significant that in France the analyses of internal events within both mathematics and physics have been always seen in a complementary manner, with, moreover, already the awareness—as Antoine-Augustin Cournot warned with an uncommon epistemic sense—that their "renewing crises are the only crises that are useful for the renewal of philosophy" [20, vol. I, p. 23]. This chapter of French thought, still little explored, it is characterised by an intense debate about the structure and the historical dimension of mathematics and science in general already in a post-positivistic perspective. The very birth of the *Revue de métaphysique et de morale*, in 1893, is justified in this context, so much so that we can say that it was, at least in its first 15 years of life, almost exclusively a workshop on philosophy and history of mathematics per se (Fig. 1). It is no coincidence that in those years at the École Normale Supérieure in Paris there was what has been called "cross-fertilization" between *savants-philosophes*, such as Charles Hermite and Henri Poincaré, and various philosophers of different orientations, who debated there, among other topics, the theoretical meaning of mathematics, up to the formation of actual *philosophes-savants*.[3]

2 The philosophie mathématique

The intense, complex debate primarily focussed on the way of being and producing *connaissance* after the advent of non-Euclidean geometries, which gave rise to the "pathology of

[3] Besides Hermite and Poincaré–also on the occasion of the first Congrès International de Philosophie held in Paris in 1900 thanks precisely to the founders of the *Revue de métaphysique et de morale*–Frege, Hilbert, Russell, Giuseppe Peano and some exponents of his school, such as Pieri, Burali-Forti and Enriques himself also took part in these debates, as is well known. Among the *philosophes-savants* or rather the protagonists of the French *philosophie mathématique* of those years we should mention Milhaud, Brunschvicg, Winter, Le Roy, Couturat, Boutroux, P. and J. Tannery for mathematics, and for other disciplines Rey, Meyerson, Ribot (in Italy, apart from *savant-philosophe* Federigo Enriques, the only *philosophe-savant* figure in this area was Giovanni Vailati). For more on this, see [2] and [46, xxxi].



Fig. 2 Jean Cavaillès in uniform (Creative Commons Attribution-Share Alike 3.0 Unported license)

mathematics" [11, p. 27]. However, even a cursory analysis of the contributions of the different protagonists of the time shows a growing interest in Bernhard Riemann's work and thought—already translated into French by 1872 (published by Darboux on the *Bulletin des sciences mathématiques*) and then in full in 1898—and in Klein's Erlangen programme, something that would be decisive for the subsequent development of French epistemology and of that particular strand of philosophy of science, the *philosophie mathématique*, which would acquire its ultimate meaning in the now classic 1912 work *Les étapes de la philosophie mathématique* by Léon Brunschvicg, teacher at the École of Lautman as well as of Gaston Bachelard and Jean Cavaillès (Fig. 2). Even though it has its distant origins in Cartesian thought and the Enlightenment, Brunschvicg identified its first solid anchor in the work of Lagrange, thus lying "at the foundation of Comte's positivism" [13, p. 292], a fact that is often forgotten in history books of philosophy. But non-Euclidean geometries, and especially the prospects opened by Riemann and the subsequent work by Klein, required a different and more comprehensive approach, to understand their conceptual novelty, no longer classifiable in old, gnoseological patterns; this is why the term *raison mathématique*[4] is relied upon to clarify the truth-affirming and dynamic character of the new theories, for the increasingly important role played in the evolution of mathematics by the processes of generalisation and unification. The *nouvelle philosophie mathématique* was meant to create the basis of a different epistemic attitude to signify its different way of being actual *connaissances* with specific "contents", against those trends that confined themselves to take note of their plurality by emphasising almost exclusively the conventional character and the necessity of their reduction to logical-formal procedures. The same consequential axiomatic process, while considered a further decisive step in mathematics, to the point of being considered itself *pensée* by Lautman himself, had to be supplemented by a historical vision that, starting from concrete cases and from what Federigo Enriques would call the "position of problems", should be able to "show the direction of mathematics: they develop in a natural way under the impulse of experience, but sooner or later hide under logical constructs the origin of their concepts" [42, p. 11] (Fig. 2).

Precisely to combat these trends on a theoretical ground, some scholars, such as the well-known Brunschvicg and others less well-known, such as Gaston Milhaud[5] and Maximilien Winter[6], gave rise to different lines of research aimed at laying the foundations of a comprehensive reflection about *la méthode en philosophie des mathématiques*, with the aim of clarifying on the epistemological level the structural changes taking place in late-19th-century mathematics, up to the construction of a specific, not only historiographical strategy, significantly called *la méthode historico-critique* [51, chap. III]. *Philosophie mathématique*, conceptually enriched by such a tool, was first conceived as the product of a self-reflection internal to mathematics, addressed to an understanding of the processes of production and generalisation, which Riemann in his few philosophical fragments called "progressive conceptualisations" with the goal of

[4] This term often occurs in the writings of French mathematicians of the late nineteenth century, such as Hermite and Gaston Darboux, authors who were later held in mind by Kurt Gödel in his considerations on resuming a certain Platonism in an anti-Carnap function.

[5] Mathematician, historian and philosopher of mathematics, Gaston Milhaud (1858–1918) translated into French in 1887 P. Du Bois-Reymond's *Allgemeine Functionentheorie*, in whose introduction epistemological questions involving the situation of the time are covered. He was the author of several studies in the history of Greek mathematics and the history of logic, wrote several essays dedicated to the philosophical consequences of non-Euclidean geometries, collected in 1898 *Le rationnel* and in 1911 in *Nouvelles études sur l'histoire de la pensée scientifique*; he was the first to obtain in France, at the Sorbonne, in 1909 a chair in philosophy of science, then called *Histoire de la philosophie dans ses rapports avec les sciences*, thus inaugurating a line of historical-epistemological investigations that would remain constant in the new century. About Milhaud, see [12].

[6] Maximilien Winter (1871–1935) is completely unknown even today in France; he was one of the founders of the *Revue de métaphysique et de morale* and was the author, until the 1930s, of several studies published in this journal, devoted especially to functional analysis and number theory, as well as essays in philosophy of logic, algebra, physics, some of which collected in [51]. He had close relations with Italian mathematician Federigo Enriques; for a first study about the relationship between the two, see [2].

explaining that "foolish thought of arriving at the explanation of phenomena by means of abstraction" that makes it possible to go "beyond the surface of phenomena" [48]. These lines of research deemed relevant on the epistemic level the crucial issue of the particular historical character of mathematics, in order to better grasp its truth-affirming aspect, to avoid falling into those "pathological" views of the history of mathematical truths that appears in various areas, precisely because of the lack of an adequate reflection on this aspect. It is not insignificant that *Philosophia mathématique* founds its roots by placing at the centre of its investigations the topic of the particular historical and theoretical dimension they assumed to have produced, since Riemann and Grassmann, a different conceptual physiognomy that is no longer compatible with classical gnoseological schemes. This meant addressing the topic of their growth as knowledge, making the creative and dynamic aspect explicit, their being *pensée* tout court and at the same time giving the right theoretical dimension to the relationship between mathematics and physics, to their ability of *penser le réel physique*, to "go beyond the surface of phenomena" by abstraction in its various phases, as Riemann had already shown programmatically. This is an attitude that, as Ludovico Geymonat said in the 1960s, has been thoughtlessly overlooked by most philosophers of science of other orientations [23, p. 185].

3 The internal "dialectic" of mathematical thought

These issues, which had already emerged in all their cruciality in the late nineteenth century and which constituted the core concepts of the *nouvelle philosophie mathématique* in France for the entire twentieth century as well, form the circle within which young Lautman was to act. In the wake of Winter's works already aiming at creating the foundations of a *philosophie mathématique* more appropriate to current issues, he believed that there was an autonomous space for the philosopher,[7], distinguished from actual mathematical activity. As he wrote in 1937:

mathematics, and especially modern mathematics—algebra, group theory, topology—seem to tell, together with the constructions that interest the mathematician, another more hidden story, made for the philosopher. A dialectic action acts in their depth, that our six chapters tend to highlight [35, p. 131].

Such dialectic action is justified by the huge amount of knowledge produced with qualitatively different and increasingly complex tools, where entire fields of mathematical thought, while autonomous, are to cross creating the basic conditions for further conceptualisations. This is seen to be operating in what he calls "real mathematics",[8] and especially in modern mathematics from Riemann to the 1930s, where he finds further elements, first thanks to a joint study of the works of Hilbert and Weyl in his two theses [32, 33] and later Gödel's and Gentzen's in his 1939 work. Lautman highlights the need to enter into their works to make explicit the "dialectic, the structure and the unity of modern mathematics" and the specific "creative dialectic", to use an expression of the mathematician and Spanish translator of his works Fernando Zalamea [52, 54]; this choice is then turned in a polemic spirit to stage the associated problems of "the nature of reality in mathematics" [35, pp. 127–132], where many of the contemporary paths of philosophy of mathematics seemed to fail, to such an extent that they to offer what he unhesitatingly calls a "depressing sight". This is also due to having uncritically contrasted research paths, such as those followed by Hilbert and Weyl, considered as two almost irreconcilable points of view, whereas a closer analysis, guided by a more comprehensive historical and conceptual perspective, would have allowed a more adequate dialectic integration.

In Lautman's writings there is a constant invitation to the philosopher to make such a commitment by various means, when he wishes to go into the field of scientific thought with the difficult task of bringing out of what Moritz Schlick in his 1918 *Allgemeine Erkenntnislehre* already called "the philosophical soul of science", not always within easy reach [49]. In agreement with his young friend Jean Cavaillès and the older Bachelard, Lautman believed that one must still settle in what they considered "the thankless country of philosophy of science" [8, p. 117–27] for the "heroic" task of bringing out the implicit thought and structural complexity of the entire corpus of mathematics and understanding in the correct dimension what Federigo Enriques in 1906 had called the "discontinuous qualitative meaning" [22, p. 45] that had occurred with the advent of non-Euclidean geometries, without falling into "pathological" views of their

[7] In this regard, to better understand the effort by Lautman and the first and few French philosophers of science (in the sense that this term was taking in the late 1930s after the establishment of standard philosophy of science, but committed to tracing the lines of a different path), it is useful to keep in mind some remarks by Gaston Bachelard, made after reading Lautman's two 1937 theses [32, 33] and formulated in a 1938 letter: "in every page there is matter to reflect on. Avoiding the calculations you have been able to bring out [mathematical] thought. And this is not the least merit. Every philosopher of science knows that this is an almost insurmountable difficulty ... With great sympathy, I hail in you a representative of the younger generation that is bringing philosophy back to the heroic tasks of difficult thinking. That we could be at least a dozen! But we need to work hard to change all of this" [8] p. 129.

[8] As is known, the expression "real mathematics" has later entered the lexicon, thanks to G.H. Hardy in his 1940 text *A Mathematician's Apology*.

truth. It is no coincidence that the first chapters of Lautman's 1937 thesis cover group theory as it had come to restructure itself with the work of Hermann Weyl, the attempts of unification of algebra and analysis by Hilbert, the role of non-Euclidean metrics in analytical functions, the algebra of non-commutative structures, Pfaffian forms. This allowed him to grasp the structural and unified face of mathematics, to give importance to its hidden mechanisms and to insist on creative processes. For this reason Lautman, from the beginning of his speculations, which were carried out within the encounter with the contemporary rich literature produced by the Vienna Circle, had taken a clear position against some of the trends therein:

> thus, any logical attempt aimed at controlling a priori the development of mathematics would disavow the essential nature of mathematical truth, since it is linked to the creative activity of the spirit and participates in its temporal character [35], p. 129.

This is also because, in his view, "the mathematical being as we conceive it has a certain analogy with the living being" [35], p. 140.

But to better understand the views of Lautman and his constant questioning about the meaning of "mathematical truth", we have to note his uncommon and thorough reading of Hermann Weyl's works, which in France Bachelard had already begun to do a few years earlier,[9] with the aim of giving a more organic shape to the "general philosophy" mentioned by the German mathematician in *Das Kontinuum*. Such a need, as Weyl himself would point out, had emerged "after turning my back on conventionalism" and after vainly chasing that "school chimera" of eliminating any reference to the "notion of natural number" despite "a very extended and more and more complicated formalisation" begun on several fronts. Such "general philosophy" aimed at giving back to mathematics that creative ability that found one of its reasons in the internal dialectic between infinite and finite, despite Dedekind's "theory of chains", with the further objective of distinguishing it from logic "as a science with distinct characteristics" at a time when it became increasingly difficult "to draw the boundaries between mathematics and formal logic", as Weyl wrote in 1918 [50].[10] This is also why in France the notion of "mathematical thought" comes to acquire, more than in other research traditions, its full epistemic meaning, for its historical and conceptual richness and, more generally, as scientific thought, which no philosophical stance, even though anchored in well-established and metabolised scientific results, can presume to exhaust and rigidly frame without falling into what Winter had already called "metascientific extension of a doctrine ... maybe in line with the development of the sciences", inevitably leading to in a "form of scholasticism" [51], p. 51. The observations by Bachelard and Lautman, strong in their being anchored to that line that goes from Riemann and Klein to Weyl, always claimed the need for an "open philosophy" [5] of sciences, by means of theoretical antibodies able to fight such temptations, the always present "metascientific extensions" of scientific results, also in the contemporary neo-positivist movement, which addressed by other means the crucial theme of the unity of science. However, the results of this orientation, especially in its analytical *tournant*, had already been defined as "totalitarian trends" in the late 1930s by mathematician Arend Heyting, due to the reduction of mathematics to language, when instead "the task of science lies not in the study of the language nor the ideas it seeks to express, but in the creation of these same ideas" [30, p. 75]. One of these antibodies lay precisely in focussing their efforts on understanding this process of creativity of mathematics, especially as "thought"[11] for its particular "dialectic structure" [35, pp. 235–257], both theoretical and historical. This occurred just at a time when what Weyl called "the empire of mathematics" (an expression he used in his 1929 *Philosophie der Mathematik und Naturwissenschaft* and in several other works), while seeming to stand on an unwavering rock, was discovering inside itself what Bachelard and young Lautman would later call *mobilité* or *dialectique*.

For Lautman, only within a Weyl-like "general philosophy", a "dialectical philosophy" or a "open philosophy" in Bachelard's sense, was there more space for a research aimed at justifying the autonomy of mathematical thought, its "essence" as endowed with a specific "inexhaustible content", as often stated by Weyl [50], with its own, mostly hidden dynamics, as Alexander Grothendieck would later strongly emphasise in several writings; it is constantly

[9] See [4]. Furthermore, it should be emphasized that Weyl's thought has been *le grand oublié* of much 20th-century philosophy of mathematics. His major work, 1918 *Raum, Zeit, Materie* was translated into French in 1922 and later French philosophy of science cannot be fully understood if this constant critical confrontation with his thought is not kept in mind, another unique fact in the epistemological landscape of the twentieth century, as was the case with Riemann; on this see [47] and the various contributions in [ACRR].

[10] We must contextualise the fact that Weyl uses "general philosophy" to denounce those now prevalent stances aiming at privileging the important logical-formal aspect of mathematical thought; on this, see [43, 47].

[11] Still in recent times, in France, against certain trends that tend to downplay this aspect, Dominique Lecourt, in the wake of Bachelard, claims that it is necessary "to restore the rightful importance to the great forgotten by both scientism and anti-science: scientific thought" [40], p. 452; see also [41]. In Italy, as is known, only Federigo Enriques in his works of an epistemological nature constantly put the idea of science as "thought" at the centre, precisely to oppose scientist positions and at the same time neo-idealistic philosophies. It is no coincidence that Enriques was appreciated and followed on this ground by Bachelard and by Lautman himself.

characterised by an internal and creative tension between structure and existence, finite and infinite, being and becoming, the whole and the part, local and global, continuous and discontinuous, perfect and imperfect, quantitative and qualitative, where what are called *schémas de génèse* and *schémas de structure* alternate [35], p. 132, and then lead to Riemannian "progressive conceptualisations". In light of these instruments, in Lautman's secondary thesis [33], there is also an analysis of Galois theory, the dynamic role played in Riemann by the notions of multiplicity and surface, Grassmann's processes of generalisation, the theory of Betti numbers, the Alexander duality theorem. With a language that was deliberately irritating to those who approached the continent "mathematics", an important notion was introduced, the result of his reflection on modern mathematics: *la montée vers l'absolu*. On the one hand, this expresses what, in Gauss's wake, was considered one of its internal needs, namely the fact for mathematical entities of assuming a given "form" at a particular time, one which no one can counteract. On the other hand, abiding by being immersed in a historical process leads to what he calls "the exceptional character of [their] existence" [35, p. 211], despite being imperfect or in a state of imperfection with the inherent ability to continuously strive toward perfection.[12]

Here an essential role is precisely played by this *montée vers l'absolu*, a decisive force in the creative transition from one state to another, as in the case of Hermann G. Grassmann. Even if references to his work in the two theses [31, 33] are relatively few compared to those to Riemann's work, they help to better understand in their most correct sense some methodological indications expressed in the Introduction to *Die lineale Ausdehnungslehre*, about the way in which "a system transforms into another", the "combinations between algebra and geometry", and the "generating acts" of more global viewpoints [27, p. XIV-XV].[13] This is highlighted even more by Lautman thanks to the analysis of surface and the role of algebraic function from Riemann to Poincaré and Paul Koebe:

The solidity of the whole and of its parts, the reduction of the relation properties to intrinsic properties, the passage from imperfection to the absolute, here is a whole series of attempts at structural organisation that give mathematical entities a movement towards completeness by which we can say that they exist. But this existence is not only manifested because the structure of these entities mimics the ideal structures to which they allow themselves to be compared; it follows that the completeness of an entity is at the same time the genesis of other entities, and here are logical relationships between essence and existence where the outline of new creations takes shape [35, p. 178].

These remarks by Lautman, derived from an unusual critical analysis of the "contents" of mathematics between the 19th and the twentieth centuries, not only allow him to take a more global view of them and to give a more "theoretical" voice to the "philosophical concerns" of scholars such as Weyl, Heyting and Bernays, who openly expressed their dissent from the prevailing analytical position, and, ahead of his time, of Gödel himself, who only in the 1950s took up a position in this sense, opting for a non-naive Platonistic position, to be precisely understood in the sense advanced by Lautman[14] to understand the objectivity of mathematical reality. However, as claimed by Dieudonné, they have a "prophetic" capacity in the sense that they seem to be relevant for the path and the "philosophical concerns" of a later mathematician[15] such as Alexander Grothendieck (who, as is known, belonged to the Bourbakist movement and then distanced himself from it for reasons internal to his project). Some of these concerns, as shown by the Colombian mathematician Fernando Zalamea, can be better understood by taking into account the instruments of that particular historical-dialectic *philosophie mathématique* deployed by Lautman, or "synthetic philosophy" of mathematics [54, ch. IV]. What Zalamea called "not just Copernican, but Einsteinian revolution" brought about by Grothendieck in Weyl's "empire of mathematics" is rooted in the constant, deep reflections on the development of ideas and in the generation of increasingly general concepts, on the great mathematical creativity; these philosophical concerns about the nature and structure of the *connaissance* produced by mathematics are not random, but have accompanied Grothendieck's work of discovery, and are not only present in the unpublished but circulating *Récoltes et semailles*, but also in other, more programmatic writings such as 1984 *Esquisse d'un programme*, where he spoke in almost Lautmanian and even Bachelardian terms of "work of discovery by the thought that probes and discovers a new world", of a mathematics that continually "feeds on *rêves* and visions". This way, he wrote, he did that "work of reflection on the fundamentals

[12] This Lautmanian idea of "perfection" and "imperfection" has striking analogies with the line of research of Jean-Yves Girard, engaged in the field of linear logic, the return to the geometric and laying the foundations of a view of logic and mathematics not referable to formal propensity.

[13] It should be noted that in France in the second half of the twentieth century, in continuity with Lautman's approach, Grassmann's work was at the centre of a systematic epistemological analysis by Gilles Châtelet.

[14] Jean Petitot, in the wake of Lautman, talks about Gödel of "transcendental Platonism" to distinguish it from other interpretations, such as those given, for instance, by Quine, Burgess, Resnik, Connes; see [45].

[15] But the same could be said for other contemporary mathematicians like Lawvere, and Alain Connes and Jean-Yves Girard for the French area; on Grothendieck see [25].

of (co)homological non-commutative algebra", which later led to the two volumes of 1986 *À la poursuite des champs*, volumes that were to constitute in its intentions "the first one of a wider series with the common title *Réflexions mathématiques*" [29]. *Mutatis mutandis*, Lautman almost seems to offer him, ahead of his time, the basics and the language itself, the conceptual coordinates to conduct his reflections within a philosophical view that allows him more easily to enter that Weyl-like "open world" that is mathematics, thanks to their cognitive history; moreover, a Lautmanian reading of his writings helps to better understand, on an epistemological level, what can be called an explosion of creativity, those "dynamic-procedural basements", in the words of Jean-Yves Girard [26, p. 210], which allowed him to achieve remarkable results. Such a reading can also be useful to remove from this multifaceted figure that aura of mystique and mystery that often surrounds him, and to shed light on a not insignificant reason for his move away from Bourbaki.

4 For an epistemology of physical–mathematical contents

All this allowed Lautman to lay the foundations of a theoretical path aiming, as Paul Bernays already aptly remarked, at dealing with the contents of the mathematical world up to a veritable epistemology of the contents that does not find analogues in other areas for putting at the centre of his analysis "what mathematical knowledge *actually* is", an issue deemed unavoidable by "thinkers like Poincaré, Husserl, Weyl, Borel, Lebesgue, Veronese, Enriques, Cavaillès, Gonseth or Gödel in his later life" [44, p. 214].[16] This was possible thanks to Lautman's ability to combine the historical dimension of mathematics, inherited from his teacher and better highlighted on the epistemic ground by Federigo Enriques[17] and Gaston Bachelard, with its structural dimension, as well as his ability to grasp the methodological scope implicit in the early works of the Bourbakists. This is enhanced by a global, synoptic look able to offer a particular reading of modern mathematics from Riemann to the 1930s, a historical view apparently absent in the attitudes of his Bourbakist friends and considered strategic for understanding the conceptual articulation of the problems. While the Bourbakists, in his view, had the merit of dealing in France with German mathematics, and in particular with that of Hilbert, they nevertheless ended up losing its theoretical value, especially as connected to the overall sense of axiomatic thought, reduced to a mere logical-procedural skeleton and analytical-formal function, as indeed, even for other purposes, was happening in the contemporary interpretations given especially to Hilbert's work by the members of the Vienna Circle, who deemed themselves to be the only faithful interpreters of his programme [38].

As it has been pointed out by the few scholars who have studied him, Lautman's thinking, especially in his two theses of 1937, offers a different interpretation of Hilbert's programme as well, deviating from the interpretations given by his Bourbakist friends who, despite having identified the central role assigned therein to the notion of structure, read it in a unilateral manner, just as they were doing with the complex and appreciated work of Weyl.[18] In Lautman's view, they were underestimating its more general cognitive scope and gave little credit to what they called "philosophical concerns" by both Hilbert and especially Weyl, remaining silent about the fact that they were able to offer the more comprehensive philosophical platform that the "mathematical continent" needed for its increasingly rich expressions. Instead, these "philosophical concerns" became points of reference for Lautman who, starting from a critical commentary of Paul Bernays's 1935 essay "Sur le platonisme en mathématique" [10, pp. 83–96], did not skip over the *vexata quaestio* of Platonism in mathematics, restored it to the appropriate epistemological dimension overcoming its naive versions, enriched it with the Kantian views, not hesitating to resort to categories borrowed from Heidegger's contemporary philosophy to find an alternative route to the agnostic attitude prevalent in certain mathematical milieux, starting with the Bourbakists themselves, and to that inadequate conception of mathematics allegedly present in analytic philosophy. What we could roughly call, in the wake of Weyl, their "conventionalism", for their insufficient critical consideration of the real contents of mathematics, an attitude shared with some members of the Vienna Circle, also explains the subsequent lack of interest in the achievements of young Kurt Gödel with his incompleteness theorems, considered by Lautman, to be understood in their most correct dimension, in need of a very different approach, to be explored with conceptually

[16] We might add to this list, in the French area, first Winter and Bachelard and then mathematicians like Grothendieck and Alain Connes.

[17] Young Lautman, like other philosophers of science of the Francophone area of the time, knew very well the historical-epistemological works by Enriques and, reviewing the proceedings of the first Congrès de Philosophie scientifique in 1935, wrote: "Enriques ... in his communication to the congress has proved how the history of science is for the study of scientific truth an instrument no less necessary than logical formalism" [38], pp. 72–73.

[18] With regard to this, it is sufficient to read the preface, written as late as 1964 by another Bourbakist, G. Th. Guilbaud, to the French translation of Weyl's *Symétrie et mathématiques modernes*, in which Weyl's "philosophical concerns" are considered to be chimeras and the axiomatic method is perceived only in its technical and analytical-formal aspect, losing sight of its more general value for thought.

richer tools. It is no coincidence that they were subjected to initial philosophical investigations not only by Lautman, but also by a few other French-language scholars such as his friend Jean Cavaillès through Husserl and Ferdinand Gonseth,[19] who in the late 1930s would put at the centre of his reflections the same topics: the unity of mathematics, its inventive character and its processes of abstraction and generalisation, its relations with reality, the dialectical aspect, the relationships between mathematics and physics.

The search for unity and the contents of the open world of mathematics further strives towards physical–mathematical unity, again in the wake of Riemann and Weyl and in the light of the analysis of the most significant events of the early twentieth century, such as relativity and quantum mechanics. Lautman, once again critically distancing himself from his Bourbakist friends who had underestimated the problem of the relationship between mathematics and physics and their applicability to physical reality, considered it to be fundamental and constitutive of the philosophy itself of science. Just as analytical philosophy had arisen from the study of the cognitive dimension of mathematical logic, considered strategic for the new "scientific philosophy", to be taken as a model of knowledge to which other sciences had to conform, so in Lautman, as in Bachelard, contemporary mathematical physics is a real rational institution with a well-defined conceptual structure that poses a fundamental epistemological problem. Even though Lautman would deal with this topic in more detail in a short but dense essay *Symétrie et dissymétrie en mathématiques et en physique* [34], which appeared after his premature death, it was already programmatically present at the beginning of his research project:

> There is a physical reality and the miracle to be explained is that we need the most advanced mathematical theories to understand it ... It would be totally devoid of interest, that philosophy of science that were not entirely based on the study of this solidarity between fields of reality and methods of investigation ... The central problem of the philosophy of science is undoubtedly that of the relations between mathematical theory and physical experience. We would like to be able, one day, to prove the fact that the concepts about the physical universe are nothing but a concrete representation of notions that can only be defined within a mathematical theory ... The real underlying problem is rather to know how a differential geometry can become a theory of gravitation, a proof of the intelligibility of the universe [35, pp. 46 and 295].

Thus, tensor calculus, the concept of space by Henri Cartan, the theories of abstract algebra, Lie group theory, and Pfaff theory are investigated to lay the foundation of a true epistemology of mathematical physics, where the purely linguistic vision of mathematics is overcome, together with those points of view aimed at reducing the problem of the relations with physical reality in terms of their simple applicability. In Ludovico Geymonat's words, we can say about Lautman and, more generally, French epistemology.

> that there was not on the one hand the revolution in physics and on the other that in mathematics (set theory, relativity), but a revolution in mathematical physics as a whole; and the French school has been able to recognise the importance of mathematical physics, which is precisely, at the same time, physics and mathematics [23, p. 73].[20]

Like Bachelard before him, Lautman captured this basic unity in its full epistemic depth, thanks to his attitude of remaining, as far as possible, "faithful" to the "mathematical world", as Bernays said [9, p. 22] or, better, to the physical–mathematical world as it came to be configured in the early twentieth century. Using conceptual tools borrowed from both the philosophical tradition and the contemporary thought, he delved into the philosophical concerns of various scientists, considering them not to be marginal, while others deemed them to be pure chimeras. He had no hesitation in considering the physical–mathematical world as having its own ontological dimension, so much so that he claimed that "this approach between metaphysics and mathematics is a necessary and not contingent fact" [35, p. 237], to be probed in its different aspects. This is why he concentrated his efforts on giving a more organic theoretical voice to these concerns that were latent in scientific thought, also creating the conditions for what we have called an active form of resistance against the reductive visions of the mathematical world present in the 1930s, not just in epistemology [18]. In this way he also gave us true epistemic reasons to make the mystery of mathematics less irrational in its fundamental role in the physical field. Ahead of his time, his visions may provide significant answers to the subsequent literature

[19] Ferdinand Gonseth (1890–1975) is the author of several writings including *Les mathématiques et la réalité* (1936) and *Philosophie mathématique* (1939); in 1938 he organised the *Entretiens de Zürich* devoted to the methods and foundations of mathematics, inviting mathematicians and philosophers—such as Enriques, George Pólya, Bernays and Gödel himself, who declined the invitation—to create the foundations of a philosophy of mathematics with a different orientation from that of standard philosophy of science, also thanks to the collaboration of Heyting; in 1947 he founded the journal *Dialectica* together with Bernays. The first reflections by Lautman, Cavaillès and Gonseth on Gödel are more meaningful if they are re-read in the light of Gödel's unpublished writings of the 1950s included in his now available *Nachlass*.

[20] On this, see also [1, 16–18].

produced by eminent physicists in the second half of the twentieth century, about the so-called "unreasonable effectiveness of mathematics".

We like to close with a remark by Grassmann that, although written almost a hundred years earlier, seems to apply to Lautman:

> The inkling [*Ahnung*] seems alien to the field of pure science, especially in the field of mathematics. However, without it, it would be impossible to find any new idea ... If it is well activated, it is the gaze that captures in one stroke the whole path leading to the new truth, but with some moments not yet made explicit; for this reason the inkling, at the beginning, can only be obscure [27], p. 31.
>
> Translated from the Italian by Daniele A. Gewurz.

Appendix. Albert Lautman

Albert Lautman was born in Paris on 8 February 1908 into a family of Jewish origin: his father was Samy Lautman, a physician who volunteered in the Foreign Legion in September 1914 and was mutilated in World War I.

After graduating from the lycée Condorcet, in 1926 he was admitted to the École Normale Supérieure where he was a pupil of the idealist philosopher Léon Brunschvicg. Although influenced by pacifist and socialist thought, at the École Normale Supérieure he successfully followed military training for officers. He became an agrégé of philosophy and spent his doctorate period in Paris. Immediately after serving in the army, in September 1931, he married Suzanne Perreau-Detrie; the couple then moved for two years to Japan, where Lautman taught French literature and philosophy at the Institute of Western Languages in Osaka.

Faced with the rise of Nazism, at the beginning of 1938, he volunteered for a cycle of training for reserve officers, spending a period in the Suippes camp. At the time of the mobilisation against the Nazis, Lautman held the rank of lieutenant and was given the command of a battery. With the outbreak of World War II, he became captain of the 1021e Batterie 404 R.A.D.C.A. with which, by 1940, he shot down seven German planes. He was taken prisoner on the Belgian border and sent to Camp Oflag IV-D, in Germany, from where he fled on 14 October 1941 together with other officers, managing to reach the free zone on 28 October 1941.

On his return, he decided to join the Resistance, where he saw to the escape to Spain of American, British and Canadian airmen passing through Toulouse. Perhaps reported by the owner of a restaurant in Toulouse and victim of a trap by the Gestapo, Albert Lautman was arrested on 15 May 1944 while trying to get news about his brother Jules, also a member of the Resistance and interned in Saint-Michel prison in Toulouse.

Deported to the Souge camp in Martignas-sur-Jalle, near Bordeaux, Lautman was shot along with 49 comrades on 1 August 1944. For his activity in the Resistance he received, posthumously, the Médaille de la Résistance, the British Medal Order and the Medal of Freedom from US President Dwight D. Eisenhower.

Thanks to his friendship with mathematician Jacques Herbrand (also a student at the École Normale Supérieure), Lautman was a keen and knowledgeable scholar of German mathematics from Hilbert to Weyl, from Emmy Noether to Artin.

He also had close contacts with some of the mathematicians who founded the Bourbaki group, defending mathematical Platonism:

The reality inherent in mathematical theories derives from the fact that they are part of an ideal reality that is preponderant in relation to mathematics, but that is only knowable through it [31, p. 143].

Albert Lautman defended his doctoral thesis and the complementary thesis: Essai sur les notions de structure et d'existence en mathématiques and Essai sur l'unité des sciences mathématiques dans leur développement actuel [31, 33]. In 1939 he published Nouvelles recherches sur la structure dialectique des mathématiques. Other essays appeared posthumously, such as Symétrie et dissymétrie en mathématiques et en physique. Le problème du temps [34] and, together with Jean Cavaillès, "La pensée mathématique" [19].

The work of Albert Lautman has played a crucial role in contemporary French philosophy and his ideas has had a great influence on the thought of philosophers such as Gilles Deleuze and Alain Badiou, for whom he represented a milestone in the development of their philosophical and scientific interest in mathematics.

References

1. Alunni, C.: Albert Lautman et le souci brisé du mouvement. Revue de synthèse. **126**, 283–311 (2005)
2. Alunni, C.: Maximilien Winter et Federigo Enriques: des harmonies exhumées. In: André, Y. (ed.) Federigo Enriques o le armonie nascoste della cultura europea. Tra scienza e filosofia, C. Alunni, pp. 101–147. Edizioni della Normale, Pisa (2015)
3. Alunni, C., Castellana, M., Ria, D., Rossi, A. (eds.): Albert Einstein et Hermann Weyl 1955–2005. Questions épistémologiques ouvertes. Barbieri-Selvaggi-ENS Éditions Rue d'Ulm, Manduria (2010)
4. Bachelard, G.: Essai sur la connaissance approchée. Vrin, Paris (1928)
5. Bachelard, G.: La philosophie du non. P.U.F., Paris (1940)

6. Barot, E.: Lautman. Les Belles Lettres, Paris (2009)
7. Bell, E.T.: The development of mathematics. McGraw-Hill, New York (1945)
8. Benis-Sinaceur, H.: Lettres inédites de Jean Cavaillès à Albert Lautman et de Gaston Bachelard à Albert Lautman. Revue d'histoire des sciences. **XL**, 117–129 (1987)
9. Bernays, P.: Albert Lautman. Essai sur les notions de structure et d'existence en mathématiques [Review]. J. Symb. Logic. **5**, 20–22 (1940)
10. Bernays, P.: Sur le platonisme en mathématique. In: Bernays, P. Philosophie des mathématiques, pp. 83–98. Vrin, Paris (2003)
11. Bourbaki, N.: Éléments d'histoire des mathématiques. Hermann, Paris (1969)
12. Brenner, A., Petit, A. (eds.): Science, histoire et philosophie selon Gaston Milhaud. Vuibert, Paris (2009)
13. Brunschvicg, L.: Les étapes de la philosophie mathématiques (1912). Blanchard, Paris (1972)
14. Castellana, M.: La filosofia della matematica in Albert Lautman. Il Protagora. **114**, 12–24 (1978)
15. Castellana, M.: Alle origini della 'nuova epistemologia'. Il Congrès Descartes del 1937. Il Protagora. **17–18**, 7–100 (1992)
16. Castellana, M.: Razionalismi senza dogmi. Per una epistemologia della fisica matematica. Rubbettino, Soveria Mannelli (2004)
17. Castellana, M.: Les mathématiques et l'expérience selon Albert Lautman. In: Barbin, E., Cléro, J.P. (eds.) Les mathématiques et l'expérience, pp. 337–364. Hermann, Paris (2015)
18. Castellana, M. Introduzione. [38], pp. 7–47 (2017)
19. Cavaillès, J., Lautman, A.: La pensée mathématique (lecture with A. Lautman at the Société française de Philosophie in February 1939). Bulletin de la Société française de Philosophie 40, 1–39 (1946). Reprinted as: Cavaillès, Lautman, J., A.: La pensée mathématique. In J. Cavaillès (eds.), Œuvres complètes de philopsophie des sciences, pp. 593–630. Hermann, Paris (1994). English trans. available at: https://www.urbanomic.com/document/mathematical-thought/
20. Cournot, A.: Essai sur les fondements de nos connaissances et sur les caractères de la critique philosophique. Voll. I-II. Hachette, Paris (1851)
21. Dieudonné, J.: Préface. In [35], pp. 35–38 (1977)
22. Enriques, F.: Problemi della scienza. Zanichelli, Bologna (1985)
23. Geymonat, L.: Filosofia e filosofia della scienza, 4th edn. Feltrinelli, Milano (1968)
24. Geymonat, L.: Tre domande per Ludovico Geymonat. In: Castellana, M., Bertani, G. Due culture a confronto: la filosofia della scienza in Francia e in Italia nel Novecento, pp. 72–75. Bertani, Verona (1986)
25. Giorello, G., et al.: Matematica ribelle. Le due vite di Alexander Grothendieck. Corriere della Sera, Milano (2014)
26. Girard, J.-Y.: Le point aveugle. I Cours de logique. Vers la perfection. Hermann, Paris (2006)
27. Grassmann, H.G.: La science de la grandeur extensive, Flament, D. (ed.) Blanchard, Paris (1994)
28. Grothendieck, A.: Récoltes et Semailles: réflexions et témoignages sur un passé de mathématicien. Unpublished (1985)
29. Grothendieck, A.: À la poursuite des champs. Hermann, Paris (1986)
30. Heyting, A.: Les fondements des mathématiques. In: Gonseth, F. Philosophie mathématique, pp. 69–77. Hermann, Paris (1939)
31. Lautman, A.: Communication au IXe Congrès international de philosophie. Actualités **535**, Fascicule VI, 140–143 (1937)
32. Lautman, A.: Essai sur les notions de structure et d'existence en mathématiques. Ph.D. thesis, Faculté des lettres de l'Université de Paris: (1937). Reprinted in: Essai sur les notions de structure et d'existence en mathématiques et divers écrits. Union Générale d'Editions, Paris (1977)
33. Lautman, A.: Essai sur l'unité des sciences mathématiques dans leur développement actuel. In: Hermann, Cie Reprinted in: Essai sur les notions de structure et d'existence en mathématiques et divers écrits. Union Générale d'Editions, Paris (1977), Paris (1938)
34. Lautman, A.: Symétrie et dissymétrie en mathématiques et en physique: Le problème du temps. Actualités scientifiques et industrielles 102. Hermann, Paris (1946). Reprinted in: Les mathématiques les idées et le réel physique, pp. 265-306, F. Zalamea, ed., Vrin, Paris (2006)
35. Lautman, A.: Les mathématiques les idées et le réel physique. Zalamea, F. (ed.), Vrin, Paris (2006)
36. Lautman, A.: Ensayos sobre la dialéctica, estructura y unidad de las matemáticas modernas. Zalamea, F. (ed.), Universidad Nacional de Colombia, Bogotá (2011)
37. Lautman, A.: Mathematics, Ideas and the Physical Real. S. Duffy, B. (ed.), Continuum International Publishing Group, London/New York (2011)
38. Lautman, A.: La matematica come resistenza. Castelvecchi, Roma (2017)
39. Lawvere, F.W.: Categories of space and quantity. In: Díez, A., et al. (eds.) Structures in mathematical theories, pp. 14–30. De Gruyter, Berlin (1992)
40. Lecourt, D.: La philosophie dans les sciences. Revue de synthèse. **126**, 451–454 (2005)
41. Lecourt, D. (ed.): Dictionnaire d'histoire et de philosophie des sciences. P.U.F., Paris (1999)
42. Milhaud, G.: Introduction. In: Du Bois-Reymond, P.: Théorie générale des fonctions, G. Milhaud (ed.) pp. 4–15. Imprimerie niçoise, Nice (1887)
43. Patras, F.: La pensée mathématique contemporaine. P.U.F., Paris (2001)
44. Petitot, J.: Idéalités mathématiques et réalité objective. Approche transcendantale. Hommage à Jean-Toussaint Desanti, Auroux, S. (ed.), pp. 213–279, T.E.R., Mauvezin (1991)
45. Petitot, J.: Continu et objectivité. La bimodalité objective du continu et le platonisme transcendantal. In: Salanskis, J.M., Sinaceur, H. (eds.) Le labyrinthe du continu. Springer, Paris (1992)
46. Pont, J.C.: De l'absolu au relatif, destin du XXe siècle. In: Pour comprendre le XIXe. Histoire et philosophie des sciences à la fin du siècle, J.-C. Pont et al. Leo S. Olschki, Firenze, eds., pp. ix-xlviii (2007)
47. Ria, D.: L'unità fisico-matematica nel pensiero epistemologico di Hermann Weyl. Congedo Ed., Galatina (2005)
48. Riemann, B.: Gesammelte mathematische Werke und wissenschaftlicher Nachlass. R. Dedekind, H. Weber, (eds), Teubner, Leipzig (1876). Reprint: Dover, New York (1953)
49. Schlick, M.: Allgemeine Erkenntnislehre. Verlag von Julius Springer, Berlin (1918)
50. Weyl, H.: Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis. Walter de Gruyter & Co., Berlin (1932)
51. Winter, M.: La méthode en philosophie des mathématiques. Alcan, Paris (1911)
52. Zalamea, F.: Albert Lautman et la dialectique créatrice des mathématiques modernes. [35, pp. 17–33] (2006)
53. Zalamea, F.: Estudio introductorio. [36], pp. 13–64 (2011)
54. Zalamea, F.: Synthetic philosophy of contemporary mathematics. Urbanomic, Falmouth (2012)



Mario Castellana is professor of philosophy of science and of epistemology and history of science at the University of Salento (Lecce, Italy). His research interests focus on the French and Italian epistemology of the 20th century, and in particular on the debates on the nature of mathematics and physics and on the relationship between epistemology and history of science. He has investigated the thought of Gaston Bachelard, Federigo Enriques, Ferdinand Gonseth, Albert Lautman, Simone Weil, Hélène Metzger, Suzanne Bachelard, Jean Desanti, Gilles Châtelet, Michel Serres, Jean Piaget and Raymond Boudon, about whom he has authored monographs and articles published in Italian and international volumes and journals. He edited and wrote the forewords to the Italian editions of the works of some of these. He is a member of the Laboratoire Disciplinaire "Pensée des sciences" of the École Normale Supériere in Paris. His latest publications include: *Razionalismi senza dogmi. Per una epistemologia della fisica matematica* (Rubbettino, 2004); "Il metodo storico in filosofia della scienza" (2007); *Dossier on Federigo Enriques* (Kairos. Journal of Philosophy and Science, 2014); "Les mathématiques et l'expérience selon Albert Lautman" (2015).