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Commuting Groups and the Topos of Triads

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Abstract. The goal of this article is to clarify the relationship between the topos of triads and the neo-Riemannian PLR-group. To do this, we first develop some theory of generalized interval systems: 1) we prove the well known fact that every pair of dual groups is isomorphic to the left and right regular representations of some group (Cayley's Theorem), 2) given a simply transitive group action, we show how to construct the dual group, and 3) given two dual groups, we show how to easily construct sub dual groups. Examples of this construction of sub dual groups include Cohn's hexatonic systems, as well as the octatonic systems. We then enumerate all $\mathbb{Z}_{12}$-subsets which are invariant under the triadic monoid and admit a simply transitive PLR-subgroup action on their maximal triadic covers. As a corollary, we realize all four hexatonic systems and all three octatonic systems as Lawvere–Tierney upgrades of consonant triads.

Keywords: PLR-group, duality, sub dual groups, hexatonic systems, octatonic, topos, topos of triads, triadic monoid

1 Introduction

In [1], Richard Cohn coined the term "the overdetermined triad" in order to draw the reader's attention away from the celebrated acoustic attractiveness of the major and minor triads to some surprising algebraic properties of the family of the consonant triads. The starting point for the present investigation is a puzzling ramification of this over-determination of the triads even within the realm of algebraic music theory. In particular, the musical and music-theoretical prevalence of three tone-collections — major/minor mixture, hexatonic, and octatonic — emerges from the study of triads in quite different algebraic ways.

In the context of a group-theoretic approach (see [2]) one obtains these collections as carrier sets, which are covered by particular families of triads, which

are orbits under certain group actions. Under this perspective it is the interplay of several triads, which makes these collections prevalent among others.

In the context of a monoid-theoretic approach (see [3]) one obtains these collections as extensions of a single triad. The stability properties of the set $\{0, 4, 7\}$ within $\mathbb{Z}_{12}$ can be studied in terms of a monoid of 8 affine transformations. On top of this monoid one obtains three interesting Lawvere–Tierney topologies, which also imply the prevalence of the three sets $\{0, 3, 4, 7\}$, $\{0, 3, 4, 7, 8, 11\}$ and $\{0, 1, 3, 4, 6, 7, 9, 10\}$ as extensions of a single triad.

How do these two approaches fit together? The present article prepares the group-theoretical grounding for a comprehensive answer and anticipates the remaining work on the topos-theoretical side in the concrete case of $\mathbb{Z}_{12}$.

Though this article does not require any topos theory, we have included an Appendix on Topos Theory in Section 5 for any readers who want to recall the main notions.

2 Dual Groups and Cayley's Theorem

We first briefly recall the notion of dual groups in transformational analysis, give the fundamental example of left and right regular representations, and show how to construct general dual groups in view of the left and right regular representations. As an example, we reconstruct the well-known PLR-group as the dual group to the T/I-group acting on consonant triads.

Definition 2.1 (Dual groups in the sense of Lewin). Let $\mathrm{Sym}(S)$ be the symmetric group on the set S. Two subgroups G and H of the symmetric group $\mathrm{Sym}(S)$ are called *dual* if their natural actions on S are simply transitive and each is the centralizer of the other, that is,

$$C_{\mathrm{Sym}(S)}(G) = H \quad \text{and} \quad C_{\mathrm{Sym}(S)}(H) = G.$$

Example 2.2 (Left and right regular representations). If G is any group, the left and right regular representations $\lambda, \rho\colon G \to \mathrm{Sym}(G)$ are defined by $\lambda_g(h) = gh$ and $\rho_g(h) = hg^{-1}$. As is well known, λ and ρ are injective and the natural actions of $\lambda(G)$ and $\rho(G)$ are simply transitive. The images $\lambda(G)$ and $\rho(G)$ clearly commute, since $\lambda_{g_1}\rho_{g_2}(h) = g_1 h g_2^{-1} = \rho_{g_2}\lambda_{g_1}(h)$. In fact, $\rho(G)$ is the centralizer of $\lambda(G)$ in $\mathrm{Sym}(G)$, for if $f\colon G \to G$ is a bijection satisfying $f(gh) = gf(h)$ for all g and h in G, then $f(h) = f(he) = hf(e)$ and f is the same as $\rho(f(e)^{-1})$. Similarly, $\lambda(G)$ is the centralizer of $\rho(G)$ in $\mathrm{Sym}(G)$. The groups $\lambda(G)$ and $\rho(G)$ provide the most basic example of dual groups. All other examples of dual groups are isomorphic to this one, as we see in the following construction.

Construction 2.3 (Construction of the dual group). Given a group G acting simply transitively on a set S, we may construct the dual group as follows. We consider G as a subgroup of $\mathrm{Sym}(S)$ via the homomorphism $G \to \mathrm{Sym}(S), g \mapsto (s \mapsto gs)$, which is an injection by simple transitivity. We suggestively call this

subgroup $\lambda(G)$, although it is not the same as $\lambda(G)$ in Example 2.2. Fix an element $s_0 \in S$. Again by simple transitivity, the function $G \to S, h \mapsto hs_0$ is a bijection. The natural action of $\lambda(G)$ on S is essentially the same as left multiplication: $g(hs_0) = (gh)s_0$. We now define a second injection $G \to \mathrm{Sym}(S)$ by $g \mapsto (hs_0 \mapsto hg^{-1}s_0)$ and call the image $\rho(G)$. The group $\rho(G)$ is the dual group to $\lambda(G)$, which we sought to construct. The bijection $G \to S, h \mapsto hs_0$ and the injection $G \to \mathrm{Sym}(S), g \mapsto (hs_0 \mapsto hg^{-1}s_0)$ depend on the choice of s_0, but the group $\rho(G)$ does not.

Example 2.4 (Construction of PLR-group). If in Construction 2.3 we take G to be the T/I-group, S to be the set of major and minor triads, and $s_0 = \{0, 4, 7\}$, then $\rho(G)$ is the neo-Riemannian PLR-group. For example, the operations parallel, leading tone exchange, and relative, denoted P, L, and R respectively, correspond to right multiplication by I_7, I_{11}, and I_4.

$$\begin{aligned}
P:&\ T_n\{0,4,7\} \mapsto T_nI_7\{0,4,7\} \quad \text{and} \quad I_n\{0,4,7\} \mapsto I_nI_7\{0,4,7\}\\
L:&\ T_n\{0,4,7\} \mapsto T_nI_{11}\{0,4,7\} \quad \text{and} \quad I_n\{0,4,7\} \mapsto I_nI_{11}\{0,4,7\}\\
R:&\ T_n\{0,4,7\} \mapsto T_nI_4\{0,4,7\} \quad \text{and} \quad I_n\{0,4,7\} \mapsto I_nI_4\{0,4,7\}.
\end{aligned}$$

The embedding $\rho\colon T/I \to \mathrm{Sym}(S)$ takes the generators I_1 and I_8 to L and R respectively (recall $I_1 \circ I_8 = T_{1-8} = T_5$). If we choose a different s_0, then we obtain another isomorphism of the T/I-group with the PLR-group.

3 Sub Dual Groups

We next give a very practical method for constructing sub dual groups from dual groups. This method minimizes computation: instead of checking commutativity of all functions on every element, it suffices to evaluate a putative commuting function on the single element s_0 and check if the output lies in S_0.

Theorem 3.1 (Construction of sub dual groups). *Let $G, H \leq \mathrm{Sym}(S)$ be dual groups, G_0 a subgroup of G, and s_0 an element in S. Let S_0 be the orbit of s_0 under the action of G_0. Then the following hold.*

(i) The group G_0 acts simply transitively on S_0.
(ii) If $g \in G$ and gs_0 is in S_0, then $g \in G_0$. In particular, if $g \in G$ and gs_0 is in S_0, then g preserves S_0 as a set, that is, $g(S_0) \subseteq S_0$.
(iii) Let H_0 denote the subgroup of H consisting of those elements $h \in H$ with $hs_0 \in S_0$. Then H_0 acts simply transitively on S_0.
(iv) Restriction from S to S_0 embeds G_0 and H_0 in $\mathrm{Sym}(S_0)$. We denote their images in $\mathrm{Sym}(S_0)$ by $G_0|_{S_0}$ and $H_0|_{S_0}$.
(v) If $g \in \mathrm{Sym}(S_0)$ and g commutes with $H_0|_{S_0}$, then g admits a unique extension to S which belongs to G. This extension necessarily belongs to G_0 by (ii). Similarly, if $h \in \mathrm{Sym}(S_0)$ commutes with $G_0|_{S_0}$, then h admits a unique extension to S which belongs to H. This extension necessarily belongs to H_0.
(vi) The groups $G_0|_{S_0}$ and $H_0|_{S_0}$ are dual in $\mathrm{Sym}(S_0)$.

Proof. (i) By the definition $S_0 = G_0 s_0$, we have $gs \in S_0$ for all $g \in G_0$ and $s \in S_0$. The orbit of s_0 is all of S_0, so the action of G_0 on S_0 is transitive. Simplicity of the G_0-action on S_0 follows from the simplicity of the G-action on S.

(ii) Suppose $g \in G$ and $gs_0 = s \in S_0$. By the simple transitivity of G_0 acting on S_0 from (i), there is a unique $\overline{g} \in G_0$ such that $\overline{g}s_0 = s$. But on the other hand, g and $\overline{g}$ are in G, which acts simply transitively on S, so $g = \overline{g} \in G_0$.

(iii) Suppose $h \in H$ and $hs_0 \in S_0 = G_0 s_0$. An arbitrary element of S_0 has the form gs_0 for some $g \in G_0$, and $h(gs_0) = g(hs_0) \in G_0 s_0 = S_0$. Thus $h(S_0) \subseteq S_0$, and the action of H_0 on S restricts to an action on S_0. The orbit of s_0 under H_0 is S_0, for if $gs_0 \in S_0$ is an arbitrary element, then there exists $h \in H$ such that $hs_0 = gs_0 \in S_0$, which implies $h \in H_0$. Simplicity follows from the simplicity of the H-action on S.

(iv) Suppose $g, g' \in G_0$ and $g|_{S_0} = g'|_{S_0}$. Then $gs_0 = g's_0$ and $g = g'$ by the simple transitivity of the G-action on S. The injectivity of the H_0-embedding is similar.

(v) Suppose $g \in \mathrm{Sym}(S_0)$ and g commutes with $H_0|_{S_0}$. By the simple transitivity of the G-action on S, there is a unique $\overline{g} \in G$ such that $\overline{g}s_0 = gs_0$. By (iii) a general element of S_0 has the form hs_0 with $h \in H_0$. We have $\overline{g}hs_0 = h\overline{g}s_0 = h|_{S_0}gs_0 = gh|_{S_0}s_0 = ghs_0$. Thus $\overline{g}$ and g agree on S_0 and $\overline{g}$ is a true extension of g. The proof for the analogous statement concerning elements in $\mathrm{Sym}(S_0)$ which commute with $G_0|_{S_0}$ is similar.

(vi) We already have simply transitive group actions of $G_0|_{S_0}$ and $H_0|_{S_0}$ on S_0 by (i) and (iii). The groups $G_0|_{S_0}$ and $H_0|_{S_0}$ commute because G and H do. Suppose $h \in \mathrm{Sym}(S_0)$ commutes with $G_0|_{S_0}$. Then by (v), the bijection h admits an extension to $\overline{h} \in H_0 \subseteq \mathrm{Sym}(S)$, which says precisely $h \in H_0|_{S_0}$. This proves $C_{\mathrm{Sym}(S_0)}(G_0|_{S_0}) = H_0|_{S_0}$. The proof $C_{\mathrm{Sym}(S_0)}(H_0|_{S_0}) = G_0|_{S_0}$ is similar.

□

Remark 3.2. In Theorem 3.1, the subgroups G_0 and H_0 of $\mathrm{Sym}(S)$ commute, since they are subgroups of the dual groups G and H, respectively. However, G_0 and H_0 are *not* dual groups in $\mathrm{Sym}(S)$ when $G_0 \lneqq G$, though the restrictions $G_0|_{S_0}$ and $H_0|_{S_0}$ are dual in $\mathrm{Sym}(S_0)$. First of all, G_0 and H_0 do not act simply transitively on S. Secondly, the centralizer of G_0 is larger than H_0, as the centralizer contains at least all of H. The isomorphism type of $C_{\mathrm{Sym}(S)}(G_0)$ may be found using the methods of [4].

Corollary 3.3 (Transforming the orbits of G_0). *Let $G, H \leq \mathrm{Sym}(S)$ be dual groups, G_0 a subgroup of G, and s_0 an element in S. Let S_0 be the orbit of s_0 under the action of G_0, and H_0 the subgroup of H consisting of those elements $h \in H$ with $hs_0 \in S_0$. Then the following hold.*

(i) If $k \in H$, then the G_0-orbit of ks_0 is kS_0. Thus, if we transform s_0 to ks_0 in Theorem 3.1, then S_0 transforms to kS_0.

(ii) If $k \in H$, then kH_0k^{-1} is precisely the subgroup of H consisting of those $h \in H$ such that $h(ks_0) \in kS_0$. Thus, if we transform s_0 to ks_0 in Theorem 3.1, then H_0 transforms to kH_0k^{-1}.

(iii) *If $k \in H$, then the restrictions of G_0 and kH_0k^{-1} to kS_0 are dual in* $\mathrm{Sym}(kS_0)$.

(iv) *Each orbit of G_0 can be written as kS_0 for some $k \in H$. Moreover, the group H acts transitively on the orbits of G_0 by $h(kS_0) = (hk)S_0$.*

Proof. (i) The element $k \in H$, commutes with each element in G_0, so $G_0ks_0 = kG_0s_0 = kS_0$.

(ii) If $kh_0k^{-1} \in kH_0k^{-1}$, then $kh_0k^{-1}(ks_0) = kh_0s_0 \in kS_0$ since $h_0s_0 \in S_0$. On the other hand, if $h \in H$ and $hks_0 = gks_0$ for some $g \in G_0$, then $k^{-1}hks_0 = gs_0$ and $k^{-1}hk \in H_0$, so $h \in kH_0k^{-1}$.

(iii) This follows directly from Theorem 3.1 (vi).

(iv) This follows directly from the transitivity of the H-action on S and (i).

□

3.1 Hexatonic Systems

To see the utility of Theorem 3.1, we construct the hexatonic systems of Cohn [5] and the dual pair of Clampitt [6]. In Theorem 3.1, we take S to be the set of major and minor triads, G to be the neo-Riemannian PLR-group, H to be the T/I-group, and G_0 to be the PL-group, which is generated by P and L, is isomorphic to S_3, and therefore has only 6 elements.

Let s_0 be $E\flat$ (we write major chords with upper-case letters and minor chords with lower-case letters). Beginning with $E\flat$, and evaluating P and L alternately, we obtain the 6 chords $\{E\flat, e\flat, B, b, G, g\}$. We have now found the complete orbit S_0 of $E\flat$ and may stop: by Theorem 3.1 (i), G_0 acts simply transitively on S_0, which means $|S_0| = |G_0| = 6$ by the Orbit-Stabilizer Theorem.

To find H_0, we must only find those transpositions and inversions which map $E\flat$ into S_0, or in other words, we must only solve the six equations

$$\begin{aligned} T_iE\flat &= E\flat & \qquad I_\ell E\flat &= e\flat \\ T_jE\flat &= G & \qquad I_mE\flat &= g \\ T_kE\flat &= B & \qquad I_nE\flat &= b. \end{aligned}$$

Clearly, i, j, and k are 0, 4, and 8 respectively, and less clearly, ℓ, m, and n are 1, 5, and 9. In cycle notation, the group $G_0|_{S_0}$ is

$$\begin{array}{lll} & \mathrm{Id} = () & LP|_{S_0} = (E\flat\ G\ B)(e\flat\ b\ g) \\ G_0|_{S_0}: & P|_{S_0} = (E\flat\ e\flat)(G\ g)(B\ b) & PL|_{S_0} = (E\flat\ B\ G)(e\flat\ g\ b) \\ & L|_{S_0} = (E\flat\ g)(G\ b)(B\ e\flat) & PLP|_{S_0} = (E\flat\ b)(G\ e\flat)(B\ g) \end{array}$$

and the group $H_0|_{S_0}$ is

$$\begin{array}{lll} & T_0|_{S_0} = () & I_1|_{S_0} = (E\flat\ e\flat)(G\ b)(B\ g) \\ H_0|_{S_0}: & T_4|_{S_0} = (E\flat\ G\ B)(e\flat\ g\ b) & I_5|_{S_0} = (B\ b)(G\ e\flat)(E\flat\ g) \\ & T_8|_{S_0} = (E\flat\ B\ G)(e\flat\ b\ g) & I_9|_{S_0} = (G\ g)(E\flat\ b)(B\ e\flat), \end{array}$$

exactly as in Clampitt's article [6]. The groups $G_0|_{S_0}$ and $H_0|_{S_0}$ are dual in $\mathrm{Sym}(S_0)$ by Theorem 3.1 (vi).

We may now use Corollary 3.3 and the above computation to find the other three hexatonic systems and the corresponding duals to the restricted PL-groups. In Corollary 3.3, we take $k = T_1$, T_2, and T_3. Starting with T_1, we find the PL-orbit

$$T_1 S_0 = \{E, e, C, c, A\flat, a\flat\}$$

and (extended) dual group

$$T_1 H_0 T_1^{-1} = \{T_0, T_4, T_8, I_3, I_7, I_{11}\}.$$

With T_2 we obtain the PL-orbit

$$T_2 S_0 = \{F, f, C\sharp, c\sharp, A, a\}$$

and (extended) dual group

$$T_2 H_0 T_2^{-1} = \{T_0, T_4, T_8, I_5, I_9, I_1\}.$$

With T_3 we obtain the PL-orbit

$$T_3 S_0 = \{G\flat, f\flat, D, d, B\flat, b\flat\}$$

and (extended) dual group

$$T_2 H_0 T_2^{-1} = \{T_0, T_4, T_8, I_7, I_{11}, I_3\}.$$

We know we have all orbits and may now stop because the union of the orbits thus far gives us all of S. For a direct computation of all four hexatonic systems, see the exposition [7].

3.2 Octatonic Systems

As another application of Theorem 3.1 and Corollary 3.3, we construct the three octatonic systems. In Theorem 3.1, we take S to be the set of major and minor triads, G to be the neo-Riemannian PLR-group, H to be the T/I-group, and G_0 to be the PR-group, which is generated by P and R, is isomorphic to D_4, and therefore has only 8 elements.[1]

Let s_0 be the C chord. Beginning with C, and evaluating P and R alternately, we obtain the 8 chords $\{C, c, E\flat, e\flat, G\flat, g\flat, A, a\}$. We have now found the complete orbit S_0 of C and may stop: by Theorem 3.1 (i), G_0 acts simply transitively on S_0, which means $|S_0| = |G_0| = 8$ by the Orbit-Stabilizer Theorem.

[1] To see that the PR-group is dihedral of order 8, we note that alternately applying P and R to C gives $C, c, E\flat, e\flat, G\flat, g\flat, A, a, C$, so that $s := RP$ has order 4 (recall P and R commute with T_1). The element $t := P$ has order 2, and the equation $tst = s^{-1}$ holds true. Thus the PR-group is generated by $\{s, t\}$ and s and t satisfy the relations of D_4.

To find H_0, we must only find those transpositions and inversions which map C into S_0, or in other words, we must only solve the eight equations

$$\begin{aligned} T_iC &= C & I_mC &= c \\ T_jC &= E\flat & I_nC &= e\flat \\ T_kC &= G\flat & I_oC &= g\flat \\ T_\ell C &= A & I_pC &= a. \end{aligned}$$

Clearly, i, j, k, and ℓ are 0, 3, 6, and 9 respectively, and with a little more work, m, n, o, and p are 7, 10, 1, and 4. The groups $G_0|_{S_0}$ and $H_0|_{S_0}$ are dual in $\mathrm{Sym}(S_0)$ by Theorem 3.1 (vi).

We may now use Corollary 3.3 and the above computation to find the other two octatonic systems and the corresponding duals to the restricted PR-groups. In Corollary 3.3, we take $k = T_1$ and T_2. Starting with T_1, we find the PR-orbit

$$T_1S_0 = \{D\flat, d\flat, E, e, G, g, B\flat, b\flat\}$$

and (extended) dual group

$$T_1H_0T_1^{-1} = \{T_0, T_3, T_6, T_9, I_9, I_0, I_3, I_6\}.$$

With T_2 we obtain the PR-orbit

$$T_2S_0 = \{D, d, F, f, A\flat, a\flat, B, b\}$$

and (extended) dual group

$$T_2H_0T_2^{-1} = \{T_0, T_3, T_6, T_9, I_{11}, I_2, I_5, I_8\}.$$

We know we have all orbits and may now stop because the union of the orbits thus far gives us all of S. For a direct computation of all three octatonic systems, see [7].

4 The Topos of Triads and the neo-Riemannian *PLR*-Group

We first recall the topos of triads $\mathbf{Set}^{\mathcal{T}}$ from [3] and then present the enumeration. In the present article we use the semi-tone encoding of pitch classes, and not the circle-of-fifths encoding used in [3]. Consequently, some formulas will differ, though the objects under consideration are exactly the same.

The *triadic monoid* $\mathcal{T}$ of [3] is the collection of affine maps $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$ which preserve the C-triad $\{0, 4, 7\}$ as a set. A map $\psi\colon \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ is *affine* if it has the form $z \mapsto mz + b$ for some $m, b \in \mathbb{Z}_{12}$. There are 144 affine maps $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$, the invertible ones are characterized by $m \in \{1, 5, 7, 11\}$. A function ψ *preserves* $\{0, 4, 7\}$ *as a set* if $\psi(\{0, 4, 7\}) \subseteq \{0, 4, 7\}$. This is weaker than preserving $\{0, 4, 7\}$ *pointwise*. The set $\mathcal{T}$ forms a *monoid* under function composition, that is, function composition is an associative, unital operation on $\mathcal{T}$ (functions in $\mathcal{T}$ are not

necessarily invertible). The triadic monoid $\mathcal{T}$ has 8 elements and is generated by the affine maps $f(z) := 3z + 7$ and $g(z) := 8z + 4$. The other maps in $\mathcal{T}$ are $f \circ f(z) = 9z + 4$, $g \circ g(z) = 4z$, the three constant maps 0, 4, 7, and the identity $\mathrm{Id}_{\mathbb{Z}_{12}}$.

If $\mathcal{S} \subseteq \mathbb{Z}_{12}$ is closed under the action of $\mathcal{T}$, that is $\mathcal{T}\mathcal{S} \subset \mathcal{S}$, then $\mu|_{\mathcal{S}} \colon \mathcal{T} \times \mathcal{S} \to \mathcal{S}$ is called a *subaction of* μ and $\mathcal{S}$ is called the *carrier set of the subaction* $\mu|_{\mathcal{S}}$.

The *topos of triads* is the category $\mathbf{Set}^{\mathcal{T}}$, its objects are sets equipped with a $\mathcal{T}$-action and its morphisms are $\mathcal{T}$-equivariant maps. Like any presheaf category, this category is a *topos*. See [3] for a complete investigation of the triadic monoid and the topos of triads. An exposition of [3] may be found in [8]. See the Appendix of the present paper for a recollection of topos-theoretic concepts used in this article.

Our aim in the present article is to clarify the relationship between the neo-Riemannian PLR-group, the triadic monoid, and the topos of triads, so we focus on subactions of the natural action $\mu \colon \mathcal{T} \times \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ which are covered by consonant triads. A tremendous difference between the triadic monoid action and the PLR-group action is that the triadic monoid acts on pitch classes, whereas the PLR-group acts on aggregate chords; the PLR-group simply *does not act on individual pitch classes*. However, we may use Theorem 3.1 to induce an action of certain transpositions and inversions on the pitch classes of a monoid subaction in the case that the carrier set is covered by an orbit of a PLR-subgroup.

For example, consider the major-minor mixture $\{0, 3, 4, 7\}$. A quick computation verifies that $\mathcal{T}\{0, 3, 4, 7\} \subseteq \{0, 3, 4, 7\}$, as it suffices to check if the two generators f and g of $\mathcal{T}$ preserve $\{0, 3, 4, 7\}$. The carrier set $\{0, 3, 4, 7\}$ of this monoid subaction is covered by the consonant triads C and c, while $\{C, c\}$ is an orbit of the PLR-subgroup $\{\mathrm{Id}, P\}$. Theorem 3.1 now implies that the dual subgroup in $\mathrm{Sym}(\{C, c\})$ is $\{\mathrm{Id}, I_7\}$. The identity and the inversion operator I_7 *do act on individual pitch classes*, and since Id and I_7 preserve the set $\{C, c\}$, they will also preserve the set of underlying pitch classes. This is how we move from PLR-subgroup actions on chords to T/I-subgroup actions on pitch classes. It is a coincidence in this very small example that I_7 and P agree on the set $\{C, c\}$.

We may now enumerate all $\mathcal{T}$-sets $\mathcal{S}$ which are covered by triads and admit an action of a PLR-subgroup on the maximal cover $\mathbb{S}$, see Figure 1 and Figure 2. In Figure 1, the operation Q_6 transposes major chords up by 6 semitones and minor chords down by 6 semitones (this is the same as T_6, but we call it Q_6 to emphasize that we think of it as element of the PLR-group). The notation $\mathrm{S}\ell$ indicates the *slide transformation* on major and minor chords, which holds the third constant and moves the root and the fifth in parallel by a semitone.

Theorem 4.1 (Enumeration). *Let $\mathcal{T}$ be the triadic monoid, that is, $\mathcal{T}$ is the monoid of affine functions $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$ which preserve the C-chord $\{0, 4, 7\}$ as a set. Suppose $\mathcal{S} \subseteq \mathbb{Z}_{12}$ is closed under the natural $\mathcal{T}$-action in the sense that $\mathcal{T}\mathcal{S} \subseteq \mathcal{S}$. Assume further that $\mathcal{S}$ is covered by consonant triads, and let $\mathbb{S}$ be the collection of consonant triads Y with $Y \subseteq \mathcal{S}$. If $\mathbb{S}$ admits a simply transitive*

action of a PLR-subgroup G, then $\mathcal{S}$ must be one of the sets enumerated in Figure 1.

Carrier Set $\mathcal{S}$	Type	Maximal Covering $\mathbb{S}$	PLR-Subgroup
$\{0,4,7\}$	Major Chord	C	$\{\mathrm{Id}\}$
$\{0,3,4,7\}$	Major-Minor Mixture	C, c	$\{\mathrm{Id}, P\}$
$\{0,3,4,7,8,11\}$	Hexatonic	$C, c, A\flat, a\flat, E, e$	$\langle P, L\rangle$
$\{0,1,3,4,6,7,9,10\}$	Octatonic	$C, c, E\flat, e\flat, G\flat, g\flat, A, a$	$\langle P, R\rangle$
$\{0,1,4,6,7,10\}$	Major Triad Tritone Mixture	$C, G\flat$	$\{\mathrm{Id}, Q_6\}$
$\{0,1,2,4,6,7,8,10\}$	Prometheus Tritone Mixture	$C, d\flat, G\flat, g$	$\{\mathrm{Id}, Q_6, \mathrm{S}\ell, Q_6\mathrm{S}\ell\}$
$\mathbb{Z}_{12}$	Chromatic Scale	All Consonant Triads	PLR-group

Fig. 1. Enumeration of all $\mathcal{T}$-Subactions with Maximal Cover Admitting a Simply Transitive Action of a PLR-Subgroup

Carrier Set $\mathcal{S}$ in $\mathbb{Z}_{12}$	Carrier Set $\mathcal{S}$ in Musical Notation
$\{0,4,7\}$	$\{C, E, G\}$
$\{0,3,4,7\}$	$\{C, E\flat, E, G\}$
$\{0,3,4,7,8,11\}$	$\{C, E\flat, E, G, A\flat, B\}$
$\{0,1,3,4,6,7,9,10\}$	$\{C, D\flat, E\flat, E, G\flat, G, A, B\flat\}$
$\{0,1,4,6,7,10\}$	$\{C, D\flat, E, G\flat, G, B\flat\}$
$\{0,1,2,4,6,7,8,10\}$	$\{C, D\flat, D, E, G\flat, G, A\flat, B\flat\}$
$\mathbb{Z}_{12}$	$\{C, D\flat, D, E\flat, E, F, G\flat, G, A\flat, A, B\flat, B\}$

Fig. 2. Carrier Sets of Figure 1

Proof. Suppose G is a PLR-subgroup which acts simply transitively on $\mathbb{S}$.
Case 1: G contains P. Assume $P \in G$. Our case-by-case method is to consider the orbit of $C = \{0,4,7\}$ under each PLR-subgroup which contains P, and to test if the underlying pitch-class set is closed under $\mathcal{T}$. By Theorem 3.1 (i), each PLR-subgroup acts simply transitively on its C-orbit.

We may quickly determine all PLR-subgroups containing P via the explicit structure of the PLR-group, described for instance in [9] and [2]. The PLR-group has two generators, Q_1 of order 12 and P of order 2, where Q_1 transposes major chords up by 1 and minor chords down by 1. The bijection Q_k is defined similarly, and satisfies $Q_k = (Q_1)^k$. The 24 elements of the PLR-group are $\{Q_k | k \in \mathbb{Z}_{12}\} \cup \{PQ_k | k \in \mathbb{Z}_{12}\}$, for example $L = PQ_4$ and $R = PQ_9$. For these reasons, any PLR-subgroup containing P is generated by P and some Q_i. The group $\{Q_k | k \in \mathbb{Z}_{12}\}$ is cyclic, hence all of its subgroups are cyclic, and for our computation of the

PLR-subgroups containing P it suffices to check $\langle P, Q_i\rangle$ where Q_i is a generator of a cyclic subgroup, that is, $Q_i \in \{\mathrm{Id}, Q_1, Q_2, Q_3, Q_4, Q_6\}$.

The smallest PLR-subgroup containing P is $\{\mathrm{Id}, P\}$, which produces the C-orbit $\{C, c\}$ with underlying pitch-class set $\{0, 3, 4, 7\}$. This is closed under the $\mathcal{T}$-action, as $f(3) = 4 = g(3)$.

The next larger subgroup is $\langle P, Q_6\rangle = \{\mathrm{Id}, Q_6, P, PQ_6\}$, which produces the C-orbit $\{C, c, G\flat, g\flat\}$ with underlying pitch-class set $\{0, 1, 3, 4, 6, 7, 9, 10\}$, the octatonic. The group $\langle P, Q_6\rangle$ does not act simply transitively on the set of all consonant triads in the octatonic, so a hypothesis of the theorem is not satisfied, and there is nothing to check.

We arrive next at $\langle P, Q_4\rangle = \{\mathrm{Id}, Q_4, Q_8, P, PQ_4, PQ_8\}$, which is the same as $\langle P, PQ_4\rangle = \langle P, L\rangle$. The C-orbit is a hexatonic system of Cohn $\{C, c, A\flat, a\flat, E, e\}$, see [5] and Section 3.1 of the present paper. The underlying pitch-class set $\{0, 3, 4, 7, 8, 11\}$ is closed under $\mathcal{T}$, since $f(8) = 7$, $f(11) = 4$, $g(8) = 8$, and $g(11) = 8$.

The group $\langle P, Q_3\rangle = \{\mathrm{Id}, Q_3, Q_6, Q_9, P, PQ_3, PQ_6, PQ_9\}$ is the same as $\langle P, PQ_9\rangle = \langle P, R\rangle$. The C-orbit is an octatonic system

$$\{C, c, E\flat, e\flat, G\flat, g\flat, A, a\},$$

see Section 3.2. The underlying pitch-class set $\{0, 1, 3, 4, 6, 7, 9, 10\}$ is closed under $\mathcal{T}$, since $f(1) = 10$, $f(6) = 1$, $f(9) = 10$, $f(10) = 1$, and $g(1) = 0$, $g(6) = 4$, $g(9) = 4$, $g(10) = 0$.

The group $\langle P, Q_2\rangle$ has $\mathbb{Z}_{12}$ as the underlying pitch-class set for its C-orbit because $\{0, 3, 4, 7\}$ has a representative from each coset $\mathbb{Z}_{12}/6\mathbb{Z}_{12}$. That is, iterated applications of Q_2 to $\{0, 4, 7\}$ and $\{0, 3, 7\}$ will reach all of $\mathbb{Z}_{12}$. However, the group $\langle P, Q_2\rangle$ does not act simply transitively on the set of all consonant triads in $\mathbb{Z}_{12}$, so a hypothesis of the theorem is not satisfied, and there is nothing to check.

The group $\langle P, Q_1\rangle$ is the entire PLR-group, and its C-orbit is the entire collection of major and minor triads. Its underlying pitch-class set is of course $\mathbb{Z}_{12}$, which is clearly closed under the $\mathcal{T}$-action.

We have considered all PLR-subgroups containing P and determined which ones produce a C-orbit with underlying pitch-class set closed under the $\mathcal{T}$-action.

Case 2: G does not contain P. Assume $P \notin G$ and G acts simply transitively on $\mathbb{S}$.

If G is trivial, then the C-orbit is simply $\{C\}$ with underlying pitch-class set $\{0, 4, 7\}$, which is closed under the $\mathcal{T}$-action by definition of $\mathcal{T}$.

For the rest of Case 2, we assume G is nontrivial and argue using the sub group $H \leq T/I$ which maps $C = \{0, 4, 7\}$ to triads in $\mathbb{S}$. The sub group H was called H_0 in Theorem 3.1, and is defined via the action on consonant triads. We will use the H-action on individual pitch classes to conclude that the only two possibilities for $\mathcal{S}$ are the Major Triad Tritone Mixture $\{0, 1, 4, 6, 7, 10\}$ and the Prometheus Tritone Mixture $\{0, 1, 2, 4, 6, 7, 8, 10\}$ whenever $P \notin G$.

We first remark $3 \notin \mathcal{S}$, for if 3 were in $\mathcal{S}$, then $\{0, 3, 7\}$ and $\{0, 4, 7\}$ would be in $\mathbb{S}$, which would necessitate $P \in G$. Based on this observation, we may

conclude that the only transposition in H is T_6, as follows. Since $3 \notin \mathcal{S}$, we have $T_3 \notin H$, and consequently $T_9 \notin H$. Similarly, $T_8 \notin H$, for if T_8 were in H, then $T_8\{0,4,7\} = \{8,0,3\}$ would be a subset of $\mathcal{S}$, a contradiction. Consequently, $T_2, T_4, T_{10} \notin H$. Clearly, $T_1, T_5, T_7, T_{11} \notin H$, for if any of these were in H, all transpositions would be in H. Thus, the only transposition that can be in H is T_6. The transposition T_6 must be in H, since $H \leq T/I$ is nontrivial by Theorem 3.1 (iii), the composite of two inversions is a transposition, and the only allowable transposition is T_6.

At this point, we note that $\{\mathrm{Id}, T_6\}$ acts simply transitively on $\{C, G\flat\}$, and its dual group $\{\mathrm{Id}, Q_6\}$, determined by Theorem 3.1, is a possibility for G. The underlying pitch-class set is the Major Triad Tritone Mixture $\{0,1,4,6,7,10\}$, which is closed under the $\mathcal{T}$-action because $f(1) = 10$, $f(6) = 1$, $f(10) = 1$, and $g(1) = 0$, $g(6) = 4$, $g(10) = 0$, so the Major Triad Tritone Mixture is a possibility for $\mathcal{S}$.

Let us return to general G with $P \notin G$. We claim $9 \notin \mathcal{S}$. Since $T_6 \in H$ and H preserves $\mathbb{S}$, so also $\mathcal{S}$, we have $T_6\{0,4,7\} = \{6,10,1\} = G\flat \subseteq \mathcal{S}$. If 9 were in $\mathcal{S}$, then $g\flat = \{6,9,1\}$ would be a subset of $\mathcal{S}$, which would necessitate $P \in G$, a contradiction. Thus $9 \notin \mathcal{S}$.

We also claim $5 \notin \mathcal{S}$. If 5 were in $\mathcal{S}$, then $f \circ f(5) = 1$ and $g(5) = 8$ must also belong to $\mathcal{S}$. Then we have the parallel triads $\{1,4,8\}$ and $\{1,5,8\}$ both in $\mathbb{S}$, which is not allowed by $P \notin G$. Hence, $5 \notin \mathcal{S}$.

Suppose for the remainder of the proof that H properly contains $\{\mathrm{Id}, T_6\}$. Our next goal is to show that the other elements of H are exactly I_2 and I_8. The other elements of H must be inversions, as the only transposition in H is T_6 as we proved above. Recall that the inversion $I_{k+\ell}$ is the unique inversion which interchanges k and ℓ. Since $3, 5, 9 \notin \mathcal{S}$ as we saw above, we can exclude from H all inversions that interchange 0, 4, or 7 with 3, 5, or 9. Thus, $I_3, I_7, I_{10}, I_5, I_9, I_0, I_9, I_1, I_4 \notin H$. The composite of T_6 with any of these cannot be in H, so we also have $I_{11}, I_6 \notin H$. The only two inversions that can be in H are I_2 and I_8. But if either is in H, then so is the other, so both I_2 and I_8 must be in H. Thus $H = \{\mathrm{Id}, T_6, I_2, I_8\}$.

At this point, we note that $\{\mathrm{Id}, T_6, I_2, I_8\}$ acts simply transitively on the chord set $\{C, d\flat, G\flat, g\}$ and the dual subgroup is $\{\mathrm{Id}, Q_6, \mathrm{S}\ell, Q_6\mathrm{S}\ell\}$ by Theorem 3.1. The underlying pitch-class set $\{0,1,2,4,6,7,8,10\}$ is closed under the $\mathcal{T}$ action, since $f(1) = 10$, $f(2) = 1$, $f(6) = 1$, $f(8) = 7$, $f(10) = 1$, and $g(1) = 0$, $g(2) = 8$, $g(6) = 4$, $g(8) = 8$, $g(10) = 0$. We conclude that the Prometheus Tritone Mixture $\{0,1,2,4,6,7,8,10\}$ is the final possibility for $\mathcal{S}$. □

We may now consider upgrades by the Lawvere–Tierney topologies

$$j_{\mathcal{P}}, j_{\mathcal{L}}, j_{\mathcal{R}} \colon \Omega \to \Omega$$

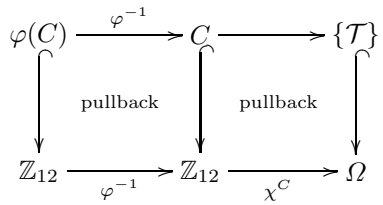
in the topos $\mathbf{Set}^{\mathcal{T}}$, found on page 117 of [3]. Recall that if ν is a sub $\mathcal{T}$-action of a $\mathcal{T}$-action σ and its characteristic morphism is $\chi \colon \sigma \to \Omega$, then the *$j$-upgrade* of ν has carrier set equal to the pre-image of the point $\mathcal{T} \in \Omega$ under $j \circ \chi$, that is, the j-upgrade has carrier set $(j \circ \chi)^{-1}(\mathcal{T})$.

Corollary 4.2 (Maximal covers of Lawvere–Tierney upgrades admit PLR-subgroup actions). *Let $\varphi\colon \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ be an element of the T/I-group and let $(\mathbb{Z}_{12}, \mu^\varphi)$ denote the $\mathcal{T}$-action given by the φ-conjugation of the natural $\mathcal{T}$-action, that is, the generators f and g of $\mathcal{T}$ act by $\varphi f \varphi^{-1}$ and $\varphi g \varphi^{-1}$. Then the maximal covers of the $j_{\mathcal{P}}$-, $j_{\mathcal{L}}$-, and $j_{\mathcal{R}}$-upgrades of the subaction $(\varphi(C), \mu^\varphi|_{\varphi(C)})$ admit simply transitive actions by PLR-subgroups. In fact, the upgrades and their maximal covers are the φ-images of the upgrades of C and their respective covers. The PLR-subgroups remain the same. See Figure 3.*

Upgrade	Carrier Set	Type	PLR-Subgroup
$j_{\mathcal{P}}$	$\varphi\{0,3,4,7\}$	Major-Minor Mixture	$\langle P\rangle$
$j_{\mathcal{L}}$	$\varphi\{0,3,4,7,8,11\}$	Hexatonic	$\langle P,L\rangle$
$j_{\mathcal{R}}$	$\varphi\{0,1,3,4,6,7,9,10\}$	Octatonic	$\langle P,R\rangle$

Fig. 3. Upgrades of the Subaction $(\varphi(C), \mu^\varphi|_{\varphi(C)})$ by the Lawvere–Tierney Topologies $j_{\mathcal{P}}$, $j_{\mathcal{L}}$, and $j_{\mathcal{R}}$

Proof. Let $\chi^C\colon \mathbb{Z}_{12} \to \Omega$ denote the characteristic morphism of the subaction $\{0,4,7\}$ of the natural $\mathcal{T}$-action on $\mathbb{Z}_{12}$. Then we have the following two pullbacks (see the Appendix in Section 5 for the notion of pullback).



The composite of these two pullback squares is also a pullback, so the characteristic morphism of $(\varphi(C), \mu^\varphi|_{\varphi(C)})$ is $\chi^C \circ \varphi^{-1}$. Thus, the j-upgrade of $(\varphi(C), \mu^\varphi|_{\varphi(C)})$ is φ of the j-upgrade of (C, μ^{Id}), that is,

$$(j \circ \chi^{\varphi(C)})^{-1}(\mathcal{T}) = (j \circ \chi^C \circ \varphi^{-1})^{-1}(\mathcal{T}) = \varphi\left((j \circ \chi^C)^{-1}(\mathcal{T})\right).$$

We recall from page 123 of [3] that the $j_{\mathcal{P}}$-, $j_{\mathcal{L}}$-, and $j_{\mathcal{R}}$-upgrades of C are the major-minor mixture, hexatonic, and octatonic appearing in Figure 1. The maximal cover of the φ-image is clearly the φ-image of the maximal cover, so we are precisely in the situation of Corollary 3.3. Sections 3.1 and 3.2 further illustrate the point. □

In future work, we will study triadic coverings of sets which are simultaneously invariant under multiple conjugated $\mathcal{T}$-actions. Another goal is to better understand the role of the three groups $\langle P\rangle$, $\langle P,L\rangle$, $\langle P,R\rangle$ within the framework of other groups which act simply transitively on triadic coverings.

5 Appendix on Topos Theory

We recall some basic notions from topos theory. For explanations, examples, and proofs we refer to the excellent textbook of Mac Lane–Moerdijk [10]. Only a very small fraction of this Appendix is actually used in the present paper, so readers do not need to understand this Appendix in order to follow the present paper. A topos is a special kind of category, so we begin with categories.

A *category* $\mathbf{C}$ consists of a class $\mathrm{Obj}\,\mathbf{C}$ of *objects*, a set $\mathrm{Mor}_{\mathbf{C}}(A,B)$ of *morphisms from A to B* for each pair of objects A and B, and a *composition*

$$\mathrm{Mor}_{\mathbf{C}}(B,C)\times\mathrm{Mor}_{\mathbf{C}}(A,B)\xrightarrow{\circ}\mathrm{Mor}_{\mathbf{C}}(A,C)$$

for each triple of objects A, B, and C. The composition is required to be *associative* and *unital*. For instance, the category **Set** has sets as objects and functions as morphisms. An prominent example from Section 4 is the category $\mathbf{Set}^{\mathcal{T}}$, in which an object is a set equipped with an action of the triadic monoid $\mathcal{T}$ and a morphism is a $\mathcal{T}$-equivariant function. Groups and group homomorphisms form a third example of a category.

A *topos* is a category $\mathbf{C}$ such that the following three axioms hold.

(i) The category $\mathbf{C}$ admits all *finite limits*.
(ii) The category $\mathbf{C}$ admits a *subobject classifier* $*\to\Omega$. This is an object Ω equipped with a universal monomorphism from the terminal object $*$ to Ω, which is universal in the sense that any other monomorphism is a pullback of it in a unique way. Often, one simply calls Ω the *subobject classifier*.
(iii) The category $\mathbf{C}$ is *Cartesian closed*, that is, for any two objects B and C there is an object C^B in $\mathbf{C}$ and for any object A of $\mathbf{C}$ a bijection

$$\mathrm{Mor}_{\mathbf{C}}(A\times B,C)\cong\mathrm{Mor}_{\mathbf{C}}(A,C^B)$$

natural in A, B, and C.

To understand these axioms, we consider how the category **Set** is a topos. This category admits all finite limits. For instance, the diagram

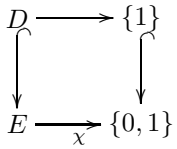
$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

has finitely many objects and morphisms, and $L:=\{(a,b)\in A\times B\mid f(a)=g(b)\}$ makes the diagram

$$\begin{array}{ccc} L & \xrightarrow{\mathrm{pr}_2} & B \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} \tag{1}$$

commute, and moreover there is a unique morphism from L to any analogous L' in such a way the two triangles involving L, L', A, and B also commute. This L is called a *limit* of the diagram, and the limit for this special diagram is called a *pullback*. The projection pr_1 is called a *pullback* of g and the projection pr_2 is called a *pullback* of f.

For the discussion of the subobject classifier of **Set**, we first point out that the monomorphisms in **Set** are precisely the injective maps. The subobject classifier in **Set** is the inclusion $\{1\} \hookrightarrow \{0,1\}$. If D is a subset of E, then the inclusion $D \hookrightarrow E$ is a pullback of $\{1\} \hookrightarrow \{0,1\}$ in a unique way: there is only one function χ that makes the following diagram a pullback.

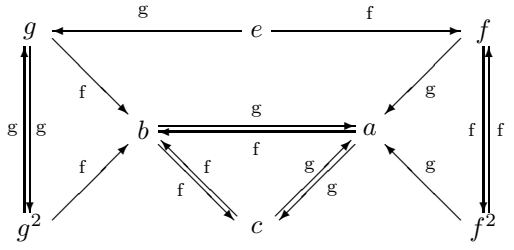


This function χ is necessarily the usual *characteristic function*, which takes the value 1 for inputs in D and the value 0 for inputs in $E \backslash D$. Note that $\chi^{-1}(1) = D$.

For the Cartesian closedness of **Set**, the object C^B is defined to be the set of functions $B \to C$, that is, $C^B = \mathrm{Mor}_{\mathbf{Set}}(B, C)$. The bijection in (iii) sends a function $f\colon A \times B \to C$ to the function $A \to C^B$ defined by $a \mapsto (b \mapsto f(a,b))$. Thus, **Set** is a topos.

Many other examples of topoi are given by presheaf categories. A *presheaf* on a category **D** is a functor from the opposite category of **D** to **Set** (here **D** is not required to be a topos). For any category **D**, the category of presheaves on **D** is a topos. In particular, if G is a group or a monoid, then the category of G-sets is a topos (when G is viewed as a category with one object, a presheaf on G^{op} is the same as a set equipped with a G-action, and a morphism of presheaves is a G-equivariant map). In particular, for the triadic monoid $\mathcal{T}$, the category $\mathbf{Set}^{\mathcal{T}}$ of $\mathcal{T}$-sets and their morphisms is a topos.

In the topos $\mathbf{Set}^{\mathcal{T}}$, the underlying set of the subobject classifier Ω is the set of left ideals in $\mathcal{T}$. For instance, the full monoid $\mathcal{T}$ and the empty set $\emptyset$ are both left ideals in $\mathcal{T}$, so $\mathcal{T}, \emptyset \in \Omega$. To obtain the other left ideals it is convenient to inspect a Cayley graph of $\mathcal{T}$ with respect to its generators f and g.



The nodes labelled $e, f, f^2, g, g^2, a, b, c$ represent the 8 elements of $\mathcal{T}$. The left ideals are node sets where no arrow starts that leads outside of this set. One

may directly check by eye that there are four non-empty proper subsets $\mathcal{B} \subset \mathcal{T}$ satisfying this condition, namely

$$\mathcal{C} = \{a, b, c\},\ \mathcal{L} = \{a, b, c, f, f^2\},\ \mathcal{R} = \{a, b, c, g, g^2\},\ \mathcal{P} = \{a, b, c, f, f,^2, g, g^2\}.$$

The $\mathcal{T}$-action on the subobject classifier $\Omega = \{\emptyset, \mathcal{C}, \mathcal{L}, \mathcal{R}, \mathcal{P}, \mathcal{T}\}$ is

$$m \cdot \mathcal{B} := \{n \in \mathcal{T} \mid n \cdot m \in \mathcal{B}\}$$

for $m \in \mathcal{T}$ and $\mathcal{B} \in \Omega$, and the universal monomorphism is $\{T\} \hookrightarrow \Omega$.

A *Lawvere–Tierney topology* in the topos $\mathbf{Set}^{\mathcal{T}}$ is a $\mathcal{T}$-equivariant map $\Omega \to \Omega$ such that

(i) $j(\mathcal{T}) = \mathcal{T}$,
(ii) $j \circ j = j$,
(iii) $j(\mathfrak{r}) \cap j(\mathfrak{s}) = j(\mathfrak{r} \cap \mathfrak{s})$ for all $\mathfrak{r}, \mathfrak{s} \in \Omega$.

The six Lawvere–Tierney topologies of $\mathbf{Set}^{\mathcal{T}}$ are determined in Sections 2.3 and 2.4 of [3]. Given a sub $\mathcal{T}$-set D of a $\mathcal{T}$-set E, one may *upgrade* it by a Lawvere–Tierney topology j in the following way. By definition of subobject classifier, the inclusion $D \hookrightarrow E$ corresponds uniquely to a morphism $\chi\colon E \to \Omega$, its characteristic morphism. The *j-upgrade of D* is then the sub $\mathcal{T}$-set of E which has characteristic morphism $j \circ \chi$. The underlying set of the upgrade is $(j \circ \chi)^{-1}(\mathcal{T})$.

For example, the characteristic morphism of the C-major chord $\{0, 4, 7\}$ in $\mathbb{Z}_{12}$ with the natural $\mathcal{T}$-action is

t	0	1	2	3	4	5	6	7	8	9	10	11
$\chi^{\{0,4,7\}}(t)$	$\mathcal{T}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{T}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{L}$

as can be determined by precomposing the characteristic morphism for $\chi^{\{0,1,4\}}$ on page 121 of [3] with multiplication by 7. Using this, and the tables on pages 117 and 118 of [3], the upgrades of $\{0, 4, 7\}$ are found to be as follows.

Upgrade	Carrier Set	Type
$j_{\mathcal{T}}$	$\{0, 4, 7\}$	Major Chord
$j_{\mathcal{P}}$	$\{0, 3, 4, 7\}$	Major-Minor Mixture
$j_{\mathcal{L}}$	$\{0, 3, 4, 7, 8, 11\}$	Hexatonic
$j_{\mathcal{R}}$	$\{0, 1, 3, 4, 6, 7, 9, 10\}$	Octatonic
$j_{\mathcal{C}}$	$\mathbb{Z}_{12}$	Chromatic Scale
$j_{\mathcal{F}}$	$\mathbb{Z}_{12}$	Chromatic Scale

The numbers in this table appear to differ from those on page 123 in [3] because we are using the semitone encoding in the present article, rather than the circle-of-fifths encoding. The two tables are related by multiplication with 7.

A surprising discovery of [3] is that these music-theoretic collections are the Lawvere–Tierney upgrades of the C-major chord. One of the goals of the present paper is to thicken this music-theoretic meaning by way of the PLR-group in Theorem 4.1.

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