

UNIVERSITY OF
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Homogeneous models and their toposes of supported sets

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This dissertation is submitted for the degree of Doctor of Philosophy

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my thesis has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text.

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Summary

Title: Homogeneous models and their toposes of supported sets
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This thesis studies homogeneous models by looking at their topos of finitely supported sets. Of various well-known toposes, such as cubical sets and simplicial sets, it is shown that they can be presented as a topos of this kind. It is argued that these examples can be understood as coming from a homogeneous model, in the sense of model theory, but formulated in topos-theoretical terms. It is shown that the connecting structure between the two is a factorizing prime site and the notion of a principal model. This is further substantiated by demonstrating that the Fraïssé-Hrushovski construction can be formulated in terms of such sites. The notion of finite support is abstracted and a general calculus of orbits for toposes of supported sets is developed. It is shown that these form a factorizing prime site and that they can be compared to the age of structure as in classical model theory, and can be given a topos-theoretic characterization in terms of the hyperconnected-localic factorization of a geometric morphism. The theory is then applied by revisiting and generalizing the Ryll-Nardzewski theorem to the new setting and by constructing a very general class of Fraenkel-Mostowki models of intuitionistic set theory. Lastly, a notion of a Kan fibration for such toposes is studied and shown to generalize the familiar notions in simplicial sets and cubical sets. Under particular assumptions, it is proved that these Kan fibrations are closed under dependent product and hence interpret a version of Martin-Löf intensional dependent type theory. For simplicial sets, a different approach using simplicial Moore paths and its connection to the present work is announced.

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Preface

A famous example of a homogeneous model is given by Erdős and Rényi's *Random Graph*. It can be constructed in the following way: starting with a countable set of vertices, enumerate all pairs of vertices, and, for each of them, decide with probability 1/2 whether to connect them by an edge or not. Miraculously, Erdős and Rényi showed that the graph obtained by this method is, with probability 1, *unique* up to isomorphism [Cam97]. This uniqueness can be attributed to a single characteristic property. Namely that, for any disjoint pair U, V of finite sets of vertices, there exists a vertex x which connects to all the vertices in U, but not to those in V. One can verify that there can be at most one countable graph with this property, and that, with probability 1, the Random graph is an example.

There are more possible constructions of a graph with this property. For instance, take as vertices the (countable) universe of finite sets V_ω, and let there be an edge between x and y if either $x \in y$ or $y \in x$. Given that V_ω is a model of set theory without infinity, it is easy to verify the condition. To define an even more explicit construction, one can combine this with a well-known interpretation of set theory without infinity in peano arithmetic: if every element of V_ω is labelled

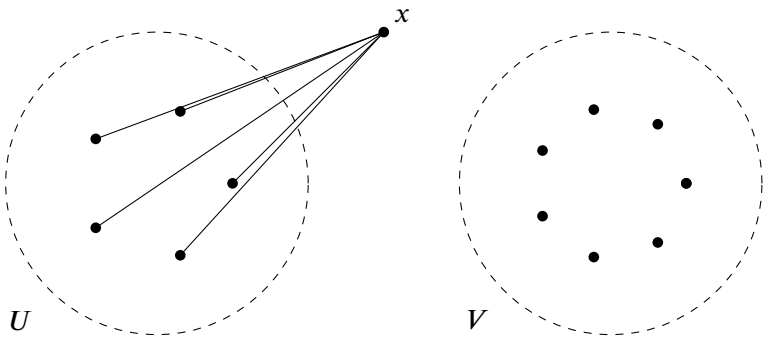


Figure 0.1: The characteristic property of the Random graph

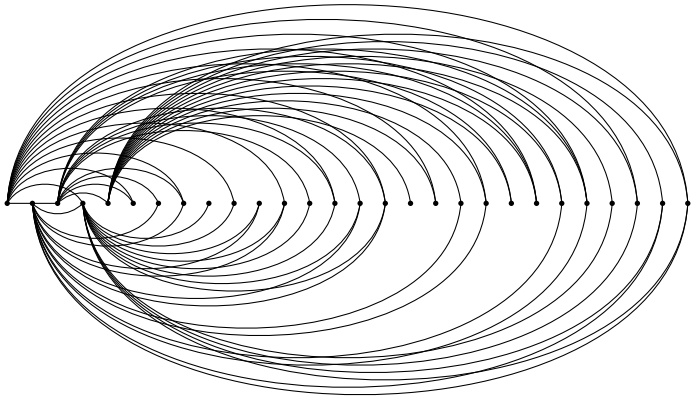


Figure 0.2: The first 28 vertices and 64 edges of Rado's 'random graph'

by a natural number, then a finite set $\{a_1, \ldots, a_n\}$ is uniquely encoded by a natural number:

$$2^{a_1} + \cdots + 2^{a_n} \tag{1}$$

In this way, Rado constructed a graph where the vertices are natural numbers, and where for $n < m$, n and m are connected by an edge precisely when the n-th bit in the binary representation of m is equal to 1. This is illustrated in Figure 0.2. By uniqueness, the resulting graph, upon completion, is isomorphic to the random graph.

The characteristic property of the random graph can be put in model-theoretic terms. For a first-order structure M with some underlying language and a tuple $\mathbf{m} := m_1, \ldots, m_n$ of elements of M, the *type* $T(\mathbf{m})$ of $\mathbf{m}$ is the set of all statements $\phi(\mathbf{m})$ that are true about $\mathbf{m}$ in that language. A countable structure M is called *strongly* $\aleph_0$*-homogeneous* if whenever two tuples $\mathbf{m}$, $\mathbf{n}$ have the same type, there exists an isomorphism from M to itself which maps $\mathbf{m}$ to $\mathbf{n}$. In other words, for M, the tuples $\mathbf{m}$ and $\mathbf{n}$ are completely interchangeable. It is an exercise in the celebrated 'back-and-forth' construction of model theory to show that when the language consists of only one 2-ary relation symbol $E(x, y)$, meaning 'there is an edge between x and y', then the Random graph is a strongly $\aleph_0$-homogeneous structure.

Now consider a structure M, but suppose $\mathbf{m}$ and $\mathbf{n}$ are tuples such that the type of $\mathbf{m}$ is merely *included* in the type of $\mathbf{n}$: $T(\mathbf{m}) \subseteq T(\mathbf{n})$. This can happen when types only contain formulas of a certain (restricted) logic, such as regular, coherent or geometric logic. In that case, it no longer makes sense to ask whether $\mathbf{m}$ and $\mathbf{n}$ are interchangeable. Indeed, replacing $\mathbf{m}$ by $\mathbf{n}$ in each formula of $T(\mathbf{m})$ preserves validity, but not vice versa. Call M *homogeneous* when, for all $\mathbf{m}$, $\mathbf{n}$ in the above

situation, there is a *homomorphism* $f : M \to M$ which sends $\mathbf{m}$ to $\mathbf{n}$. The above intuition has now become unilateral: $\mathbf{m}$ is a placeholder for $\mathbf{n}$, but not the other way around. Categorically speaking, the *action* of substitution is no longer an action by a group, but an action by a monoid.

This thesis studies sets equipped with such a 'unilateral' substitution action, coming from some structure M. This does not just mean a set equipped with a monoid action, for one has to make sense of the fact that the elements of such a set are only sensitive to certain 'finite' substitutions. This will be taken care of by the notion of *finite support*. Moreover, the action does not necessarily have to come from a monoid, as will be seen in Chapter 4. In general, the sets of interests will be referred to as finitely supported $\mathbf{M}$-sets, of which I give a preliminary definition in Section 1.1.

Finitely supported $\mathbf{M}$-sets have not been investigated as much as sets equipped with a 'finitely supported' group action: For groups, finitely supported is the same as *continuous* for a certain topology, and sets equipped with a continuous group action have of course been widely studied. When considering general $\mathbf{M}$-sets, one is soon forced to abandon this topological viewpoint, as I will argue in the next chapter. Instead, (finite) support is the primary notion that underpins the type of structure studied in this thesis. In Chapter 4, I define a more general notion of 'support' of which the elementary definition is a special case; this is analogous to the fact that 'finite support' corresponds to only a certain topology on a group, whereas one can naturally consider various other topologies.

What I present here on (finitely) supported $\mathbf{M}$-sets is written in the language of topos theory. This means that I will also formulate the connections and applications to mathematical logic from a topos-theoretic perspective. Should one insist, I could summarize the main thesis as follows:

> The *age* of a structure is a factorizing prime site.

What is meant by this statement will be gradually unfolded over the course of Chapters 2 to 4. As explained in Chapter 3, it addresses an issue that arises when comparing two construction in model theory: the Fraïssé construction and its adaptation due to Hrushovski. Further, it provides the fundamental connection to toposes of (finitely) supported $\mathbf{M}$-sets, which are characterized by factorizing prime sites.

The results obtained in this thesis should be compared to topos-theoretic work that relate similar themes. As the introduction suggests, this is mainly the theme of homogeneous structures in model theory. For topos-theoretic accounts of the Fraïssé construction, I have provided references in the introduction of chapter 3, most notably Caramello [Car14] and Kubiś [Kub14]. The completeness result for prime-generated toposes obtained in Chapter 4 (Theorem 4.4.9) can be considered an alternative to Forssell [For12], who uses topological groupoids, although the condition of prime-generatedness makes the theory here a bit less general, so a similar remark to the one made in 4.3.6 below applies.

The topos-theoretic literature on (finitely) supported **M**-sets appears mostly due to Makkai [Mak81]; [Mak82]. His definition of support is a special case of what I propose in Chapter 4 and is discussed in Example 4.1.17. In recent work, Makkai's presentation has appeared in Coumans [Cou12]. In these sources, the category **M** is intended to be a class of sufficiently 'special' or 'saturated' models, whose existence mostly relies on forms of the completeness theorem of classical logic. By contrast, I show in the below that in many cases, much more explicit examples of **M** can be found. I discuss some examples in Section 1.2 below. For instance, the topos of simplicial sets can be presented as a topos of finitely supported **M**-sets, for a very reasonable (but not unique) monoid **M**. I see the model and topos-theoretic explanations of these surprisingly friendly examples as one of the main contributions of this thesis.

The first such example I encountered was obtained by Andrew Pitts for the topos of cubical sets [Pit15]. After I discussed this with him, he encouraged me to think about simplicial sets, for which I found the presentation stated in example 1.2.10 below. It was this initial result that has motivated me to understand the nature of these examples better. All the while I have been aware of Pitts' motivation to use toposes of finitely supported **M**-sets as models of homotopy type theory. In that direction, the example of simplicial sets was even more encouraging. Yet, the many subtleties in this presentation have forced me to tread slowly. In the last chapter, I discuss a proposed definition of Kan fibration for a quite general class of toposes, which generalizes Kan fibrations for cubical sets and simplicial sets. If one thinks of the underlying monoid as a 'substitution monoid' on a set of names, then the idea is roughly the same as the notion of fibration for cubical sets in Cohen et al. [Coh+18]. Under some additional assumptions I am able to show some initial results for these fibrations. Unfortunately, these assumptions do not hold for simplicial sets. However, it is still possible to develop a constructive theory of simplicial Kan fibrations based on this definition of fibration. This is the content of recent joint work with Benno van den Berg, of which the last section will give a brief impression.

Overview of the chapters

- In the first chapter, after this introduction, I give an elementary definition of a finitely supported **M**-set, where **M** is just a monoid of functions on a set. This humble beginning allows for a clear initial discussion of some of the properties of the structures involved. For instance, I will show that a finitely supported **M**-set is not generally the same as a continuous **M**-set. I end the first chapter with a section of motivating examples, which can be understood using the elementary definitions.

- The second chapter introduces the categorical language. The key definitions are that of a prime-generated site and that of a factorizing site. I believe the latter notion is in itself not

studied before and it will be motivated on model-theoretic grounds in Chapter 3. The rest of the chapter discusses related work on prime-generated toposes, such as Makkai's 'topos of types' [Mak81].

- Chapter 3 starts with an overview of the model-theoretic phenomenon of a *homogeneous structure*. By taking a categorical perspective, it will be shown that a category of structures naturally gives rise to a factorizing prime-generated site, which will hence be the primary object of study. I will then be able to describe homogeneous structures, and in particular the Fraïssé-Hrushovski construction using the language of Chapter 2.

- In Chapter 4, I return to the definitions of the first chapter concerning finite support. In the abstracted language, I will formulate a new notion of support which neatly covers all examples, but allows for a much cleaner theory which incorporates the theory of Chapter 3. Finally, I will be able to give categorical substance to the model-theoretic notion of *age* of a structure. In the last two section I discuss two classical applications to logic, namely the Ryll-Nardewski theorem and models of set theory.

- The last chapter ties up the contents of the thesis with one of my initial purposes for studying finitely supported **M**-sets: models of intensional Martin-Löf type theory. Although it seems quite a long stretch given the main content of the thesis, the intuition that comes with **M**-sets suggests a notion of fibration that not only proves workable, but also generalizes the known cases of simplicial sets and cubical sets. I use this in Section 5.2 to study a new class of examples, for which I am able to give a constructive interpretation of Kan dependent products. This class of examples includes the algebraic models that were also discussed in Chapter 4.

  The last section of Chapter 5 is devoted to the specific case of simplicial sets and is based on joint work with Benno van den Berg. It describes how the suggested fibration relates to Garner and Van den Berg's simplicial Moore path functor [BG12]. Mostly, this section gives an impression of how the ideas of this thesis relate to the project I have since started with Van den Berg. This project has grown so extensively over the past year that I can only hope to leave the reader with an interesting preview of this work.

1 Finitely supported **M**-sets

1.1 A first definition

I start by defining categories of finitely supported **M**-sets similar to those defined by Pitts for 'monoids of substitutions' **M** (cf. [Pit15], Definition 2.1). The starting data is a set M together with a submonoid $\mathbf{M} \subset \mathrm{Set}(M, M)$ of functions (or homomorphisms) $M \to M$. A finite tuple of elements of M will be denoted $\mathbf{m} = m_1, \ldots, m_n$. For a finite tuple $\mathbf{m}$ and a homomorphism $f : M \to M$ in $\mathbf{M}$, the values on each component yields another tuple, written $f\mathbf{m}$.

An **M**-*set* is a set X equipped with an **M**-action, which is the same thing as a functor $A : \mathbf{M} \to \mathrm{Set}$. A function $f : X \to Y$ is said to be *equivariant* if it commutes with the **M**-action, and such a function is the same thing a natural transformation from X to Y as functors. This defines the category of **M**-sets and equivariant maps as the topos of functors $[\mathbf{M}, \mathrm{Set}]$. When there is no chance of confusion, the same symbol will be used for an **M**-set as set and as functor, e.g. for $X : \mathbf{M} \to \mathrm{Set}$ it makes sense to write $x \in X$ for an element of X as a set. For such an element $x \in X$, and an endomorphism $f \in \mathbf{M}$, the value of the action of f on x is denoted $f \cdot x$, or sometimes fx.

Definition 1.1.1. Let X be an **M**-set, and $x \in X$. A (finite) *support* for x is a finite tuple $\mathbf{m}$ of elements of M satisfying for all $f, g \in \mathbf{M}$ that whenever $f\mathbf{m} = g\mathbf{m}$, it follows that $f \cdot x = g \cdot x$. In the presence of a support, it is said that x is *finitely supported*.

The **M**-set X is finitely supported when all its elements are. The finitely supported **M**-sets constitute a full subcategory of $[\mathbf{M}, \mathrm{Set}]$ denoted by $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$.

Remark 1.1.2. If **M** is a *group*, then it is well known that a finitely supported **M**-set is the same thing as an **M**-set for which the action is *continuous*: here M is given the discrete topology and $\mathbf{M} \subset M^M$ the subspace topology on the space of functions M^M equipped with the product topology. However, this no longer holds when **M** is a monoid: consider the monoid **M** of functions $\mathbb{N} \to \mathbb{N}$

which are constantly 0 on an initial segment of $\mathbb{N}$ and the identity outside of this segment[1]. As a category, this monoid is the same as the monoid consisting of one object $\bullet$ and $\mathrm{Hom}(\bullet, \bullet) = \mathbb{N}$, where composition is given by $\max(-,-) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. It can be readily verified that for this monoid, *every* **M**-set is continuous. However, an **M**-set is only finitely supported if for every element the action is eventually constant.

The definitions of continuous and finitely supported are separated by a swapping of quantifiers, which can be remedied by instead equipping **M** with the structure of a *uniform space*. Its use for this thesis will be limited, since the notion of finite support will be replaced by a more general notion of support in Chapter 4. Consequently, the 'spatial' characterisation of finitely supported **M**-sets will be subsumed by a more general topos-theoretic property. Indeed, by applying Corollary 4.2.3 to Example 4.1.14, the category $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$ arises as the hyperconnected-localic image of a given geometric morphism. Still, the characterisation as uniformly continuous **M**-sets stated in Proposition 1.1.4 below is of interest here since I haven't been able to find this result anywhere else. It requires the following definition, which may be safely skipped as far as the rest of this thesis is concerned.

Definition 1.1.3 ([Bou98], § II.1)**.** A *uniform structure* on a set X is a set $\mathfrak{U}$ of subsets of $X \times X$ such that:

(i) The set $\mathfrak{U}$ contains $X \times X$
(ii) If $U, V \in \mathfrak{U}$, there exists $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$
(iii) If $U \subseteq V$ and $U \in \mathfrak{U}$, then $V \in \mathfrak{U}$
(iv) Every $U \in \mathfrak{U}$ contains the diagonal $\Delta : X \to X \times X$
(v) For every $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that $V^{\mathrm{op}} \subseteq U$, where $-^{\mathrm{op}}$ takes the opposite relation.
(vi) For every $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that $V \circ V \subseteq U$, where $- \circ -$ is ordinary relational composition.

The elements of $\mathfrak{U}$ are often called *entourages*, a set X equipped with a set of entourages is called a *uniform space*. A function $f : X \to Y$ between uniform spaces is called *uniformly continuous* if for every entourage V of Y, there exists an entourage U of X such that f maps U into V.

A uniform structure is a bit like a metric, where every $U \in \mathfrak{U}$ consists of those pairs of points that are within distance 'U' from each other. Indeed, every metric space has a (canonical) uniform structure. In similar vein, every set can be equipped with a *discrete* uniform structure, by letting the uniform structure consist of all sets that contain the diagonal. The product of two uniform spaces

[1] This example was communicated to me by Martijn den Besten.

X, Y is formed by taking the coarsest uniform structure on $X \times Y$ for which the projection maps are uniformly continuous.

For a monoid $\mathbf{M}$ as above, one can define a family of subsets $U_{\mathbf{m}} \subseteq \mathbf{M} \times \mathbf{M}$ indexed over all finite tuples $\mathbf{m}$ in M defined by (1.1):

$$U_{\mathbf{m}} := \{(f, g) \in \mathbf{M} \mid f\mathbf{m} = g\mathbf{m}\} \tag{1.1}$$

It is easy to check that the set $\mathfrak{U}$ of sets V such that $U_{\mathbf{m}} \subseteq V$ for some $\mathbf{m}$ defines a uniform structure on $\mathbf{M}$.

Proposition 1.1.4. *Suppose* $\mathbf{M}$ *is a monoid as above and* X *an* $\mathbf{M}$*-set. Then* X *is finitely supported if and only if the action* $\mathbf{M} \times X \to X$ *is uniformly continuous, where* X *carries the discrete uniform structure, and* $\mathbf{M} \times X$ *the product uniform structure.*

Proof. The action $\mathbf{M} \times X \to X$ is uniformly continuous if and only if both parametrized actions (for fixed $f \in \mathbf{M}$, $x \in X$) are uniformly continuous (e.g. see Chapter II of [Bou98], Proposition 2.6.9). Since X is discrete, this is equivalent to $(-) \cdot x : \mathbf{M} \to X$ being uniformly continuous for every $x \in X$. So for the diagonal $\Delta \subset X \times X$, there must be an entourage $U_{\mathbf{m}}$ such that $(f, g) \in U_{\mathbf{m}}$ implies that (fx, gx) is in the diagonal, i.e. $fx = gx$. This is equivalent to $\mathbf{m}$ being a finite support for x. □

For now, I will stick with the definition of finite support as in Definition 1.1.1 until it is subsumed by Definition 4.1.2, which has, as already announced, Corollary 4.2.3 as corresponding 'spatial' characterisation. Before getting to the examples of interest, I will establish some properties of the category of finitely supported $\mathbf{M}$-sets.

Proposition 1.1.5. *The inclusion* $i : [\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}} \to [\mathbf{M}, \mathrm{Set}]$ *has a right adjoint and, moreover, preserves finite limits.*

Proof. As to the first claim, let for an $\mathbf{M}$-set A, $A_{\mathrm{fs}} \subset A$ be the subset of finitely supported elements of A. Observe that if $a \in A$ is supported by $\mathbf{x}$, fa is supported by $f\mathbf{x}$ for every $f \in \mathbf{M}$. So A_{fs} is closed under the $\mathbf{M}$-action and defines a finitely supported $\mathbf{M}$-set. Similarly, when $\mu : A \to B$ is a natural transformation of $\mathbf{M}$-sets, and $x \in A$ is finitely supported, then $\mu(x)$ is finitely supported. Consequently the operation $(-)_{\mathrm{fs}}$ extends to a functor

$$(-)_{\mathrm{fs}} : [\mathbf{M}, \mathrm{Set}] \to [\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}. \tag{1.2}$$

It is easy to see that $(-)_{\mathrm{fs}}$ is right adjoint to i.

For the second part, it is easy to see that the construction of finite limits in $[\mathbf{M}, \mathrm{Set}]$ preserves finitely supported $\mathbf{M}$-sets, so that in fact $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$ is closed under finite limits constructed in $[\mathbf{M}, \mathrm{Set}]$. □

Corollary 1.1.6. *The category* $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$ *is a topos.*

Proof. By the previous proposition, $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$ is comonadic over $[\mathbf{M}, \mathrm{Set}]$ where the comonad is left exact. Therefore $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$ is a topos. □

In fact, $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$ is a Grothendieck topos. This can be seen directly by identifying a separating family of *orbits*.

Definition 1.1.7. For a tuple $\mathbf{x} = x_1, \ldots, x_n$ of elements of M, the *orbit* of $\mathbf{x}$ is the **M**-set (1.3) equipped with the component-wise action on tuples.

$$O(\mathbf{x}) = \{(\alpha x_1, \ldots, \alpha x_n) \mid \alpha \in \mathbf{M}\} \tag{1.3}$$

For A a finitely supported **M**-set, an equivariant map $f : O(\mathbf{x}) \to A$ is uniquely defined by its value on $\mathbf{x}$:

$$f\alpha\mathbf{x} = \alpha f\mathbf{x}$$

Moreover, it is required that the corresponding value $f\mathbf{x} \in A$ is *supported* by $\mathbf{x}$. It is not hard to see that these conditions are sufficient; indeed an equivariant map $f : O(\mathbf{x}) \to A$ corresponds precisely to an element a of A supported by $\mathbf{x}$. This has an important consequence.

Proposition 1.1.8. *The set of orbits generate the topos* $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$.

Proof. For a finitely supported **M**-set A, any $a \in A$ is supported by some tuple $\mathbf{x}$. Then one can define $O(\mathbf{x}) \to A$ by sending $\alpha\mathbf{x}$ to αa, which is equivariant since a is supported by $\mathbf{x}$. In this way, the orbits clearly form a separating family. □

By Giraud's theorem, the full subcategory of orbits, together with the canonical topology, forms a site for $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$. A morphism of orbits $O(\mathbf{x}) \to O(\mathbf{y})$ is given by a tuple $\alpha\mathbf{y}$ for some $\alpha \in \mathbf{M}$, such that $\alpha\mathbf{y}$ is supported by $\mathbf{x}$. The morphism between the orbits then sends $\mathbf{x}$ to $\alpha\mathbf{y}$. Such an endomorphism f induces an epimorphism when there exists β such that $\beta \circ \alpha\mathbf{y} = \mathbf{y}$.

The following proposition establishes an equivalent site of definition. The construction of this site, and its equivalence, is part of a more general phenomenon described by Lemma 4.1.12. In Example 4.1.14 I show how the proposition below follows from this lemma.

The equivalent site, call it $\mathcal{I}$, is defined from the site of orbits $\mathcal{O}$ by taking all objects, but restricting the *morphisms* in a certain way and defining a subcoverage; i.e. a coverage on $\mathcal{I}$ whose covers are also covers in $\mathcal{O}$. From the general form of the comparison lemma, given in Theorem 2.2.4 below, it follows that the inclusion $\mathcal{I} \hookrightarrow \mathcal{O}$ induces an equivalence of toposes if for every morphism $\alpha : O(\mathbf{x}) \to O(\mathbf{y})$ in $\mathcal{O}$, there exists a cover $O(\mathbf{z}_\alpha) \to O(\mathbf{x})$ in $\mathcal{I}$ such that the composite:

$$O(\mathbf{z}_\alpha) \to O(\mathbf{x}) \to O(\mathbf{y}) \tag{1.4}$$

is in $\mathcal{I}$.

In the present case, $\mathcal{I}$ is defined as follows. Observe that for $\mathbf{x}$, $\mathbf{y}$ tuples as in the above, $\alpha\mathbf{y}$ is supported by $\mathbf{x}$ in particular when

$$\{\alpha\mathbf{y}\} \subseteq \{\mathbf{x}\} \tag{1.5}$$

as sets, i.e. when $\alpha\mathbf{y}$ is simply among $\mathbf{x}$. Now, let $\mathcal{I}$ be the subcategory of $\mathcal{O}$ consisting of morphisms $\alpha : O(\mathbf{x}) \to O(\mathbf{y})$ for which this relation holds. Define a *cover* in $\mathcal{I}$ to be such a morphism represented by $1_M : O(\mathbf{x}) \to O(\mathbf{y})$.

Proposition 1.1.9. *The category $\mathcal{I}$ with the above class of covers is a* site*. Moreover, the inclusion $\mathcal{I} \hookrightarrow \mathcal{O}$ is* dense*, i.e. satisfies the conditions of the comparison lemma (Theorem 2.2.4) and hence defines an equivalent topos of sheaves.*

Proof. It is not hard to verify the coverage condition. For density, suppose $\alpha : O(\mathbf{x}) \to O(\mathbf{y})$ is a morphism in $\mathcal{O}$. Then one can just cover $\pi : O(\mathbf{x}, \alpha\mathbf{y}) \to O(\mathbf{x})$ where π has the identity as underlying morphism. The composite with α is clearly in the described category. □

An advantage of using $\mathcal{I}$ is that it is essentially the opposite category of a category of sets and functions. A disadvantage is that (sub)canonicity of the topology is lost. The category $\mathcal{I}$, consisting of finite 'substructures' of $\boldsymbol{M}$, can be compared to what has been called the *age* of a structure in classical Fraïssé theory. Indeed, a construction in Section 3.3 suggests such an interpretation.

1.2 Examples

In this section I will lay out the main examples of toposes of finitely supported $\mathbf{M}$-sets that motivate this thesis. The goal is to provide a feel for how examples may be constructed, starting with a particular site, and therefore proofs will be largely sketched. For a rigorous proof, one can apply the results of the relevant subsequent chapter.

Example 1.2.1 (Object classifier)**.** Consider an infinite set $\mathbb{A} = \{x, y, z, \dots\}$, which I will suggestively call a set of *names* (see Example 1.2.3). Denote by FinSet the category of finite subsets of $\mathbb{A}$, referred to as the (small) category of finite sets. Define an *embedding* of finite sets to be an injective map. This class of embeddings forms a coverage on $\mathrm{FinSet}^{\mathrm{op}}$. Since every morphism out of the empty set is an embedding, it follows that the full subcategory $\mathrm{FinSet}^{\mathrm{op}}_{\neq\emptyset}$ on non-empty sets is *dense*[2]. Further, because every embedding $I \rightarrowtail J$ between non-empty sets is the section of some function $J \to I$, every presheaf on the induced subsite is a sheaf. So the topos of sheaves is the

[2] As part of the general theory, this will be introduced on page 30 below.

presheaf topos $[\mathrm{FinSet}_{\neq\emptyset}, \mathrm{Set}]$. This topos is a classifier for the theory $\mathbb{O}_\bullet$ of non-empty objects, which consists of an empty language and the sequent (1.6).

$$\vdash \exists x.\top \tag{1.6}$$

Let $\mathbf{M} := \mathrm{Set}(\mathbb{A}, \mathbb{A})$ be the monoid of all endomorphisms $\mathbb{A} \to \mathbb{A}$. Then by Proposition 1.1.9, the topos $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$ is a topos of sheaves on finite subsets $I \subset \mathbb{A}$ and restrictions of endomorphisms $\mathbb{A} \to \mathbb{A}$ between them (which could be any function), where covers are generated by inclusions $I \subseteq J$. Hence, this is the same as the site given above. Consequently, there is an equivalence of toposes:

$$\mathcal{S}[\mathbb{O}_\bullet] \cong [\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}} \tag{1.7}$$

Here the left-hand side denotes the classifying topos of the theory $\mathbb{O}_\bullet$.

Remark 1.2.2. Example 1.2.1 silently introduces a recurring theme in this thesis, namely that of a site equipped with a notion of embedding. In Chapter 2, this will be introduced formally as a factorizing (prime-generated) site. The important bit to realize for later on is that even though the coverage here is (almost) completely trivial, this particular choice of covers still contains information. Indeed, it determines the theory that it classifies in terms of the model M. This is contained in Theorem 4.4.8 below and has more examples in Section 4.4.2.

Example 1.2.3 (Nominal sets)**.** Let $\mathbb{A}$ be the same set of names, but denote by $\mathcal{I}$ the category of finite subsets of $\mathbb{A}$ and *injective* maps, with the same notion of embedding. So every morphism is an embedding and the topos of sheaves on $\mathcal{I}^{\mathrm{op}}$ is an atomic topos. This topos is known as the *Schanuel topos* or the topos of *nominal sets* by Gabbay and Pitts [GP01], for which see also [Pit13].

For $\mathbf{M} := \mathrm{Set}_{1\text{-}1}(\mathbb{A}, \mathbb{A})$ the monoid of *injections* $\mathbb{A} \to \mathbb{A}$, one can again use Proposition 1.1.9 to conclude that $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}} \cong \mathrm{Sh}(\mathcal{I}^{\mathrm{op}})$.

Remark 1.2.4. In most literature, nominal sets are presented as finitely supported (or continuous) sets with an action coming from the *group* of permutations of $\mathbb{A}$, which is a proper submonoid of $\mathbf{M}$ of Example 1.2.3. Hence from this example it follows that, in general, the property of $\mathbf{M}$ being a group cannot be a property of the topos $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$. This is another reason why this thesis is not much concerned with groups. Note that the class of toposes of finitely supported $\mathbf{M}$-sets for monoids is also strictly larger than for groups: as Example 1.2.1 shows (and many below), the former may contain non-Boolean toposes, whereas toposes in the latter class are always Boolean.

Example 1.2.5 (Non-example: Decidable objects)**.** Comparing the two examples above, one may be tempted to think that the topos of nominal sets classifies *decidable* objects. However, this is not the case. For decidable sets, the only embeddings are isomorphisms, and therefore the topos

classifying decidable sets is the presheaf topos $[\mathcal{I}^{\mathrm{op}}, \mathrm{Set}]$. In the case of nominal sets, the fact that an injection $I \rightarrowtail J$ is an *embedding* even though it is not an embedding of logical structures reveals the fact that I and J are not models of the theory that the topos classifies. This is an important observation that will become more rigorous in Chapter 3. Indeed, the topos of nominal sets classifies *infinite* decidable objects, which has signature $\{\#\}$ and validates the sequents:

$$x\#x \vdash_x \bot \tag{1.8}$$

$$\vdash_{x,y} x = y \vee x\#y \tag{1.9}$$

$$\vdash_{x_1,\ldots,x_n} \exists x(x\#x_1) \wedge \ldots \wedge (x\#x_n) \text{ for all } n \tag{1.10}$$

Note that by the last statement, (1.6) is also satisfied.

From the 'nominal' perspective, 1.2.3 starts a theme with many variations, as proposed by Pitts in [Pit15]. Indeed, cubical sets are such a variation, but the essence of this variation is already captured by the following example, which appears in chapter 9 of [Pit13] as the category of *nominal restriction sets*. Its presentation as a topos of presheaves is attributed to S. Staton.

Example 1.2.6 (Nominal restriction sets)**.** Suppose $\mathbf{M}$ is the monoid of *partial* injections $\mathbb{A} \to \mathbb{A}$. This monoid can also be described as the monoid of total functions $f : \mathbb{A} \cup \{\mathsf{u}\} \to \mathbb{A} \cup \{\mathsf{u}\}$, where $\mathsf{u} \notin \mathbb{A}$, that preserve u and are injective on $f^{-1}\mathbb{A}$.

For this monoid, the topos of finitely supported $\mathbf{M}$-sets is equivalent to the functor category $[\mathbf{pInj}, \mathrm{Set}]$ where $\mathbf{pInj}$ is the category of finite subsets $I \subset \mathbb{A}$ and functions $I \to J \cup \{\mathsf{u}\}$ injective on $f^{-1}J$. Indeed, it is easy to see that every cover splits, so the sheaf condition trivializes.

In Chapter 9 of [Pit13], it is shown that $[\mathbf{pInj}, \mathrm{Set}]$ is monadic over nominal sets and as such is called the category of *nominal restriction sets*.

Since the viewpoint in this thesis is model-theoretic, I make it a habit of finding the theory that a topos of finitely supported $\mathbf{M}$-sets classifies. Let $\{u, \#\}$ be a signature consisting of a constant symbol u and a relation symbol # like for decidable sets, and consider the following sequents:

$$\vdash_x x\#u \tag{1.11}$$

$$x\#x \vdash_x x = u \tag{1.12}$$

$$x\#y \vdash_{x,y} y\#x \tag{1.13}$$

$$\vdash_{x,y} x = y \vee x\#y \tag{1.14}$$

These make up a theory $\mathbb{T}_{\mathrm{du}}$ of *decidable sets with an undefined element*. Since the language comes with a constant symbol, every model is non-empty. Also the sequent (1.13) is redundant; but since it plays a role in finding a suitable cartesian fragment below, I have stated it explicitly.

Indeed, to find the classifying topos, take as cartesian fragment the sequents (1.11), (1.12), and (1.13). The category of finitely presented Set-models corresponding to this theory is precisely the category of finite undirected pointed graphs with only one loop at the point u, and an edge between every vertex and u. Morphisms are graph homomorphisms. Denote this category by $\mathbf{G}_u$.

The sequent (1.14) puts a topology on $\mathbf{G}_u^{\mathrm{op}}$. One can now use similar reasoning as in [Joh02b], D3.2.7. A co-cover on a graph G is generated by any pair of morphisms $f : G \to H$, $g : G \to K$, one of which draws an edge between two unconnected vertices, and one of which equates two vertices. Note that a morphism cannot equate two connected vertices, unless it equates them both to u. By composing these coverings, one can deduce that every graph G is covered by the family $f_E : (G, \#) \to (G/E, \#_E)$ where E ranges over equivalence relations on the vertices of G and $\#_E$ is the complement of the diagonal joined with $\{(u, u)\}$. It follows that the full subcategory of complete graphs is dense, and the induced topology is trivial. This category is equivalent to **pInj**: It consists of finite sets (of *named* vertices), and morphisms map two vertices x, y either to two distinct vertices (it must preserve edges), or both to u, which has the only loop. Moreover, the element u is always preserved. This result can be summarised in the following proposition:

Proposition 1.2.7. *The topos* [**pInj**, Set]*, and therefore the topos of finitely supported* **M***-sets of Example 1.2.6 classifies the theory* $\mathbb{T}_{du}$ *of decidable objects with an undefined element.* □

Example 1.2.8 (Cubical sets I)**.** The following example advances nominal restriction sets by only one step: namely, consider decidable sets with *two* undefined elements, named 0 and 1. Then, let **M** be the monoid of functions $\mathbb{A} \cup \{0, 1\} \to \mathbb{A} \cup \{0, 1\}$ that preserve $0, 1$ and are injective on $f^{-1}\mathbb{A}$.

The topos of finitely supported **M**-sets is the topos of cubical sets of Bezem, Coquand and Huber [BCH14] ([Pit15], theorem 2.9). It can be presented as the presheaf topos [**Cb**, Set] where **Cb** is the category with finite subsets $I \subset \mathbb{A}$ as object, and morphisms $I \to J$ given by functions $f : I \cup \{0, 1\} \to J \cup \{0, 1\}$ preserving $0, 1$ and injective on $f^{-1}J$. It can be readily checked that this is the category of (pre-)sheaves on the category of orbits, from which the result follows. From the previous example, it is not hard to determine the theory that it classifies. In this case, the signature consists of two constants and a relation: $\{\#, 0, 1\}$. The axioms are then given by the following sequents:

$$1 = 0 \vdash \bot \tag{1.15}$$

$$\vdash_x x\#0 \wedge x\#1 \tag{1.16}$$

$$x\#x \vdash_x x = 0 \vee x = 1 \tag{1.17}$$

$$x\#y \vdash_{x,y} y\#x \tag{1.18}$$

$$\vdash_{x,y} x = y \vee x\#y \tag{1.19}$$

Again, note that (1.18) actually follows from the other axioms.

Example 1.2.9 (Cubical sets II)**.** Recall the alternative category $\mathbf{Cb}_{\mathrm{dM}}$ of *cubes* of Cohen et al. [Coh+18]. Objects consists of finite sets of *names* $I \subset \mathbb{A}$ as in the examples above, and a morphism $J \to I$ is a *function* $I \to \mathrm{dM}\, J$, where dM is the free de Morgan algebra monad on Set. Morphisms are composed using Kleisli composition.

This category is the same as the category of free finite de Morgan algebras, and as the category of presheaves on its opposite category, cubical sets classify *flat de Morgan algebras*, as observed by Spitters [Spi15] by drawing from earlier work of Johnstone-Wraith [JW78].

I show in Theorem 4.4.11 that for any finitary algebraic theory, non-empty flat models are classified by the topos of finitely supported $\mathbf{M}$-sets for $\mathbf{M}$ the monoid of endomorphisms on the free countably infinite model. In the case of cubical sets, this is the free de Morgan Algebra on $\mathbb{A}$. Hence this example extends the result of Pitts [Pit15] to this enhanced version of cubical sets.

The discovery of the following example has initially motivated me to study finitely supported $\mathbf{M}$-sets in more depth. As I described in the beginning, it resulted from a conversation I had with Andrew Pitts on the possible reach of the 'substitution sets' described in [Pit15].

Example 1.2.10. Recall that the topos of *simplicial sets* is given by presheaves on the *simplex category* Δ. It is well known[3] that this is equivalent to the topos $[\mathbf{I}, \mathrm{Set}]$, where $\mathbf{I}$ is the category of finite *intervals*. An interval is a linear order with distinguished and distinct *top* and *bottom* elements. So the finite ordinals $> [0]$ form the objects of this category, and morphisms are order-preserving maps that also preserve the top and bottom elements.

As a site, the (opposite) category of intervals is easily seen to be equivalent to a site of orbits. For the ordinal $\omega + 1$, let $\mathbf{M}$ be the monoid of *interval* endomorphisms $\omega + 1 \to \omega + 1$. An orbit is given by a sequence $a_1 \leq \cdots \leq a_n$ of ordinals $\leq \omega$; which can always be completed to an interval $0 \leq a_1 < \cdots < a_n \leq \omega$ if it does not have some endpoints already. A morphism of orbits:

$$[f] : O(a_1 \leq \cdots \leq a_n) \to O(b_1 \leq \cdots \leq b_m) \tag{1.20}$$

is just an interval morphism:

$$f : \{0 \leq b_1 \leq \cdots \leq b_m \leq \omega\} \to \{0 \leq a_1 \leq \cdots \leq a_n \leq \omega\} \tag{1.21}$$

and every such interval morphism can be extended to an endomorphism on $\omega + 1$. So the site of orbits is equivalent to finite intervals with interval morphisms. Since every cover splits, the topos of finitely supported $\mathbf{M}$-sets is a presheaf topos and $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}} \cong [\mathbf{I}, \mathrm{Set}]$ is the topos of simplicial sets.

As remarked above, the category of simplicial sets classifies intervals, and the monoid $\mathbf{M}$ of Example 1.2.10 is the monoid of endomorphisms on a sufficiently 'generic' interval. It is by no

[3] This observation is attributed to A. Joyal. It can be used to show that the topos of simplicial sets classifies the theory of *intervals*. A full proof of this fact seems to have appeared first in Johnstone [Joh79].

means unique, since for instance one could have taken the interval $\mathbb{Q} \cap [0, 1]$, which is not isomorphic to $\omega + 1$. I will show in Chapter 3 that these different choices of a 'generic interval' correspond to different filtered colimits of embedded finite structures (Proposition 3.4.9). The following example is a simple variation on Example 1.2.10.

Example 1.2.11 (Cyclic sets)**.** The category **K** of *abstract circles* can be presented in various ways [Moe96]; here I will present the simplest one for our current purposes which I attribute to T. Trimble [Tri07]. Think of a finite abstract circle as a finite interval $x_0 < x_1 < \cdots < x_n < x_0$ whose endpoints are identified, or a 'necklace' consisting of beads in a certain orientation. A morphism between abstract circles preserves the order of the 'beads' non-strictly. Concretely, a finite abstract circle is a finite model of the theory which has one ternary relation symbol $C(x, y, z)$ meaning 'walking from x in the clockwise direction, one first encounters y and then z'. The theory thus consists of the following sequents:

$$C(x, y, z) \vdash C(z, x, y) \tag{1.22}$$

$$\vdash C(x, y, z) \vee C(x, z, y) \tag{1.23}$$

$$\vdash C(x, y, x) \tag{1.24}$$

$$C(x, y, z) \wedge C(z, y, w) \vdash C(x, y, w) \tag{1.25}$$

$$C(x, y, z) \wedge C(z, y, x) \vdash x = y \vee x = z \vee y = z \tag{1.26}$$

By identifying endpoints, every finite interval $[n + 1]$ is a presentation of the unique abstract circle with n beads. A morphism of abstract circles $[n] \to [m]$ is a morphism preserving the relation $C(x, y, z)$.

Similarly, the countable abstract circle can be presented by the ordinal $\omega + 1$. Let **M** be its monoid of endomorphisms. A morphism between orbits $O(\mathbf{x}) \to O(\mathbf{y})$ is the same thing as a morphism between abstract circles $[f] : \mathbf{y} \to \mathbf{x}$. Moreover, the site of orbits is a presheaf site since every epimorphism splits. Therefore $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}} \cong [\mathbf{K}, \mathrm{Set}]$ is the topos of *cyclic sets* [Moe96].

Other examples

Many other examples are covered in this thesis. In Chapter 3, I spend much time on the topos of sheaves on the site of finite *Hrushovski* $(k + 1)$*-graphs* (see Example 3.2.10 and Theorem 3.3.18). This example is of conceptual importance, for it illustrates that when trying to understand the Hrushovski construction, and the corresponding notion of age, one cannot restrict to atomic toposes. Further, as announced in Example 1.2.9, Section 4.4.2 presents a large class of examples, namely the classifying toposes of non-empty, flat and free models of finitary algebraic theories.

2 Prime-generated Toposes

This chapter serves mostly to remind the reader of the concepts from category and topos theory that this thesis relies on, and in particular of those aspects that are defined and treated in a way less often met with. For instance, sites are not assumed to be *regular* in this thesis, in the sense suggested by Makkai [1] [Mak81]. This goes even further in that sites are not taken to be finitely complete. Presently, these common assumptions are superseded by an orthogonal factorization system on the site. In the next chapter, it will be seen that there are interesting examples of sites that satisfy the latter, but neither of the former two conditions; in fact the simplex category encountered in the introduction is already an example. The present chapter sets the scene for subsequent chapters, but by no means does it serve as a proper, textbook-style introduction to these notions or to topos theory in general. For most topos-theoretic results I will refer to Johnstone's Elephant [Joh02a]; [Joh02b]. A good, shorter introduction to the subject which also covers much of the logic is Moerdijk-Maclane [MM94]. For internal notions, perhaps the good old Johnstone [Joh77] is still the best place. The notion of prime object and prime-generated topos is due to Makkai [Mak81]; [Mak82].

2.1 Sites and coverages

In this section I recall the precise relation between families of arrows in a (small) category and coverages, for this will be especially important in connecting prime-generated sites to orthogonal factorization systems. Since this is neither the time nor the place for a review of sheaf theory, I will only briefly pause at the notion of a sheaf, providing some references for the uninitiated. In similar vein, the treatment in the next section on morphisms between sites is rather cursory and directs

[1] Namely, that the underlying category is finitely complete, and that the coverage is subcanonical, generated by finite covering families, such that every morphism factors as a single cover followed by a monomorphism. This can be viewed as a slight abstraction of a regular category equipped with the *regular coverage*, see e.g. Example A2.1.11 in [Joh02a].

attention to only the most urgent of details. Below, all definitions are usually assumed to be over some base topos $\mathcal{S}$ with a natural numbers object. See also the remarks in the beginning of section 2.2.

Suppose $\mathcal{C}$ is a category. For an object U of $\mathcal{C}$, consider the class $\mathrm{Fam}(U)$ consisting of families (sets) of arrows $\{u_i : U_i \to U\}$, where the i ranges over some index set, and define a pre-order on this class by (2.1):

$$\{u_i : U_i \to U\} \preceq \{v_j : V_j \to U\} \text{ if every } u_i \text{ factors through } v_j \text{ for some } j \tag{2.1}$$

For a family $\mathcal{V} \in \mathrm{Fam}(V)$ and an arrow $g : V \to U$, one can define a family $g_!\mathcal{V}$ as follows:

$$g_!\mathcal{V} := \{g \circ v : V' \to U \mid v : V' \to V \in \mathcal{V}\} \tag{2.2}$$

Suppose one is given, for every object U in $\mathcal{C}$, a collection of families:

$$\mathrm{Cov}(U) = \{\mathcal{U}_j \in \mathrm{Fam}(U) \mid j \in \mathcal{J}\} \tag{2.3}$$

Then the following definition is standard (cf. [Joh02b], C2.1.1):

Definition 2.1.1. A collection of families (2.3) for every object of a category $\mathcal{C}$ forms a *coverage* if whenever $\mathcal{U} \in \mathrm{Cov}(U)$ and $g : V \to U$ is an arrow, then there exists $\mathcal{V} \in \mathrm{Cov}(V)$ such that $g_!\mathcal{V} \preceq \mathcal{U}$. This condition is referred to as the *coverage condition*. A category equipped with a choice of coverage is called a *site*. The elements of $\mathrm{Cov}(C)$ are called *covering families*.

When $\mathcal{C}$ is a *small* category, or an internal category in $\mathcal{S}$, an arrow $g : V \to U$ in $\mathcal{C}$ and a family $\mathcal{U} \in \mathrm{Fam}(U)$ define a family in $\mathrm{Fam}(V)$:

$$g^*\mathcal{U} = \{v : V' \to V \mid g \circ v \text{ factors through } u \text{ for some } u \in \mathcal{U}\} \tag{2.4}$$

It is easy to check that $g_! : \mathrm{Fam}(V) \to \mathrm{Fam}(U)$ is left adjoint to g^*:

$$g_!\mathcal{V} \preceq \mathcal{U} \Leftrightarrow \mathcal{V} \preceq g^*\mathcal{U} \tag{2.5}$$

Denote the category of pre-orders by PreOrd. The above implies the following:

Proposition 2.1.2. *The operation* $\mathrm{Fam} : \mathcal{C}^{\mathrm{op}} \to \mathrm{PreOrd}$*, taking objects* U *to* $\mathrm{Fam}(U)$ *and morphisms* $g : V \to U$ *in* $\mathcal{C}$ *to* $g^* : \mathrm{Fam}(U) \to \mathrm{Fam}(V)$ *defines a pseudofunctor.*

Proof. Since $g_!$ is functorial, this follows from (2.5). □

By means of the Grothendieck construction, Proposition 2.1.2 yields a (cloven) *Grothendieck fibration*:

$$\mathrm{fam}_{\mathcal{C}} : \mathbf{Fam}_{\mathcal{C}} \to \mathcal{C} \tag{2.6}$$

This is called the *fibration of families*. A family of the form $\mathcal{V} = g^*\mathcal{U}$ (2.4) is special in that it satisfies:

$$v : V' \to V \in \mathcal{V} \text{ and } v' : V'' \to V' \Rightarrow v \circ v' \in \mathcal{V} \tag{2.7}$$

A family $\mathcal{V}$ satisfying the property (2.7) is called a *sieve* on D. For sieves $\mathcal{V}', \mathcal{V}$, one has $\mathcal{V}' \preceq \mathcal{V}$ if and only if $\mathcal{V}' \subseteq \mathcal{V}$. Since $1_U^*\mathcal{U}$ is always a sieve, it follows that the full subcategory of $\mathbf{Fam}_{\mathcal{C}}$ on sieves is its poset reflection:

$$\mathbf{Siev}_{\mathcal{C}} \xrightarrow{\sim} \mathbf{Fam}_{\mathcal{C}} \tag{2.8}$$

Hence there is a *functor* $\mathrm{Siev} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Poset}$ sending an object C to its partial order of sieves and with re-indexing given by (2.4). By the Grothendieck construction, this can be turned into a *split Grothendieck fibration* $\mathrm{siev}_{\mathcal{C}} : \mathbf{Siev}_{\mathcal{C}} \to \mathcal{C}$ equivalent to **Fam**:

$$\begin{array}{ccc} \mathbf{Siev}_{\mathcal{C}} & \xrightarrow{\sim} & \mathbf{Fam}_{\mathcal{C}} \\ & \searrow^{\mathrm{siev}_{\mathcal{C}}} & \downarrow{\scriptstyle \mathrm{fam}_{\mathcal{C}}} \\ & & \mathcal{C} \end{array} \tag{2.9}$$

Now suppose one is given a family (2.3) satisfying the condition of Definition 2.1.1. Such a family defines a full subcategory $I : \mathbf{Cov} \to \mathbf{Siev}_{\mathcal{C}}$ by letting, for every sieve $S \in \mathrm{Siev}(U)$:

$$S \in \mathbf{Cov}(C) \Leftrightarrow \mathcal{U} \preceq S \text{ for some } \mathcal{U} \in \mathrm{Cov}(U) \tag{2.10}$$

By this definition, it follows that this family satisfies that $g^*\mathcal{U} \in \mathbf{Cov}$ whenever $\mathcal{U} \in \mathrm{Cov}(U)$. As a direct consequence, the following proposition provides an equivalent definition of site (2.1.1) for small categories:

Proposition 2.1.3. *Let $\mathcal{C}$ be a small category. A collection of families* (2.3) *forms a* coverage *if the embedding*

$$I : \mathbf{Cov} \to \mathbf{Siev}_{\mathcal{C}} \tag{2.11}$$

is cartesian*, i.e. it induces a Grothendieck fibration over $\mathcal{C}$*

$$\mathbf{Cov} \xrightarrow{I} \mathrm{Siev}_{\mathcal{C}} \to \mathcal{C} \tag{2.12}$$

where I preserves cartesian arrows. A small category equipped with this structure is called a small site.

The following defines a way in which a small site may induce a coverage on another small category. If $F : \mathcal{D} \to \mathcal{C}$ is a functor, then F extends to a functor $F : \mathrm{Fam}(\mathcal{D}) \to \mathrm{Fam}(\mathcal{C})$ which sends a family $\mathcal{U} = \{u_i : U_i \to U\}$ to $F\mathcal{U} = \{Fu_i : FU_i \to FU\}$. Given a coverage $\mathrm{Cov}_{\mathcal{C}}$ on $\mathcal{C}$, one can define:

$$\mathrm{Cov}_{\mathcal{D}} = \{\mathcal{V} \in \mathrm{Fam}(D) \mid F\mathcal{V} \in \mathrm{Cov}_{\mathcal{C}}(FD)\} \tag{2.13}$$

This does not always define a coverage, but there clearly is a least *cartesian* full subcategory $\overline{\mathbf{Cov}_{\mathcal{D}}} \to \mathrm{Siev}_{\mathcal{D}}$ which contains $\mathbf{Cov}_{\mathcal{D}}$. This is called the *induced coverage*. Note that this coverage is not given by the family (2.13). The family of all fibres of $\overline{\mathbf{Cov}_{\mathcal{D}}}$ would be a valid defining family, but it is not a very convenient one. However, there are cases where (2.13) defines a coverage, such as in Theorem 2.2.4 below, and the induced coverage is a useful notion.

If $\mathcal{C}$ is a site, and $i : \mathcal{D} \hookrightarrow \mathcal{C}$ is a subcategory, then i is called *dense* (cf. Johnstone [Joh02b], definition D2.2.1) if:

(i) For every object C, there exists $\mathcal{U} \in \mathrm{Cov}_{\mathcal{C}}(C)$ such that for all $u \in \mathcal{U}$, the domain of u lies in $\mathcal{D}$.

(ii) For every morphism $f : D' \to D$ in $\mathcal{C}$ between objects in $\mathcal{D}$, there exists $\mathcal{U} \in \mathrm{Cov}_{\mathcal{C}}(D')$ such that $f_!\mathcal{U} \in \mathrm{Fam}_{\mathcal{D}}(D)$ (that is, all arrows in $f_!\mathcal{U}$ are in the category $\mathcal{D}$).

Examples of sites abound in this thesis. A topos $\mathcal{T}$ admits a *canonical coverage* consisting of its jointly epimorphic families. A bounded topos over Set always admits a small subcategory which is dense for the canonical coverage. For a bounded topos over $\mathcal{S}$, something similar holds, namely that there exists an internal separating family over an object from $\mathcal{S}$, which generates an internal full subcategory, and this will be a suitable site. For details, see section B3.3 of Johnstone [Joh02b]. Here, these details do not matter so much, and one can pretend that the situation is presented as explicitly as for $\mathcal{S} = \mathrm{Set}$.

So with the induced topology, this subcategory forms a small site. The bounded topos is the topos of *sheaves* for this site:

Definition 2.1.4. Suppose $\mathcal{C}$ is a site, and $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is a functor. Then F is called a *sheaf* (e.g. [Joh02b], definition C2.1.2) when for every $\mathcal{U} = \{u_i : U_i \to U\}_i \in \mathrm{Cov}_{\mathcal{C}}(U)$ and every family of elements $x_i \in F(U_i)$ that satisfies:

(i) Whenever a pair $g : V \to U_i$ and $h : V \to U_j$ satisfies $u_i \circ g = u_j \circ h$, then $x_i \cdot g = x_j \cdot h$

then there exists a *unique* $x \in F(U)$ such that $x \cdot u_i = x_i$ for all i.

For more background on sheaves and coverages, I refer to chapter C2 of Johnstone [Joh02b]. The important conclusion is the following theorem. For more on the sheafification functor, the first chapter of Makkai and Reyes [MR77] is very explicit and detailed.

Theorem 2.1.5. *When $\mathcal{C}$ is a small site, the full subcategory* $\mathrm{Sh}(\mathcal{C})$ *of* $[\mathcal{C}^{\mathrm{op}}, \mathcal{S}]$ *on sheaves is a topos, has all small limits and colimits and is reflective. The reflector*

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{S}] \xrightarrow{\mathbf{a}} \mathrm{Sh}(\mathcal{C}) \tag{2.14}$$

preserves finite limits and hence forms the left adjoint of a geometric inclusion. This functor is called sheafification.

2.2 Morphisms of sites

As mentioned above, the (small) sites $\mathcal{C}$ studied in this thesis are regarded as being internal to some base topos $\mathcal{S}$ with natural numbers object. One of the marvels of topos theory is that these can be studied while pretending that $\mathcal{S}$ is the topos of sets, provided that one only uses intuitionistic reasoning. Since it is by far the easiest approach, I will proceed in this way. An external approach could be given by using the language of $\mathcal{S}$-indexed categories, see for instance chapter B2 of the Elephant [Joh02a], but it would not be justified by the present aims.

A difference between the theory set out here and most of the literature is that the sites in this thesis do not generally have finite limits. With an eye towards the model-theoretic perspective on sites that is laid out in Chapter 3, this would be too much of a restriction. This difference requires caution when adopting existing theory on sites, since most common sources only treat the case where sites have finite limits. An example is the notion of morphism between sites, which is usually only defined only for left exact sites (c.f. definition C2.3.1 in [Joh02b], §1 in [BM87]).

So what is a morphism of sites? In general, one wants a morphism of sites to be a functor between the underlying categories which induces, in a prescribed way, a geometric morphism between the toposes of sheaves. For a given functor $F : \mathcal{C} \to \mathcal{D}$, this should entail at least the following:

(i) The composition

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{S}] \xrightarrow{F^*} [\mathcal{D}^{\mathrm{op}}, \mathcal{S}] \xrightarrow{\mathbf{a}} \mathrm{Sh}(\mathcal{D}) \tag{2.15}$$

preserves finite limits. Here F^* is defined by left Kan extension of $y^{\mathcal{D}} \circ F$ along the Yoneda embedding $y^{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{S}]$ and $\mathbf{a}$ is given by sheafification (Theorem 2.1.5). Since this composition has a right adjoint, where the right adjoint to F^* is given by restriction along $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$, (2.15) is the left adjoint part of a geometric morphism.

(ii) The right adjoint to (2.15) factors through sheaves, i.e. there is a morphism f_* as in in the below, making the diagram commute:

$$\begin{array}{ccc} \mathrm{Sh}(\mathcal{D}) & \overset{f_*}{\dashrightarrow} & \mathrm{Sh}(\mathcal{C}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ [\mathcal{D}^{\mathrm{op}}, \mathcal{S}] & \xrightarrow{F_*} & [\mathcal{C}^{\mathrm{op}}, \mathcal{S}] \end{array} \tag{2.16}$$

Here F_* is the right-adjoint to F^* as in (i). Note that in the situation of the diagram, f_* is automatically right adjoint to the composite:

$$f^* : \mathrm{Sh}(\mathcal{C}) \xrightarrow{i} [\mathcal{C}^{\mathrm{op}}, \mathcal{S}] \xrightarrow{F^*} [\mathcal{D}^{\mathrm{op}}, \mathcal{S}] \xrightarrow{\mathbf{a}} \mathrm{Sh}(\mathcal{D}) \tag{2.17}$$

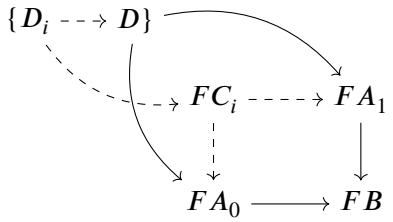
These conditions are respectively achieved by requiring that (i) the functor F is *flat* and (ii) *preserves* covering families, or rather that whenever $\mathcal{U} \in \mathrm{Cov}(C)$ is covering, there exists a $\mathcal{V} \in \mathrm{Cov}(FC)$ such that $\mathcal{V} \preceq F\mathcal{U}$. The accompanying notion of flatness, which, as in (i), does not mention the coverings in $\mathcal{C}$, is then as follows:

Definition 2.2.1. A functor $F : \mathcal{C} \to \mathcal{D}$ between sites is *flat* when:

(i) For every object D of $\mathcal{D}$, there exists a covering family $\{D_i \to D\}$ such that for every i, there exists an object A_i in $\mathcal{C}$ and a morphism $D_i \to FA_i$.

(ii) For every pair of morphisms $a : D \to FA$, $b : D \to FB$, there exists a covering family $\{g_i : D_i \to D\}$ such that for every i, there is an object C_i with maps $f : C_i \to A, h : C_i \to B$ and a morphism $c : D_i \to FC_i$ such that $F(f) \circ c = a \circ g_i$ and $F(h) \circ c = b \circ g_i$.

(iii) For every commutative square as in the solid part of the diagram (2.18), there exists a covering family $\{D_i \to D\}$ such that for every i there exists a span $A_0 \leftarrow C_i \to A_1$ in $\mathcal{C}$ and a morphism $D_i \to FC_i$ making the diagram commute:



(2.18)

Remark 2.2.2. In most sources, a notion of *flat functor* is derived from a standard definition of a flat functor $F : \mathbb{C} \to \mathcal{S}$ from some category to the category of sets. It is important to realize the difference between this definition and 2.2.1 above. Indeed, a functor

$$F : \mathbb{C} \to \mathcal{S} \tag{2.19}$$

is a flat functor in the traditional sense if and only if it is flat in the sense of definition 2.2.1 when $\mathcal{S}$ is construed as a site with the *canonical* (so, jointly epimorphic families) topology. The subtle difference lies in the fact that the object D in (i)-(iii) could be the empty set. However, let me emphasize that it is not the case that there are different notions of flat functors in the literature: one just has to be careful about what the underlying objects are, namely either sites or categories. In Definition 2.2.1 I have chosen to work with the former.

For a site $\mathcal{C}$, the 'canonical' model of $\mathcal{C}$ in $\mathrm{Sh}(\mathcal{C})$ which factors as the Yoneda embedding followed by sheafification is denoted as follows:

$$\epsilon : \mathcal{C} \to \mathrm{Sh}(\mathcal{C}) \tag{2.20}$$

As is well known, ϵ is a morphism of sites when $\mathrm{Sh}(\mathcal{C})$ is equipped with the canonical topology.

From now on, I say that a functor $\mathcal{C} \to \mathcal{D}$ between sites *preserves* covering families if for all $\mathcal{U} \in \mathrm{Cov}(C)$, there exists a $\mathcal{V} \in \mathrm{Cov}(FC)$ such that $\mathcal{V} \preceq F\mathcal{U}$.

Theorem 2.2.3. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a flat functor which preserves covering families. Then the functor* (2.15) *is the left adjoint part of a geometric morphism which also satisfies condition (ii), so that F induces a geometric morphism:*

$$f : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C}) \tag{2.21}$$

Moreover, the left adjoint f^* makes the following diagram commute up to isomorphism:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle\epsilon}\downarrow & & \downarrow{\scriptstyle\epsilon} \\ \mathrm{Sh}(\mathcal{C}) & \xrightarrow{f^*} & \mathrm{Sh}(\mathcal{D}) \end{array} \tag{2.22}$$

Proof. This follows from the standard theory since Definition 2.2.1 means precisely that as a functor $\mathcal{C} \to \mathrm{Sh}(\mathcal{D})$, F is flat in the internal logic of $\mathrm{Sh}(\mathcal{D})$ in the traditional sense, i.e. its category of elements is filtered. So F defines a model of $\mathcal{C}$ in $\mathrm{Sh}(\mathcal{D})$, hence a geometric morphism $\mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$. Commutativity of the diagram follows from the definitions. □

From now on, a *morphism of sites* will be a flat functor which preserves covering families. The following lemma is then the version of the celebrated 'comparison lemma' (due to Grothendieck) appropriate for this thesis. Note it is more general than the one in Proposition 1.2 of [BM87] since the sites need not be left exact.

Theorem 2.2.4 (Comparison lemma)**.** *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a morphism of sites, and assume the following conditions hold:*

(i) If for $\mathcal{U} \in \mathrm{Fam}_{\mathcal{C}}(C)$ *there exists* $\mathcal{V} \in \mathrm{Cov}_{\mathcal{D}}(F(C))$ *such that* $\mathcal{V} \preceq F\mathcal{U}$ *then there exists* $\mathcal{U}' \preceq \mathcal{U}$ *such that* $\mathcal{U}' \in \mathrm{Cov}_{\mathcal{C}}(C)$

(ii) Every object D *of* $\mathcal{D}$ *can be covered by a family of the form* $\{f_i : FA_i \to D\}$

(iii) For any morphism $f : FA \to FB$*, there is a covering family* $\{A_i \to A\}$ *such that all composites* $FA_i \to FA \to FB$ *are of the form* Ff_i *for some* $f_i : A_i \to B$*.*

The properties (i) and (iii) of F *are respectively called co-continuous and locally full. Then the induced geometric morphism* $\mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ *is an equivalence.*

Proof. With the new notion of flatness, this follows from the ordinary comparison lemma (e.g. Theorem C2.2.3 in [Joh02b]) since F carves out a dense subcategory of $\mathcal{D}$ by (ii) and (iii) (see above) and $\mathcal{C}$ carries the induced coverage by (i). □

As should be expected, the canonical model of equation (2.20) satisfies (i)-(iii) of theorem 2.2.4. A detailed reference is the first chapter of Makkai-Reyes [MR77]. Local fullness, for instance, is proved in Lemma 1.3.8.

Remark 2.2.5. During the final draft of this thesis, Caramello has released a preprint [Car19] which addresses the theory of this section in more detail. This preprint starts out with a notion of morphism between sites, which are not assumed to have finite limits, and which seems to agree with what I have presented here.

2.3 Prime objects and prime sites

Suppose $\mathcal{C}$ is any site. An object P of $\mathcal{C}$ is called *prime* (cf. Makkai [Mak82] definition 2.1, Makkai [Mak81], § 1 and Barr-Makkai [BM87], § 3) if whenever $\mathcal{U} \in \mathrm{Cov}_{\mathcal{C}}(P)$, there exists a singleton $\{f\} \in \mathrm{Cov}_{\mathcal{C}}(P)$ such that $\{f\} \preceq \mathcal{U}$. A site $\mathcal{C}$ is *prime-generated* (ibid.) if the full subcategory on prime objects is *dense* (see page 30). A topos $\mathcal{T}$ is *prime-generated* if it is prime-generated as a canonical site.

Proposition 2.3.1. *If a site* $\mathcal{C}$ *is prime-generated, then* $\mathrm{Sh}(\mathcal{C})$ *is prime-generated.*

Proof. Using property (iii) of Theorem 2.2.4 for the canonical embedding $\epsilon : \mathcal{C} \to \mathrm{Sh}(\mathcal{C})$, it is straightforward to verify that every representable sheaf on a prime object (i.e. an object isomorphic to ϵP for a prime object P) is prime. Hence the statement follows. □

A partial converse to the above proposition is given in 2.4.5. First, some more definitions are needed. If $\mathcal{E}$ is a class of arrows in $\mathcal{C}$, then it defines a collection of families:

$$\mathrm{Cov}_{\mathcal{E}}(C) = \{\{f : D \to C\} \mid f \in \mathcal{E}\} \tag{2.23}$$

Definition 2.3.2. A *prime site*, or a *site of prime objects* is a category $\mathcal{C}$ together with a class of arrows $\mathcal{E}$ such that $\mathrm{Cov}_{\mathcal{E}}$ forms a coverage and for every C the set $\mathrm{Cov}_{\mathcal{E}}(C)$ is non-empty.

The following is straightforward:

Proposition 2.3.3. *If $\mathcal{C}$ is a prime site for some class of arrows $\mathcal{E}$, then every object of $\mathcal{C}$ is prime. Conversely, if $\mathcal{C}$ is a site and every object of $\mathcal{C}$ is prime, then there exists a class of arrows $\mathcal{E}$ which turns $\mathcal{C}$ (as a mere category) into a site $\mathcal{C}_{\mathcal{E}}$ such that the inclusion:*

$$\mathcal{C}_{\mathcal{E}} \to \mathcal{C} \tag{2.24}$$

satisfies the conditions of the Comparison lemma (Theorem 2.2.4), i.e. induces an equivalence on sheaves. □

2.4 Factorizations

Recall that for a site $\mathcal{C}$, a *model* of $\mathcal{C}$ in a (bounded) topos $\mathcal{T}$ over $\mathcal{S}$ is a morphism of sites $\mathcal{C} \to \mathcal{T}$ where $\mathcal{T}$ has the canonical coverage. Equivalently, it is a geometric morphism $\mathcal{T} \to \mathrm{Sh}(\mathcal{C})$ over $\mathcal{S}$. When $\mathcal{C}$ is the syntactic site $\mathcal{C}_{\mathbb{T}}$ of some theory $\mathbb{T}$ (see Example 2.4.6 below), a model of $\mathcal{C}$ in $\mathcal{S}$ corresponds to an ordinary set-model of this theory.

In classical model theory, one distinguishes three types of morphisms between models: homomorphisms, embeddings and elementary embeddings. These notions make sense for classical, first-order logic but become conflated or ill-defined when considering, for instance, the coherent fragment of first-order logic, or, as in this thesis, a wide range of logics: regular, coherent, geometric and so on. In Chapter 3, I show how the structure of factorization defined in this section equips a site $\mathcal{C}$ with a notion of 'substructure' that sits in between a homomorphism and an elementary embedding. The nature of $\mathcal{C}$ and the choice of this extra structure determines whether it looks more like one or the other. This extra structure allows for a unified treatment of the Fraïssé-Hrushovski construction from model theory. For the Hrushovski construction in particular, this treatment is new and suggests interesting examples, one of which is worked out in Example 3.3.13 below.

Suppose $\mathcal{C}$ is a category. Recall that an arrow $f : D \to C$ in C is *left orthogonal* to another arrow $g : A \to B$ if every commutative square such as (2.25) has a unique diagonal filler making the two triangles commute:

$$\begin{array}{ccc} D & \longrightarrow & A \\ {\scriptstyle f}\downarrow & \overset{\exists!}{\nearrow} & \downarrow{\scriptstyle g} \\ C & \longrightarrow & B \end{array} \tag{2.25}$$

Equivalently, it is said that g is *right orthogonal* to f.

A common situation is the following:

Definition 2.4.1. Let $\mathcal{C}$ be a category. An orthogonal factorization system is a pair $(\mathcal{E}, \mathcal{M})$ of classes of arrows of $\mathcal{C}$ such that:

(i) $\mathcal{E}$ is the class of those arrows that are left orthogonal to all arrows in $\mathcal{M}$
(ii) $\mathcal{M}$ is the class of those arrows that are right orthogonal to all arrows in $\mathcal{E}$
(iii) Every arrow $f : A \to B$ in $\mathcal{C}$ can be factored as an arrow in $\mathcal{E}$ followed by an arrow in $\mathcal{M}$:

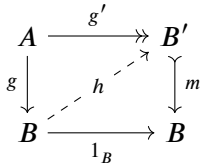
$$f := m \circ e : A \twoheadrightarrow B' \rightarrowtail B \tag{2.26}$$

For a category $\mathcal{C}$ equipped with an orthogonal factorization system, the arrows in $\mathcal{E}$ will be denoted by $\twoheadrightarrow$, and the arrows in $\mathcal{M}$ by $\rightarrowtail$. An orthogonal factorization system is called *standard* when all arrows in $\mathcal{M}$ are monomorphisms.

In this thesis, the following lemma is often used to prove that a given pair $(\mathcal{E}, \mathcal{M})$ is an orthogonal factorization system.

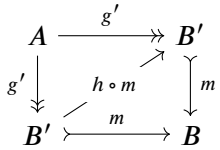
Lemma 2.4.2. *Let $\mathcal{C}$ be a category. Suppose $\mathcal{E}$ is a class of arrows in $\mathcal{C}$ closed under composition with isomorphisms, and let $\mathcal{M} = \mathcal{E}^{\pitchfork}$ be the class of arrows which are right-orthogonal to all arrows in $\mathcal{E}$. Then if every arrow $f : A \to B$ in $\mathcal{C}$ factors as an arrow in $\mathcal{E}$ follows by an arrow in $\mathcal{M}$, the pair $(\mathcal{E}, \mathcal{M})$ is an orthogonal factorization system on $\mathcal{C}$.*

Proof. Only condition (i) of Definition 2.4.1 needs to be verified. So suppose $g : A \to B$ is an arrow which is left-orthogonal to every arrow in $\mathcal{M}$. It needs to be shown that $g \in \mathcal{E}$. By (iii), g can be factored as in the following diagram:



(2.27)

By the assumption on g, there exists a unique factoring morphism h as drawn, such that $m \circ h = 1_B$. Since $g' \in \mathcal{E}$, there is a unique diagonal for the square:



(2.28)

Since $h \circ m$ and $1_{B'}$ both define a filler for this square, it follows that $h \circ m = 1_{B'}$. Hence, m is an isomorphism and $g = m \circ g' \in \mathcal{E}$. □

Note that the method used to prove the preceding lemma can also be used to show that the factorization (2.26) is always unique up to isomorphism.

In this thesis, the typical example of Definition 2.4.1 is a *regular* category, where $\mathcal{E}$ is the class of regular epimorphisms and $\mathcal{M}$ is the class of monomorphisms. Regular categories often appear when studying 'small' models of theories in logic, such as finitely presented models. When this category of models is rich enough, it is often co-regular, i.e. its opposite category is regular. However, this is not always the case. Below I will argue that there are interesting examples where this opposite is not regular, but does admit an orthogonal factorization system. Retaining precisely this structure, in addition to the structure of a *site*, which abstracts from another property of regular categories, enables a unified treatment of some famous constructions in model theory. This is further discussed in Chapter 3. For other examples and a more general introduction to orthogonal factorization systems, see for instance [Bou77] or Chapter 14 of [AHS90].

In this thesis, factorization systems are combined with coverages:

Definition 2.4.3. A site $\mathcal{C}$ is *factorizing* if the set

$$\mathcal{E} = \{e : D \to C \mid \{e\} \in \mathrm{Cov}_{\mathcal{C}}(C)\} \tag{2.29}$$

forms the left class of an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$. A factorizing site is called *standard* if this orthogonal factorization system is standard.

Note that for a (factorizing) *prime* site (Definition 2.3.2), the definition above can be alternatively formulated as a condition on an orthogonal factorization system, i.e. one where the left class satisfies the coverage condition.

For a factorizing site, the class $\mathcal{E}$ is closed under composition. This has the following consequence:

Lemma 2.4.4. *Suppose P is a prime object in a factorizing site $\mathcal{C}$. If $f : P \twoheadrightarrow E$ is an arrow in $\mathcal{E}$, then E is also prime.*

Proof. If $\mathcal{U} \in \mathrm{Cov}_{\mathcal{C}}(E)$, then by the coverage condition, there exists a cover $\mathcal{V}$ on P such that $f_!\mathcal{V} \preceq \mathcal{U}$. Since P is prime, there is $g \in \mathcal{E}$ such that $\{g\} \preceq \mathcal{V}$. So it follows that $f_!\{g\} = \{f \circ g\} \preceq \mathcal{U}$, but since $f \circ g \in \mathcal{E}$, it follows that E is prime. □

For standard factorizing sites there is a converse to Proposition 2.3.1. This is stated in the following proposition, which extends Proposition 3.2 in Barr-Makkai [BM87] to sites which may not be regular.

Proposition 2.4.5. *Suppose $\mathcal{C}$ is a standard factorizing site, and assume* $\mathrm{Sh}(\mathcal{C})$ *is prime-generated. Then $\mathcal{C}$ is prime-generated.*

Proof. Recall the representable sheaf functor ϵ from (2.20).

Let C be an object of $\mathcal{C}$. Since $\mathrm{Sh}(\mathcal{C})$ is prime-generated, ϵC can be covered by primes. By epi-mono factorization and Lemma 2.4.4, it can be assumed that such a cover consists of monomorphisms $f_i : E_i \rightarrowtail \epsilon C$, with E_i prime. Further, there exists for each i an object C_i in $\mathcal{C}$ a single covering arrow $\epsilon C_i \twoheadrightarrow E_i$, such that the composites $\epsilon C_i \twoheadrightarrow E_i \rightarrowtail \epsilon C$ are the images $\epsilon(g_i)$ family of arrows $g_i : C_i \to C$: this holds since ϵ is locally full (condition (iii) of Theorem 2.2.4). Using factorization, each of these can be factored as in (2.30)

$$g_i : C_i \twoheadrightarrow C_i' \rightarrowtail C \tag{2.30}$$

Now since ϵ preserves monomorphisms, and the arrows in $\mathcal{M}$ are monomorphisms, this results in a regular epi-mono factorization in $\mathrm{Sh}(\mathcal{C})$:

$$\epsilon(g_i) : \epsilon C_i \twoheadrightarrow \epsilon C_i' \rightarrowtail \epsilon C \tag{2.31}$$

Therefore one must have $\epsilon C_i' \cong E_i$. Consequently, each C_i' is prime, and C, which was arbitrary, is covered by them. □

The following example generates a class of very common examples of sites that are factorizing. On their own they do not justify the generalized (non left-exact) setting of the above, but in Chapter 4 I will show that categories of models of such sites naturally give rise to other factorizing sites which need not be left exact. For details on the construction of syntactic categories, see for instance D1.4 of the Elephant [Joh02b].

Example 2.4.6. For any theory $\mathbb{T}$ that is at least *regular* (e.g. regular, coherent, geometric), the syntactic category $\mathcal{C}_\mathbb{T}$ corresponding to that theory admits a regular epi - mono orthogonal factorization system, and all regular epis are covers. Hence every such category is 'canonically' equipped with the structure of a standard factorizing site.

The following two examples are instances of both definition 2.4.3 and 2.3.2, so they are factorizing prime sites. In some sense these two span the range of factorizing prime sites that are covered in this thesis.

Example 2.4.7. If $\mathcal{C}$ is any category, one can choose an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ consists only of isomorphisms, and $\mathcal{M}$ of all arrows. Since the arrows in $\mathcal{E}$ also satisfy the covering conditions of Definition 2.3.2, this constitutes a factorizing prime site. The corresponding topos of sheaves is the presheaf topos $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$.

Example 2.4.8. If $\mathcal{C}$ is a category satisfying the *right Ore condition*, i.e. every cospan can be completed to a commutative square, then $\mathcal{C}$ can be made into a prime site by defining every arrow

to be covering. By taking the class of isomorphisms as the right class $\mathcal{M}$, this yields a factorizing prime site. Without the, in this case trivial, orthogonal factorization, such sites are called *atomic* and generate the class of *atomic* toposes, which forms a subclass of prime-generated toposes, all of which are boolean.

This thesis covers a wide range of examples within these two bounds, both common and obscure. A common, straightforward example is the syntactic category $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ (cf. Example 2.4.6) when the theory $\mathbb{T}$ is *regular*.

Example 2.4.9. For a regular theory $\mathbb{T}$, $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ is an orthogonally factorizing site as in example 2.4.6, but in addition the regular epimorphisms generate all covers, and clearly satisfy the conditions of 2.3.2. So $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ is always a factorizing prime site.

Lastly, the following example from Chapter 1 emphasizes the two main new ideas in this chapter, namely that sites need not have finite limits, and that an orthogonal factorization system is additional structure on a site:

Example 2.4.10. The simplex category Δ (see example 1.2.10) admits an orthogonal factorization system consisting of a class $\mathcal{E}$ of surjections (between partial orders) and injections $\mathcal{M}$. Moreover, the class $\mathcal{E}$ satisfies the conditions of Definition 2.3.2. In this way, Δ is a *standard* factorizing prime site. But since every arrow in $\mathcal{E}$ splits, its category of sheaves is just the presheaf topos $[\Delta^{\mathrm{op}}, \mathcal{S}]$, the category of simplicial sets. Observe how this differs from Example 2.4.7, where the orthogonal factorization was not standard.

2.5 Types and prime filters

The purpose of this section is to discuss Makkai's 'prime completion' in the context of factorizing sites. It is mostly intended for those who know about this construction and may naturally wonder how it could apply here. I will point out that this construction has some limitations and that the 'topos of orbits' construction discussed in Chapter 4 is in some sense more fundamental. This section can be safely skipped if the reader feels so inclined. Some definitions, like that of prime filters and types, can be referred to later.

In Chapter 4 it will be shown how to construct factorizing *prime* sites from factorizing sites $\mathcal{C}$ using certain categories of models. I have called this construction 'the topos of orbits' (see e.g. Proposition 4.1.11 and Lemma 4.1.12). The 'prime completion' due to Makkai [Mak81] mentioned above aims to 'complete' a (regular) site $\mathcal{C}$ to a prime site $\mathcal{P}(\mathcal{C})$ which enjoys a certain universal property. The defined topos is called the 'topos of types'. Although this construction is syntactic, it relies on very strong, non-constructive model-theoretic results to show that it defines a

prime site (cf. (1.5) in [Mak81])). Indeed, using these results he shows in Theorem 2.3 that his prime-completed topos (but in fact something stronger[2] holds for the site $\mathcal{P}(\mathcal{C})$) is equivalent to a topos of finitely supported **M**-sets. In the language that should make sense by Chapter 4, this means that one cannot do the topos of types construction if there is not already a sufficiently 'complete' (in the model-theoretic sense) category of 'saturated' models (Proposition 2.2 in [Mak81]). Hence, the topos of types construction is, at least for Makkai's approach, a special case of the topos of orbits construction that I put forward in Chapter 4. A better means to the topos of types construction is given by Coumans [Cou12], who, for coherent sites, defines the topos of types as the topos of sheaves on an internal locale in $\mathrm{Sh}(\mathcal{C})$ and thereby circumvents the model theory. It remains to be seen whether this construction can be adapted to general factorizing sites, something that would be desirable for this thesis although not quite germane to the main theme. About this I make some more remarks at the end of this chapter.

Definition 2.5.1. Suppose $\mathcal{C}$ is a site with orthogonal factorization $(\mathcal{E}, \mathcal{M})$. For an object C of $\mathcal{C}$, a *prime filter* on C is a subset $\rho \subset \mathcal{M}$ of arrows with codomain C satisfying the properties (i)-(iv) below.

(i) (Non-empty) Every isomorphism is in ρ
(ii) (Upward closed) if $q \circ p \in \rho$ then $q \in \rho$.
(iii) (Filter) Whenever $p, q \in \rho$, there exists a commutative square $p \circ r_1 = q \circ r_2$ such that the diagonal is in ρ, that is $p \circ r_1 \in \rho$.
(iv) (Prime) When $p : P \rightarrowtail C \in \rho$ and $f_i : U_i \rightarrowtail P$ is a covering family, then there exists i such that $f_i \circ p \in \rho$.

Observe that property (iv) also ensures that a prime filter can not contain a morphism $P \to C$ if P is covered by the empty family.

Example 2.5.2 (Principal filters)**.** Suppose C is an object of $\mathcal{C}$ and $m : A \rightarrowtail C$ is an arrow in $\mathcal{M}$ where A is prime. Then the upward closure (viz. under condition (ii) of 2.5.1) of the set of all morphisms $m' : A' \rightarrowtail C$ which factor through m via a cover (hence an isomorphism) defines a prime filter on C. A prime filter of this type is called *principal* and m is called its *generator*. It will be noted by $\langle m \rangle$ or $\langle A \rangle$ in the below.

Example 2.5.3 (Types)**.** Suppose $\mathcal{C}$ is a site with orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ and let $M : \mathcal{C} \to \mathcal{S}$ be a model of $\mathcal{C}$. Then for every object $c \in M(C)$, the *type* $\tau(c)$ of c is defined as the following family of arrows with codomain C:

$$\tau(c) := \{m : A \rightarrowtail C \mid m \in \mathcal{M} \text{ and } M(m)(a) = c \text{ for some } a \in M(A)\} \tag{2.32}$$

[2] Namely that there is a morphism of sites, satisfying (i)-(iii) of Theorem 2.2.4, between the site of orbits and P(C).

It is straightforward to check that $\tau(c)$ is a prime filter on $\mathcal{C}$.

Makkai's prime completion consists in an effort to turn prime filters into a category and subsequently into a site. As I will argue, there are some problems with this, specific to the usage of *prime* filters. Sites of (non-prime) filters have been studied by for instance Moerdijk [Moe95] and Pitts [Pit83]. The following construction of the category of prime filters is similar to these sources, although absence of finite limits requires a different notion of morphism that I introduce below.

The following is an adaptation of Makkai's definitions [Mak81]. For ρ a prime filter on C and D an object of $\mathcal{C}$, a *germ* of morphisms $(C,\rho) \to D$ is given by a $\sim$-equivalence class of pairs (f,r) where $r \in \rho$ and $f : \operatorname{dom} r \to D$. The equivalence $(f,r) \sim (g,q)$ holds when there exists a commutative square $r \circ p_1 = q \circ p_2 \in \rho$ such that $f \circ p_1 = g \circ p_2$. An arrow $r \circ p \in \rho$ is called a *refinement* of r in ρ. Therefore one can alternatively say that $(f,r) \sim (g,q)$ if f, g agree on a common refinement of p and q. One can readily check that this is an equivalence relation. A germ of morphisms represented by (f,r) is denoted by $[(f,r)]$, and the set of germs of functions $(C,\rho) \to D$ is denoted by $\mathrm{Germ}((C,\rho), D)$. A germ of morphisms $(C,\rho) \to D$ for which there exists a representation by a function $f : C \to D$ defined on the whole of C is called *total*, and the set of total germs is denoted $\mathrm{Germ}_{\mathrm{tot}}((C,\rho), D)$.

Lemma 2.5.4. *Suppose* $[(f,r)] : (C,\rho) \to D$ *is a germ of morphisms. Then the image* (2.33) *is a prime filter on* D*, where* $\uparrow$ *denotes the upward closure, i.e. the closure under condition (ii) of 2.5.1.*

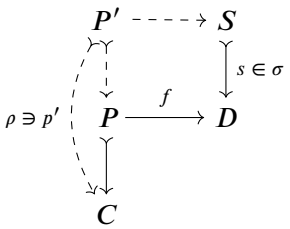
$$f \cdot \rho :=\uparrow \{\mathrm{im}(f' \circ q) \rightarrowtail D \mid q : Q \rightarrowtail \operatorname{dom} r,\ r \circ q \in \rho\} \tag{2.33}$$

Proof. Straightforward verification. □

Definition 2.5.5. For a factorizing site $\mathcal{C}$, the *category of prime filters* $\mathcal{P}(\mathcal{C})$ is defined as follows:

(i) Objects are pairs (C,ρ) where ρ is a prime filter on C.

(ii) A morphism $(C,\rho) \to (D,\sigma)$ is a germ of morphisms $[(f,p)] : (X,\rho) \to D$ satisfying the condition that for every s in σ there exists $p' \in \rho$ factoring through p such that the composition $P' \to D$ factors through S as in the diagram below.



(2.34)

One can verify that this condition does not depend on the choice of representative.

(iii) Two germs $[(f,p)] : (C,\rho) \to (D,\sigma)$, $[(g,q)] : (D,\sigma) \to (E,\tau)$ can be composed in the obvious way, since we can assume that the representative f factors through q - then the resulting composition with g yields a composed arrow $[(g,q)] \circ [(f,p)] : (C,\rho) \to (E,\tau)$.

A morphism $[(f,r)] : (C,\rho) \to (D,\sigma)$ in $\mathcal{P}(\mathcal{C})$ is a *cover* if for every $p : P \rightarrowtail \operatorname{dom} r$ such that $r \circ p \in \rho$, we have $\operatorname{im}(f \circ p) \in \sigma$. It follows from property (ii) of prime filters that this holds for every representative f whenever it holds for one, so again it is a property of $[(f,r)]$ (note, this uses the orthogonal factorization system). This set of covers on $\mathcal{P}(\mathcal{C})$ is denoted by $\mathcal{E}_{\mathcal{P}(\mathcal{C})}$.

Lastly, $\mathcal{P}(\mathcal{C})$ can be restricted to its subcategory on *total* morphisms, denoted $\mathcal{P}_{\mathrm{tot}}(\mathcal{C})$.

Note that it is not the case (in general) that the class of covers in $\mathcal{P}(\mathcal{C})$ form a coverage. As noted in the beginning of section 2.5, this fact relies on non-constructive results from classical model theory, most notably Gödel's completeness theorem and the construction of saturated models (see (1.3) in Makkai [Mak81]). Of course it could be completed to a 'least coverage' containing all defined covers in a manner similar to the induced coverage of Section 2.1. About this coverage almost nothing can be known constructively, but it makes it possible to talk about sheaves as for instance in Proposition 2.5.8 below (but not about $\mathcal{P}(\mathcal{C})$ as a 'prime completion', because it is not clear if $\mathcal{P}(\mathcal{C})$ is prime-generated). The generating class of covers does have some nice properties, such as the following. The right-class $\mathcal{M}$ consists of morphisms that Moerdijk [Moe95] calls 'embeddings' in his sites of filters.

Proposition 2.5.6. *Covers in $\mathcal{P}(\mathcal{C})$ form the left class of an orthogonal factorization system, where the right class $\mathcal{M}_{\mathcal{P}(\mathcal{C})}$ consists of the pairs $[(f,r)]$ s.t. f admits a restriction, along an element in ρ, to an arrow in $\mathcal{M}$.*

Proof. Observe that the definition of $\mathcal{M}_{\mathcal{P}(\mathcal{C})}$ does not depend on the choice of representative.

Suppose $[(f,r)] : (C,\rho) \to (D,\sigma)$ is a morphism in $\mathcal{P}(\mathcal{C})$. Let ρ' be the restriction of ρ along r. Then $(D, f \cdot \rho')$ is a prime filter on D by lemma 2.5.4 and $[(f,r)]$ factors through it as a cover followed by an arrow in $\mathcal{M}_{\mathcal{P}(\mathcal{C})}$

It remains to show that $(\mathcal{E}_{\mathcal{P}(\mathcal{C})}, \mathcal{M}_{\mathcal{P}(\mathcal{C})})$ is an orthogonal factorization system. So suppose (2.35) is a given square with vertical arrows in the respective classes. From the properties of prime filters, one can produce the diagram (2.36) in $\mathcal{C}$, where the objects indicated by $(-)'$ are in the corresponding prime filters.

$$\begin{array}{ccc} (F,\varpi) & \longrightarrow & (D,\sigma) \\ \downarrow & & \downarrow \\ (E,\tau) & \longrightarrow & (C,\rho) \end{array} \qquad (2.35)$$

$$\begin{array}{ccc} F' & \longrightarrow & D' \\ & \searrow \quad \overset{\exists!}{\nearrow} & \\ \downarrow & G & \downarrow \\ & \swarrow_{\in \tau} & \\ E' & \longrightarrow & C \end{array} \qquad (2.36)$$

The left arrow factors as indicated, image in τ. By the factorization in $\mathcal{C}$ there exists a unique $G \to D'$ as drawn, which defines a morphism $(E, \tau) \to (D, \sigma)$: indeed any further restriction $D'' \rightarrowtail D'$ in σ just produces another diagram like 2.36, factoring through the original one.

Uniqueness of the constructed diagonal $(E, \tau) \to (D, \sigma)$ is easy using the definition of a germ and uniqueness of the lifts for $(\mathcal{E}, \mathcal{M})$. □

Having defined $\mathcal{P}(\mathcal{C})$ as a category, the sets $\mathrm{Germ}((C, \rho), D)$ of germs of morphisms form a profunctor $\mathrm{Germ} : \mathcal{P}(\mathcal{C}) \nrightarrow \mathcal{C}$:

$$\mathrm{Germ} : \mathcal{P}(\mathcal{C})^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set} \tag{2.37}$$

with a transpose:

$$G : \mathcal{C} \to [\mathcal{P}(\mathcal{C})^{\mathrm{op}}, \mathrm{Set}] \tag{2.38}$$

Proposition 2.5.7. *The functor G of* (2.38) *is a morphism of sites, where* $[\mathcal{P}(\mathcal{C})^{\mathrm{op}}, \mathrm{Set}]$ *carries the canonical topology.*

Hence there is an induced geometric morphism:

$$g : [\mathcal{P}(\mathcal{C})^{\mathrm{op}}, \mathrm{Set}] \to \mathrm{Sh}(\mathcal{C}) \tag{2.39}$$

Proof. Flatness is an easy check by making repeated use of the property (iii) of definition 2.5.1 and the fact that we can restrict prime filters to a prime filter on any of its elements.

Suppose $\{f_i : C_i \twoheadrightarrow D\}$ is a covering family in $\mathcal{C}$. Then for a germ of morphisms $(E, \sigma) \to D$, there is some representative $g : A \to D$ defined on a domain $A \rightarrowtail E$ in σ.

By the covering condition, there exists a covering family $\{h_j : A_j \rightarrowtail A\}$ on A such that each $g \circ h_j$ factors through f_i for some i. By (iv) of definition 2.5.1, there exists j such that A_j is in σ. By restricting the domain to A_j, the germ $[g] : (E, \sigma) \to D$ factors through the $C_i \to D$ corresponding to A_j.

So $\{\mathrm{Germ}(-, f_i) : \mathrm{Germ}(-, C_i) \to \mathrm{Germ}(-, D)\}$ is a point-wise epimorphic family, which proves the claim. □

Similarly, there is also a functor

$$G_{\mathrm{tot}} : \mathcal{C} \to [\mathcal{P}_{\mathrm{tot}}(\mathcal{C})^{\mathrm{op}}, \mathrm{Set}] \tag{2.40}$$

The decision in Makkai [Mak81] to work with non-total morphisms in $\mathcal{P}(\mathcal{C})$ has to do with the fact that (2.40) is not flat in general, or in Makkai's case, does not preserve finite limits. However, this concern is unnecessary: upon completing $\mathcal{P}_{\mathrm{tot}}(\mathcal{C})$ to a *site*, so that covers satisfy the coverage condition, one can show similarly to Proposition 2.5.7 that the composition

$$\mathcal{C} \xrightarrow{G_{\mathrm{tot}}} [\mathcal{P}_{\mathrm{tot}}(\mathcal{C})^{\mathrm{op}}, \mathrm{Set}] \xrightarrow{\mathbf{a}} \mathrm{Sh}(\mathcal{P}_{\mathrm{tot}}(\mathcal{C})) \cong \mathrm{Sh}(\mathcal{P}(\mathcal{C})) \tag{2.41}$$

is indeed flat and continuous. From Definition 2.5.5 it is not hard to see that the inclusion $\mathcal{P}_{\mathrm{tot}}(\mathcal{C}) \hookrightarrow \mathcal{P}(\mathcal{C})$ is dense already for the generating class of covers. One can use this to adapt the proof of Proposition 2.5.7 to:

Proposition 2.5.8. *The functor* (2.41) *is a morphism of sites and hence induces a geometric morphism*

$$\mathrm{Sh}(\mathcal{P}_{\mathrm{tot}}(\mathcal{C})) \to \mathrm{Sh}(\mathcal{C}) \tag{2.42}$$

□

2.5.1 The prime completion?

Unfortunately, this is as much as I am able to say about the $\mathcal{P}(-)$ construction. To prove anything more, for instance that in some cases it yields a 'prime completion' in some sense and classifies a certain class of $\mathcal{C}$-models, much more assumptions on $\mathcal{C}$ are needed in addition to completeness results. Besides that the site structure of $\mathcal{P}(\mathcal{C})$ is very difficult, the construction itself does not have good categorical properties. For instance, it does not seem to extend to a functor on sites and morphisms of sites. Its use here rather appears to be restricted to a specific choice of site. For instance, it provides an alternative way to think about prime objects in a prime-generated site:

Proposition 2.5.9. *Consider the functor G in* (2.38)*. If C is prime, then $G(C)$ is representable. When $\mathcal{C}$ is prime-generated, the converse holds as well.*

Proof. The first claim is easy, since if C is prime, it generates a trivial prime filter on itself (Example 2.5.2), representing $G(C)$.

For the converse, suppose $G(C)$ is represented by (D, σ). Then the identity on (D, σ) corresponds to a germ of morphisms $(D, \sigma) \to C$ through which every other germ of morphisms uniquely factors. Considering its image as in Lemma 2.5.4, it follows that (D, σ) is isomorphic to a prime filter (C, σ') on C.

Now let $\{a_i : A_i \to C\}$ be a cover of C by primes. Each element of this cover induces a germ of morphisms $\langle A_i \rangle \to C$, and hence each of them factors through (C, σ'); but since the filters $\langle A_i \rangle$ are principal, each a_i factors through $\lim_{\leftarrow} \sigma'$ in $\mathrm{Sh}(\mathcal{C})$:

$$a_i : A_i \to \lim_{\leftarrow} \sigma' \to C \tag{2.43}$$

Since this family is a cover, it follows that $\lim_{\leftarrow} \sigma' \to C$ is a cover. But then σ' consists only of isomorphisms, from which it follows that C is prime. □

The paper by Coumans [Cou12] gives more hope of making the prime completion functorial: for coherent sites, this can be inferred from theorem 26 in that paper. In general it suggests to look

not at categories of prime filters, but categories of downward-closed, join irreducible sets of prime filters, ordered by reverse inclusion.

3 Homogeneous models

3.1 Introduction

The first part of this chapter aims to arrive at prime-generated sites from a point of departure inside classical model theory. This point is centred on the notion of *age* of a structure and the accompanying notion of *homogeneous structure*. These notions were introduced by Roland Fraïssé and have become an important topic in the more than half a century that has followed. In particular a small, but powerful, extension of Fraïssé's theory by Hrushovski [Hru93] has been used to disprove several long-standing model-theoretic conjectures and has since become an important tool in stable model theory [Wag94]. Outside of model theory, applications are to be found in group theory and combinatorics, for instance in infinite permutation groups and Jordan groups [Bha+97] [Mac94]. A good recent survey of what has been done since Fraïssé's seminal work from the 1950s is given by Macpherson [Mac11].

Every mathematical field emphasizes certain phenomena and conceals other ones. It is therefore important to think first about what the phenomena are that model theory cares about and that one likes to see reflected in a categorical theory. The basic concept in classical model theory is that of a set equipped with a *relation* on its elements, such as the rational numbers $\mathbb{Q}$ equipped with their natural order. Everything else, from formulas to sentences and theories, indeed *syntax*, is auxiliary for model theorists, or at least for the school of model theorists that follows the philosophy of Roland Fraïssé. In short, model theory focuses on semantics rather than syntax: it starts with the models themselves.

Recall that in Section 1.1, the starting data was a mere set M and a submonoid $\mathbf{M} \subseteq \mathrm{Set}(M, M)$ Similarly in Section 3.3 below, I start the investigation with a concrete category of models. As in Chapter 1, these models are not presented as models of some underlying theory, but rather given immediately. This raises the question of what structure there can be to work with when these models have no syntactical attributes. This comes down to the mere notion of *finite substructure*.

By contrast, the syntactic approach to categorical logic starts with a (predicate) logic and builds a category reflecting this logical structure. When the logic contains equality and conjunctions, this is reflected in the presence of equalizers and finite products. Starting from semantics, one should therefore expect something else. In Section 3.3 it is made apparent that the notion of model and finite substructure is reflected in the notion of a factorizing prime site.

This is further tested in Section 3.5 for the Fraïssé-Hrushovski theory on homogeneous structures. I will show there that it is possible to develop this theory abstractly using the notion of a factorizing prime (-generated) site. This is built on the observation in Section 3.4 which asserts that a factorizing site can be seen as a site equipped with a notion of embedding for its models. Had one started with a syntactical description of these models for some logic (regular, geometric, first-order), then this notion specializes to the familiar notions from classical logic.

The discussion of the Fraïssé-Hrushovski construction in the categorical setting draws from previous literature on this topic. Building on earlier work [DG92], the work of Caramello [Car14] is probably the closest to what is presented in this thesis, since it also adopts a topos-theoretic perspective. Other recent work is Kubiś[Kub14], from which I have mostly drawn the analysis on cofinal sequences that build a homogeneous structure. However, these sources do not cover most of the examples of chapter one. Indeed, the former only concerns *atomic toposes*. Furthermore, they do not home deeper into the extended construction due to Hrushovski. As I point out, the structure of 'age', or class of embedded substructures fits, for a typical example of the Hrushovski construction, the mould of a factorizing prime site, but not that of an atomic site. The example is special since all arrows in this site are epimorphisms, yet not all of them are covers. For this example, Theorem 4.5.4 below will show that also the induced topos is non-atomic, so the site is not even equivalent to an atomic site.

Hence this shows that there are important examples of the Hrushovski construction which are not understood using atomic sites. The main novelty here is that, in model-theoretic terms, the amalgamation condition should be replaced by a non-symmetric *forcing condition*, which is dual to the coverage condition for prime sites. Cameron and Nešetřil [CN06] seem to partially realize this by presenting an asymmetric amalgamation property in a setting where every arrow is a monomorphism, but they do not mention the analogy with the Hrushovski construction. The theory presented in this chapter covers this theory as well. In the special case when the traditional amalgamation condition is satisfied, the material below also presents a categorical account of the Fraïssé theory of atomic or boolean toposes. The example taken from the Hrushovski construction appears between these two and can be presented as a boolean cover of a non-atomic topos.

3.2 Homogeneous first-order structures

3.2.1 Structures and models

I start the content of this chapter by giving the picture of both Fraïssé's and Hrushovski's theory as in the standard literature. I should emphasize that this is a cursory picture, intended to situate the content inside classical model theory, but not to give a self-contained account. This means that I do not go into the many subtleties that arise when defining these very foundational structures.

The standard picture of model-theoretic structure starts with a set Σ of function and relation symbols of a certain arity (and, in the many-sorted case, a type) as well as constant symbols (or 0-ary functions). Such a set is called a *signature*. A *structure* for Σ is a set M which interprets each symbol of the signature for what it signifies, e.g. a function, relation or constant of the correct type and arity.

Definition 3.2.1. For M, N two structures of a certain signature Σ, a *homomorphism* is a function $f : M \to N$ which preserves the interpretation of every function, constant and relation symbol, e.g.:

$$\text{If } R^M(m_1, \dots, m_n), \text{ then } R^N(fm_1, \dots, fm_n) \tag{3.1}$$

when R^M and R^N are the interpretations of some relation symbol R in M, N respectively. The same holds for n-ary function symbols (viewed as $n+1$-ary relations) and constants.

A homomorphism $f : M \to N$ is called an *embedding* if it is injective and the converse direction of (3.1) holds for all symbols in the signature. To distinguish embeddings from homomorphisms $M \to N$, they are usually denoted $M \rightarrowtail N$.

On top of signatures, first-order logic adds a language, consisting of *formulas* $\phi(\mathbf{x})$, theories and a system for the interpretation of these in structures. There are many different types of such systems that one may wish to consider. When I distinguish different languages and theories, I am referring to those distinguished in chapter D1 of Johnstone's Elephant [Joh02b], such as regular, coherent and geometric theories. As to the underlying system, I stick to the Gentzen-style sequent calculi that are also used by Johnstone, rather than the more classical Hilbert-style axiomatization common in model theory. This leads to some subtle differences, for instance that in general, I will allow for empty structures when the theory permits this. Indeed, this is also common practice among the more semantics-oriented model theorists, such as Hodges [Hod93].

I should emphasize that none of this really matters from Section 3.3 onwards. There, the starting data is completely void of syntax and only specifies a *category* of models. The 'structure' is whatever is 'preserved' by the morphisms in this category. Moreover, starting in Section 3.4, the underlying signature and language is represented by a (factorizing) *site* $\mathcal{C}$, which naturally also covers the multi-sorted case. All classical theory is just intended for demonstrative purposes.

As a last remark, note that this does not remove the language-dependence of the notions of homomorphism and embeddings. Recall that one can always add to a given signature $\{R_1, \dots, R_n\}$ a set of symbols $\overline{R}_1, \dots, \overline{R}_n$ which are interpreted as the complements of each respective relation. In doing so, every homomorphism becomes an embedding as per Definition 3.2.1. One can even go further and add symbols $C_\phi(\mathbf{x})$, $D_\phi(\mathbf{x})$ for every *formula* $\phi(\mathbf{x})$, formally representing $\phi(\mathbf{x})$ and $\neg\phi(\mathbf{x})$. As a consequence every homomorphism will be an *elementary embedding*, which preserves and reflects every formula. This is related to a method in which first-order logic can be interpreted in coherent logic, called *Morleyization* (see e.g. [Joh02b] , lemma D1.5.13). Analogously, the constructions presented in this chapter depend on a choice of underlying site. For now, I leave this discussion of language or site dependence aside. An interesting research topic would be to study the above constructions on languages in the abstract setting below, since they are connected to important model-theoretic concepts such as *model-completions* and *model-companions*. A good starting point for this is Macintyre [Mac77].

3.2.2 Classical homogeneity

Definition 3.2.2 below defines homogeneous structures as in standard works on model theory, such as Hodges [Hod93]. It is also the definition maintained by Macpherson [Mac11]. Following this notion, I define Fraïssé classes and state Fraïssé's theorem. A comparison with an example from Chapter 1 (Example 3.2.7), from which it becomes clear that the classical notion of homogeneity does not adequately explain the phenomena observed there, carries the investigation forward.

Next, I state the modified version of Fraïssé's construction that has become known as the Hrushovski construction. This provides further motivation for generalizing homogeneity to a wider categorical setting. Although it is not the object of this thesis to fully integrate all known instances of Hrushovski's construction in the new theory, a typical example gives some hope that such a project might be possible.

Definition 3.2.2 (e.g. [Hod93], §7.1)**.** A structure M is *ultrahomogeneous* if whenever $\mathbf{a}$, $\mathbf{b}$ are tuples of the same length which satisfy the same atomic formulas (that is, the mapping $\mathbf{a} \mapsto \mathbf{b}$ is an isomorphism of substructures), then there exists an automorphism $f : M \to M$ such that $f\mathbf{a} = \mathbf{b}$.

Recall that for a tuple of elements $\mathbf{a}$ of a structure M, the first-order *type* of $\mathbf{a}$ is the set $\mathrm{tp}_M(\mathbf{a})$ of all first-order formulas $\phi(\mathbf{x})$ for which $\phi(\mathbf{a})$ is satisfied in M. One can show by induction on the number of quantifiers in a formula that if $f : M \to M$ is an automorphism, then:

$$\mathrm{tp}_M(\mathbf{a}) = \mathrm{tp}_M(f\mathbf{a}).$$

Therefore it follows that if M is an ultrahomogeneous structure and $\mathbf{a}$, $\mathbf{b}$ are tuples that satisfy the same atomic formulas, then they have the same type. This can be regarded as a form of

quantifier elimination for M. This observation can be used to motivate a more general definition of homogeneity:

Definition 3.2.3 (e.g. 'strongly $\aleph_0$-homogeneous' [Hod93], Chapter 10)**.** A structure M is *strongly homogeneous* if whenever $\mathbf{a}$, $\mathbf{b}$ are tuples of the same length with the same first-order types:

$$\mathrm{tp}_M(\mathbf{a}) = \mathrm{tp}_M(\mathbf{b}),$$

then there exists an automorphism $f : M \to M$ such that $f\mathbf{a} = \mathbf{b}$.

Clearly every ultrahomogeneous structure is strongly homogeneous. For an example of a structure which is strongly homogeneous but not ultrahomogeneous, one can take the linear order $(\mathbb{Z}, <)$. The Fraïssé construction as commonly found in the literature concerns ultrahomogeneous structures, and will be discussed in the rest of this section. However, in this thesis it is a relaxation of definition 3.2.3 to endomorphisms that is generalized by the categorical Fraïssé construction of Section 3.5. For boolean, finitary geometric theories $\mathbb{T}$, this result yields a Fraïssé construction for Definition 3.2.3 by applying this construction to a prime-generated sub-site of the syntactic site $\mathcal{C}_{\mathbb{T}}$.

I now return to ultrahomogeneous structures in the following discussion of the Fraïssé construction as it is found in the literature. For a *countable* structure M, the *age* $\mathfrak{A}(M)$ of M (e.g. [Mac11] § 2) is defined as the set of all finite substructures of M, i.e. finite subsets with the inherited structure. If the signature contains function symbols, one should rather take *finitely generated* substructures. Again, these are subtleties that will not arise with the categorical language that starts in Section 3.4.

Definition 3.2.4. For a given signature, a *Fraïssé class* is a countable set $\mathfrak{A}$ of finite structures such that the following conditions hold:

(i) The set $\mathfrak{A}$ is closed under finite substructures

(ii) For finite structures $A, B \in \mathfrak{A}$, there is a set $C \in \mathfrak{A}$ together with *embeddings* $i_A : A \to C$, $i_B : B \to C$ i.e. homomorphisms preserving and reflecting atomic formulas:

$$\begin{array}{ccc} & & A \\ & & \rotatebox{90}{$\rightarrowtail$} \\ & & \exists \\ & & \rotatebox{90}{$\rightarrowtail$} \\ B & \rightarrowtail \exists \rightarrow & C \end{array} \tag{3.2}$$

(iii) A span of embeddings between elements in $\mathfrak{A}$, as in the diagram (3.3) below, can be completed to a commutative square where the arrows are embeddings and $D \in \mathfrak{A}$.

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \rotatebox{90}{$\rightarrowtail$} & & \rotatebox{90}{$\rightarrowtail$} \\ \exists & & \exists \\ \rotatebox{90}{$\rightarrowtail$} & & \rotatebox{90}{$\rightarrowtail$} \\ C & \rightarrowtail \exists \rightarrow & D \end{array} \tag{3.3}$$

Condition (ii) is known as the *joint embedding property* (JEP) and condition (iii) as the *amalgamation property* (AP).

Remark 3.2.5. When the language only contains relation symbols, property (ii) follows from property (iii) by taking the empty model for A. However, there is no general analogue of this fact in the categorical version of the Fraïssé theory in Section 3.5. There, condition (iii) is relaxed to a *forcing condition* which incorporates the non-atomic case into the theory. This condition is conceptually very different from (ii).

The following theorem is the classic result that Fraïssé [Fra53] first obtained on the structure of the rational numbers. For broader discussion, see chapter 7 of Hodges [Hod93]. The sketch of the proof below is from the Macpherson [Mac11].

Theorem 3.2.6 (Fraïssé's theorem)**.** *Suppose M is a countable ultrahomogeneous structure. Then $\mathfrak{A}(M)$ is a Fraïssé class. Conversely, if $\mathfrak{A}$ is a Fraïssé class, then there exists a countable ultrahomogeneous structure M such that $\mathfrak{A}(M) = \mathfrak{A}$. Moreover, if N is another countable ultrahomogeneous structure with $\mathfrak{A}(N) = \mathfrak{A}$, then M and N are isomorphic.*

Proof. The set of finite subsets form an ideal under inclusion, hence the conditions (i)-(iii) of Definition 3.2.4 follow. For the second part, the idea for constructing M from $\mathfrak{A}$ is to build it in stages, e.g. ensuring at even stages that every $A \in \mathfrak{A}$ is a substructure of M, and at odd stages that for every embedding $f : A \to M$ and $A \subseteq B$, f extends to an embedding $f : B \to M$. Once M is constructed, it can be shown using a back-and-forth argument that it is ultrahomogeneous. Uniqueness can be shown in the same way. □

Example 3.2.7. Consider the language $\{0, 1, \leq\}$ of intervals with endpoints. It is well known that a dense linear order with endpoints, e.g. $\mathbb{Q} \cap [0, 1]$, is a ultrahomogeneous structure for this theory. Recall from the remarks under example 1.2.10 that for $\mathbf{M}$ the monoid of endomorphisms on $\mathbb{Q} \cap [0, 1]$, $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$ is equivalent to the topos of simplicial sets. However, in the actual example, it was shown that this is also satisfied for the monoid of endomorphisms $\omega + 1 \to \omega + 1$. This is also an ultrahomogeneous structure, although one that does not admit any non-trivial automorphisms.

Clearly $\mathbb{Q} \cap [0, 1]$ and $\omega + 1$ are not the same, but when one relaxes automorphisms to endomorphisms, they give rise to the same topos. Either ultrahomogeneity or strong homogeneity seem too rigid to give a satisfactory explanation for this.

A subtle but extremely powerful generalization of Fraïssé's construction was presented by Hrushovski [Hru93] as a method for constructing a new class of $\aleph_1$-categorical structures that refutes an important conjecture on such structures by Zilber ([Zil84], but see Hrushovski [Hru93]). This generalization has found many applications since, for instance in the the theory of Jordan

groups. For an introduction, see chapter 15 of [Bha+97], the discussion of Evans [Eva94] or Macpherson [Mac94]. The ingredients look a lot like what I will put forward as a generalized Fraïssé construction in Section 4.4.1 and, indeed, I will show in Example 3.3.13 below that a version of Hrushovski's intended class of models actually also fits in the setting presented here. First, I briefly introduce the Hrushovski construction as in the literature.

For a class of finite structures, Hrushovski's construction deviates from Fraïssé's by restricting the way in which they are to be embedded in their homogeneous amalgamation. For a class of finite structures $\mathfrak{A}$ on some underlying signature, a class $\mathcal{M}$ of *well-embeddings* is a class of embeddings $f : A \to B$ between structures in $\mathfrak{A}$ such that :

(i) all isomorphisms are in $\mathcal{M}$
(ii) $\mathcal{M}$ is closed under composition
(iii) if $f : A \to B$ is in $\mathcal{M}$ factoring through some substructure $C \subseteq B$ in $\mathfrak{A}$, then $f : A \to C$ is also in $\mathcal{M}$.

The notion of well-embedding enables the following generalization of Fraïssé classes:

Definition 3.2.8 (cf. [Eva94], §2.1.3)**.** A countable class of finite structures $\mathfrak{A}$ together with a class of well-embeddings $\mathcal{M}$ is a *Hrushovski class* if the following conditions hold:

(i) The set $\mathfrak{A}$ is closed under finite substructures
(ii) For finite structures $A, B \in \mathfrak{A}$, there is a $C \in \mathfrak{A}$ together with *well-embeddings* $A \rightarrowtail C$, $B \rightarrowtail C$ i.e. homomorphisms preserving and reflecting atomic formulas:

$$\begin{array}{ccc} & & A \\ & & \rotatebox{90}{$\prec$} \\ & & \exists \in \mathcal{M} \\ & & \rotatebox{90}{$\succ$} \\ B & \overset{\exists}{\underset{\in \mathcal{M}}{\prec\!\!-\!\!\succ}} & C \end{array} \tag{3.4}$$

(iii) A span of well-embeddings between elements in $\mathfrak{A}$, as in the diagram (3.5) below, can be completed to a commutative square of well-embeddings, where $D \in \mathfrak{A}$.

$$\begin{array}{ccc} A & \overset{\in \mathcal{M}}{\rightarrowtail} & B \\ \rotatebox{90}{$\prec$} & & \rotatebox{90}{$\prec$} \\ \in \mathcal{M} & & \exists \in \mathcal{M} \\ \rotatebox{90}{$\succ$} & & \rotatebox{90}{$\succ$} \\ C & \overset{\exists}{\underset{\in \mathcal{M}}{\prec\!\!-\!\!\succ}} & D \end{array} \tag{3.5}$$

Further, for a structure M and a finite structure A in a Hrushovski class $\mathfrak{A}$, an embedding $f : A \to M$ is a *well-embedding* if it factors through a finite substructure $C \subseteq M$ via a well-embedding $f' : A \to C$.

I have taken the term *well-embedding* from the specific example in chapter 15 of [Bha+97]. With an eye towards Example 3.2.10, derived from the same source, this terminology seems natural here. The following theorem is Hrushovski's refinement of Theorem 3.2.6 above.

Theorem 3.2.9 (Hrushovski Construction, cf. [Eva94], theorem 2.10)**.** *Suppose $\mathfrak{A}$ together with $\mathcal{M}$ is a Hrushovski class. Then there exists a countable structure $\boldsymbol{M}$ such that:*

(i) $\mathfrak{A}$ *is the class of well-embedded substructures of* $\boldsymbol{M}$
(ii) $\boldsymbol{M}$ *is a union of* $\mathcal{M}$*-embedded finite substructures*
(iii) If $f : A \to \boldsymbol{M}$ *is a well-embedding and* $g : A \to \boldsymbol{B}$ *is in* $\mathcal{M}$*, then there exists a well-embedding* $f' : \boldsymbol{B} \to \boldsymbol{M}$ *such that* $f' \circ g = f$*.*

Further, every well-embedding $f : A \to \boldsymbol{B}$ between well-embedded substructures of $\boldsymbol{M}$ extends to an automorphism of $\boldsymbol{M}$. Lastly, if $\boldsymbol{N}$ is another countable structure satisfying the properties (i)-(iii) above, then $\boldsymbol{N}$ is isomorphic to $\boldsymbol{M}$.

Proof. The same as for Fraïssé's theorem. □

The following example is important and will be followed up in 3.3.13 below.

Example 3.2.10 ([Bha+97], def. 15.5)**.** This example is a simpler version of Hrushovski's [Hru93], but captures very well those aspects that make the Hrushovski construction interesting from a model-theoretic point of view. These also concern some applications in this thesis. In Example 3.3.13 it is shown that, without any modification, this special case can be given the structure of a factorizing prime site, where the left class is precisely the class of well-embeddings. This is remarkable since, typical though the example may be, there is more structure in it than Definition 3.2.8 entails.

Consider the signature with one $k+1$-ary relation symbol $\boldsymbol{R}$. The class of structures is restricted to the sets which, equipped with an interpretation of $\boldsymbol{R}$, are (simple) $k+1$-hypergraphs, where $\boldsymbol{R}(x_1, \dots, x_{k+1})$ means that there is an edge between $x_1, \dots, x_{k+1}$; so any interpretation of $\boldsymbol{R}$ is closed under $k+1$-permutations of its terms. For a finite hypergraph $\boldsymbol{K}$, let $|E|$ be its number of edges, and define:

$$\delta(\boldsymbol{K}) = |K| - |E| \tag{3.6}$$

For A a finite substructure of some structure $\boldsymbol{M}$, define

$$d_{\boldsymbol{M}}(A) = \min\{\delta(\boldsymbol{B}) \mid \boldsymbol{B} \text{ finite}, A \subseteq \boldsymbol{B} \subseteq \boldsymbol{M}\} \tag{3.7}$$

We now let $\mathfrak{H}$ be the set of finite structures $\boldsymbol{K}$ such that for every substructure $\boldsymbol{K}' \subseteq \boldsymbol{K}$ the following holds:

$$\delta(\boldsymbol{K}') \geq \min\{|K'|, k\} \tag{3.8}$$

This means there are at least k more vertices than edges in such a substructure, unless $|K'| < k$ itself, in which case there are just no edges. For $k = 1$, for instance, this means that graphs in $\mathfrak{H}$ do not contain any cycles.

Between finite substructures, an embedding $f : A \to B$ can be viewed as an inclusion, which I sometimes write as $A \leq B$. A *well-embedding* is now defined as an embedding $f : A \to B$ which, viewed as a substructure $A \leq B$, satisfies $d_A(A) = d_B(A)$, i.e. $\delta(K) \geq \delta(A)$ for every $A \leq K \leq B$. It is easy to see that these satisfy properties (i) and (iii) of well-embeddings. In Figure 3.1 I have drawn an example of a well-embedding for the case $k = 2$.

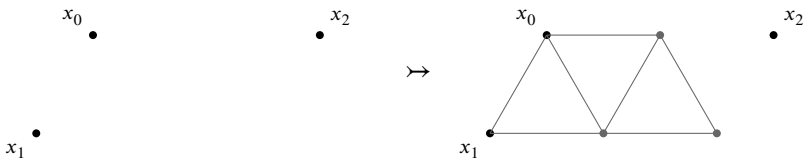


Figure 3.1: A well-embedding of Hrushovski 3-graphs

For the following lemma I will give the proof as a way of gaining some intuition for these structures.

Lemma 3.2.11 (cf. [Bha+97], Lemma 15.8)**.** *Well-embeddings are closed under composition.*

Proof. Suppose $A \leq B$ and $B \leq C$ are well-embeddings, and suppose $A \leq K \leq C$. Consider the process of adding vertices from K to B. Since $B \leq C$ is well-embedded, this cannot add more edges than vertices. Hence adding vertices from K to $B \cap K$ also cannot add more edges than vertices, i.e. $\delta(K) \geq \delta(B \cap K)$. But clearly $\delta(A) \leq \delta(B \cap K)$, so $\delta(K) \geq \delta(A)$; hence $A \leq C$ is a well-embedding. □

In Lemma 15.15 of [Bha+97] it is shown that the class $\mathfrak{H}$ satisfies (AP) and, since one can amalgamate over the empty structure, (JEP). The proof is essentially the same as that of Lemma 3.3.17 below. Together with the above, $\mathfrak{H}$ is a Fraïssé class. It has interesting properties as discussed in the rest of Chapter 15 of the book, some of which are also of interest to the matter in this thesis. Example 3.3.13 goes further into this.

From the definition of the Hrushovski construction it is not clear in what sense the homogeneous structures it yields generalize ultrahomogeneous structures. Further, generalizing to *strongly homogeneous* structures (Definition 3.2.3) does not remedy this, because the only definition of a *type* is that of a type of a finite substructure, i.e. just a finite tuple of elements, so it is not clear in what sense the extension property of Theorem 3.2.9 is really of the kind of Definition 3.2.3. One way to solve this categorically is to throw away all the non-well-embeddings, which leaves one with

an atomic site, and the Hrushovski construction reduces to the (categorical) Fraïssé construction. The problem is that this changes the underlying language, and a lot of the interesting model-theoretic properties get lost. For instance Example 3.3.13 exhibits through the Hrushovski construction a homogeneous structure which is not $\aleph_0$-categorical in the language of $(k+1)$-hypergraphs, but this property would immediately get lost by removing the non-well-embeddings, leaving behind a structure of yet unknown signature.

One of the aims of this chapter is to approach this question more carefully. First, again, I present a classical approach to the notion '*strongly homogeneous*' which makes intuitive sense for other notions of *type* (regular, coherent, geometric). However, Example 3.2.10 only satisfies the broader definition of homogeneous model when one switches to the site-theoretic definition in Section 3.5 below. Also, I will skip a corresponding 'quantifier-free' version which would directly generalize ultrahomogeneity, for here similar remarks as above definition 3.2.3 apply. The latter is used by Mašulović and Pech, for which see Remark 3.2.14.

Since in Definition 3.2.3 types include formulas obtained by negation, one has that whenever $\mathrm{tp}_M(\mathbf{a}) \subseteq \mathrm{tp}_M(\mathbf{b})$, equality $\mathrm{tp}_M(\mathbf{a}) = \mathrm{tp}_M(\mathbf{b})$ follows automatically. In generalizing to (Horn, regular, geometric) types, homogeneity loses this symmetry. The following definition captures this.

Definition 3.2.12. For a logic (Horn, regular, geometric), a structure M is *homogeneous* if whenever $\mathbf{a}$, $\mathbf{b}$ are tuples of the same length satisfying

$$\mathrm{tp}_M(\mathbf{a}) \subseteq \mathrm{tp}_M(\mathbf{b}) \tag{3.9}$$

for their (Horn, regular, geometric) types, then there exists a *homomorphism* $f : M \to M$ such that $f\mathbf{a} = \mathbf{b}$.

Remark 3.2.13. At first sight Definition 3.2.12 does not generalize Definition 3.2.3, even when considering classical, first-order types, since the extension $f : M \to M$ is not required to have an inverse. However, it is a result of the 'back-and-forth' construction that in the countable case, for first-order types, the above definition is actually equivalent to Definition 3.2.3; that is, one can always choose an extension in such a way that it is invertible.

In the language of topos theory, this becomes manifest in a result on *atomic sites* proven in proposition 3.5.6 below. There, I show that for countable models, embeddings between finite substructures of countable models can always be extended to isomorphisms between the whole models.

This further relates to a Remark 1.2.4, where it was observed that invertibility of the arrows in $\mathbf{M}$ is not a property of the topos $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$. Rather, the induced property is that the topos is *atomic*, so that one can find a representation $[\mathbf{M}, \mathrm{Set}]_{\mathrm{fs}}$ where $\mathbf{M}$ is a group if one wishes. But this runs a bit ahead at this point.

Remark 3.2.14. Mašulović and Pech [MP11] use the quantifier-free version of the above notion of homogeneous model for the positive fragment of first-order logic and call it *homomorphism-homogeneous*. Their paper also contains a discussion on this quantifier-freeness similar to the remarks above Definition 3.2.3 above. In this thesis, I have chosen to just call the notion of Definition 3.2.12 *homogeneous* because it is general and also applies to boolean theories.

Example 3.2.15. All the examples of models M with a monoid of endomorphisms $\mathbf{M}$ of Chapter 1 are homogeneous in the sense of Definition 3.2.12. To contrast with Example 3.2.7, observe that $\omega + 1$ and $\mathbb{Q} \cap [0, 1]$ are both homogeneous, since any homomorphism between embedded substructures can be extended to an endomorphism of the whole. Note it is important here that $\leq$ is interpreted as weak inequality in $\omega + 1$, otherwise it would not work.

In order to state a Fraïssé theorem for definition 3.2.12, one needs a corresponding notion of *age* for a (Horn, regular, geometric) structure. This notion will be gradually developed over the next sections and takes on its final shape in Section 4.2. The next section serves as an introduction to and motivation for the categorical approach presented there and departs from a category of models for classical theories rather than sites. From this initial study of models and their substructures, one naturally arrives at factorizing prime sites.

3.3 The category of finite types of a concrete category

I suspend for the moment the study of a *single* structure M that is homogeneous in the sense of Definition 3.2.12 and, by observing that the input data essentially consists of a monoid of homomorphisms $M \to M$ and a class of finite substructures $A \subset M$, will consider general categories of sets and functions $M \to N$. This is called a *concrete* category and is more formally given by a faithful functor $U : \mathbf{M} \to \mathrm{Set}$ from some category $\mathbf{M}$ into Set.

A concrete category can be viewed as a *semantical* notion of a theory, a 'category of examples'. Below I define the category of *finite types* for such a theory. This category admits an orthogonal factorization system which satisfies an additional *forcing* condition. It turns out that this orthogonal factorization system precisely dictates what it means for a homomorphism of models to be an embedding, even though the category of examples started only with homomorphisms. The forcing property is completely dual to the notion of a *coverage* and enables the construction of a site so that the original category is a category of models for it. Moreover, these models are jointly conservative for this site and the models are *prime* (Definition 3.5.2), or homogeneous in a generalized sense - hence a generalized picture of the Fraissé-Hrushovski construction emerges. But this runs ahead to later sections and will only be understood by Chapter 4. The following definition, used to explore some initial structures in this section, is covered by Example 4.1.14 in that chapter.

Definition 3.3.1. Suppose $\mathbf{M}$ is a concrete category, i.e. a category equipped with a faithful functor $U : \mathbf{M} \to \mathrm{Set}$. To simplify notation, we will just identify the objects M with their images $U(M)$. The category $\mathbf{T}(\mathbf{M})$ of *finite types* is defined as follows:

(i) Objects (A, M) are given by subsets $A \subseteq M$, where M is an object of $\mathbf{M}$ and $|A|$ finite
(ii) Morphisms $f_A : (A, M) \to (B, N)$ given by functions $f_A : A \to B$ in Set for which there exists $f : M \to N$ such that f_A is the restriction of f to A.

In the definition above, one can view the (A, M) as *finitely generated* substructures of M. Morphisms between such substructures are homomorphisms, provided that they allow for an extension to the whole of M. The following proposition gives an interpretation to the use of factorization systems in this setting. In Chapter 4, this will be a consequence of a general result (Proposition 4.1.11) and its application to Example 4.1.14.

Proposition 3.3.2. *Suppose* $\mathbf{M}$ *is a concrete category. Then there exists an orthogonal factorization system* $(\mathcal{E}, \mathcal{M})$ *on* $\mathbf{T}(\mathbf{M})$ *where* $\mathcal{E}$ *is the class of those*

$$f_A : (A, M) \to (B, N)$$

for which the underlying function $f_A : A \to B$ *is surjective; in particular, every morphism in* $\mathcal{E}$ *is an epimorphism.*

Proof. For $f_A : (A, M) \to (B, N)$, there exists a factorization

$$(A, M) \to (\mathrm{im}\, f, N) \rightarrowtail (B, N) \tag{3.10}$$

where im f is the image of f_A in Set and the last arrow is just inclusion. Clearly the first arrow is an epimorphism. To finish the proof, it is by lemma 2.4.2 enough to show that inclusions $(B', N) \rightarrowtail (B, N)$ have the right lifting property with respect to epimorphisms. But this follows straight from the epi-mono factorization system in Set. □

Definition 3.3.3. For a category of finite types as in Proposition 3.3.2, the class $\mathcal{M}$, defined as those morphisms that have the right lifting property with respect to $\mathcal{E}$, is called the class of *embeddings*. As in Chapter 2, arrows in this class are denoted $A \rightarrowtail B$.

Proposition 3.3.4. *The category* $\mathbf{T}(\mathbf{M})$ *has the following property. Whenever* $m : T \rightarrowtail S$ *is an embedding and* $f : T \to U$ *any morphism, they can be completed to a square:*

$$\begin{array}{ccc} T & \xrightarrow{f} & U \\ {\scriptstyle m}\downarrow & & \vdots\,{\scriptstyle m'} \\ S & \xrightarrow[f']{} & V \end{array}, \tag{3.11}$$

where m' is an embedding.

Proof. By the factorization (3.10), $m : T \rightarrowtail S$ is isomorphic to an arrow $(A, M) \rightarrowtail (A', M)$ where $A \subseteq A'$. Writing $U = (B, N)$, the arrow $f : (A, M) \to (B, N)$ has an underlying morphism $f : M \to N$ for which the restriction to A' has an image $\mathrm{im}_f A' \subseteq B$. This defines f'. Clearly the arrow $m' : (B, N) \to (\mathrm{im}_f A', N)$ is in $\mathcal{M}$ and completes the square (3.11). □

Definition 3.3.5. A category equipped with an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ has the *forcing property* when the property of Proposition 3.3.4 is satisfied.

As announced, the forcing property is dual to the property of a factorizing prime site (see the remark under Definition 2.4.3). Therefore the following conclusion holds:

Corollary 3.3.6. *The category* $\mathbf{T}(\mathbf{M})^{\mathrm{op}}$ *equipped with the orthogonal factorization system inherited from its dual (as in Proposition 3.3.2) is a factorizing prime site.* □

The forcing property for $\mathbf{T}(\mathbf{M})$ justifies the intuition that for a concrete category of models $\mathbf{M}$, the category $\mathbf{T}(\mathbf{M})$ can be viewed as a category of types. Consider the case where $\mathbf{M}$ is some subcategory of a category of models $\mathrm{Mod}(\mathbb{T})$ of some (Horn, regular, geometric) theory $\mathbb{T}$. If one associates to a finite tuple $\mathbf{a}$ of elements of M of such a model M its (Horn, regular, geometric) *type*, $\mathrm{tp}_M(\mathbf{a})$, then clearly all quantifier-free formulas in $\mathrm{tp}_M(\mathbf{a})$ are preserved by any morphism $f : (\{\mathbf{a}\}, M) \to (\{\mathbf{b}\}, N)$. But what, for instance, if $\mathrm{tp}_M(\mathbf{a})$ contains a formula $\exists x \phi(\mathbf{a}, x)$, but $\{\mathbf{a}\}$ contains no witness to show for? Then f cannot be said to *preserve* this formula, but it does have an extension, along an embedding, to a morphism which does. This follows from the forcing condition: if c is a witness such that $\phi(\mathbf{a}, c)$, the below diagram can be completed:

$$\begin{array}{ccc} (\{\mathbf{a}\}, M) & \xrightarrow{\quad f \quad} & (\{\mathbf{b}\}, N) \\ \big\downarrow & & \big\vdots \\ (\{\mathbf{a}, c\}, M) & \overset{f'}{\dashrightarrow} & (\{\mathbf{b}, d\}, N) \end{array} \tag{3.12}$$

The following example illustrates the type of information stored in the orthogonal factorization system:

Example 3.3.7 (Decidable sets)**.** Consider the category $\mathbf{Inj}_\kappa$ of sets of cardinality $\leq \kappa$ and injective functions between them. This can be regarded as the category of models of cardinality bounded by κ of the geometric theory consisting of a single relation symbol $\{\#\}$ and axioms which interpret this as decidable inequality, i.e. (1.8)-(1.9) above. Therefore I will call it the category of *decidable* sets of cardinality bounded by κ. Now form the (essentially) small category $\mathbf{T}(\mathbf{Inj}_\kappa)$ of *finite* types. Then the left class $\mathcal{E}$ of definition 3.3.3 is defined as those arrows $(A, M) \twoheadrightarrow (B, N)$ for which

the underlying arrow $A \to B$ is an epimorphism and hence, a bijection. Yet, as a morphism $(A, M) \twoheadrightarrow (B, N)$ in $\mathbf{T}(\mathbf{Inj})_\kappa$, there is an arrow in the other direction only when M and N have the same cardinality. When that is the case, the morphism is an isomorphism, and indeed: any morphism $f : (A, M) \to (B, M)$ is in the *right* class $\mathcal{M}$ precisely when M and N have the same cardinality.

Suppose one finds a way to stitch together a family of types $(\{\mathbf{a}_i\}, M_i)$ in $\mathbf{T}(\mathbf{M})$ in a 'finitely consistent' manner, and in such a way that all formulas in $\mathrm{tp}_{M_i}(\mathbf{a}_i)$ are forced. In ordinary model theory, the resulting object is another model of our theory. In the current situation, there is no (a priori) definition of model: there is only a category of 'examples', $\mathbf{M}$.

Recall that for a site $\mathcal{C}$ over the category of sets, a *model* of $\mathcal{C}$ is a morphism of sites $\mathcal{C} \to \mathrm{Set}$, which is a continuous flat functor. Working over Set, this means that as a presheaf , on $\mathcal{C}^{\mathrm{op}}$, this functor is a filtered colimit of representables (defined by its category of elements) and satisfying an extra continuity condition. This can be spelled out:

Definition 3.3.8. Let $\mathbf{M}$ be a concrete category such that $\mathbf{T}(\mathbf{M})^{\mathrm{op}}$ is a small category. Then a functor $M : \mathbf{T}(\mathbf{M})^{\mathrm{op}} \to \mathrm{Set}$ defines a *model* if two conditions hold:

(i) M is a filtered colimit of a diagram $\mathcal{I} \to \mathbf{T}(\mathbf{M})$ of representable presheaves:

$$y_{(A_i,M_i)} : \mathbf{T}(\mathbf{M})^{\mathrm{op}} \to \mathrm{Set}$$

(ii) This diagram satisfies the *forcing condition*: whenever $f_i : y_{(A_i,M_i)} \to M$ is an arrow of the limiting cocone, and $m : (A_i, M_i) \rightarrowtail (B, N)$ is an embedding, then there is a morphism $y_{(B,N)} \to M$ making the diagram commute:

$$\begin{array}{ccc} y_{(A_i,M_i)} & \xrightarrow{f_i} & M \\ \downarrow{\scriptstyle m} & \nearrow & \\ y_{(B,N)} & & \end{array} \tag{3.13}$$

A sanity check for this definition would be to verify that when $\mathbf{M}$ is a category of models (in the sense of logic) of a theory $\mathbb{T}$, then any 'model' defined by the above is also a model of $\mathbb{T}$. This is the result of Theorem 3.3.10.

Remark 3.3.9. Comparing condition (ii) in Definition 3.3.8 to (iii) in 3.2.9, one can begin to understand why the 'age' of a model should have the structure dual to a site. But this looks ahead to Sections 3.4 and 4.2.

The following theorem implies that $\mathbf{T}(\mathbf{M})^{\mathrm{op}}$ classifies $\mathbb{T}$-models which satisfy conditions (i)-(ii) of Definition 3.3.8. In Theorem 4.4.8 I will state and prove a similar result, in the case when $\mathbf{M}$ are *principal* with respect to $\mathbb{T}$, to say more about the 'theory' of $\mathbf{M}$ with respect to $\mathbb{T}$ (i.e. all statements in the language of $\mathbb{T}$ that are true for $\mathbf{M}$).

Theorem 3.3.10. *Suppose* $\mathbf{M}$ *is a small subcategory of the category of models of some (Horn, regular, geometric) theory* $\mathbb{T}$*. Then for a functor* $M : \mathbf{T}(\mathbf{M})^{\mathrm{op}} \to \mathrm{Set}$ *satisfying the conditions of definition 3.3.8, the set*

$$\underline{M} = \varinjlim A_i \tag{3.14}$$

formed as the colimit of the diagram $\mathcal{I} \to \mathbf{T}(\mathbf{M}) \to \mathrm{Set}$ *where the last arrow sends* (A_i, M_i) *to the underlying sets, is a model of* $\mathbb{T}$*.*

Moreover, if all homomorphisms between $\mathbb{T}$*-models are morphisms in* $\mathbf{M}$*, then* M *is the functor which sends* (A, N) *to the set of functions* $f_A : A \to M$ *for which there exists a homomorphism* $f : N \to M$ *extending* f_A*.*

Proof. I will sketch a proof. First one has to provide an interpretation for the underlying signature of $\mathbb{T}$, a set of relation symbols $\{R, P, Q, \dots\}$. For $\mathbf{a} : \{\mathbf{a}\} \to M$ a tuple in $\underline{M}$, say that $R(\mathbf{a})$ holds if $\mathbf{a}$ factors as $f : \{\mathbf{a}\} \to B_i$ followed by a cocone morphism $B_i \to M$ and $R(f(\mathbf{a}))$ holds in B_i, i.e. in its underlying structure.

By induction, one can show that for any formula $\phi(\mathbf{m})$ that holds in $\underline{M}$, there is some type $(\{\mathbf{a}\}, M_i)$ in $\mathbf{T}(\mathbf{M})$ which satisfies $\phi(\mathbf{a})$. Note it is enough to check this for atomic formulas and to show closure under conjunction. For a pair of formulas $\phi(\mathbf{a}, \mathbf{b})$, $\psi(\mathbf{a}, \mathbf{c})$, this follows exactly from filteredness of the diagram. Together, this shows that the interpretation satisfies any sequent (3.15) in $\mathbb{T}$, where $\psi(\mathbf{a})$ is quantifier-free.

$$\phi(\mathbf{x}) \vdash_{\mathbf{x}} \psi(\mathbf{x}) \tag{3.15}$$

Now to satisfy an existential sequent

$$\phi(\mathbf{x}) \vdash_{\mathbf{x}} \exists y \psi(\mathbf{x}, y) \tag{3.16}$$

one can apply the forcing condition to some type in which $\phi(\mathbf{x})$ is satisfied. Note that a similar argument can be given for a disjunction of formulas.

The last claim is quite easy since every model N in $\mathbf{M}$ is a filtered colimit of its own diagram of types (A, N). So a homomorphism $N \to \underline{M}$ is given by some cocone on the types of N types, and therefore some $f_A : A \to M$ is a morphism if it can be extended to a cocone. But since it has a finite image (B, K) in M for some other model K in $\mathbf{M}$, this is equivalent to an extension of f_A to a homomorphism $N \to K$. So if $\mathbf{M}$ contains all homomorphisms, there is the desired correspondence. □

The forcing property plus the fact that every model in $\mathbf{M}$ is a filtered colimit of the filtered diagram of all its substructures entail that the models in $\mathbf{M}$ used to construct $\mathbf{T}(\mathbf{M})$ are models of $\mathbf{T}(\mathbf{M})^{\mathrm{op}}$ themselves in the sense of Definition 3.3.8. A natural question is whether these are in any way special. Recall that in the Hrushovski construction (Theorem 3.2.9) the constructed homogeneous model is a union of *embedded* substructures. This suggests:

Definition 3.3.11. A model M of $\mathbf{T}(\mathbf{M})^{\mathrm{op}}$ is *principal* if it is a filtered colimit as in (i) of definition 3.3.8 where all the arrows are *embeddings*, i.e. arrows in $\mathcal{M}$, and also condition (ii) holds for this particular diagram.

Clearly, the models coming from $\mathbf{M}$ are principal models of $\mathbf{T}(\mathbf{M})^{\mathrm{op}}$. This would further suggest that a Fraïssé theorem for $\mathbf{T}(\mathbf{M})$ would say something about these principal models being unique up to isomorphism, but recall that viewing models up to automorphisms does not suit the direction set out in Definition 3.2.12. Rather, it will become clear that, were one to repeat the $\mathbf{T}(-)$ construction for *all* principal models of $\mathbf{T}(\mathbf{M})^{\mathrm{op}}$, this would define the same class of models as $\mathbf{T}(\mathbf{M})$. In what follows, this will become manifest in an equivalence of toposes; see Theorem 4.3.4.

In addition, the monoid case has another special property, which can be compared to the second property in the definition (3.2.8) of a Hrushovski class. If $\mathbf{M}$ is some monoid of endomorphisms $M \to M$, then the embeddings of $\mathbf{T}(\mathbf{M})$ form a *filtered* subcategory. Hence:

Definition 3.3.12. A category $\mathcal{C}$ with orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ has the *joint embedding property* (JEP) when for any pair of objects A, B there exists an object C together with embeddings:

$$\begin{array}{ccc} A & & \\ & \searrow & \\ & & C \\ & \nearrow & \\ B & & \end{array} \tag{3.17}$$

In Section 3.6 I will show that the above property, in combination with the forcing condition and factorization, defines a unique topos: this means that any other model of this topos with the same properties as $\mathbf{M}$ defines the same topos. This is the extent of the uniqueness property of the general Fraïssé-Hrushovski construction and will be better understood through Proposition 4.3.1.

Before I get there, I start in the next section by dualizing the properties of orthogonal factorization and forcing, thereby making a factorizing site $\mathcal{C}$, which is also assumed to be prime-generated, the starting point of the categorical investigation. The factorization allows for a definition of a 'well-embedding' between models of $\mathcal{C}$ that would otherwise have remained hidden.

To end this section, I return to Example 3.2.10 and show that the well-embeddings defined there indeed fit this mould, and thus form a motivating example for what follows.

Example 3.3.13. The goal is to show that the class of structures presented in Example 3.2.10 can actually be equipped with the structure of a category with an orthogonal factorization system, with the forcing property and the (JEP).

So recall the class of finite $(k+1)$-hypergraphs $\mathfrak{H}$ which satisfy the condition (3.8) for all subgraphs. Let $\mathcal{H}_{k+1}$ denote the *category* of such graphs, where morphisms are embeddings in the traditional sense, so those maps which preserve and reflect the single relation. I call this the category of *Hrushovski k-hypergraphs*. Define $\mathcal{M}$ to be the class of *well-embeddings* defined earlier, i.e. inclusions $A \leq B$ that satisfy $d_A(A) = d_B(A)$. The following lemma generalizes Lemma 15.9 in [Bha+97], but note that the statement can be compared to axiom (C4) in the generalized treatment of Wagner [Wag94].

Lemma 3.3.14. *Consider a pullback diagram of a well-embedding along another arrow, e.g.:*

$$\begin{array}{ccc} B \cap C & \dashrightarrow & C \\ \vdots & & \downarrow \\ B & \longrightarrow & D \end{array} \tag{3.18}$$

Then $B \cap C \to B$ is a well-embedding.

Proof. Suppose K is such that $B \cap C \leq K \leq B$. Since $C \leq D$ is well-embedded, $\delta(C \cup K) \geq \delta(C)$. So adding the vertices in $K \setminus C$ to C adds more vertices than edges. Hence adding them to $B \cap C$ adds even fewer edges since the edges now must fully lie in K. It follows that $\delta(K) \geq \delta(B \cap C)$. So $B \cap C \to B$ is well-embedded. □

For every arrow $f : A \to B$, there is a well-embedding $g : B' \rightarrowtail B$ such that f factors through g: one can just minimize $\delta(K)$ over all finite substructures $A \leq K \leq B$. Moreover there is a unique *least* such; this follows from the above lemma and the fact that well-embeddings are closed under composition. So:

Lemma 3.3.15. *Every arrow $f : A \to B$ factors as:*

$$g \circ f' : A \twoheadrightarrow B' \rightarrowtail B \tag{3.19}$$

where g is a well-embedding and f' has the property that whenever it factors through some well-embedding $K \rightarrowtail B$, one has $K \cong B$.

In fact, arrows f' with this property are left orthogonal to well-embeddings.

Proof. By the above discussion, only left orthogonality is new. Given a commutative square $f' \to h$ for some well-embedding $h : C \rightarrowtail D$, by Lemma 3.3.14 the pullback $B' \cap C \rightarrowtail B'$ is a well-embedding and f' factors through it. So $B' \cap C \cong B'$, i.e. $B' \leq C$ as substructures of D. □

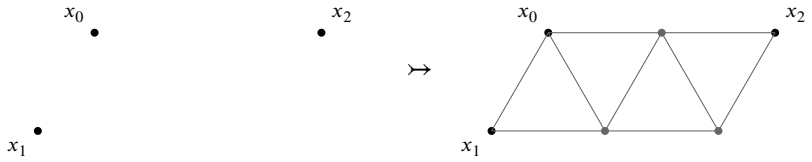


Figure 3.2: A $\mathcal{E}$-embedding of Hrushovski 3-graphs

Remark 3.3.16. Lemma 3.3.15 yields an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{H}_{k+1}$, where $\mathcal{M}$ consists of well-embeddings. As in the proofs of Lemmas 3.2.11 and 3.3.17 below, one can think of a well-embedding as a process which adds more vertices than edges. Conversely, an arrow in $\mathcal{E}$ can be thought of as a process that 'mostly adds edges'. However, this is slightly deceiving, since arrows in $\mathcal{E}$ can in fact add an arbitrarily large number of vertices; for an example, see Figure 3.2. The 'path' between x_0, x_1 and x_2 can be made arbitrarily long, and this argument works for all dimensions. A consequence is that the homogeneous structure M constructed from $\mathfrak{H}$ is not $\aleph_0$-categorical ([Bha+97], Lemma 15.17). I expand further on this in Example 4.5.3. For now, a way to think of the arrows in $\mathcal{E}$ is as substructures $A \leq B$ where B is 'algebraic', or definable, over A. This fits precisely with it being left orthogonal to well-embeddings if one thinks of the latter as preserving all definable structure, and is essential to the main idea of Hrushovski [Hru93]. Another thing to observe is that the arrows in $\mathcal{E}$ are not epimorphic, which makes this example, when dualized, non-standard as per Definition 2.4.3.

That the category $\mathcal{H}_{k+1}$ has the (JEP) was already mentioned in 3.2.10, but is also easily derived by adapting the below proof. So the last ingredient is the forcing condition, which, I recall, replaces the amalgamation property, which was proven in Lemma 15.15 of [Bha+97]. It is interesting to note that although the below statement is neither weaker nor stronger than the amalgamation property, the proof presented in [Bha+97] works without modification: the fact that the inclusions in the amalgamation are well-embeddings only uses the fact that the opposite arrow is a well-embedding.

Lemma 3.3.17. *The class of well-embeddings in $\mathcal{H}_{k+1}$ has the forcing property.*

Proof. Suppose given an arrow $A \to B$ and a well-embedding $A \rightarrowtail C$. Define $D := B \oplus_A C$ as the (Set) pushout inheriting the edges from B and C; so no edge connects vertices in $C \setminus A$ to vertices in $B \setminus A$. We show that the inclusion $B \rightarrowtail D$ is a well-embedding.

So suppose $B \leq K \leq D$. This means that $K \setminus B$ consists of vertices from $C \setminus A$. Consider the process of adding vertices from $K \setminus B$ to B. Any edge that is added in this process lies completely in C since no edge connects vertices from C to vertices from B. So the process is the same as adding vertices from K to A within C. Since the left arrow is well-embedded, this cannot add more edges than vertices. So it follows that $\delta(K) \geq \delta(B)$. □

The example can be summarized in the following theorem

Theorem 3.3.18 (Hrushovski hypergraphs)**.** *For $k \geq 1$, the category $\mathcal{H}_{k+1}$ of finite Hrushovski $(k+1)$-hypergraphs admits an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{M}$ is the class of well-embeddings. In addition, the class $\mathcal{M}$ has the forcing property and the (JEP).*

Therefore, $\mathcal{H}_{k+1}^{\mathrm{op}}$ is a factorizing prime site. Moreover, the topos of sheaves $\mathrm{Sh}(\mathcal{H}_{k+1}^{\mathrm{op}})$ is a subtopos of the classifying topos of (decidable) simple $(k+1)$-hypergraphs:

$$\mathrm{Sh}(\mathcal{H}_{k+1}^{\mathrm{op}}) \hookrightarrow [\mathcal{G}_{k+1}, \mathcal{S}] \tag{3.20}$$

Proof. Only the last claim has not been addressed yet. Let $\mathcal{G}_{k+1}$ be a small category equivalent to the category of (decidable) finite simple $(k+1)$-hypergraphs. It is an exercise in the theory of classifying toposes that $[\mathcal{G}_{k+1}, \mathcal{S}]$ classifies $(k+1)$-hypergraphs.

One can put a co-coverage on $\mathcal{G}_{k+1}$ by letting every hypergraph that is not in the class $\mathfrak{H}$ (so does not satisfy the condition defined in (3.8)) be co-covered by the empty family, and define the other co-coverings as in $\mathcal{H}_{k+1}$. By the nature of the condition (3.8), this defines a coverage, and it is clear that $\mathcal{H}_{k+1}^{\mathrm{op}}$ is a dense subcategory and carries the induced coverage. So indeed, there is a corresponding inclusion (3.20) of toposes. □

Remark 3.3.19. This example leaves open whether the many more Hrushovski-type constructions (see e.g. [Wag94]), with slightly different axiomatics than this example, also satisfy a version of Theorem 3.3.18. This is a highly interesting question, but since the Hrushovski construction has mostly been applied ad-hoc, there is no way of giving a positive answer straight away. The example chosen here, though, is typical and gives much hope in that direction.

Remark 3.3.20. Theorem 3.3.18 shows that the topos of sheaves on $\mathcal{H}_{k+1}^{\mathrm{op}}$ classifies some theory in the language of $(k+1)$-hypergraphs, but it remains hidden what this theory actually says. A beginning of an answer is given by Theorem 4.4.8, which basically says that this topos classifies filtered colimits of *well-embedded* subgraphs of the homogeneous Hrushovski graph constructed from the Hrushovski class $\mathfrak{H}$ of example 3.2.10.

3.4 Embedded substructures

In the previous sections I have outlined connections to classical model theory, formulated a question regarding the relationship between the Fraïssé construction and Hrushovski's adaptation, and explored the structure of a category of finite types of models in order to motivate the shift in mathematical language that marks the beginning of this section and resumes the theory started in Chapter 2. Therefore the starting data is a factorizing, prime-generated site $\mathcal{C}$ as defined there. In

fact, for the definitions in this section, it is enough that $\mathcal{C}$ is a factorizing site. Below, I show how this definition yields a notion of embedding that, when viewed dually, corresponds precisely to the notion of embedding between models that was found in Section 3.3. Note in the latter all objects were prime, but this is not an assumption in general. For a site $\mathcal{C}$, the orthogonal factorization system is denoted by $(\mathcal{E}, \mathcal{M})$ as in Chapter 2, which swaps these symbols with respect to the dual situation of Section 3.3.

The shift in language also entails that from now on sites are understood as internal to a base topos $\mathcal{S}$, which is assumed to possess a natural numbers object. This requires extra scrutiny when it comes to non-intuitionistic reasoning, but will otherwise be left implicit.

A *model* always means a morphism of sites (as in Section 2.2)

$$M : \mathcal{C} \to \mathcal{S} \tag{3.21}$$

where $\mathcal{S}$ carries the canonical coverage. Denote by $h^{(-)}$ the co-Yoneda embedding $\mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathcal{S}]$. It is easy to check that every $h^P : \mathcal{C} \to \mathcal{S}$ is *flat* (Definition 2.2.1). If $M : \mathcal{C} \to \mathcal{S}$ a model of $\mathcal{C}$, one can consider an element $p \in M(P)$ as a morphism between flat functors $p : h^P \to M$. The cases of interest are where h^P can be viewed as a 'finite' substructure of M.

Observe that for $e : D \to C$ and $m : A \to B$ arrows in $\mathcal{C}$, the property of having a unique diagonal filler for every square (3.22) is equivalent to the naturality square (3.23) being a pullback:

$$\begin{array}{ccc} D & \longrightarrow & A \\ {\scriptstyle e}\downarrow & \overset{\exists!}{\nearrow} & \downarrow{\scriptstyle m} \\ C & \longrightarrow & B \end{array} \tag{3.22}$$

$$\begin{array}{ccc} h^C(A) & \xrightarrow{h^C(m)} & h^C(B) \\ {\scriptstyle h^e_A}\downarrow & & \downarrow{\scriptstyle h^e_A} \\ h^D(A) & \xrightarrow{h^D(m)} & h^D(B) \end{array} \tag{3.23}$$

With the class $\mathcal{E}$ representing *embeddings*, this observation suggests an extension of the notion of embedding to one between flat functors $M, N : \mathcal{C} \to \mathcal{S}$, such that the class $\mathcal{E}$ is precisely the class of representable embeddings between representable flat functors. This approach has been used by Barr [Bar86] for a categorical notion of 'elementary embedding' and can also be compared to the direct translation of the model theoretic notion in topos-theoretic semantics, as for instance by Johnstone [Joh02b], D1.2.10(b).

Definition 3.4.1 (cf. 'elementary embedding' in Barr [Bar86])**.** Suppose $\mathcal{C}$ is a site with orthogonal factorization system $(\mathcal{E}, \mathcal{M})$, and suppose $M, N : \mathcal{C} \to \mathcal{S}$ are flat functors. A natural transformation $\eta : M \to N$ is an *embedding* if for each $m : A \rightarrowtail B$ in $\mathcal{M}$, the naturality square (3.24) is a pullback.

In diagrams, arrows that are embeddings are drawn as $\eta : M \rightarrowtail N$.

$$\begin{array}{ccc} M(A) & \xrightarrow{M(m)} & M(B) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ N(A) & \xrightarrow{N(m)} & N(B) \end{array} \tag{3.24}$$

Example 3.4.2. Observe that indeed, by the above remarks, a natural transformation $\eta : h^C \to h^D$ between representable flat functors $h^C, h^D : \mathcal{C} \to \mathcal{S}$, which is always represented by h^f for some $f : D \to C$, is an embedding in the above sense precisely when $f \in \mathcal{E}$.

Example 3.4.3. For atomic sites, where $\mathcal{E}$ is the class of all morphisms and $\mathcal{M}$ is the class of isomorphisms, every morphism between models is an embedding. This accords with the fact that in Definition 3.2.1, every homomorphism between models of first-order (boolean) theories is an embedding. That is, for their translation to a *geometric* signature, which must include negations of all relations. Indeed, this appears as one of the conditions for a classifying topos $\mathcal{S}[\mathbb{T}]$ of a coherent theory $\mathbb{T}$ to be boolean, see e.g. proposition D3.4.9 in [Joh02b]. Note that in that case, every homomorphism between models is even an elementary embedding, a phenomenon known as being *model-complete* [Mac77].

Example 3.4.4. For the *site* $\mathcal{H}_{k+1}^{\mathrm{op}}$ of Hrushovski $(k+1)$-graphs of Example 3.3.13, Definition 3.4.1 gives a much better definition of a *(well-)embedding* $h^A \rightarrowtail M$ than what is contained in the definition of a Hrushovski class (Definition 3.2.8). The latter relies on the assumption that the objects of the Fraïssé class are (literally) substructures of M, i.e. they come with a 'canonical' well-embedding into M. The categorical definition above shows that this is not needed and that well-embeddings can be defined generally, as long as the Hrushovski class has the structure dual to a factorizing prime site.

The following notion appears in the context of regular sites as 'principal elements' in Barr-Makkai [BM87].

Definition 3.4.5. Suppose $M : \mathcal{C} \to \mathcal{S}$ is a model, C an object of $\mathcal{C}$ and $c \in M(C)$. For $m : A \rightarrowtail C$ in $\mathcal{M}$, I write $c \in M(A)$[1] if there exists $a \in M(A)$ such that $m \cdot a = c$.

I further write $c \in_> M(A)$ if $m : A \rightarrowtail C$ is the *least* such that $c \in M(A)$. In this case, it follows from unique factorization that the witness $a \in M(A)$ is unique up to some automorphism $A \to A$.

[1] In a flagrant abuse of notation, since after the first mention of $c \in M(C)$, the notation $c \in \ldots$ changes meaning. However it feels quite natural to write this, and since the arrows m are close to being monomorphisms, it is close to being completely fine.

Lemma 3.4.6. *Suppose $M : \mathcal{C} \to \mathcal{S}$ is a model, $A \rightarrowtail C$ an arrow in $\mathcal{M}$ and $c \in M(A)$. Then $c : h^A \to M$ is an embedding if and only if $c \in_> M(A)$.*

Proof. Note that it can be assumed without loss of generality that $A \rightarrowtail C$ is the identity arrow. So suppose $c : h^C \rightarrowtail M$ is an embedding, and let $m : A \rightarrowtail C$ be such that $c \in M(A)$. Then since we have a pullback,

$$\begin{array}{ccc} h^C(A) & \longrightarrow & M(A) \\ {\scriptstyle h^C(m)}\downarrow & & \downarrow{\scriptstyle M(m)} \\ h^C(C) & \xrightarrow{c} & M(C) \end{array} \tag{3.25}$$

it follows that there exists a unique $f : C \to A$ such that $m \circ f = \mathrm{id}_A$. Hence m is an isomorphism.

Conversely, suppose $c \in_> M(C)$, and consider the following square, for some $m : A \rightarrowtail B$ in $\mathcal{M}$:

$$\begin{array}{ccc} h^C(A) & \xrightarrow{(\text{-})\cdot c} & M(A) \\ {\scriptstyle h^C(m)}\downarrow & & \downarrow{\scriptstyle M(m)} \\ h^C(B) & \xrightarrow{(\text{-})\cdot c} & M(B) \end{array} \tag{3.26}$$

We have to prove that it is a pullback. So suppose $a \in M(A)$ and $f : C \to B$ are such that $m \cdot a = f \cdot c$. Then since M is flat, there is a square (3.27) in its category of elements:

$$\begin{array}{ccc} d \in M(D) & \longrightarrow & a \in M(A) \\ {\scriptstyle e}\downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle m} \\ c \in_> M(C) & \xrightarrow[f]{} & m \cdot a \in M(B) \end{array} \tag{3.27}$$

But since $c \in_> M(C)$, the left-hand arrow $e : D \to C$ must be in $\mathcal{E}$, hence the square has a unique diagonal filler $g : C \to A$, satisfying $g \cdot c = a$ since the upper triangle commutes. Hence (3.26) is a pullback. □

Lemma 3.4.7. *Suppose $M : \mathcal{C} \to \mathcal{S}$ is a model, $A \rightarrowtail C$ an arrow in $\mathcal{M}$, $c \in_> M(A)$, and $f : C \to D$ any morphism. Then $f \cdot c \in_> M(B)$, where $B \rightarrowtail D$ is the image of the composite:*

$$A \rightarrowtail C \xrightarrow{f} D \tag{3.28}$$

Proof. Without loss of generality, it can be assumed that $A = C$ and that f is an arrow in $\mathcal{E}$. Suppose $A' \rightarrowtail D$ is such that $f \cdot c \in M(A')$. Since M is a model, there is a square (3.29) and $e \in M(E)$ such that $g \cdot e = c$. Since $c \in_> M(C)$, we must have $g \in \mathcal{E}$, so there is a diagonal filler

as drawn, sending c to $f \cdot c \in M(A')$. By commutativity, it follows that m be in $\mathcal{E} \cap \mathcal{M}$, hence an isomorphism.

$$\begin{array}{ccc} E & \longrightarrow & A' \\ {\scriptstyle g}\downarrow & \nearrow_{f} & \downarrow{\scriptstyle m} \\ C & \longrightarrow & D \end{array} \tag{3.29}$$

□

Lemma 3.4.8. *Suppose $M : \mathcal{C} \to \mathcal{S}$ is a model and $c \in_> M(A)$. Then A is prime.*

Proof. If $\{f_i : U_i \to A\}$ is a covering family, then there is an i such that for some $u \in M(U_i)$, $f_i \cdot u = c$. But then the image of f_i must cover A, i.e. $f_i : U_i \to A$ alone is covering, hence A is a prime object. □

When $M : \mathcal{C} \to \mathcal{S}$ is a model, one can look at the category $\mathrm{El}_>(M)$ of *embedded elements*, i.e. embeddings $c : h^A \rightarrowtail M$ and compatible arrows $h^f : h^B \rightarrowtail h^A$ between them, which are necessarily embeddings. There is a forgetful functor

$$U : \mathrm{El}_>(M) \to \mathcal{C}^{\mathrm{op}} \tag{3.30}$$

which by Lemma 3.4.8 carves out a full subcategory of prime objects.

The following proposition corresponds to theorem 16 in [Bar86], but note that I have dropped the assumption that $\mathcal{C}$ is left exact. The formulation here is close to proposition 3.6 in [BM87], but again there the left exactness is held on to. Thus again I emphasize that proposition 3.4.9 is a statement about *semantics*: The site $\mathcal{C}$ may be any of the examples suggested by the previous sections.

Proposition 3.4.9. *Let $\mathcal{C}$ be a site with orthogonal factorization system $(\mathcal{E}, \mathcal{M})$. Suppose $M : \mathcal{C} \to \mathcal{S}$ is a flat functor. Then the following statements are equivalent:*

(i) The category $\mathrm{El}_>(M)$ is filtered and M is its colimit as a diagram in $[\mathcal{C}, \mathcal{S}]$:

$$M \cong \operatorname{colim} \mathrm{El}_>(M) : \mathcal{C} \to \mathcal{S} \tag{3.31}$$

(ii) M is a filtered colimit of a diagram of embedded substructures, whose transition arrows are embeddings

(iii) For every element $c \in M(C)$, there exists $m : A \rightarrowtail C$ in $\mathcal{M}$ such that $c \in_> M(A)$

(iv) For every element $c \in M(C)$, its type $\tau(c)$ is a principal filter (see Examples 2.5.2, 2.5.3).

Proof. (ii) ⇒ (iii): Suppose $c : h^C \to M$ is an element, then since $[\mathcal{C}, \mathcal{S}]$ is locally presentable, there is a substructure $h^D \rightarrowtail M$ such that c factors through h^D for some $f : D \to C$. By factoring f as $D \twoheadrightarrow A \rightarrowtail C$ we obtain a factorization of c:

$$h^C \twoheadrightarrow h^A \rightarrowtail M \tag{3.32}$$

Hence $c \in_> M(A)$ by Lemma 3.4.6.

(iii) ⇒ (i): As always, M is the colimit of co-representables on its category of elements. By the assumption and Lemma 3.4.6, the embedded structures $h^C \rightarrowtail M$ form a co-filtered, final subcategory thereof, but this is precisely the category $\mathrm{El}_>(M)$.

(i) ⇒ (ii) and (iii) ⇔ (iv) are immediate. □

The previous proposition captures the same class of models that were singled out in Definition 3.3.11 and which I will repeat. The term 'principal' extends the use of Barr-Makkai [BM87] of this term for principal models of regular sites (see their Proposition 3.6).

Definition 3.4.10. A model $M : \mathcal{C} \to \mathcal{S}$ satisfying one of the equivalent conditions of Proposition 3.4.9 will be called a *principal* model of $\mathcal{C}$.

Before moving on to the next section, I record the following fact, which just says that prime-generatedness is a necessary condition for the argument used in Chapter 4 for proving the result of Theorem 4.4.9 there.

Proposition 3.4.11. *Suppose $M_i : \mathcal{C} \to \mathcal{S}$ is a family of jointly conservative principal models. Then $\mathcal{C}$ is prime-generated.*

Proof. For every object C and every i, the family

$$\{M_i(m) : M_i(A) \to M_i(C) \mid c \in_> M_i(A)\}_{c \in M_i(C)} \tag{3.33}$$

is covering since M_i is principal. By Lemma 3.4.8, each A in (3.33) is prime. Since the M_i are jointly conservative, the totality of such families is a covering of C by prime objects. □

3.5 The Fraïssé-Hrushovski construction

Having established a rigid notion of embedding between models of factorizing sites, it is finally possible to define what it means for a model to be *homogeneous* in the sense of Definition 3.2.12, that is, what it means to extend a morphism between embedded substructures to an endomorphism of the whole.

Definition 3.5.1. A model $M : \mathcal{C} \to \mathcal{S}$ is called *homogeneous* if for every pair of embedded substructures $c : h^C \rightarrowtail M$, $d : h^D \rightarrowtail M$ and every morphism $g : C \to D$ in $\mathcal{C}$, there is an endomorphism $\eta : M \to M$ that makes the diagram (3.34) commute.

$$\begin{array}{ccc} h^D & \xrightarrow{g} & h^C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle c} \\ M & \xrightarrow{\eta} & M \end{array} \tag{3.34}$$

However, it turns out that the above definition doesn't quite capture the essence of structures that arise from a categorical Fraïssé construction on general sites. This is because, when considering multiple models, there is no reason for the extension of a morphism $f : h^D \to h^C$, where $h^D \rightarrowtail M$ is an (embedded) substructure, to map back into M.

The following definition gives the appropriate strengthening of Definition 3.5.1 and will be called *prime*. I show below that the categorical Fraïssé-Hrushovski construction indeed yields models that satisfy this stronger notion. In Chapter 4, the consequences of this will be explored in the context of **M**-sets. The predicate 'prime' has been used by Barr-Makkai [BM87] for models of regular theories that satisfy a similar property with respect to the types of their elements. It also appears as 'principal prime' in Makkai [Mak90], again denoting something similar for regular theories. The term itself is widely used in model theory, sometimes denoting different concepts, see e.g. Hodges [Hod93] §7.2, §8.3 and §12.2. It is also sometimes called 'atomic', which would make sense for classical first-order logic. Here, 'prime' seems the most appropriate.

Definition 3.5.2. A model $M : \mathcal{C} \to \mathcal{S}$ is called *prime* if whenever $N : \mathcal{C} \to \mathcal{S}$ is another model, $d : h^D \rightarrowtail M$ is an embedding and $c : h^C \to N$ any element, every arrow $g : C \to D$ in $\mathcal{C}$ extends to a natural transformation $\eta : M \to N$ that makes the diagram (3.35) commute:

$$\begin{array}{ccc} h^D & \xrightarrow{g} & h^C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle c} \\ M & \xrightarrow{\eta} & N \end{array} \tag{3.35}$$

In most examples of the Fraïssé-Hrushovski construction, the Fraïssé or Hrushovski classes are at most countable. This is what makes the back-and-forth argument, used to build isomorphisms between homogeneous models, work. In the current setting one faces a similar limitation, although of course, for specific cases, one can always look and see whether the techniques presented here for the countable world may apply to the specific case in mind. One could alternatively develop a rigorous infinitary theory of Fraïssé-Hrushovski constructions, but all that does is axiomatize the countable case with respect to some underlying calculus of cardinals. I will refrain from that here,

pointing to the much more insightful alternative that there is always an open, surjective, localic cover of the base topos $\mathcal{S}$ where Proposition 3.5.3 below holds (Theorem 3.5.12). In a way this is just like building a theory around infinite cardinals, since it just switches to a conservative (for first-order logic), propositional extension of the universe where the old structures may be treated as countable, and in the same sweep it takes care of the requirement of choice in proposition 3.5.3.

In the following proposition, I show that when $\mathcal{S}$ satisfies the axiom of dependent choice and a model $M : \mathcal{C} \to \mathcal{S}$ is principal for a *countable* diagram of embedded substructures, it can be written as a particularly nice colimit which can be used in back-and-forth constructions. Note that it is not necessary to assume that $\mathcal{S}$ is boolean here. There is a striking resemblance between this statement and Theorem 3.2.9, even though the starting data for both constructions do not quite align. For the statement of the proposition, I have drawn from Kubiś notion of *Fraïssé sequence* [Kub14], which is developed there for the general infinitary case. The *forcing condition* (i) of Proposition 3.5.3 extends his *amalgamation property* (A), a change which should feel familiar by now.

Proposition 3.5.3. *Suppose $\mathcal{S}$ satisfies the axiom of dependent choice. Then when $M : \mathcal{C} \to \mathcal{S}$ is a principal model for a countable diagram of embedded substructures, there exists a diagram of prime objects in $\mathcal{C}$:* (3.36)*:*

$$M_0 \twoheadleftarrow M_1 \twoheadleftarrow M_2 \twoheadleftarrow \dots \text{ such that } M \cong \operatorname{colim} h^{M_i} \tag{3.36}$$

satisfying for every $i \geq 0$ the forcing condition:

(i) Whenever $f : D \twoheadrightarrow M_i$ is a cover, there exists $j \geq i$ and an arrow $g : M_j \to D$ such that $f \circ g = m_i^j$.

where $m_i^j : M_j \twoheadrightarrow M_i$ is the unique transition arrow composed along the sequence.

Proof. First, M is a filtered colimit of embedded substructures:

$$M \cong \operatorname{colim} h^{C_i} \text{ where } C_{(-)} : \mathcal{I} \to \mathcal{C} \tag{3.37}$$

where $\mathcal{I}$ is a countable category. It is well-known (e.g. [AR94], 1.6) that such a colimit can be rewritten as the colimit over a chain, but using dependent choice in $\mathcal{S}$, we can actually find a final functor $m_{(-)} : \omega \to \mathcal{I}$ in stages. Enumerate the objects of $\mathcal{I}$ by $i_0, i_1, \dots$ and enumerate its morphisms by $e_0, e_1, \dots$, in such a way that $\operatorname{dom} e_n = C_k$ for some $k \leq n$.

Let $m_0 := i_0$. Then at stage $2n+1$, one has the following diagram in $\mathcal{I}$:

$$\begin{array}{ccc} i_k & \rightarrowtail & m_{2n} \\ {\scriptstyle e_n}\downarrow & & \downarrow \\ i_l & \dashrightarrow & m_{2n+1} \end{array} \tag{3.38}$$

Which since $\mathcal{I}$ is directed can be completed to a square for a choice of m_{2n+1}.

At stage $2n+2$, one chooses m_{2n+2} in $\mathcal{I}$ as:

$$\begin{array}{ccc} & & m_{2n+1} \\ & & \rotatebox{90}{$\rightarrowtail$} \\ & & \vdots \\ & & \rotatebox{90}{$\rightarrow$} \\ i_{n+1} & \rightarrowtail & m_{2n+2} \end{array} \tag{3.39}$$

Then it is easily verified that $m_{(-)} : \omega \to \mathcal{I}$ is final. The forcing condition (i) follows from the fact that M is a model. □

The following can be considered a categorical version of the 'forth' method of constructing homomorphisms in model theory.

Proposition 3.5.4. *Suppose $M, N : \mathcal{C} \to \mathcal{S}$ are a pair of models such that:*

(i) M satisfies the conclusion of Proposition 3.5.3

(ii) N, as a filtered colimit of its elements, has explicit witnesses for the forcing condition.

Then the rest of the statement of Definition 3.5.2 holds, i.e. every homomorphism $g : h^D \to h^C$, for an embedded substructure $h^D \rightarrowtail M$ and an element $h^C \to N$, extends to a natural transformation $\eta : M \Rightarrow N$, i.e. such that $\eta d = gc$.

When $\mathcal{S}$ satisfies the axiom of dependent choice, condition (ii) drops out, in which case M is a prime model as in Definition 3.5.2.

Proof. First, write $N = \operatorname{colim}_j h^{N_j}$ for $N_{(-)} : J \to \mathcal{C}$. Then c factors through $c' : h^C \to h^{N_{j_0}}$ for some j_0. Similar, the embedding $d : h^D \rightarrowtail M$ factors through some $h^{M_k} \rightarrowtail M$ via an embedding $d : h^D \rightarrowtail h^{M_k}$. So for d, g, c as in diagram (3.34), there is a diagram:

$$\begin{array}{ccccc} h^D & \xrightarrow{g} & h^C & \xrightarrow{c'} & h^{N_{j_0}} \\ {\scriptstyle d'}\downarrow & & & & \downarrow{\scriptstyle n_0^1} \\ h^{M_k} & & \xrightarrow{h_0} & & h^{N_{j_1}} \end{array} \tag{3.40}$$

Using successively the covering condition and the forcing condition (i) for $N_{(-)}$, we obtain the dashed arrows, where j_0^1 is some transition arrow, and thus extend $g : h^D \to h^C$ to $h_0 : h^{M_k} \to h^{N_{j_1}}$.

We will keep extending this until we exhaust M.

Concretely, we inductively define a sequence of arrows $h_i : h^{M_{k+i}} \to h^{N_{j_i}}$ for $0 \leq i < \omega$, with h_0 as above, such that all diagrams of the form (3.41), for $0 \leq i < i' < \omega$, commute.

$$\begin{array}{ccc} h^{M_{k+i}} & \xrightarrow{h_i} & h^{N_{j_i}} \\ {\scriptstyle m_{k+i}^{k+i'}}\downarrow & & \downarrow{\scriptstyle n_{j_i}^{j_{i+i'}}} \\ h^{M_{k+i'}} & \xrightarrow{h_{i+i'}} & h^{N_{j_{i+i'}}} \end{array} \tag{3.41}$$

If h_i has been defined, then h_{i+1} can be defined by combining the covering condition with condition (i):

$$\begin{array}{ccc} h^{M_{k+i}} & \xrightarrow{h_i} & h^{N_{l_i}} \\ {\scriptstyle m_{k+i}^{k+i+1}}\downarrow & & \downarrow{\scriptstyle n_{j_i}^{j_{i+1}}} \\ h^{M_{k+i+1}} & \xrightarrow{h_{i+1}} & h^{N_{l_{i+1}}} \end{array} \tag{3.42}$$

Clearly, family of morphisms constructed in this way satisfies (3.41).

It remains to verify that the h_i together define a cocone:

$$\eta : M \cong \operatorname{colim}_{i \geq k} h^{M_i} \Rightarrow N, \tag{3.43}$$

But this is immediate. Commutativity of the resulting diagram (3.34) is also clear, so we have extended h to a natural transformation $\eta : M \Rightarrow N$.

The last statement is easy since dependent choice just proves assumption (ii) for any model N. □

The following corollary is of course immediate:

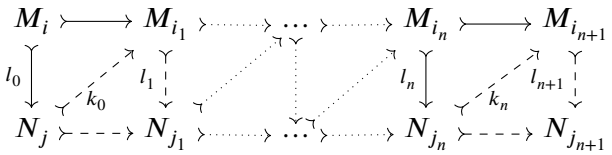
Corollary 3.5.5. *Suppose $M : \mathcal{C} \to \mathcal{S}$ is a model satisfying the resulting statement of Proposition 3.5.3. Then M is homogeneous.* □

3.5.1 Uniqueness of models

Early on, in Example 3.2.15, it has been clarified that a homogeneous model in the weak sense of Definition 3.5.1 is in no way unique. However, there are cases where this happens nevertheless. Since this is essentially the theory of Caramello [Car14]; [Car12] as well as Kubiś [Kub14], I will be brief, but it does have some implications for the Hrushovski construction which I discuss in Section 3.7 below. The proof method may also satisfy any reader who feels that, after the exposition of the 'forth' argument in Proposition 3.5.4, the full 'back-and-forth' argument has been held in suspense. Note that the premiss of the proposition below is essentially the requirement that M and N are countable models of $\mathcal{C}$, since every model of an atomic site is principal.

Proposition 3.5.6. *Assume $\mathcal{C}$ is an atomic site, i.e. all arrows between prime objects are covers. Suppose further that M_i, N_i are sequences as in* (3.36)*, satisfying for every $i \geq 0$ the forcing condition (i) of Proposition 3.5.3. Then every embedding $l_0 : h^D \rightarrowtail h^C$ between embedded substructures of M, N respectively extends to an isomorphism $\lambda : M \Rightarrow N$.*

Proof. The idea of the proof is the same as in Proposition 3.5.4, but this time one constructs two (inverse) homomorphisms in a double induction. By the covering condition it can be assumed that $D = M_i$, $C = N_j$ for some i, j. Drawing the arrows in $\mathcal{C}^{\mathrm{op}}$, we obtain a diagram:



(3.44)

By going back and forth as drawn, using the covering condition and the forcing condition, making fruitful use of the fact that all arrows remain covers, we are ensuring at each stage that $k_n \circ l_n = m^{i_n}_{i_{n+1}}$ and $l_{n+1} \circ k_n = n^{j_n}_{j_{n+1}}$, yielding inverse natural transformations $\lambda : M \Rightarrow N$, $\kappa : N \Rightarrow M$ between the colimits. □

3.5.2 Existence of models

Clearly, Proposition 3.5.3 has a converse, namely that any diagram of shape (3.36), satisfying the forcing condition, yields a principal model; this follows straight from Definition 3.4.10. So under dependent choice, existence of a countable principal model is equivalent to existence of a sequence as in Proposition 3.5.3, and in the case when $\mathcal{C}$ is *separable*, i.e. it has a countable set of objects and countable hom-sets, and when $\mathcal{C}$ has enough prime objects, not much different from existence of any point for the topos $\mathrm{Sh}(\mathcal{C})$. This is shown explicitly in Proposition 3.11 in Barr-Makkai [BM87], but for slightly stronger assumptions on the site. For a proof under the present assumptions, one may as well extend the approach by Kubiś and derive the result using a Baire category argument, which in addition to dependent choice, requires classical logic:

Proposition 3.5.7 (cf. Kubiś [Kub14], corollary 3.8)**.** *Assume that $\mathcal{S}$ is a boolean topos which validates the axiom of dependent choice. Suppose $\mathcal{C}$ is a separable, prime-generated site and let A be a prime object of $\mathcal{C}$. Then there exists a diagram of prime objects* (3.36) *satisfying the forcing condition (i) of Proposition 3.5.3 and such that $M_0 := A$.*

Proof. Recall that under classical logic and dependent choice, any Polish space is a Baire space. Observe that functors $M_{(-)} : \omega \to \mathcal{C}^{\mathrm{op}}$ mapping into prime objects and covers and subjected to $M_0 := A$ form closed subspace $\mathcal{F}$ of the Polish space $C_1^{\{\langle n,m\rangle \mid n \leq m\}}$, where C_1 is the (countable)

set of arrows of $\mathcal{C}^{\mathrm{op}}$. For every cover $f : D \twoheadrightarrow C$ of $\mathcal{C}$, and every $i < \omega$, the subspace $V_{i,f} \subset \mathcal{F}$ defined by

$$V_{i,f} := \{ M_{(-)} \in \mathcal{F} \mid (M_i = C) \Rightarrow (\exists j \geq i)(\exists g : M_j \to D) f \circ g = m_i^j \} \tag{3.45}$$

is open and dense, where $m_i^j : M_j \twoheadrightarrow M_i$ is the unique transition arrow. Indeed, if $M : \omega \to \mathcal{C}^{\mathrm{op}}$ is a functor in $V_{i,f}$, as witnessed by some j, g, then any sequence that agrees with M up to j is also an element of $V_{i,f}$, hence it is open. Similarly, if $M : \omega \to \mathcal{C}^{\mathrm{op}}$ is any sequence in $\mathcal{F}$ and $M \notin V_{i,f}$, and $i \geq k$ is any number, then $M_i = C$, and we can use the coverage condition on primes:

$$\begin{array}{ccc} M_i = C & \xleftarrow{m_i^k} & M_k \\ {\scriptstyle f}\uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\ D & \leftarrow\!\!\!\!\!\!- & M'_{k+1} \end{array} \tag{3.46}$$

to define an initial segment $M'_1, \ldots, M'_{k+1}$ which agrees with $M_{(-)}$ up to k and defines an open subset entirely within $V_{i,f}$.

By the statement in the beginning, there must exists a sequence in the intersection of all $V_{i,f}$, which hence satisfies all properties in Proposition 3.5.3. □

Remark 3.5.8. Theorem 3.5.7 is the version of the theorems of Fraïssé and Hrushovski that has been anticipated throughout the preceding sections. In particular it applies to the category $\mathcal{H}_{k+1}$ of Theorem 3.3.18, producing a homogeneous $(k+1)$-hypergraph with respect to well-embeddings. However, here only h^A is necessarily well-embedded into this graph, which will be improved to *all* finite h^C in Corollary 3.6.3 below.

Remark 3.5.9. Observe that the previous proposition implies a weak version of the *omitting types theorem* (cf. Hodges [Hod93], §7.2). Since the constructed model is principal, all its types are principal (proposition 3.4.9). So any non-principal prime filter in $\mathcal{C}$ is omitted.

Remark 3.5.10. To show that Theorem 3.5.7 is sometimes the best one can get, I recall the example of a connected atomic topos without points (Johnstone [Joh02b], D3.4.14). This is an atomic site, satisfying even the JEP (see below), so it fits all definitions above (although, of course, it is not separable), but it has no models in Set.

As discussed in the beginning, there may be situations when $\mathcal{C}$ is not separable or when dependent choice does not hold, but when there is nevertheless a principal model $M : \mathcal{C} \to \mathcal{S}$. In those cases one may be able to prove directly that M is homogeneous, but it may also not be provable at all. For those cases, I recall the following topos-theoretic result:

Lemma 3.5.11 ([Joh02b], Lemma C5.1.11)**.** *Suppose $\mathbb{C}$ is a filtered internal category in $\mathcal{S}$. Then there exists an open localic surjection $p : \mathcal{S}' \to \mathcal{S}$ and a cofinal functor $\omega \to p^*\mathbb{C}$ in $\mathcal{S}'$, where ω is the natural numbers object equipped with the natural order relation.* □

In the above, open and surjective implies that p^* preserves and reflects first-order logic; so change of base along p can be seen as a conservative extension of the universe; since it is localic it is a propositional extension. So:

Theorem 3.5.12. *Suppose $M : \mathcal{C} \to \mathcal{S}$ is a principal model of a (small) site $\mathcal{C}$. Then there exists an open localic surjection $p : \mathcal{S}' \to \mathcal{S}$ such that $p^*M : p^*\mathcal{C} \to \mathcal{S}'$ is a principal homogeneous model of $p^*\mathcal{C}$.*

Proof. Since M is principal, $\mathrm{El}_>(M)$ is a filtered category. By Lemma 3.5.11, there exists an open localic surjection $p : \mathcal{S}' \to \mathcal{S}$ and a final functor $\omega \to p^* \mathrm{El}_>(M)$ in $\mathcal{S}'$. Since p is open and surjective, p^* preserves the fact that M is a principal model of $\mathcal{C}$ and $\mathrm{El}_>(p^*M) \cong p^* \mathrm{El}_>(M)$.

So in $\mathcal{S}'$, there is a cofinal functor $\omega \to \mathrm{El}_>(p^*M)^{\mathrm{op}}$ and since p^*M is a model, it satisfies the conditions of Proposition 3.5.3 (as in the proof of that proposition). So by Proposition 3.5.4, p^*M is homogeneous. □

Remark 3.5.13. Since the notion of a principal (and homogeneous) model depends on the underlying site, Theorem 3.5.12 does not say anything about possible 'principal' geometric morphisms $\mathcal{S} \to \mathrm{Sh}(\mathcal{C})$. Of course, this raises many questions about the scope and naturality of the notion of a principal model, but these are left for the future.

3.6 Conservative models

With Proposition 3.5.7 I have presented a categorical Fraïssé-Hrushovski construction which yields homogeneous models for prime-generated sites with an orthogonal factorization system. Under the stated assumptions, it is however inherently non-constructive. Often, one can just find a (countable) principal model, and verify using Corollary 3.5.4 that it is homogeneous. This is the case for all examples from Chapter 1. In addition, the examples from Chapter 1 are all *conservative*:

Definition 3.6.1. A model $M : \mathcal{C} \to \mathcal{S}$ is *conservative* if whenever $\mathcal{U} = \{u_i : U_i \to U\}$ is a family such that $M\mathcal{U} = \{Mu_i : M(U_i) \to M(C)\}$ is jointly epimorphic, then there exists a family $\mathcal{V} \in \mathrm{Cov}_{\mathcal{C}}(C)$ such that $\mathcal{V} \preceq \mathcal{U}$.

This condition was called *co-continuous* in Theorem 2.2.4. One can show that the resulting geometric morphism $\mathcal{S} \to \mathrm{Sh}(\mathcal{C})$ is conservative in the topos-theoretic sense, namely that the left adjoint is fully faithful.

The previous section raises the question of precisely when the sequence (3.36) gives rise to a conservative model. To answer this, the condition (U) isolated by Kubis [Kub14] can be adapted to the principal models in this thesis:

(U) For every prime object P of $\mathcal{C}$, there exists a $k \geq 0$ and a cover $M_k \twoheadrightarrow P$.

It can be easily seen from the proof of 3.5.3 that this condition can be guaranteed if the category $\mathcal{C}$ satisfies the following additional condition, which dual to the *joint embedding property* (JEP) from classical Fraïssé theory and was already encountered in Definition 3.3.12. This dual will be called the *joint covering condition* (JCC).

(JCC) For any two prime objects A, B of $\mathcal{C}$, there exists a prime object C in $\mathcal{C}$ such that there are covers $C \twoheadrightarrow A$, $C \twoheadrightarrow B$.

Proposition 3.6.2. *If the category $\mathcal{C}$ satisfies (JCC), then, under the same premiss, the sequence* (3.36) *of Proposition 3.5.3 can be improved to also satisfy the condition (U) above.* □

Corollary 3.6.3. *Assume that $\mathcal{S}$ satisfies the axiom of dependent choice, and $\mathcal{C}$ is a separable, prime-generated site which satisfies (JCC). Then there exists a principal, homogeneous and conservative model of $\mathcal{C}$.*

Proof. This follows from Propositions 3.5.7 and 3.6.2, but of course the proof of Proposition 3.5.7 can also be modified directly. □

Proposition 3.6.4. *Suppose $\mathcal{C}$ is a separable, prime-generated, factorizing site, and let M_i be a sequence as in* (3.36)*. If this sequence satisfies both the forcing condition (i) of Proposition 3.5.3 and the property (U) above, then the model M constructed as its colimit is conservative.*

Proof. By factorization, is enough to prove that if $g : A \rightarrowtail C$ is an arrow in $\mathcal{M}$ such that $M(g)$ is an isomorphism, then g is an isomorphism. So assume g is such that $M(g)$ is an isomorphism. By condition (U), and prime-generatedness, there exists a covering family $f_i : M_{k_i} \to C$ of C. By the assumption on $M(g)$, there exists a family $f_i' : M_{l_i} \to A$, $l_i \geq k_i$, such that $g \circ f_i' = f_i \circ m_{k_i}^{l_i}$. Now the family on the right-hand side of this equality is covering; hence g must be a cover. □

Combining the above with the full back-and-forth argument for atomic sites, this yields a categorical version of $\aleph_0$-categoricity for complete atomic theories. This is also obtained by Barr-Makkai [BM87], theorem 3.29. Note that they also prove a converse to this statement.

Corollary 3.6.5. *If $\mathcal{C}$ is a separable atomic site satisfying (JCC), then a principal model M defined by a sequence* (3.36) *which satisfies the forcing condition (i) as well as condition (U) above is unique up to isomorphism.*

Proof. For M, N two such models, there is by condition (U) an embedding $h^{N_0} \rightarrowtail h^{M_k}$. By Proposition 3.5.6, this extends to an isomorphism between M and N. □

3.7 Boolean covers

Lastly, I would like to briefly address an issue that arises when comparing Examples 3.2.10 and 3.3.13 carefully. Namely, that the class of embeddings $\mathcal{M}$ in $\mathcal{H}_{k+1}$, for which the amalgamation property has been 'replaced' by the forcing property, does anyway satisfy the amalgamation property in the sense of the classical Hrushovski construction: Every cospan (3.47), drawn as solid arrows, can be completed to a square:

$$\begin{array}{ccc} C & \rightarrowtail & A \\ \downarrow & & \vdots \\ D & \dashrightarrow & B \end{array} \tag{3.47}$$

In other words, if one removes all non-isomorphism arrows in $\mathcal{E}$ from $\mathcal{H}_{k+1}$, one is left with the opposite of an atomic site. This is not a coincidence, in fact it is a consequence of a straightforward construction of boolean covers for conservative models. Indeed if $\mathcal{C}$ is a factorizing site and M a principal, conservative model, then a subcategory of $\mathcal{C}$ can be defined by taking all objects in the image of the forgetful functor $U : \mathrm{El}_{>}(M) \to \mathcal{C}^{\mathrm{op}}$ described in (3.30) and all *covers* between them. This yields a subcategory $\mathcal{A}$ where all arrows are covers. Since M is *flat*, this subcategory satisfies the coverage condition, since any span in this subcategory can be presented as lying in the image of U^{op} (since M is continuous), and $\mathrm{El}_{>}(M)$ is a filtered category (since M is principal). For the inclusion $i : \mathcal{A} \to \mathcal{C}$, the induced map:

$$\mathcal{C}(i(-), -) : \mathcal{C} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{S}] \tag{3.48}$$

is continuous (this is the coverage condition on $\mathcal{C}$) and flat (immediate), so it induces a geometric morphism:

$$j : \mathrm{Sh}(\mathcal{A}) \to \mathrm{Sh}(\mathcal{C}) \tag{3.49}$$

The model M, as a functor $\mathcal{S} \to \mathrm{Sh}(\mathcal{C})$ factors through it, so when M is conservative, j is a geometric surjection. The connection with the Hrushovksi construction is the following corollary:

Corollary 3.7.1. *Suppose $\mathcal{C}$ is a factorizing site and M a principal, conservative model, presentable as in Proposition 3.5.3. Then every embedding $h^C \rightarrowtail h^D$ between embedded substructures of M extends to an automorphism of the restricted model $M : \mathcal{A} \to \mathcal{S}$ defined above. If $\mathcal{S}$ satisfies the axiom of dependent choice, then M is $\aleph_0$-categorical as a model of $\mathcal{A}$.*

Proof. From the above discussion and Proposition 3.5.6. □

Example 3.7.2. For the category of Hrushovski graphs $\mathcal{H}_{k+1}$, the conservative, principal, homogeneous model M, obtained by either Corollary 3.6.3 or the classical Hrushovski construction (Theorem 3.2.9) is not $\aleph_0$-categorical. I had announced this already in Remark 3.3.16 and I will say a bit more about it in Example 4.5.3.

For this situation, Corollary 3.7.1 gives the last part of Theorem 3.2.9 for free, even though it is not covered by the categorical Fraïssé Hrushovski construction that is Corollary 3.6.3. Hence in general, this construction gives extensions of embeddings to automorphisms for free. See also Remark 3.2.13.

4 Support

In Chapter 1 I have presented various examples of toposes defined as finitely supported **M**-sets, where **M** is a monoid of endomorphisms of some structure. In Chapter 3, I have shown how these structures can be understood as homogeneous structures, in the sense of model theory, for an underlying factorizing site. Having thus situated the model theory of Chapter 3 in the realm of topos theory, this chapter revisits **M**-sets and the notion of support. The structure of support brings forth a notion of *age* for a category of models. This leads to the main thesis that an *age* is a factorizing prime site.

The first section below explores a definition of *support* that encompasses all previous examples. As it is defined here, it allows for a clean 'calculus of orbits' using the language of factorizing prime sites. In the next section, I show how the topos of supported **M**-sets it defines also arises as the hyperconnected-localic image of a geometric morphism. The notion of *prime* or *homogeneous* model returns when comparing this to the surjection-inclusion factorization. Section 4.4 combines the theory of Chapter 3 with the new class of toposes and yields the results from which all equivalences of toposes from the first chapter can be derived. This includes a digression into flat and free models of finitary algebraic theories, which finds an application for the topos of cubical sets. As an application in classical logic, I investigate and prove a formulation of the Ryll-Nardzewski criterion from classical model theory for the homogeneous structures from Chapter 3, entirely within the new abstract setting. In the last section, I briefly look at models of intuitionistic set theory. The result is that a presheaf interpretation of intuitionistic set theory with atoms (IZFA) due to Scott restricts to hereditarily (finitely) supported sets. Hence, from the theory presented in this chapter, a new class of models for IZFA can be obtained, which generalize the well-known Fraenkel-Mostowski models for ZFA.

4.1 The topos of (finitely) supported M-sets

I start by defining a notion of *support* for arrows in arbitrary categories. For any arrow $f : A \to B$ in a category with finite limits, recall that the *kernel pair* of f is defined as the pullback:

$$\begin{array}{ccc} \ker f & \xrightarrow{k_0} & A \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array} \tag{4.1}$$

By the universal property of the pullback, this defines a subobject $\ker f \hookrightarrow A \times A$.

Proposition 4.1.1. *For $h : A' \to A$, $f : A \to B$ arrows in a category with finite limits, one has $\ker(f \circ h) = (h \times h)^* \ker f$ as subobjects of $A' \times A'$.*

Proof. This is an easy verification using the two defining universal properties of $\ker f \circ h$ and $(h \times h)^* \ker f$. □

Definition 4.1.2. Suppose $\mathbb{C}$ is a category with finite limits. Then an arrow $f : A \to B$ is *supported* by an arrow $g : A \to C$ if $\ker g \subseteq \ker f$ as subobjects of $A \times A$.

Remark 4.1.3. In fact, Definition 4.1.2 yields a notion of support for any (small) category: one can say f is supported by g whenever y_f is supported by y_g in $[\mathbb{C}^{\mathrm{op}}, \mathcal{S}]$, and of course this can also be explicitly spelled out to yield a notion for large categories.

Proposition 4.1.4. *Suppose $f : A \to B$ is an arrow in a category that admits a decomposition:*

$$f = h \circ f' \tag{4.2}$$

Then f is supported by f'. If $h : A' \to B$ is a monomorphism, then f' is supported by f.

Proof. The first part follows since $\ker f' \subseteq \ker f$. The second part uses:

$$\ker f = (f' \times f')^* \ker h = (f' \times f')^* \ker 1_{A'} = \ker f' \tag{4.3}$$

□

Proposition 4.1.5. *Suppose $\mathbb{C}$ is a small category. For an object A of $\mathbb{C}$, the notion of* support *defines a preorder on the arrows with domain A. Equipped with this preorder, the co-representable presheaves form a $\mathbb{C}$-indexed preorder:*

$$h^{(-)} : \mathbb{C}^{\mathrm{op}} \to \mathrm{PreOrd} \tag{4.4}$$

Proof. The first part is immediate. For the second part, Proposition 4.1.1 yields that if $f : A \to B$ is supported by $g : A \to C$ and $h : A' \to A$, then $f \circ h$ is supported by $g \circ h$, hence the preorder relation is preserved under re-indexing. □

For functors $X, Y : \mathbf{M} \to \mathcal{S}$, an object M of $\mathbf{M}$ and elements $x \in X(M)$, $y \in Y(M)$, say that x is *supported* by y if for the corresponding arrows, $x : h^M \to X$ is supported by $y : h^M \to Y$.

Definition 4.1.6. Suppose $\mathbf{M}$ is a small category and let

$$\Phi : \mathbb{C} \to [\mathbf{M}, \mathcal{S}] \tag{4.5}$$

be a functor. A functor $X : \mathbf{M} \to \mathcal{S}$ is *supported* by Φ if every element $x \in X(M)$ is supported by an element $s \in \Phi(C)(M)$ for some C, in other words the element $x : h^M \to X$ is supported by an object in $h^M \Downarrow \Phi$. Such an element s is called a *support* for x. The full subcategory of $[\mathbf{M}, \mathcal{S}]$ on functors supported by Φ is denoted $[\mathbf{M}, \mathcal{S}]_\Phi$.

Proposition 4.1.7. *If* $\mathbf{M}$ *is a small category and* Φ *a functor as in* (4.5)*, then the inclusion*

$$[\mathbf{M}, \mathcal{S}]_\Phi \hookrightarrow [\mathbf{M}, \mathcal{S}] \tag{4.6}$$

has a right adjoint.

Proof. For every $X : \mathbf{M} \to \mathcal{S}$, define the $X_\Phi : \mathbf{M} \to \mathcal{S}$ by:

$$X_\Phi(M) = \{x \in X(M) \mid x \text{ is supported by some } s \in \Phi(C)(M) \text{ for some } C\} \tag{4.7}$$

By Proposition 4.1.5, the $\mathbf{M}$-action of X restricts to an $\mathbf{M}$-action on X_Φ and hence defines a functor. By Proposition 4.1.4, natural transformations $X \Rightarrow Y$ preserve supports of elements, hence they restrict to a natural transformation $X_\Phi \Rightarrow Y_\Phi$. For the same reason, there is a bijective correspondence

$$[\mathbf{M}, \mathcal{S}](F, X) \cong [\mathbf{M}, \mathcal{S}]_\Phi(F, X_\Phi) \tag{4.8}$$

where F is supported by Φ, which establishes the adjunction. □

Proposition 4.1.8. *Suppose* $\mathbf{M}$ *and* Φ *are as in Proposition 4.1.7. Then the following statements are equivalent:*

(i) For every object M *of* $\mathbf{M}$*, and any finite family of arrows*

$$\{s_i : h^M \to \Phi(C_i)\} \tag{4.9}$$

there is a single arrow $s : h^M \to \Phi(C)$ *such that* s *supports* s_i *for every* i*.*

(ii) The inclusion $[\mathbf{M}, \mathcal{S}]_\Phi \hookrightarrow [\mathbf{M}, \mathcal{S}]$ *creates finite limits*

When either one of these conditions are satisfied, the category $[\mathbf{M}, \mathcal{S}]_{\Phi}$ *is a topos.*

Proof. (i) '⇒' (ii): Observe that the terminal object of $[\mathbf{M}, \mathcal{S}]$ is supported by Φ if and only if for every M, there exists C such that $\Phi(C)(M)$ is non-empty. This follows from (i) by taking the empty family.

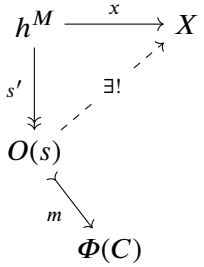
Second, suppose $X, Y : \mathbf{M} \to \mathcal{S}$ are functors with $x \in X(M)$, $y \in Y(M)$ supported by $s_0 \in \Phi(C_0)(M)$, $s_0 \in \Phi(C_1)(M)$ respectively. By (i), there exists $s \in \Phi(C)(M)$ which supports s_0 and s_1. Hence x and y are both supported by s, so the pair (x, y) is supported by s. It follows that the subcategory (4.6) is closed under pullbacks. So the inclusion creates all finite limits.

(ii) '⇐' (i): For a finite family $\{s_i : h^M \to \Phi(C_i)\}$, the induced element of the product $s : h^M \to \Pi_i \Phi(C_i)$ must by assumption be supported by some $t : h^M \to \Phi(C)$. By Proposition 4.1.4, the s_i are supported by s, hence they are all supported by t.

The last statement follows since the inclusion is comonadic for a (by (ii)) cartesian comonad (see e.g. theorem A4.2.1 of Johnstone [Joh02a]). In the present case, it would also follow from Lemma 4.6.4 below. □

Definition 4.1.9. For $\mathbf{M}$ a small category, a functor Φ as in (4.5) is called *supportive* if the two equivalent conditions of Proposition (4.1.8) are satisfied.

Suppose $X : \mathbf{M} \to \mathcal{S}$ is a functor, and suppose $x \in X(M)$ is supported by $s : h^M \to \Phi(C)$. By factoring $s = m \circ s'$ as an epimorphism followed by a monomorphism in $[\mathbf{M}, \mathcal{S}]$, one obtains the following situation:



(4.10)

Since x is supported by s, it is by Proposition 4.1.4 also supported by s'. Because in a topos epis are coequalizers of their kernel pair, this implies by definition that there must be a unique factoring morphism as drawn. The object $O(s)$ is the smallest subobject of $\Phi(C)$ containing s and is called the *orbit* of s.

Proposition 4.1.10. *Suppose* $\mathbf{M}$*,* Φ *are as above with* $\mathbb{C}$ *a small category. Then the family of* orbits *forms a small separating family of objects in* $[\mathbf{M}, \mathcal{S}]_{\Phi}$*. Hence when* Φ *is supportive (Definition 4.1.9) the latter is a Grothendieck topos.*

Proof. It is easy to see that the orbits $O(s)$ are supported by Φ. Since $\mathbb{C}$ is small and by (4.10), they form a separating family for the objects of $[\mathbf{M}, \mathcal{S}]_\Phi$. Since the inclusion (4.6) creates colimits, the conclusion follows from Giraud's theorem (see e.g. Johnstone [Joh02b], Theorem C2.2.8). □

So under the stated conditions, the full subcategory on orbits $O(s)$, equipped with the canonical coverage, forms a small *site* for which $[\mathbf{M}, \mathcal{S}]_\Phi$ is the topos of sheaves. Observe that if $\{f_i : U_i \to O(s)\}$ is jointly epimorphic, where:

$$s : h^M \twoheadrightarrow O(s) \rightarrowtail \Phi(C) \tag{4.11}$$

then there must be some i and some element $u \in U_i(M)$ such that $(f_i)_M u = s$[1]. But then the single morphism $f_i : U_i \to O(s)$ is already a cover. So the orbits are prime objects for the canonical topology, and hence it follows that the full subcategory of orbits forms a site of prime objects (Definition 2.3.2). This site is denoted $\mathcal{O}$. Note that for every morphism $f : O(s) \to O(t)$ between orbits, there is a commutative square:

$$\begin{array}{ccc} h^M & \twoheadrightarrow & O(s) \\ {\scriptstyle h^\alpha}\downarrow & & \downarrow{\scriptstyle f} \\ h^N & \twoheadrightarrow & O(t) \end{array} \tag{4.12}$$

for some $\alpha : N \to M$ in $\mathbf{M}$. Recall that the following proposition was anticipated in Chapter 3, Proposition 3.3.2.

Proposition 4.1.11. *Suppose Φ is supportive. Then the single covering arrows in $\mathcal{O}$ form the left class of an orthogonal factorization system on $\mathcal{O}$. So $\mathcal{O}$ is a factorizing prime site.*

Proof. Observe that the single covers form a class of arrows which satisfies the first requirement of lemma 2.4.2. Second, if $f : O(t) \to O(s)$ is a morphism between orbits, the composite $O(t) \to \Phi(C)$ has some orbit $O(s')$ as its image:

$$\begin{array}{ccccc} h^{M'} & \twoheadrightarrow & O(t) & \rightarrowtail & \Phi(D) \\ & & \downarrow{\scriptstyle f} & \nearrow{\scriptstyle i} & \\ & & O(s') & & \\ h^M & \twoheadrightarrow & O(s) & \rightarrowtail & \Phi(C) \end{array} \tag{4.13}$$

[1]Note that this uses that a jointly epimorphic family in $[\mathbf{M}, \mathcal{S}]_\Phi$ is also a jointly epimorphic family in $[\mathbf{M}, \mathcal{S}]$. This follows because every such family factors as a family of coproduct inclusions followed by a single epimorphism, and the former two are preserved by the (left adjoint) inclusion.

Where the arrow i is the unique arrow induced by the epi-mono factorization system. Hence every morphism between orbits factors as a cover followed by a morphism fitting in a triangle $(*)$. By Lemma 2.4.2, it remains to show that such morphisms are right orthogonal to covers. But since i has to be a monomorphism, this follows from ordinary epi-mono factorization in $[\mathbf{M}, \mathcal{S}]$. □

So far, the arrows in the category $\mathbb{C}$ have remained completely silent. An arrow $f : C \to D$ in $\mathbb{C}$ may induce a morphism between orbits in the following way: Suppose $s \in \Phi(C)(M), t \in \Phi(D)(N)$ and $\alpha : N \to M$ are such that the following diagram commutes:

$$\begin{array}{ccccc} h^M & \twoheadrightarrow & O(s) & \rightarrowtail & \Phi(C) \\ {\scriptstyle h^\alpha}\downarrow & & \vdots\,{\scriptstyle [f]} & & \downarrow{\scriptstyle \Phi(f)} \\ h^N & \twoheadrightarrow & O(t) & \rightarrowtail & \Phi(D) \end{array} \tag{4.14}$$

Then there is a unique induced arrow $[f] : O(s) \to O(t)$ between orbits.

Lemma 4.1.12. *Suppose* $\mathbf{M}$*,* Φ *are as above. Suppose* Φ *has the property that any cone* $h^M \to \Phi(C)$*,* $h^M \to \Phi(D)$ *factors through a cone in the image of* Φ*:*

$$\begin{array}{ccccc} & & h^M & & \\ & \swarrow & \downarrow & \searrow & \\ & & \Phi(E) & & \\ & {\scriptstyle \Phi(f)}\swarrow & & \searrow{\scriptstyle \Phi(g)} & \\ \Phi(C) & & & & \Phi(D) \end{array} \tag{4.15}$$

Then $\mathbf{M}$*,* Φ *satisfy condition (i) of Proposition 4.1.8, and the subcategory of* $\mathcal{O}$ *on the morphisms of the form* $[f]$ *is dense. The induced coverage is generated by arrows* $[f] : O(s) \to O(t)$ *as in* (4.14) *where* α *is the identity. Moreover, these arrows form the left class of an orthogonal factorization system on this dense subcategory.*

Proof. The first part of the statement is easy. For the second part, suppose $h : O(s) \to O(t)$ is a morphism between orbits. By the assumption, there exists a commutative diagram:

$$\begin{array}{ccccc} h^M & \twoheadrightarrow & O(r) & \rightarrowtail & \Phi(E) \\ \| & & \downarrow\!\downarrow & & \vdots\,{\scriptstyle \Phi(f)} \\ h^M & \twoheadrightarrow & O(s) & \rightarrowtail & \Phi(C) \quad {\scriptstyle \Phi(g)} \\ {\scriptstyle h^\alpha}\downarrow & & \downarrow{\scriptstyle h} & & \searrow \\ h^N & \twoheadrightarrow & O(t) & \rightarrowtail & \Phi(D) \end{array} \tag{4.16}$$

The top arrow is the cover $[f] : O(r) \twoheadrightarrow O(s)$ between orbits, and the composite is the arrow $[g] : O(r) \to O(t)$ which is of the required form. This proves density.

The same argument can be applied to show that covers of the form $[f]$, with α the identity, satisfy the coverage condition. Lastly, if h as in (4.16) is a cover, then it can be refined to a cover of the form $[g]$ in the generating family of covers. So indeed, this coverage generates the induced coverage. Lastly, the factorization follows since the arrows in the right class in Proposition 4.1.11 are of the form $[1_C]$ (note that this also implies that the induced lift for a square $([u], [v]) : [f] \to [1_C]$ is of the form $[v]$). □

Example 4.1.13. Perhaps the most obvious example of a pair $\mathbf{M}, \boldsymbol{\Phi}$ that comes to mind when looking at Definition 4.1.6 is when $\mathbf{M} = \mathbb{C}^{\mathrm{op}}$ and

$$\boldsymbol{\Phi} : \mathbb{C} \to [\mathbb{C}^{\mathrm{op}}, \mathcal{S}] \tag{4.17}$$

is the Yoneda embedding. Since this embedding is flat, it satisfies condition (i) of Proposition 4.1.8. Further, the property of lemma 4.1.12 is immediate since the Yoneda embedding is fully faithful. Orbits are epi-mono images of representable arrows:

$$y_f : y_D \twoheadrightarrow O \rightarrowtail y_C \tag{4.18}$$

The site of orbits forms a dense sub-site of the canonical site of $[\mathbb{C}^{\mathrm{op}}, \mathcal{S}]$. Therefore topos $[\mathbb{C}^{\mathrm{op}}, \mathcal{S}]_{\boldsymbol{\Phi}}$ of sheaves on these orbits is equivalent to $[\mathbb{C}^{\mathrm{op}}, \mathcal{S}]$. Indeed, this can be seen directly: every element $x : y_C \to X$ of a presheaf is supported by $1_C : y_C \to y_C$, so that the inclusion

$$[\mathbb{C}^{\mathrm{op}}, \mathcal{S}]_{\boldsymbol{\Phi}} \hookrightarrow [\mathbb{C}^{\mathrm{op}}, \mathcal{S}] \tag{4.19}$$

is (trivially) an equivalence of categories.

The example below is of much greater importance. It allows for understanding how the constructions of Sections 1.1 and 3.3 are special cases of the above. Therefore it connects all of the previous chapters.

Example 4.1.14. Suppose **M** is a *concrete category*, i.e. a category of sets and functions between them, as in Section 3.3. Define $\mathbb{C}$ as the category with finite ordinals $[n]$ as objects and morphisms $s : [n] \to [m]$ given by morphisms of finite sets $s : [m] \to [n]$. Then define:

$$\boldsymbol{\Phi} : \mathbb{C} \to [\mathbf{M}, \mathrm{Set}] \tag{4.20}$$

as follows:

$$\boldsymbol{\Phi}([k])(M) = M^{[k]} \tag{4.21}$$

where I have used the same notation for the object M and its underlying set. With the obvious action, $\Phi([k])$ defines a functor $\mathbf{M} \to \mathrm{Set}$, and for $s : [m] \to [k]$ a morphism in $\mathbb{C}$, there are induced maps:

$$M^{[m]} \to M^{[k]} \tag{4.22}$$

by precomposition.

It is easy to see that Φ satisfies condition (i) of 4.1.8. The resulting orbits are objects of the form

$$O(\mathbf{m})(N) = \{(\alpha m_1, \ldots, \alpha m_k) \mid \alpha : M \to N\} \tag{4.23}$$

for every tuple $(m_1, \ldots, m_k)$ of elements of a set M. In the case when $\mathbf{M}$ is a monoid, the resulting site of orbits is the same as defined from Definition 1.1.7, and the dense site of Lemma 4.1.12 is precisely the dense site of Proposition 1.1.9. In the multi-object case, a morphism $[s] : O(\mathbf{m}) \to O(\mathbf{n})$ between orbits in the dense site of Lemma 4.1.12 is the same thing as the restriction of a morphism $\alpha : N \to M$ to $\mathbf{n}$ with values among $\mathbf{m}$ (as can be inferred from diagram (4.14)). Hence the dense subsite of the category of orbits is essentially the category $\mathbf{T}(\mathbf{M})^{\mathrm{op}}$ defined in Section 3.3.

Remark 4.1.15. If $\mathbf{M}$ is a monoid of endomorphisms $M \Rightarrow M$, the orbits defined in the above example resemble the familiar orbits for group actions. As Mašulović and Pech point out in their paper [MP11], one should, for monoids, distinguish between *orbits* and *traces*. Indeed, they define sets like (4.23) above as the *trace* of $\mathbf{m}$ and define orbits as equivalence classes of traces ([MP11], Definition 2.1). Only if the endomorphism monoid is a group, do traces and orbits coincide and therefore they ought to be named differently in general.

However, the name *orbit* is justified because the orbits now form a category. When two tuples $\mathbf{m}$, $\mathbf{n}$ have the same trace (i.e. the *sets* (4.23) are equal), their objects of orbits are isomorphic.

Example 4.1.16. Suppose $\mathbf{M} := G$ is a *group*. In this case one may be interested in those G-sets whose elements have stabilizer subgroups of G contained in a normal filter of subgroups $\mathcal{F}$ (equivalently, continuous G-sets for some topology on G, see e.g. Freyd [Fre87]). These elements are called *symmetric* with respect to $\mathcal{F}$. This notion forms the basis for the *Fraenkel-Mostowski models* of set theory, see e.g. Blass [Bla11] or Chapter 4 of Jech [Jec73]. In Section 4.6 below I have more to say about Fraenkel-Mostowski models in the context of this chapter.

A family $\mathcal{F}$ of subgroups of G is a *normal filter* when it is a filter, i.e. closed under finite intersections and subgroups, and when it is additionally closed under conjugation by elements of G. As a poset, $\mathcal{F}$ is a filtered category. One can define a functor:

$$\Phi : \mathcal{F} \to [G, \mathrm{Set}] \tag{4.24}$$

By letting $\Phi(H) := G/H = \{gH \mid g \in G\}$ be the set of left cosets equipped with the natural left action, and $\Phi(H \subseteq K)(gH) = gK$. Since $gH = g'H$ if and only if $g'^{-1}g \in H$, either statement implies $gK = g'K$ when $H \subseteq K$ and hence this morphism is well-defined and clearly equivariant.

Now if $x \in X$ is an element of a G-set, and if x is supported by $gH \in \Phi(H)$, then it is supported by $gHg^{-1} \in \Phi(gHg^{-1})$. From this it follows that x is supported by Φ if and only if its stabilizer is contained in $\mathcal{F}$, i.e. when it is symmetric with respect to $\mathcal{F}$. Since $\mathcal{F}$ is filtered, condition (i) of Proposition 4.1.8 holds and $[G, \mathrm{Set}]_\Phi$ is a Grothendieck topos.

In Section 4.6, I will show that the interpretation of Fraenkel-Mostowski set theory defined by a pair $(G, \mathcal{F})$ can be extended to all pairs $(\mathbf{M}, \Phi)$ which satisfy the conditions of Proposition 4.1.8.

The notion of support obtained in the following example was introduced by Makkai, see e.g. Definition 2.1 of [Mak81]. The following shows that it is a special case of the construction above. In the next section, I will discuss models of $\mathcal{C}$ in more depth.

Example 4.1.17. Suppose $\mathbf{M}$ is a subcategory of models of a site $\mathcal{C}$ over a topos $\mathcal{S}$. Then the *evaluation functor*

$$\mathrm{ev} : \mathcal{C} \to [\mathbf{M}, \mathcal{S}] \tag{4.25}$$

which is the double transpose of the inclusion $\mathbf{M} \subseteq [\mathcal{C}, \mathcal{S}]$ is a *flat* functor. This means that the condition of Lemma 4.1.12 is certainly satisfied, so $[\mathbf{M}, \mathcal{S}]_{\mathrm{ev}}$ is a topos of sheaves on orbits. Orbits $O(s)$ are subobjects of $\mathrm{ev}(C)$ generated by a single element $s \in M(C)$ for some M.

When $\mathcal{C} = \mathcal{C}_\mathbb{T}$ is a syntactic category, these elements are tuples of elements $\mathbf{s} \in M\{\mathbf{x}.\phi(\mathbf{x})\}$ which validate a $\mathbb{T}$-formula $\phi(\mathbf{x})$ in some M. Since every tuple validates $\{\mathbf{x}.\top\}$, the resulting topos $[\mathbf{M}, \mathcal{S}]_{\mathrm{ev}}$ is the same as the topos $[\mathbf{M}, \mathcal{S}]_\Phi$ from Example 4.1.14. So for syntactic categories, the construction yields *finitely supported* $\mathbf{M}$-sets.

The evaluation functor is also discussed by Makkai [Mak81] for a specific class of 'special' models $\mathbf{M}$ (see [Mak81], Proposition 2.2), as well as in [Mak82] (see for instance theorem 2.9 loc. cit.) and was later studied by Coumans [Cou12]; some cases of these results are followed up in the next sections. Especially worth mentioning is a theorem by A. Joyal which can be found as theorem 6.3.5 in Makkai-Reyes [MR77] and will be discussed in Theorem 4.3.5 below.

4.2 The topos of orbits

This section treats an important special case of the functor Φ introduced in the previous section. Namely, the case when Φ induces a geometric morphism:

$$f : [\mathbf{M}, \mathcal{S}] \to [\mathbb{C}^{\mathrm{op}}, \mathcal{S}], \tag{4.26}$$

or equivalently, when Φ is *flat*. Note that one may be concerned about the case when the restricted Φ:

$$\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]_\Phi \tag{4.27}$$

is flat, but the unrestricted Φ is not. However, under the conditions of Proposition 4.1.8 this situation is ruled out, so it does not introduce any new cases in that setting. Here, it is assumed that Φ, unrestricted, is a flat functor.

In that case, Φ is always supportive and also the conditions of Lemma 4.1.12 are automatically satisfied, so $[\mathbf{M}, \mathcal{S}]_\Phi$ is a Grothendieck topos. The main result of this section is that this topos appears as the unique hyperconnected-localic image of the geometric morphism (4.26). For an introduction to this factorization, see section A4.6 of the Elephant [Joh02a]. Recall that when Φ is supportive, the inclusion $[\mathbf{M}, \mathcal{S}]_\Phi \hookrightarrow [\mathbf{M}, \mathcal{S}]$ is the left-adjoint part of a geometric morphism. This geometric morphism has the following property:

Proposition 4.2.1. *Suppose Φ is supportive. Then the geometric morphism*

$$s : [\mathbf{M}, \mathcal{S}] \twoheadrightarrow [\mathbf{M}, \mathcal{S}]_\Phi \tag{4.28}$$

induced by Proposition 4.1.7 is hyperconnected.

Proof. The left adjoint is an inclusion, so it is full and faithful. Subobjects of $\mathbf{M}$-sets supported by Φ are also supported by Φ, so the inclusion is closed under subobjects. It follows that s is hyperconnected (see e.g. proposition A4.6.6 of [Joh02a]). □

Since $\Phi : \mathbb{C} \to [\mathbf{M}, \mathcal{S}]$ factors through the inclusion $[\mathbf{M}, \mathcal{S}]_\Phi \hookrightarrow [\mathbf{M}, \mathcal{S}]$, a geometric morphism (4.26) factors through (4.28) when it is induced by Φ. Hence the following:

Proposition 4.2.2. *The geometric morphism:*

$$f : [\mathbf{M}, \mathcal{S}]_\Phi \to [\mathbb{C}^{\mathrm{op}}, \mathcal{S}] \tag{4.29}$$

induced by Φ is localic. *So $[\mathbf{M}, \mathcal{S}]_\Phi$ can be represented as a topos of sheaves on the internal locale in $[\mathbb{C}^{\mathrm{op}}, \mathcal{S}]$ given by* (4.30) *below.*

$$\begin{aligned} \Lambda(C) &\cong \mathrm{Sub}_{[\mathbf{M},\mathcal{S}]_\Phi}(\Phi(C)) \\ \Lambda(f : D \to C)(A) &:= f^* A \end{aligned} \tag{4.30}$$

Proof. By definition, orbits $O(s)$ are in particular the generating subobjects of $\Phi(C)$. So clearly every object of $[\mathbf{M}, \mathcal{S}]_\Phi$ is a quotient of a subobject of an object in the image of the left adjoint

part f^*. Hence f is localic (see e.g. [Joh02a], A4.6.1). The expression (4.30) for the corresponding internal locale now follows straight from the expression for the right adjoint (4.31) below (e.g. [Joh02b] ,C1.4.7):

$$f_*(\Omega)(C) \cong \mathrm{Hom}_{[\mathbf{M},\mathcal{S}]_{\Phi}}(\Phi(C), \Omega) \tag{4.31}$$

□

When $\mathcal{C}$ is a *site*, the above results can be strengthened to the case when $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$ is not just flat but also *continuous*. In that case, the geometric morphism (4.26) factors through the inclusion of sheaves:

$$\mathrm{Sh}(\mathcal{C}) \hookrightarrow [\mathbb{C}^{\mathrm{op}}, \mathcal{S}]. \tag{4.32}$$

Clearly, $[\mathbf{M}, \mathcal{S}]_{\Phi}$ is also localic over $\mathrm{Sh}(\mathcal{C})$.

The following corollary can be considered a semantic version of theorem 40 in Coumans [Cou12]. Indeed, one can obtain the latter from this corollary when $\mathbf{M}$ is a category of models satisfying Makkai's conditions ([Mak81], Proposition 2.2). In that case, the image of the hyperconnected-localic factorization of $[\mathbf{M}, \mathcal{S}] \to \mathrm{Sh}(\mathcal{C})$ is equivalent to Makkai's 'topos of types' as shown by Coumans. The following describes this factorization for *any* model in a presheaf topos, and shows that it is precisely the topos $[\mathbf{M}, \mathcal{S}]_{\Phi}$. In this spirit, one may call it the 'topos of orbits' of $\mathbf{M}$.

Corollary 4.2.3. *Suppose Φ induces a geometric morphism:*

$$f : [\mathbf{M}, \mathcal{S}] \to \mathrm{Sh}(\mathcal{C}) \tag{4.33}$$

Then $[\mathbf{M}, \mathcal{S}]_{\Phi}$ is defined uniquely up to equivalence as the topos over $\mathcal{S}$ appearing in the hyperconnected-localic factorization of f:

$$f : [\mathbf{M}, \mathcal{S}] \twoheadrightarrow [\mathbf{M}, \mathcal{S}]_{\Phi} \xrightarrow{l} \mathrm{Sh}(\mathcal{C}) \tag{4.34}$$

□

4.2.1 Orbits as the age of a category of models

I discussed in Example 4.1.14 how the *age* of a concrete category as defined in Section 3.3 is essentially the dual of a site of orbits. In general, a (flat, continuous) functor

$$\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}] \tag{4.35}$$

defines a notion of *age* of $\mathbf{M}$ when the objects of $\mathbf{M}$ are viewed as models of $\mathcal{C}$ through Φ. Recall that for any object M of a category $\mathbf{M}$, there is a functor:

$$M^* : [\mathbf{M}, \mathcal{S}] \to \mathcal{S} \tag{4.36}$$

which takes a functor $X : \mathbf{M} \to \mathcal{S}$ to its evaluation $X(M)$. Since this functor preserves limits, it induces a model of $\mathcal{C}$, i.e. a flat continuous functor:

$$\Phi(-)(M) : \mathcal{C} \to [\mathbf{M}, \mathcal{S}] \to \mathcal{S} \tag{4.37}$$

In this sense, an element $s : h^M \to \Phi(C)$ is truly an element in the interpretation of C in some model of $\mathcal{C}$. For $t : h^N \to \Phi(D)$ another element and a morphism between orbits $[f] : O(s) \to O(t)$ induced by $f : C \to D$ as in Lemma 4.1.12, there must exist an extension of f to an arrow $\alpha : N \to M$, i.e. such that $\alpha \cdot t = f \cdot s$. Again, this can be seen as a version of the construction in 3.3.2 with more allowed structure. Hence the site of orbits for $\mathbf{M}$ can be construed as the *age* of $\mathbf{M}$ with respect to the interpretation Φ. The structure of an age is that of a factorizing prime site (Proposition 4.1.11) or that of an internal locale in $\mathrm{Sh}(\mathcal{C})$ (Proposition 4.2.2).

4.3 Inclusions

In this section I compare the hyperconnected-localic factorization of Corollary 4.2.3 to the surjection-inclusion factorization. Recall that the latter can be given explicitly as the composite:

$$[\mathbf{M}, \mathcal{S}] \twoheadrightarrow \mathrm{Sh}(\mathcal{C}_\Phi) \hookrightarrow \mathrm{Sh}(\mathcal{C}) \tag{4.38}$$

where $\mathcal{C}_\Phi$ has the same underlying category as $\mathcal{C}$ but comes equipped with the smallest coverage that contains the coverage of $\mathcal{C}$ but turns $\Phi : \mathcal{C}_\Phi \to [\mathbf{M}, \mathcal{S}]$ into a *conservative* or *co-continuous* morphism of sites. Recall that a functor $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$ is co-continuous if, in the notation of Section 2.1, whenever $\Phi\mathcal{U}$ is covering for some family $\mathcal{U} \in \mathrm{Fam}(C)$, there exists a covering family $\mathcal{V} \in \mathrm{Cov}(C)$ such that $\mathcal{V} \preceq \mathcal{U}$. To that end, one can define covering families for $\mathcal{C}_\Phi$ as those $\mathcal{U} \in \mathrm{Fam}(C)$ for which $\Phi\mathcal{U}$ is covering.

Since a hyperconnected geometric morphism is a surjection, there is always a comparison morphism as in the following commutative square:

$$\begin{array}{ccc} [\mathbf{M}, \mathcal{S}] & \twoheadrightarrow & \mathrm{Sh}(\mathcal{C}_\Phi) \\ \downarrow & \nearrow & \downarrow \\ [\mathbf{M}, \mathcal{S}]_\Phi & \hookrightarrow & \mathrm{Sh}(\mathcal{C}) \end{array} \tag{4.39}$$

Corollary 4.3.3 gives a useful criterion on Φ and $\mathbf{M}$ for when this diagonal is an equivalence. This needs the following proposition, which will be applied to the above case by setting $\mathcal{C} := \mathcal{C}_\Phi$ in the proof of Corollary 4.3.3.

Proposition 4.3.1. *Let $\mathcal{C}$ be a site. Suppose $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$ is flat, continuous and co-continuous. Then the following statements are equivalent:*

(i) The functor Φ of (4.27) *has* dense *image and is* locally full*, i.e. satisfies conditions (ii), (iii) of Theorem 2.2.4*

(ii) For every $s \in \Phi(C)(M)$, there exists $m : A \to C$ in $\mathcal{C}$ and $t \in \Phi(A)(M)$ such that $\Phi(m)(M)(t) = s$, and satisfying the following property: When $t' \in \Phi(A)(N)$ for some N, there is a morphism $\alpha : M \to N$ such that $\alpha \cdot s = \Phi(m)(N)(t')$.

When these additional conditions are satisfied, Φ induces an equivalence of toposes:

$$[\mathbf{M}, \mathcal{S}]_{\Phi} \simeq \mathrm{Sh}(\mathcal{C}) \tag{4.40}$$

over $\mathcal{S}$.

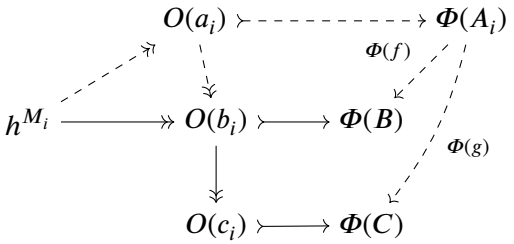
Proof. (i)⇒(ii): It is enough to use density and local fullness together with the fact that orbits are *prime* to find for every $O(s) \rightarrowtail \Phi(C)$ a cover $\Phi(A) \twoheadrightarrow O(s)$ such that their composite is $\Phi(m)$ for some $m : A \to C$. That this satisfies the properties of (ii) follows directly from the definitions.

(ii)⇒(i): For an orbit $O(s)$, the property means precisely that there exists a cover $\Phi(A) \twoheadrightarrow O(s)$ such that the composition $\Phi(A) \twoheadrightarrow O(s) \hookrightarrow \Phi(C)$ is in the image of Φ. So the image of Φ is dense.

When $\eta : \Phi(B) \to \Phi(C)$ is an arbitrary morphism, $\Phi(B)$ can be covered by a family of orbits

$$\{O(b_i) \hookrightarrow \Phi(B)\}_i \tag{4.41}$$

For $b_i \in \Phi(B)(M_i)$. This induces a diagram similar to (4.16):



(4.42)

Where the composite $O(b_i) \to \Phi(C)$ is the restriction of η to $O(b_i)$. Using flatness of Φ, one can cover every $O(b_i)$ with $O(a_i)$ such that the resulting composite $\Phi(A_i) \to \Phi(B) \to \Phi(C)$ is of the form $[g]$. Using property (ii) for $O(a_i)$, there is $\Phi(A_i') \twoheadrightarrow O(a_i)$ such that the composite

$$\Phi(A_i') \twoheadrightarrow O(a_i) \rightarrowtail \Phi(A_i) \tag{4.43}$$

is of the form $\Phi(m_i)$. Composing covers, this results in a covering family

$$\{\Phi(A_i') \twoheadrightarrow O(b_i) \rightarrowtail \Phi(B)\}_i \tag{4.44}$$

whose composites with $\Phi(B) \to \Phi(C)$ are $\Phi(g \circ m_i')$. So indeed Φ is locally full.

The conclusion follows since Φ now satisfies (i)-(iii) of Theorem 2.2.4. □

The condition (ii') below is what Makkai calls *principal prime* in [Mak90], Definition 4.2'(ii) in the case that $\mathcal{C}$ is a regular category. The following shows that it is a special case of the condition (ii) above.

Proposition 4.3.2. *When $\mathcal{C}$ is a* standard *factorizing site, the condition (ii) in Proposition 4.3.1 is equivalent to:*

(ii') For every $s \in \Phi(C)(M)$, there exists $m : A \rightarrowtail C$ in $\mathcal{M}$ and $t \in \Phi(A)(M)$ such that $\Phi(m)(M)(t) = s$, and satisfying the following property: When $t' \in \Phi(A)(N)$ for some N, there is a morphism $\alpha : M \to N$ such that $\alpha \cdot t = t$.

Proof. Clearly, the condition (ii') is stronger than (ii).

For the converse, assume (ii). By factorization, it may be assumed that $m \in \mathcal{M}$. Since Φ is flat and m is a monomorphism, $\Phi(m)$ is a monomorphism, so it is enough to show that

$$\Phi(m)(N)(\alpha \cdot t) = \Phi(m)(N)(t) \tag{4.45}$$

But for the left-hand side:

$$\Phi(m)(N)(\alpha \cdot t) = \alpha \cdot \Phi(m)(M)(t) = \alpha \cdot s \tag{4.46}$$

Hence the conclusion follows. □

Corollary 4.3.3. *Suppose $\mathcal{C}$ is a site and $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$ is flat and continuous. Further assume that Φ satisfies condition (ii) of Proposition 4.3.1. Then the factorization*

$$[\mathbf{M}, \mathcal{S}] \twoheadrightarrow [\mathbf{M}, \mathcal{S}]_\Phi \to \mathrm{Sh}(\mathcal{C}) \tag{4.47}$$

of (4.34) *is also a factorization as a surjection followed by an inclusion.*

Proof. The arrow $[\mathbf{M}, \mathcal{S}]_\Phi \to \mathrm{Sh}(\mathcal{C})$ factors as a surjection followed by an inclusion:

$$[\mathbf{M}, \mathcal{S}]_\Phi \to \mathrm{Sh}(\mathcal{C}_\Phi) \hookrightarrow \mathrm{Sh}(\mathcal{C}) \tag{4.48}$$

Where $\mathcal{C}_\Phi$ carries the coverage of $\mathcal{C}$ strengthened by $\mathbf{M}$, i.e. a family $\mathcal{U}$ is covering when $\Phi\mathcal{U}$ is covering. The first of these morphisms is induced by $\Phi : \mathcal{C}_\Phi \to [\mathbf{M}, \mathcal{S}]_\Phi$ and this now satisfies the conditions of Proposition 4.3.1 (since it is co-continuous), hence it is an equivalence. The conclusion follows. □

Proposition 4.3.1 can also be used to show that the construction of the site of orbits from Φ is 'idempotent' in the following sense. Assuming that $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$ is supportive, there is a functor:

$$\Psi : \mathcal{O} \to [\mathbf{M}, \mathcal{S}] \tag{4.49}$$

defined by the Yoneda embedding, that is:

$$\Psi(O(t))(M) := [\mathbf{M}, \mathcal{S}](h^M, O(t)) \tag{4.50}$$

Then Ψ is flat, continuous and co-continuous. Moreover, any $h^M \to O(c)$ can be extended to a square:

$$\begin{array}{ccc} h^M & \overset{a}{\dashrightarrow\!\!\!\!\rightarrow} & O(a) \\ \downarrow & & \vdots \\ O(c) & \rightarrowtail & \Phi(C) \end{array} \tag{4.51}$$

where some $O(c) \rightarrowtail \Phi(C)$ exists by definition of orbit and the rest is obtained by factorization. By the top epimorphism, one has for any $a' : h^N \to O(a)$ that there exists an $\alpha : M \to N$ such that $\alpha \cdot a = a'$. Hence Ψ satisfies the condition (ii') of Proposition 4.3.2 above, which implies proposition (ii) of Proposition 4.3.1. This discussion yields:

Theorem 4.3.4. *Suppose Φ is supportive and Ψ is defined as in* (4.49) *above. Then there is an equivalence of toposes:*

$$[\mathbf{M}, \mathcal{S}]_\Psi \cong [\mathbf{M}, \mathcal{S}]_\Phi \tag{4.52}$$

□

It is in the sense of this theorem that any concrete category $\mathbf{M}$ contains 'all' models of $\mathbf{T}(\mathbf{M})^{\mathrm{op}}$, as was hinted in Section 3.3 - i.e. $\mathbf{M}$ contains a sufficiently complete set of models of $\mathbf{T}(\mathbf{M})^{\mathrm{op}}$.

Another nice application of Proposition 4.3.1 is the following theorem, which relates it to a theorem stated as 6.3.5 in Makkai-Reyes [MR77]. It is attributed to A. Joyal by the authors. See Remark 4.3.6 below for the precise connection between the two.

Theorem 4.3.5. *Suppose $\mathbf{M}, \Phi$ satisfy the conditions of Proposition 4.3.1. Then the left adjoint:*

$$f^* : \mathrm{Sh}(\mathcal{C}) \to [\mathbf{M}, \mathcal{S}] \tag{4.53}$$

which is left Kan extension of Φ along the Yoneda embedding, is a Heyting functor*, i.e. it preserves first-order logic.*

Proof. Recall that a geometric morphism is called *open* if its left adjoint has the property stated above (e.g. theorem C3.1.7 in [Joh02b]). By Corollary 4.2.3 and Proposition 4.3.1, f is hyperconnected. Hence the claim follows from the fact that every hyperconnected geometric morphism is open (corollary C3.1.9 in [Joh02b]). □

Remark 4.3.6. Theorem 6.3.5 in [MR77] is the statement that there is a (small) category of models $\mathbf{M}$ such that the evaluation functor (4.25) satisfies the conclusion of Theorem 4.3.5. The proof they give is for the (large) category $\mathbf{M}$ of *all* models, and they leave it to the reader to show that it can be trimmed down to a small category. The argument, then, consists in showing from basic definitions that f^* preserves universal quantifiers. This involves the construction of a particular model of $\mathcal{C}$ somewhere along the way (which will hence be in $\mathbf{M}$) which proves the property at that particular instance.

With the above theorem, one can avoid this (impredicative) proof by showing that there exists a small category of jointly conservative models which satisfy condition (ii) of Proposition 4.3.1. For this condition to hold it is enough for $\mathbf{M}$ to be principal prime. This will be stated in the next section. Hence for prime-generated sites, the theorem follows from repeated applications of Proposition 3.5.7, or Corollary 3.6.3 on the connected components of $\mathcal{C}$. This also resolves the issue of cutting down $\mathbf{M}$ to a *small* site. This is then perhaps not yet as general as for separable, coherent sites, but I will leave it for another occasion to combine the construction with a 'prime completion' for coherent sites as discussed in Chapter 2.

4.4 Principal models

One can generalize Definition 3.4.10 to arbitrary models $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$:

Definition 4.4.1. Suppose $\mathcal{C}$ is a factorizing site. A flat and continuous functor $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$ is *principal* if for every $s \in \Phi(C)(M)$, there exists a *least* $m : A \rightarrowtail C$ in $\mathcal{M}$ such that there exists $s' \in \Phi(A)(M)$ for which $\Phi(m)_M(s') = s$. This is denoted $s \in_> \Phi(A)(M)$ as in Definition 3.4.5.

The following proposition just rephrases this:

Proposition 4.4.2. *A flat and continuous functor $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$ is principal if and only if every model*

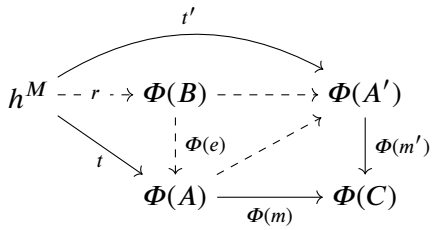
$$\mathcal{C} \xrightarrow{\Phi} [\mathbf{M}, \mathcal{S}] \xrightarrow{M^*} \mathcal{S} \tag{4.54}$$

is principal in the sense of Definition 3.4.10. □

Proposition 4.4.3. *Suppose $\mathcal{C}$ is* standard *factorizing and Φ satisfies the conditions of Proposition 4.3.1. Then Φ is principal.*

Proof. Recall that for $\mathcal{C}$, condition (ii) of Proposition 4.3.1 is equivalent to condition (ii') of Proposition 4.3.2 above. For $s \in \Phi(C)(M)$, let $m : A \rightarrowtail C$ and $t \in \Phi(A)(M)$ be as in that condition. Now suppose $m' : A' \rightarrowtail C$ and $t' \in \Phi(A')(M)$ are such that $\Phi(m)_M(t') = s$. Since Φ

is flat, this can be completed to a commutative diagram:



(4.55)

By condition (ii) of the proposition, $\Phi(e)$ is an epimorphism: indeed whenever $t'' \in \Phi(A)(N)$ for some N, there is $\alpha : M \to N$ such that $\alpha \cdot t = t''$; but then $\alpha \cdot r \in \Phi(B)(N)$ lies above t''. Since Φ is conservative, it follows that e is a cover. But then by factorization, there is a diagonal $A \to A'$ as drawn, making the diagram commute. □

By Lemma 3.4.8, any model $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$ carves out a subcategory of prime objects of $\mathcal{C}$:

Definition 4.4.4. Suppose $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$ is flat and continuous. Define the category $\mathcal{P}_\Phi$ as the category with objects $s \in_> \Phi(A)(M)$ for some A, M and morphisms $f : s \to t$, for $t \in_> \Phi(B)(N)$, as arrows $f : A \to B$ in $\mathcal{C}$.

Proposition 4.4.5. *Suppose Φ is principal. Then the covers in $\mathcal{P}_\Phi$, as inherited from $\mathcal{C}$, satisfy the coverage condition.*

Proof. This follows by combining the coverage condition on $\mathcal{C}$ with the fact that Φ is flat and continuous. □

For any category of models, the forgetful functor $U : \mathcal{P}_\Phi \to \mathcal{C}$ is clearly fully faithful. Regardless of Proposition 4.4.5, $\mathcal{P}_\Phi$ can always be equipped with the induced coverage of Section 2.1. This is assumed in the following lemma.

Lemma 4.4.6. *The functor Φ principal if and only if the essential image of*

$$U : \mathcal{P}_\Phi \to \mathcal{C}_\Phi \tag{4.56}$$

is a dense subcategory of $\mathcal{C}_\Phi$

Proof. Assume Φ is principal. Then for every $t \in \Phi(C)(M)$ for some C, M, there exists $m : A \rightarrowtail C$ such that $t \in_> \Phi(A)(M)$ and $\Phi(m)_M(t) = s$. It follows that $\Phi(C)$ is covered by the family of all $\Phi(f)$ for such $m : A \rightarrowtail C$, for all M. So the image of U is dense.

Conversely, density means that for every $s \in \Phi(C)(M)$, there exists $f : A \to C$ and $t \in_> \Phi(A)(M)$ such that $\Phi(f)_M(t) = s$. Since f can be factored as $f' \circ e : A \twoheadrightarrow A' \rightarrowtail C$, it follows that $A' \rightarrowtail C$ is such that $\Phi(f')_M(t') = s$ for some t' and $t' \in_> \Phi(A')(M)$. So Φ is principal. □

Corollary 4.4.7. *If $\boldsymbol{\Phi}$ is principal, then the forgetful functor $U : \mathcal{P}_{\boldsymbol{\Phi}} \to \mathcal{C}_{\boldsymbol{\Phi}}$ induces an equivalence of toposes*

$$\mathrm{Sh}(\mathcal{P}_{\boldsymbol{\Phi}}) \cong \mathrm{Sh}(\mathcal{C}_{\boldsymbol{\Phi}}) \tag{4.57}$$

over $\mathrm{Sh}(\mathcal{C})$. *Hence* $\mathrm{Sh}(\mathcal{P}_{\boldsymbol{\Phi}})$ *appears as the image of the surjective-inclusive factorization of* $[\mathbf{M}, \mathcal{S}] \to \mathrm{Sh}(\mathcal{C})$.

Proof. By Lemma 4.4.6, since the covers in $\mathcal{P}_{\boldsymbol{\Phi}}$ are clearly induced by $\mathcal{C}$. □

The above corollary says that the 'theory' of $\mathbf{M}$ with respect to $\mathcal{C}$, as the subtopos appearing in the image of $[\mathbf{M}, \mathcal{S}] \to \mathrm{Sh}(\mathcal{C})$, is the theory $\mathcal{P}_{\boldsymbol{\Phi}}$. The theorem below gives a characterisation of this theory. The question that it answers is the following: For principal $\boldsymbol{\Phi}$, every geometric morphism $\mathcal{S} \to \mathrm{Sh}(\mathcal{P}_{\boldsymbol{\Phi}})$ induces a geometric morphism $\mathcal{S} \to \mathrm{Sh}(\mathcal{C})$ by postcomposition. Hence $\mathrm{Sh}(\mathcal{P}_{\boldsymbol{\Phi}})$ classifies particular models of $\mathcal{C}$. The theorem below says that these models are precisely those that can be written as certain special colimits of representables

$$h^A : \mathcal{C} \to \mathcal{S}.$$

Further, the theorem states this more generally for any bounded topos $\mathcal{F}$ over $\mathcal{S}$.

In order to understand its statement in that generality, as well as the proof, it is necessary to formulate an internal notion of model for every morphism of sites

$$f : \mathcal{C} \to \mathcal{F} \tag{4.58}$$

where $\mathcal{F}$ is a topos over $\mathcal{S}$:

$$p : \mathcal{F} \to \mathcal{S} \tag{4.59}$$

carrying the canonical topology. For small sites, this internal notion is called a (continuous) $\mathcal{C}^{\mathrm{op}}$-*torsor*. Since it is only used in the proof of the following theorem, I have not introduced this notion in chapter 2. For the reader who wishes to learn about this properly, I refer to section B3.2 of the Elephant [Joh02a] or section 4.3 of the older [Joh77]. For now, I hope that a brief explanation suffices.

When $\mathcal{C}$ is small, a model (4.58) corresponds to an internal discrete fibration

$$F : \mathbb{F} \to p^*\mathbb{C}^{\mathrm{op}}$$

where $\mathbb{C}$ is the underlying category of $\mathbb{C}$, so $p^*\mathbb{C}$ comes with the induced structure of an internal category, and $\mathbb{F}$ is filtered (in the internal logic). This is called a $\mathbb{C}^{\mathrm{op}}$-*torsor* in $\mathcal{F}$. The coverage on $\mathbb{C}$ also transfers along p^*, so that continuity of f translates to an extra condition on the discrete fibration. This condition just says that reindexing along covering families is jointly epimorphic.

In terms of this torsor, the induced $\mathcal{C}$-model $\mathcal{C} \to \mathcal{F}$ is given by the below composite:

$$\mathcal{C} \xrightarrow{Y} [\mathcal{C}^{\mathrm{op}}, \mathcal{S}] \xrightarrow{U^*} [\mathcal{P}^{\mathrm{op}}_{\boldsymbol{\Phi}}, \mathcal{S}] \xrightarrow{p^{\mathcal{P}^{\mathrm{op}}_{\boldsymbol{\Phi}}*}} [p^*\mathcal{P}^{\mathrm{op}}_{\boldsymbol{\Phi}}, \mathcal{F}] \xrightarrow{F^*} [\mathbb{F}, \mathcal{F}] \xrightarrow{\varinjlim} \mathcal{F} \tag{4.60}$$

Note that this fact is part of the proof of an equivalence of categories between torsors and geometric morphisms. This is often referred to as *Diaconescu's theorem*. The proof of the theorem below essentially consists in chasing through the composite (4.60).

Theorem 4.4.8. *Suppose $\boldsymbol{\Phi}$ is principal. Then* $\mathrm{Sh}(\mathcal{P}_{\boldsymbol{\Phi}}) \cong \mathrm{Sh}(\mathcal{C}_{\boldsymbol{\Phi}}) \hookrightarrow \mathrm{Sh}(\mathcal{C})$ *classifies $\mathcal{C}$-models $\mathcal{C} \to \mathcal{F}$ in bounded toposes over $\mathcal{S}$ which are (internal) colimits of filtered diagrams of embedded substructures $h^A \rightarrowtail M$ of a model:*

$$M : \mathcal{C} \xrightarrow{\boldsymbol{\Phi}} [\mathbf{M}, \mathcal{S}] \xrightarrow{M^*} \mathcal{S} \tag{4.61}$$

in $\mathbf{M}$*, satisfying the* forcing condition*:*

(i) Whenever $s_{(-)} : \mathcal{I} \to \mathcal{C}^{\mathrm{op}}$ is such a diagram and $t \twoheadrightarrow s_i$ is a cover, then there is an arrow $s_j \to t$ such that the composition

$$s_j \to t \twoheadrightarrow s_i \tag{4.62}$$

is a transition arrow of the (opposite) diagram

Proof. Suppose $p : \mathcal{F} \to \mathcal{S}$ is an $\mathcal{S}$-topos. A geometric morphism $f : \mathcal{F} \to \mathrm{Sh}(\mathcal{P}_{\boldsymbol{\Phi}})$ over $\mathcal{S}$ corresponds to an $\mathcal{P}_{\boldsymbol{\Phi}}$-torsor $F : \mathbb{F} \to p^*\mathcal{P}^{\mathrm{op}}_{\boldsymbol{\Phi}}$. Since the geometric morphism factors through sheaves, this fibration satisfies the condition that for any cover $e : t \twoheadrightarrow s$, for $t \in_> \boldsymbol{\Phi}(M)(B)$ and $s \in_> \boldsymbol{\Phi}(N)(A)$, the induced re-indexing $\mathbb{F}_t \to \mathbb{F}_s$ is epic.

So the induced model $\mathcal{C} \to \mathcal{F}$ over $\mathcal{S}$ is given by (4.60). Therefore

$$H \cong \varinjlim F^* p^{\mathcal{P}^{\mathrm{op}}_{\mathbf{M}}*} U^* Y,$$

where Y is the Yoneda embedding and the colimit is taken internally, pointwise with respect to $\mathcal{C}$. Now for $f \in \mathbb{F}_s$ for some $s \in_> \boldsymbol{\Phi}(M)(A)$:

$$F^* p^{\mathcal{P}^{\mathrm{op}}_{\mathbf{M}}*} U^* Y(f) \cong p^{\mathcal{P}^{\mathrm{op}}_{\mathbf{M}}*} U^* Y(s) \cong h^A : \mathcal{C} \to \mathcal{F} \text{ where } s : h^A \rightarrowtail M \tag{4.63}$$

So indeed H is the (internal) colimit of a diagram of *embedded* substructures $h^A \rightarrowtail M$. The continuity condition for the torsor translates to the forcing condition (i).

Conversely, if a $\mathcal{C}$-model is an internal colimit of a filtered diagram as stated, then this precisely means that it is of the form (4.60), with the additional continuity condition. So in $\mathbf{Top}_{\mathcal{S}}/\,\mathrm{Sh}(\mathcal{C})$, where $\mathbf{Top}_{\mathcal{S}}$ denotes the category of bounded toposes over $\mathcal{S}$, $\mathrm{Sh}(\mathcal{P}_{\boldsymbol{\Phi}})$ precisely classifies the stated colimits. □

4.4.1 Principal prime models

Recall from Section 3.5 that when a model M is *principal* and *prime*, the following condition holds:

(ii") For every $c \in M(C)$, there exists $m : A \rightarrowtail C$ and $a \in_> M(A)$ such that $M(m)(a) = c$, and whenever N is a model with $a' \in N(A)$, there is a natural transformation $\alpha : M \to N$ which satisfies $\alpha \cdot a = a'$.

This can be formulated for Φ:

(ii") For every $c \in \Phi(C)(M)$, there exists $m : A \rightarrowtail C$ and $a \in_> \Phi(A)(M)$ such that $M(m)(a) = c$, and whenever $a' \in \Phi(A)(N)$, there exists $\alpha : M \to N$ which satisfies $\alpha \cdot a = a'$.

Clearly, (ii") is satisfied for Φ when every model

$$\mathcal{C} \xrightarrow{\Phi} [\mathbf{M}, \mathcal{S}] \xrightarrow{M^*} \mathcal{S} \tag{4.64}$$

is principal and prime. Second, it clearly implies condition (ii) of Proposition 4.3.1.

As remarked in 4.3.6 above, when $\mathcal{C}$ is a prime-generated, separable site, and $\mathcal{S}$ admits the axiom of dependent choice in addition to being boolean, then repeated applications of Proposition 3.5.7 or Corollary 3.6.3 allow the construction of a category $\mathbf{M}$ of *countable* principal models for which the evaluation functor

$$\mathrm{ev} : \mathcal{C} \to [\mathbf{M}, \mathcal{S}] \tag{4.65}$$

is co-continuous (or conservative). By Proposition 3.5.4, these models are all prime and hence (4.65) satisfies condition (ii") above. This proves:

Theorem 4.4.9. *Assume that $\mathcal{S}$ validates the axiom of dependent choice, and let $\mathcal{C}$ be a prime-generated, separable site. Then there exists a small category $\mathbf{M}$ of countable models of $\mathcal{C}$ such that*

$$\mathrm{Sh}(\mathcal{C}) \cong [\mathbf{M}, \mathcal{S}]_{\mathrm{ev}} \tag{4.66}$$

over $\mathcal{S}$. □

4.4.2 Flat algebraic models

This section presents a class of examples that illustrate the above generalities in slightly more tangible setting. The case is that of finitary algebraic theories. For such theories, I will show in Theorem 4.4.11 below that the stronger theory of *flat*[2] algebras (viewed as a subtopos) arises as the image in the factorization of Corollary 4.3.3 for a suitable *monoid* $\mathbf{M}$, and that moreover, the

[2]This is not an algebraic theory, see Remark 4.4.12

category of supported **M**-sets is always a presheaf topos. This fact connects this chapter to a theory on internal algebraic theories developed by Johnstone and Wraith [JW78] and finds a pleasing new application here in the specific case of cubical sets introduced in Example 1.2.9.

Recall (e.g. [Joh02b], D1.1.7(a)) that a theory $\mathbb{T}$ is *algebraic* if it is single-sorted, its signature has only function symbols, and if its axioms are all of the form (4.67), where $\phi(\mathbf{x})$ is an atomic formula, e.g. $s(\mathbf{x}) = t(\mathbf{x})$ for *terms* $s(\mathbf{x})$, $t(\mathbf{x})$:

$$\vdash_{\mathbf{x}} \phi(\mathbf{x}) \tag{4.67}$$

Like in Example 2.4.9, I denote by $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ the syntactic category of *regular* formulas, for a finitary algebraic theory $\mathbb{T}$. Recall that such formulas are constructed from atomic formulas using only $\wedge$ and $\exists$. By what is put in, the category $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ is *regular*, i.e. it has finite limits and coequalizers of kernel pairs. As such, it can be equipped with the *regular coverage*, where covers are (single) regular epimorphisms. As in Example 2.4.9, this turns $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ into a prime site with a regular epi-mono factorization system, for which the induced notion of embedding is the usual one. For more details on the precise definitions, again see section D1.4 of [Joh02b].

The following is concerned with *finitary* algebraic theories in a base topos $\mathcal{S}$ with natural numbers object, where the signature and the set of axioms are of finite cardinality. For such a theory $\mathbb{T}$, there is a left adjoint (4.68) to the forgetful functor $U : \mathbb{T}\text{-Mod}(\mathcal{S}) \to \mathcal{S}$ of $\mathbb{T}$-models in $\mathcal{S}$ to their underlying object (see Johnstone-Wraith [JW78], theorem 5.1).

$$F : \mathcal{S} \to \mathbb{T}\text{-Mod}(\mathcal{S}) \tag{4.68}$$

In other words, *free* $\mathbb{T}$-models exist in $\mathcal{S}$. In addition, $\mathbb{T}$-Mod($\mathcal{S}$) is monadic over $\mathcal{S}$ and has finite colimits.

Let $\mathbb{A} = \{x, y, z, \dots\}$ be a countably infinite decidable object of $\mathcal{S}$. Any free $\mathbb{T}$-model on a finite number of generators can be written as $F\{\mathbf{x}\}$ for some tuple $\mathbf{x}$ of elements of $\mathbb{A}$; these would correspond to representable functors $h^{\{\mathbf{x}.\top\}} : \mathcal{C}_{\mathbb{T}}^{\mathrm{reg}} \to \mathcal{S}$ in the notation of Section 3.4. The category of free finitely generated $\mathbb{T}$-models is denoted FF$\mathbb{T}$-Mod($\mathcal{S}$). Further, there is the free infinite model on $\mathbb{A}$. The latter is here denoted by M and its monoid of endomorphisms by $\mathbf{M}$. Since $\mathbb{A}$ is countable and F is a left-adjoint, $\mathbb{A}$ is clearly a colimit of the obvious diagram of free models of the form $F\{\mathbf{x}\}$, which on top of that admits a cofinal sequence of the form (3.36), satisfying the conclusion of Proposition 3.5.3. Indeed, the transition maps are *embeddings* of $\mathbb{T}$-models, for since any inclusion $\{\mathbf{x}\} \subseteq \{\mathbf{x}, y\}$ admits a co-section, so the induced $F\{\mathbf{x}\} \to F\{\mathbf{x}, y\}$ also admits one, so M is clearly a principal (and hence prime) model of $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$. Since all site-theoretic properties of $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ are explicit constructions, Proposition 3.5.4 holds, so in particular M is homogeneous. The following is immediate from Corollary 4.3.3.

Theorem 4.4.10. *Denote by $\mathcal{S}[\mathbb{T}]$ the topos of sheaves on $\mathcal{C}_{\mathbb{T}}^{reg}$, the classifying topos of $\mathbb{T}$. Then for the geometric morphism $[\mathbf{M}, \mathcal{S}] \to \mathcal{S}[\mathbb{T}]$ induced by M, the hyperconnected-localic and surjective-inclusive factorizations coincide and present $[\mathbf{M}, \mathcal{S}]_{\text{ev}}$ as their image:*

$$[\mathbf{M}, \mathcal{S}] \twoheadrightarrow [\mathbf{M}, \mathcal{S}]_{\text{ev}} \hookrightarrow \mathcal{S}[\mathbb{T}] \tag{4.69}$$

Here $[\mathbf{M}, \mathcal{S}]_{\text{ev}}$ is defined as in Example 4.1.17. □

In Example 4.1.17, it was observed that as a topos over $\mathcal{S}$, $[\mathbf{M}, \mathcal{S}]_{\text{ev}} \cong [\mathbf{M}, \mathcal{S}]_{\text{fs}}$ where the right-hand side is the topos of *finitely supported* $\mathbf{M}$-sets of Example 4.1.14. Further, in Example 1.2.1 it was shown that the theory of non-empty objects is classified by $\mathcal{S}[\mathbb{O}_\bullet] \cong [\mathbf{U}, \mathcal{S}]_{\text{fs}}$, where $\mathbf{U}$ is the monoid of all endofunctions on some infinite set. Defining $\mathbf{U}$ as the monoid of all $\mathcal{S}$-functions $\mathbb{A} \to \mathbb{A}$, there is a geometric surjection $[\mathbf{U}, \mathcal{S}]_{\text{fs}} \to [\mathbf{M}, \mathcal{S}]_{\text{fs}}$ whose inverse image is just restriction along the inclusion $F : \mathbf{U} \to \mathbf{M}$ induced by taking the free model:

$$\mathcal{S}[\mathbb{O}_\bullet] \cong [\mathbf{U}, \mathcal{S}]_{\text{fs}} \twoheadrightarrow [\mathbf{M}, \mathcal{S}]_{\text{fs}} \hookrightarrow \mathcal{S}[\mathbb{T}] \tag{4.70}$$

I have also drawn the composition into $\mathcal{S}[\mathbb{T}]$. The fact that the first morphism is a surjection follows since the induced inverse image functor is clearly faithful.

The resulting composition is the geometric morphism corresponding to the *generic* free $\mathbb{T}$-model on a non-empty object; this follows from the same reasoning as the discussion below Theorem 5.21 in Johnstone-Wraith [JW78]. Adapting their argument further, it follows that the surjection-inclusion factorization of this geometric morphism can also be written as the composite[3], where FF$\mathbb{T}$-Mod$(\mathcal{S})_{\neq\emptyset}$ denotes full subcategory on non-empty objects of FF$\mathbb{T}$-Mod$(\mathcal{S})$:

$$\mathcal{S}[\mathbb{O}_\bullet] \twoheadrightarrow [\text{FF}\mathbb{T}\text{-Mod}(\mathcal{S})_{\neq\emptyset}, \mathcal{S}] \hookrightarrow \mathcal{S}[\mathbb{T}] \tag{4.71}$$

Combining this with Theorem 4.4.10 and their Theorem 5.22 yields:

Theorem 4.4.11. *There is an equivalence of toposes*

$$[\text{FF}\mathbb{T}\text{-Mod}(\mathcal{S})_{\neq\emptyset}, \mathcal{S}] \cong [\mathbf{M}, \mathcal{S}]_{\text{ev}} \tag{4.72}$$

over $\mathcal{S}[\mathbb{T}]$. Hence $[\mathbf{M}, \mathcal{S}]_{\text{ev}}$ classifies non-empty, flat $\mathbb{T}$-models in $\mathcal{S}$.

Proof. This follows from the above discussion and the adoption of the terminology in [JW78]. □

Remark 4.4.12. Observe the caution taken in formulating Theorem 4.4.11, since empty models must be avoided. Because this is not an algebraic condition, it has to be imposed externally. Similarly,

[3] Although Johnstone-Wraith show this for the general object classifier, the reasoning for non-empty objects is the same

the notion of a 'flat' model is not algebraic. However, both notions are geometric in the sense that they are preserved along the inverse image part of a geometric morphism.

As soon as the signature of $\mathbb{T}$ contains constants, the requirement of non-emptyness is automatically satisfied and topos on the left-hand side becomes the same as the topos classifying flat $\mathbb{T}$-models in Theorem 5.22 in [JW78]. Also worth noting is that the conclusion drawn by Johnstone-Wraith in the same place accords with Theorem 4.4.8 above.

Remark 4.4.13. Looking back at Example 1.2.9, where I had anticipated the above theorem, it can now finally be shown that the topos of cubical sets is the topos of finitely supported **M**-sets for the free countably infinite de Morgan algebra. Indeed, this follows from theorem 4.4.11 by taking for $\mathbb{T}$ the algebraic theory of de Morgan algebras. Since this theory has constant symbols, every model is non-empty; hence this requirement drops out. Then all is left is the equivalence of toposes:

$$[\mathbf{Cb}_{\mathrm{dM}}, \mathcal{S}] \cong [\mathbf{M}, \mathcal{S}]_{\mathrm{ev}} \tag{4.73}$$

where $\mathbf{Cb}_{\mathrm{dM}}$ is the category of cubes of Cohen et al., and where **M** is the monoid of de Morgan algebra endomorphisms $\mathrm{dM}\,\mathbb{A} \to \mathrm{dM}\,\mathbb{A}$ of the free infinite de Morgan algebra.

Remark 4.4.14. Another difference between the formulation of Theorem 4.4.10 and the discussion surrounding Theorem 5.22 in Johnstone-Wraith [JW78] is that Johnstone and Wraith use, instead of $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$, the presheaf site of finitely presentable $\mathbb{T}$-models, which yields an equivalent topos (op. cit., Theorem 5.21). It would not have been possible to use this site here, since it is not clear whether it is factorizing, and indeed it appears it is not in general. For instance, Freese and Nation [FN16] give an example (2-9.1) showing that a finitely generated sublattice of a finitely presented lattice need not be finitely presented, hence some arrow $F\{\mathbf{x}\} \to A$ would not factor in the category of finitely presented lattices.

4.5 Oligomorphy and the Ryll-Nardzewski theorem

A famous result in classical model theory says that a countable structure M is $\aleph_0$ categorical (that is, $\aleph_0$-categorical for its own theory) if and only if all sets M^n, under the action of its automorphism group, are partitioned into finitely many orbits. A permutation group with this property is known as *oligomorphic*. The result is known as the Ryll-Nardzewski theorem or the Ryll-Nardzewski criterion. Hodges [Hod93] also credits Engeler and Svenonius; see Theorem 7.3.1 in the place cited. Naturally, the result has also turned up in topos theory, for instance in a criterion due to Blass and Ščedrov for when the classifying topos $\mathcal{S}[\mathbb{T}]$ of a finitary geometric theory is a boolean topos [BŠ83]. In the current context, this raises the question of what remains of the Ryll-Nardzewski

theorem for the case when $\mathbf{M}$ is a more general category of models, and in particular a monoid of endomorphisms.

The Ryll-Nardewski criterion is usually formulated as the equivalence of a multitude of statements, such as the statements (a)-(g) in theorem 7.3.1 of Hodges cited above. Mašulović and Pech [MP11] give versions of some of these statements for endomorphism monoids. Their notion of *oligomorphic*, namely orbit-finite on all M^n, is a straightforward generalization of the automorphism case. They end up with a Ryll-Nardewski theorem for the positive fragment of first-order logic (which is preserved by homomorphisms), losing some of the analogue statements of the traditional case.

Before diving into the topos-theoretic version, it is worthwhile to pause and speculate on the possible generalizations of a Ryll-Nardzewski theorem by reflecting on the classical case. After all Mašulović and Pech already lose *some* of the equivalent criteria of the usual Ryll-Nardzewski theorem, and it should be clarified why this happens. First, with an eye towards Proposition 3.5.6, it seems relatively easy to prove $\aleph_0$-categoricity once certain finiteness conditions are met; e.g. that the topos is connected atomic with a (countable) point. The surprising direction of the Ryll-Nardzewski theorem is rather the opposite: given that a structure is $\aleph_0$-categorical, how does one derive certain 'finiteness' properties from it? The answer to this can be inferred from Hodges' proof (Theorem 7.3.1 of [Hod93]): it is the fact that infinite boolean algebras have non-principal ultrafilters (which requires the boolean prime ideal theorem). The Ryll-Nardewski argument then goes roughly as follows:

(i) M is $\aleph_0$ categorical
(ii) All types of M are *principal*
(iii) For each n, the boolean algebra of formulas $\phi(x_1, \ldots, x_n)$ under equivalence is *finite*.
(iv) The sets M^n are orbit-finite

In the non-boolean case, it is therefore expected that (ii)⇒(iii) breaks down. If one still wishes to end up with a series of equivalent statements, every such statement must have some 'finiteness' since this no longer comes free.

The definition below specializes to the notion of Mašulović and Pech [MP11] when $\mathbf{M}$ is a monoid and $\mathcal{C}$ is the syntactic category of a coherent theory. Note the motivation in moving from a monoid to a general category $\mathbf{M}$: in the $\aleph_0$-categorical case, it would not make a difference for countable models, since they are all isomorphic. But when losing this, one should naturally consider more than one model.

Definition 4.5.1. A *model* $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$ is called *oligomorphic* when for every object C of $\mathcal{C}$, the poset of orbits:

$$O(s) \hookrightarrow \Phi(C) \tag{4.74}$$

as subobjects in $[\mathbf{M}, \mathcal{S}]$ is finite.

Recall from Example 2.5.3 that a prime filter τ on an object C is a *type* if there is some N and an element $c \in \Phi(C)(N)$ such that:

$$\tau(c) := \{m : A \rightarrowtail C \mid m \in \mathcal{M} \text{ and } c \in N(A)\} \tag{4.75}$$

The following theorem requires classical reasoning.

Theorem 4.5.2 (Ryll-Nardzewski criterion for principality)**.** *Assume $\mathcal{S}$ is a boolean topos which validates the axiom of dependent choice. Let $\mathcal{C}$ be a separable factorizing site and let $\Phi : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$ be a conservative model, i.e. flat, continuous and co-continuous. Then for the following statements:*

(i) Φ is oligomorphic
(ii) For each object C and $c \in \Phi(C)(M)$, the type $\tau(c)$ is principal
(iii) Φ is principal

One has (i)⇒(ii)⇒(iii).

Proof. (i)⇒(ii): Let C be an object of $\mathcal{C}$. Observe that when $\alpha : M \Rightarrow N$, it follows that $\tau(c) \subseteq \tau(\alpha_C(c))$ for all $c \in \Phi(C)(M)$. Since $\mathbf{M}$ is oligomorphic, this means that there are only finitely many types, represented by, say, $\tau(c_1), \dots, \tau(c_k)$.

To prove that an arbitrary $\tau(c)$ is principal, suppose $c \in M(C)$. Then for every i such that $\tau(c) \nsubseteq \tau(c_i)$, there is a $p_i : P_i \rightarrowtail C \in \tau(c)$ such that $p_i \notin \tau(c_i)$. Since there are only finitely many such, there is in fact a single $p : P \rightarrowtail C \in \tau(c)$ such that $p \notin \tau(c_i)$ whenever $\tau(c) \nsubseteq \tau(c_i)$.

I claim that $c \in_> M(P)$. Indeed, if $c \in M(A)$, then there must be $q : Q \to P$ such that (4.76) below commutes:

$$\begin{array}{ccc} c \in M(Q) & \rightarrowtail & c \in M(A) \\ {\scriptstyle q}\downarrow & \nearrow & \downarrow{\scriptstyle m} \\ c \in M(P) & \underset{p}{\rightarrowtail} & c \in M(B) \end{array} \tag{4.76}$$

By the property of p, there is no type that has P, but not Q, so since Φ is conservative, $q : Q \to P$ must restrict to a cover. Without loss of generality, it may be assumed that it is a cover. But then there is a comparison arrow $P \to A$ as drawn, which also maps between the corresponding elements of $M(C)$. It follows that $c \in_> M(P)$, hence $\tau(c)$ is generated by $p : P \rightarrowtail C$.

(ii)⇒(iii): Let C be an arbitrary object and let $\mathcal{P} = \{p_1, \dots, p_k\}$ denote the generators of all types of C. When $\mathcal{U} = \{u_i : U_i \rightarrowtail C\}$ is a covering family, it is clear that one must have $\mathcal{P} \preceq \mathcal{U}$. So C is compact. Under the assumptions, it is also clear that every model is principal. □

The following example shows (iii)⇒(i) does not generally hold:

Example 4.5.3. Recall the category $\mathcal{H}_{k+1}$ of finite Hrushovki $(k+1)$-graphs of Theorem 3.3.18. For M a principal, conservative, countable (hence prime) model, any substructure $h^A \to M$ (recall that any such morphism is an embedding in the traditional sense, so injective) factors as:

$$h^A \twoheadrightarrow h^B \rightarrowtail M \tag{4.77}$$

where the first arrow is induced by a map $A \twoheadrightarrow B$ in the class $\mathcal{E}$ (note, this is the left class for the category $\mathcal{H}_k$, which is the dual of the *site* we are working in). As an extension of the substructure A of M, B is called the *envelope* of A in M (cf. [Bha+97], Lemma 15.10). Figure 3.2 on page 64 depicts such an envelope, if one thinks of the right-hand graph as being well-embedded in some larger structure M. Recall that I remarked in 3.3.16 that such envelopes can become arbitrarily large, because the 'path' between x_0, x_1 and x_2 can be made arbitrarily long. Since every object in $\mathcal{H}_{k+1}$ admits a well-embedding into M, this shows that there are substructures A such that for every natural number, there is homomorphism $h^A \to M$ whose envelope adds that number of vertices. Since an endomorphism can only increase the size of an envelope of some substructure, it follows that there is an A whose poset of orbits is infinite. In the example, $A = \{x_0, x_1, x_2\}$ with no edges. This argument is essentially the proof of Lemma 15.17 in [Bha+97].

The reasoning in the previous example can also be used to prove the following result, announced at several occasions. Note that it is assumed that $\mathcal{S} := \mathrm{Set}$ with classical logic.

Theorem 4.5.4. *The topos of sheaves on the site $\mathcal{H}_{k+1}^{\mathrm{op}}$ of finite Hrushovski $(k+1)$-hypergraphs is not atomic.*

Proof. By Theorem 3.3.18, the principal, prime and conservative model M of Example 4.5.3 induces the following equivalent subtoposes of the classifying topos of simple $(k+1)$-graphs:

$$[\mathbf{M}, \mathcal{S}]_{\mathrm{ev}} \cong \mathrm{Sh}(\mathcal{H}_{k+1}^{\mathrm{op}}) \cong \mathcal{S}[\mathbb{T}_M] \hookrightarrow [\mathcal{G}_{k+1}, \mathcal{S}] \tag{4.78}$$

Where $\mathcal{S}[\mathbb{T}_M]$ denotes the classifying topos of the theory of M in the language of $(k+1)$-hypergraphs. The argument of Example 4.5.3 (or Lemma 15.17 of [Bha+97]) also shows that M is not $\aleph_0$-categorical by the ordinary Ryll-Nardzewski theorem. But then $\mathcal{S}[\mathbb{T}_M]$ cannot be atomic; for it already has a conservative model. □

Remark 4.5.5. Theorem 4.5.2 could be strengthened in various ways, and even altered to yield three equivalent statements. Indeed, for (i)⇒(ii) one only needs the following condition on orbits:

(i') For every C, the lattice L of orbits $O(s) \hookrightarrow \Phi(C)$ satisfies that for all $l \in L$:

$$L \setminus \{v \mid v \leq l\} \text{ is finite} \tag{4.79}$$

Subsequently the lattice of types satisfies a similar (dual) property; as well as the lattice of principal filters in $\mathcal{C}$.

One can further turn the statements into equivalences by restricting to *countable* models $\mathbf{M}$. Then since every principal model is prime, one can show that, if the lattice of principal filters in $\mathcal{C}$ satisfies the above property, (i') must hold. So by adding to each statement such 'finiteness' conditions, one can forge the theorem into three equivalent statements. However, it would also lose much of its weight; after all I had argued that (ii)⇒(i) relies on the ultrafilter principle for boolean algebras, so forcing it to work in the non-boolean case misses the point. One thing to consider is to change *oligomorphic* to the notion (i') above, which remains unaltered in the case when $\mathbf{M}$ is a group.

To connect Theorem 4.5.2 to the traditional Ryll-Nardzewski theorem, recall[4] that for M a structure for some underlying signature, the monoid of endomorphisms $\mathbf{M} := \mathrm{End}(M)$ is *oligomorphic* when for every natural number n, the set of orbits of tuples of length n is finite (equivalently, the set of *traces* is finite, see Remark 4.1.15 above and Lemma 2.2 in [MP11]). The following states a general Ryll-Nardzewski criterion, but note that by the remarks in the beginning of this section, the version for first-order theories is stronger than what can be derived from the below.

Corollary 4.5.6 (Ryll-Nardzewski criterion, general version)**.** *Suppose $\mathcal{S}$ is boolean and, internally in $\mathcal{S}$, M is an an infinite structure on some signature and let $\mathbb{T}$ be its (regular, coherent, geometric) theory, so that the syntactic category $\mathcal{C}_\mathbb{T}$ is an orthogonally factorizing site. Then:*

(i) If $\mathbf{M} := \mathrm{End}(M)$ is oligomorphic, then M is a principal model of $\mathcal{C}_\mathbb{T}$.

(ii) If, in addition, M is countable and $\mathcal{S}$ satisfies the axiom of dependent choice, then M is prime, and hence $\mathrm{Sh}(\mathcal{C}_\mathbb{T}) \cong [\mathbf{M}, \mathcal{S}]_{\mathrm{fs}}$.

Proof. For (i), it needs to be verified that the definition of oligomorphy coincides with oligomorphy of the endomorphisms of the functor $M : \mathcal{C}_\mathbb{T} \to \mathcal{S}$, which follows from the definition of the syntactic category. Clearly M is a conservative model, so Theorem 4.5.2 applies.

For the second claim (ii), the assumption implies that Proposition 3.5.3 holds, for which choice is not needed since $\mathcal{C}_\mathbb{T}$ admits a recursive description in $\mathcal{S}$. Then the assertion is a consequence of Propositions 4.3.1 and 4.3.2. □

Remark 4.5.7. The above corollary encompasses what can be inferred from Lemma 3.2 and Lemma 3.4 in Mašulović and Pech [MP11] when $\mathbb{T}$ is a coherent theory. The authors refer to this as a theory built using only *positive* formulas.

[4]This appears as Definition 2.1 in [MP11]

4.6 Hereditarily supported M-sets as a model of IZFA

This section discusses another application to logic, namely to models of intuitionistic Zermelo-Fraenkel set theory with atoms. For classical Zermelo-Fraenkel set theory (with atoms), there is a class of models referred to as *permutation models* or Fraenkel-Mostowski models, after their early appearance in [Fra22] and [Mos39]. Such models are constructed by building a cumulative hierarchy of sets equipped with a G-action for some group G. Often, this hierarchy is restricted to G-sets which are hereditarily *symmetric* with respect to a normal filter of subgroups $\mathcal{F}$. A textbook-style introduction to these models can be found in Chapter 4 of Jech [Jec73]. For a more brief account, see for instance Blass [Bla11]. Some applications in computer science and the theory of nominal sets can be found in Gabbay and Pitts [GP01] or Pitts [Pit13].

Recall from Example 4.1.16 that symmetric G-sets can be presented as a topos of $\boldsymbol{\Phi}$-supported $\mathbf{M}$-sets, where $\mathbf{M} := G$ is the group and $\boldsymbol{\Phi}$ is determined by the filter $\mathcal{F}$. This raises the question of whether there exists a similar construction of a model of set theory for any pair $(\mathbf{M}, \boldsymbol{\Phi})$ where $\boldsymbol{\Phi}$ is supportive. The goal of this section is to show in Theorem 4.6.6 below that this is indeed the case.

First, one needs to decide how to interpret sets in $[\mathbf{M}, \mathcal{S}]$. Since the topos is no longer boolean, the interpretation should generally be an interpretation of *intuitionistic* Zermelo-Fraenkel set theory with atoms (IZFA). In Appendix A.1, I have given a very cursory description of this formalism. From early results by Fourman [Fou80] and Hayashi [Hay81] it has become well-known that every Grothendieck topos admits an interpretation of IZFA. Interpretations may differ on the choice of universe; e.g. Fourman explicitly constructs a cumulative hierarchy, whereas Hayashi works with an (abstract) collection of well-founded extensional graphs. A good framework for understanding the relationship between these different methods is Shulman's *stack semantics* [Shu10]. His theory also provides connections to Joyal and Moerdijk's more abstract *algebraic set theory* [JM95]. Note that these remarks pass over the important and perhaps categorically more interesting formalisms generally referred to as *constructive set theories*. These differ from IZF(A) in that the underlying models have to obey a completely different ontology: not only is the logic required to be constructive, it is also required to be predicative. A good introduction to these formalisms, as well as an account of their topos-theoretic semantics, is Van den Berg [Ber06]. A more subtle semantics, for a larger class of (elementary) toposes, is given by Awodey et al. [Awo+14], who also discuss the interpretations of Hayashi and Fourman as special cases.

For the present purposes, I leave both the generalized semantics and constructive set theories aside and shall focus on a simple interpretation of IZFA in $[\mathbf{M}, \mathcal{S}]$ presented in a talk [5] by Dana Scott [Sco80], defined as an interpretation in presheaves. This model is essentially the same as Fourman's [Fou80] but uses a forcing-style semantics rather than subobjects. The main result of

[5] I am grateful to Andrew Pitts for sharing Dana Scott's notes of this talk.

this section is that Scott's interpretation of set theory in $[\mathbf{M}, \mathcal{S}]$ transfers along the functor

$$(-)_{\boldsymbol{\Phi}} : [\mathbf{M}, \mathcal{S}] \to [\mathbf{M}, \mathcal{S}]_{\boldsymbol{\Phi}} \tag{4.80}$$

to yield a concrete interpretation of IZFA in '$\boldsymbol{\Phi}$-supported **M**-sets'. When $\mathbf{M} := G$ is a group, and $\mathcal{F}$ is a normal filter of subgroups, the functor $\boldsymbol{\Phi}$ from (4.24) is such that the interpretation given here is the usual Fraenkel-Mostowski model for the pair $(G, \mathcal{F})$. However, it is important to emphasize that in general, the 'sets' in the interpretation will not be mere '$\boldsymbol{\Phi}$-supported **M**-sets' when **M** contains non-invertible morphisms, in the sense that the action is not defined element-wise. I will illustrate this later with an example, see Remark 4.6.7.

4.6.1 Scott's presheaf model

Scott defines his interpretation of IZFA in a topos of presheaves $\widehat{\mathbb{C}} := [\mathbb{C}^{\mathrm{op}}, \mathcal{S}]$ on an arbitrary category $\mathbb{C}$. Instead of introducing his semantics directly, I will derive it from the cumulative hierarchy defined by Fourman [Fou80] for any Grothendieck topos.

Working in ZF with individuals, or atoms, one needs to keep track of the totality of sets (non-individuals) S^α of rank $< \alpha$ while defining the set-theoretic universe V^α of objects of rank $< \alpha$ at the same time. For a choice of object I of individuals, which is just some presheaf, the definitions are as follows (Fourman [Fou80], § 2):

$$S^0 = 0, \quad V^0 = I \tag{4.81}$$

$$S^{\alpha+1} = P(V^\alpha), \quad V^\beta = I + S^\beta \tag{4.82}$$

$$S^\lambda = \bigvee_{\alpha<\lambda} S^\alpha \tag{4.83}$$

where λ ranges over limit ordinals. From this, one can take the sequence V^α as a cumulative hierarchy of sets, consisting for each α of a presheaf of *individuals* and a presheaf of *sets* of rank $< \alpha$. In Scott's interpretation, an element $a \in I(A)$ is viewed as a system:

$$a = (a_f)_{f:B\to A} \tag{4.84}$$

of 'arbitrary things' indexed by the set of morphisms with codomain A, with a restriction operation (right action)

$$(a \cdot g)_{h:D\to C} = a_{g\circ h} \tag{4.85}$$

along morphisms $g : C \to A$. This is called an individual at *stage* A. Indeed, as an element of a presheaf, an individual can be identified with the (prime!) object appearing in the image:

$$a : y_A \twoheadrightarrow O(a) \rightarrowtail I \tag{4.86}$$

So an *individual* as in (4.84) can be construed as a single *orbit* like in Section 4.1. Later, when restricting the model of set theory along the functor 4.80, these orbits will be restricted to the orbits defined by $\boldsymbol{\Phi}$.

For the interpretation of sets, recall that in a presheaf topos $[\mathbb{C}^{\mathrm{op}}, \mathcal{S}]$, the power object PX of a presheaf X can be presented in the following way. For A an object of $\mathbb{C}$, an element $w \in PX(A)$ is a system:

$$(w_f)_{f:B\to A} \tag{4.87}$$

indexed over all morphisms into A, such that $w_f \subseteq X(B)$ for all f and $x \cdot g \in w_{f\circ g}$ for all $g : C \to B$ and $x \in X(B)$. For an arrow $h : C \to A$, the right action on w is given by $(w \cdot h)_{i:D\to C} = w_{h\circ i}$.

So, similar to individuals, a *set* of rank α at *stage* A is given by a system

$$s = (s_f)_{f:B\to A} \tag{4.88}$$

indexed over all morphisms into A, such that $s_f \subseteq V^\alpha(B) = I(B) + S^\alpha(B)$ is a subset for every $f : B \to A$, satisfying the condition that $x \cdot g \in s_{f\circ g}$ for all $x \in s_f$. Here the restriction operation on sets $(s'_g)_{g:C\to B}$ is defined as in (4.85) above.

For the interpretation of formulas, Scott's forcing semantics can now be defined as follows:

Definition 4.6.1 (Presheaf semantics for IZFA [Sco80])**.** For an assignment a of elements in $V^\alpha(A)$ for some α to free variables x, x_i, y, etc., and $\phi(\mathbf{x})$ a formula of set theory, the clauses

$$A \Vdash \phi(\mathbf{x})[a] \tag{4.89}$$

are inductively defined as follows:

(i) $A \Vdash x = y[a]$ iff $a(x) = a(y)$

(ii) $A \Vdash x \in y[a]$ iff $a(y) \in V^\alpha(A)$ and $a(x) \in a(y)_{1_A}$

(iii) $A \Vdash x \in V[a]$ iff $a(x) \in V^\alpha(A)$

(iv) $A \Vdash \phi(\mathbf{x}) \wedge \psi(\mathbf{x})[a]$ iff $A \Vdash \phi(\mathbf{x})[a]$ and $A \Vdash \psi(\mathbf{x})[a]$

(v) $A \Vdash \phi(\mathbf{x}) \vee \psi(\mathbf{x})[a]$ iff $A \Vdash \phi(\mathbf{x})[a]$ or $A \Vdash \psi(\mathbf{x})[a]$

(vi) $A \Vdash \neg\phi(\mathbf{x})[a]$ iff for all $f : B \to A$ one does not have $A \Vdash \phi(\mathbf{x})[a \cdot f]$, where $a \cdot f(x) = a(x) \cdot f$

(vii) $A \Vdash \phi(\mathbf{x}) \to \psi(\mathbf{x})[a]$ iff for all $f : B \to A$, $A \Vdash \phi(\mathbf{x})[a \cdot f]$ implies $A \Vdash \psi(\mathbf{x})[a \cdot f]$

(viii) $A \Vdash (\forall x)\phi(x, \mathbf{y})[a]$ iff for all $f : B \to A$, all $b \in V^\beta(B)$ (for any β), $A \Vdash \phi(x, \mathbf{y})[a']$ where $a'(x) = b$ and $a'(\mathbf{y}) = a(\mathbf{y})$

(ix) $A \Vdash (\exists x)\phi(x, \mathbf{y})[a]$ if there exists β and $b \in V^\beta(A)$ such that $A \Vdash \phi(x, \mathbf{y})[a']$ where a' extends a by letting $a(x) = b$

Scott's presheaf model establishes the following theorem:

Theorem 4.6.2 (Scott [Sco80])**.** *For the interpretation of Definition 4.6.1, all axioms of IZFA (Definition A.1.1) are forced together with the axioms and rules of intuitionistic logic.* □

For a proof, one needs to provide several set-theoretic constructions in the model that validate the axioms. Most of these are straightforward. To illustrate a slightly more involved case, consider the interpretation of power objects. If $(s_f)_{f:B\to A}$ is an arbitrary set of rank $< \alpha$ at stage A, the above definition of power objects in a topos suggests to put:

$$t_{f:B\to A} = \{c \in V^\alpha(B) \mid c_g \subseteq s_{f\circ g} \text{ for all } g : C \to B\} \tag{4.90}$$

Indeed, one can verify that $(t_f)_{f:A\to B}$ is a set at stage A (of rank α) satisfying the power set axiom for the set $(s_f)_{f:A\to B}$.

Scott also defines a notion of *global set*, which is the assignment of a set s_A at stage A to every object A, such that $s_A \cdot f \subseteq s_B$ for all $f : B \to A$. Every global set defines a presheaf. A *global map* is a global set which defines a function between two global sets. He observes:

Proposition 4.6.3 (Scott [Sco80])**.** *There is an equivalence between the category of global sets and global maps and the category of presheaves* $[\mathbb{C}^{\mathrm{op}}, \mathcal{S}]$. □

For the above to work, it is in fact enough to take for the set of individuals those defined by the orbits of example 4.1.13. Indeed, this follows since every presheaf is a colimit of sets of such individuals.

The restricted model

Instead of going over the proof of the above results, I will turn to the case of $\boldsymbol{\Phi}$-supported **M**-sets, which, for a monoid **M** and $\boldsymbol{\Phi}$ as in Example 4.1.14, should yield a 'monoidal' Fraenkel-Mostowski type model. The argument consists in the fact that for some $\boldsymbol{\Phi} : \mathcal{C} \to [\mathbf{M}, \mathcal{S}]$, the interpretation of IZFA in $[\mathbf{M}, \mathcal{S}]$ is, in some sense, preserved along the functor:

$$(-)_{\boldsymbol{\Phi}} : [\mathbf{M}, \mathcal{S}] \twoheadrightarrow [\mathbf{M}, \mathcal{S}]_{\boldsymbol{\Phi}} \tag{4.91}$$

First, in the above $\mathbb{C}^{\mathrm{op}}$ is now replaced by **M**. For an object X in $[\mathbf{M}, \mathcal{S}]$, its power object is given by:

$$\begin{aligned} P(X)(M) := \big\{(w_\eta)_{\eta:M\to N} \ &\text{such that for all } \eta,\ w_\eta \subseteq X(N) \\ &\text{and for all } \zeta : N \to K,\ \zeta \cdot w_\eta \subseteq w_{\zeta\circ\eta}\big\} \end{aligned} \tag{4.92}$$

where the action $\zeta \cdot w_\eta$ is element-wise. The action on a family (w_η) is given by, for $\zeta : M \to K$:

$$(\zeta \cdot w)_{\vartheta:K\to L} = w_{\vartheta\circ\zeta} \tag{4.93}$$

The following lemma is well-known, but not always given as explicitly as needed here:

Lemma 4.6.4. *Suppose $\mathcal{G}$ is an elementary topos. Let $\mathcal{F}$ be a coreflective subcategory of $\mathcal{G}$ closed under finite limits and subobjects. Denote the inclusion by i and the reflection by r. Then for every $C \in \mathcal{F}$, $r \circ P \circ iC$ defines a power object for $\mathcal{C}$:*

Proof. From the natural equivalences:

$$\mathcal{F}(A, rP(iC)) \cong \mathcal{G}(iA, P(iC)) \cong \mathrm{Sub}_{\mathcal{G}}(iA \times iC) \cong \mathrm{Sub}_{\mathcal{F}}(A \times C) \tag{4.94}$$

and the definition of power object. □

Corollary 4.6.5. *Let X be a Φ-supported $\mathbf{M}$-set, where Φ is supportive. Then its powerobject $P_{\Phi}X$ in $[\mathbf{M}, \mathcal{S}]_{\Phi}$ is given by the Φ-supported elements of the $\mathbf{M}$-set defined by* (4.92)*.*

By Fourman [Fou80], the construction (4.81)-(4.83) always yields a model of IZFA, so also in the case $[\mathbf{M}, \mathcal{S}]_{\Phi}$. However, one can be a bit more explicit by combining this fact with Scott's presentation and Corollary 4.6.5. Thus the cumulative hierarchy of *hereditarily Φ-supported* $\mathbf{M}$-sets is given by:

$$S^0_{\Phi} = 0, \quad V^0_{\Phi} = I_{\Phi} \tag{4.95}$$

$$S^{\alpha+1}_{\Phi} = P(V^{\alpha}_{\Phi})_{\Phi}, \quad V^{\beta}_{\Phi} = I_{\Phi} + S^{\beta}_{\Phi} \tag{4.96}$$

$$S^{\lambda}_{\Phi} = \bigvee_{\alpha<\lambda} S^{\alpha}_{\Phi} \tag{4.97}$$

where the power object is just the power object in $[\mathbf{M}, \mathcal{S}]$ defined as in (4.92). Naturally, the individuals also need to be supported. In the light of Fraenkel-Mostowski models, it makes sense to let these consist of the orbits, i.e.

$$I(M) = \{c \in \Phi(C)(M) \mid \text{ for some } C\} \tag{4.98}$$

with the obvious action. In other words, the individuals are precisely the elements of the presheaves $\Phi(C)$ for an object C in $\mathbb{C}$. Maintaining Scott's presentation, a hereditarily Φ-supported *set* of rank α at stage M is a system:

$$\left(s_{\eta}\right)_{\eta : M \to N} \tag{4.99}$$

where $s_{\eta} \subseteq I(N) \cup V^{\alpha}_{\Phi}(N)$ such that the restriction action, given by (4.85), is supported by some $c \in \Phi(C)(M)$ for some C. Here $V^{\alpha}_{\Phi}(N)$ is the set of hereditarily Φ-supported sets of rank $< \alpha$ at stage N. The new interpretation bears the following theorem.

Theorem 4.6.6. *Restricted to the universe V_{Φ} of hereditarily Φ-supported sets, the interpretation of Definition 4.6.1 forces all axioms of IZFA (Definition A.1.1) as well as those of intuitionistic logic.*

Before saying something about the proof, some remarks are in order. The fact that the theorem is true is not the most important message here, since by virtue of its construction, this should be expected from Fourman [Fou80], Theorem 2.4, although his interpretation of the logic uses subobjects rather than forcing, which leaves some work to be done. Alternatively, one can restrict the universe of Hayashi [Hay81] to the objects of the form V^{α}_{Φ}, which would also involve a translation of Scott's forcing clauses to Hayashi's. However this is relatively straightforward, see e.g. Section 9 of Shulman [Shu10]. The interesting part here is that Theorem 4.6.6 uses the relatively easy presheaf semantics of Definition 4.6.1. Indeed, it can now be easily compared to the group case:

Remark 4.6.7. If **M** is a *group* (or a groupoid), the requirement $\zeta \cdot s_\eta \subseteq s_{\zeta\circ\eta}$ in fact implies:

$$\zeta \cdot s_\eta = s_{\zeta\circ\eta} \tag{4.100}$$

for all ζ, η, since the action is invertible. Hence the action on a set is element-wise, and a set is nothing more than a set of elements, on each of which **M**-acts. Subsequently, under taking Φ-supported sets for Φ as in Example 4.1.14, and picking M as set of individuals, this model reduces precisely to a Fraenkel-Mostowski model of hereditarily supported sets, where the atoms are (in simple cases) the elements of the underlying set of the (permutation) group.

However, when **M** is a *monoid*, sets (4.99) are *not* just sets on which **M** acts element-wise. To see why, consider a simple example, where **M** is a monoid of functions $M \to M$, and $a, b \in M$ are distinct elements, and $\eta : M \to M$ is a function in the monoid such that $\eta(a) = \eta(b)$. Then consider the set $s = \{a, b\}$, which, as a system, is just given by:

$$s_\zeta = \{\zeta a, \zeta b\} \subseteq I \tag{4.101}$$

and define a new set by separation:

$$t = \{\varnothing \mid \forall x, y \in s.x = y\} \tag{4.102}$$

Then the set t is non-empty, as witnessed by η, i.e.:

$$M \Vdash \neg\neg\exists x.x \in t \tag{4.103}$$

But as a system (4.99), t_{1_M} is the empty set, whereas t_η suddenly contains an element (with empty support!). One could say that in the **M**-set model, sets carry a 'history' of endomorphisms that have been applied to them, after which new elements may have appeared. Note that there are no interesting monoid cases where this does not hold, for if every $\eta : M \to M$ is locally invertible (i.e. on every element $c \in M(C)$), then $\mathcal{O}$ is atomic and hence **M** may be restricted to a group without changing the topos.

The proof of Theorem 4.6.6 consists in showing that the constructions on sets (4.99) that validate the axioms in the presheaf model can be restricted to $\boldsymbol{\Phi}$-supported $\mathbf{M}$-sets. For instance, when $s \in V^{\alpha}_{\boldsymbol{\Phi}}(M)$, its power set can again be defined by:

$$t_{\eta : M \to N} = \{c \in V^{\alpha}_{\boldsymbol{\Phi}}(N) \mid c_{\zeta} \subseteq s_{\zeta \circ \eta} \text{ for all } \zeta : N \to K\} \tag{4.104}$$

Indeed, if s is supported by $c \in \boldsymbol{\Phi}(C)(M)$ and $\boldsymbol{\Phi}(C)(\eta)(c) = \boldsymbol{\Phi}(C)(\eta')(c)$, then:

$$s_{\zeta \circ \eta} = (\eta \cdot s)_{\zeta} = (\eta' \cdot s)_{\zeta} = s_{\zeta \circ \eta'} \tag{4.105}$$

so

$$c_{\zeta} \subseteq s_{\zeta \circ \eta} \iff c_{\zeta} \subseteq s_{\zeta \circ \eta'} \tag{4.106}$$

So t is also supported by $c \in \boldsymbol{\Phi}(C)(M)$, as required. It is easy to check that it defines a power object. I leave the rest of the verifications to the reader.

Alternatively, as mentioned above, one can show that the definition of forcing (Definition 4.6.1) coincides with the sheaf semantics that comes with the site of orbits $\mathcal{O}$ for $[\mathbf{M}, \mathcal{S}]_{\boldsymbol{\Phi}}$, i.e. forcing clauses of the form

$$O(c) \Vdash \phi(\mathbf{x})[a] \tag{4.107}$$

for some $c \in \boldsymbol{\Phi}(C)(M)$, where a assigns a set or individual supported by c to every variable x_i. Since this is a prime site, the only rule that really changes is the existential quantifier:

(ix') $O(c) \Vdash (\exists x)\phi(x, \mathbf{y})[a]$ if there exists a cover $[\eta] : O(d) \twoheadrightarrow O(c)$ in $\mathcal{O}$ and a $b \in I + V(O(d))$ such that $O(d) \Vdash \phi(x, \mathbf{y})[a']$ where $a'(x) = \eta \cdot a(x)$ and $a'(x) = b$.

However, it is easy to see that this existential clause is the same as the old one restricted to a universe of $\boldsymbol{\Phi}$-supported $\mathbf{M}$-sets.

5 Applications to Type theory

Several examples of toposes of finitely supported **M**-sets from Section 1.2 share a property that seems a long walk from any of the theory discussed in this thesis. Namely, many of these toposes can be given the structure of a *model category* in the sense of Quillen (see e.g. [Hov99] for an introduction). Indeed, this applies saliently to simplicial sets, but also cubical sets and cyclic sets carry a *model structure*. In the recent decade, a new perspective on such categories has appeared in computer science and logic. Through the notion of *Kan fibration*, these categories interpret a version of Martin-Löf intensional dependent type theory in which identity is interpreted as a type constructor, satisfying an inductive rule which resembles a form of path lifting. This version of type theory has become known as *homotopy type theory*, or *univalent type theory* in the cases where the axiom of univalence is insisted on [Pro13].

A proper introduction to homotopy type theory would be too much of a diversion for the purpose of this last chapter, which is more like the invitation than the ball itself. Yet, it is interesting to try and see how the notion of a fibrant type connects to the themes in this thesis, and to that end I will show some initial results. The idea of such a connection first emerged from a conversation I had with Andrew Pitts about his **M**-sets presentation of cubical sets, as discussed in the introduction and Example 1.2.8 (and for which see [Pit15]). Indeed, the observation that there are many more examples, such as simplicial sets and the enhanced cubical sets of Example 1.2.9, has greatly motivated the research undertaken for this thesis. The notion of *Kan mould* that I put forward below as a class of generating cofibrations is inspired by the analogy with the 'nominal' presentation of cubical type theory by Cohen et al. [Coh+18].

I begin and end my exposition with the motivating case of simplicial sets. A long-standing open problem in the nascent field of univalent foundations is to find a way to interpret univalent type theory in simplicial sets *constructively*. The notion of Kan mould has enabled a new approach to this problem which I have undertaken in collaboration with Benno van den Berg. The key suggestion of Van den Berg was to combine Kan moulds with his and Garner's simplicial Moore paths [BG12]

in order to be able to define a uniform structure on fibrations. I have devoted the last section to demonstrating this connection between Kan moulds and Moore paths. For the rest of our theory, which has grown extensively over the past year, I refer the reader to our forthcoming work [BF], which aims to provide a fully constructive interpretation of univalent type theory in simplicial sets based on these two ingredients. For another constructive approach to the simplicial model, which was developed at around the same time, see Gambino and Henry [GH19].

Between the first section and the last, I explore Kan moulds under additional assumptions which do not hold for simplicial sets. As a consequence, a more obvious uniformity condition on fibrations proves enough to show that the resulting type theory has dependent products. The main class of examples for which this holds are the classifying toposes of Section 4.4.2, namely for non-empty, flat models of algebraic theories. This includes the enhanced cubical sets of Example 1.2.9, and so the following gives a new proof for dependent products in that case.

5.1 Kan fibrations in simplicial sets

This section aims to arrive at the definition of a Kan mould, and to show that, naively, it defines the same fibrations as ordinary Kan horns. To avoid introducing too much new notation I have kept the amount of type theory below to a minimum. It is mostly an exercise in understanding Kan fibrations by viewing the topos of simplicial sets as a topos of finitely supported $\mathbf{M}$-sets. The most intuitive choice of $\mathbf{M}$ (see below) is the monoid of $\{0, 1, \leq\}$-preserving endomorphisms of $\mathbb{Q} \cap [0, 1]$, but of course the presentation of Example 1.2.10 admits a similar interpretation. In the language of finitely supported $\mathbf{M}$-sets, degeneracy maps can be regarded as 'weakenings' which introduce a new variable:

$$s_i : O(0 \leq \cdots \leq a_i \leq a_{i+2} \leq \cdots \leq 1) \to O(0 \leq \cdots \leq a_i \leq a_{i+1} \leq a_{i+2} \leq \cdots \leq 1) \tag{5.1}$$

Each degeneracy map s_i has two sections d_i, d_{i+1} which identify a_i and a_{i+1}, a_{i+1} and a_{i+2} respectively.

For a simplicial set B, an $(n+1)$-simplex b is just a morphism:

$$b : O(0 \leq a_1 \leq \cdots \leq a_{n+1} \leq 1) \to B \tag{5.2}$$

Hence one can regard the variable a_{i+1} as a way to continuously vary between the n-simplices $b \cdot d_i$ and $b \cdot d_{i+1}$:

$$b \cdot d_i \xrightarrow{b(a_{i+1})} b \cdot d_{i+1} \tag{5.3}$$

For this reason, it is nicer to work with $\mathbb{Q} \cap [0, 1]$ since a_{i+1} actually ranges over an interval of values. However, let me emphasize that all this is completely cosmetic: the goal is just to look at things with a different hat on.

Now suppose $p : E \to B$ is a morphism of simplicial sets, b is an $(n+1)$-simplex in B, and e_1 is an n-simplex in E such that $p(e) = bd_{i+1}$. This can be expressed diagrammatically as follows:

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{e_1} & E \\ {\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta[n+1] & \xrightarrow{b} & B \end{array} \tag{5.4}$$

Here I have abbreviated the standard simplices in the ordinary way. For this situation, a diagonal filler for the diagram (5.4) amounts to a lift for the path 5.3 in B to a path in E above b, with one given endpoint:

$$\begin{array}{ccc} e_0 & \overset{e(a_{i+1})}{\dashrightarrow} & e_1 \\ \vdots & & \vdots \\ b \cdot d_i & \xrightarrow{b(a_{i+1})} & b \cdot d_{i+1} \end{array} \tag{5.5}$$

Suppose now that, in addition to the $(n+1)$-simplex b and the n-simplex e_1, a *partial* path $e(a_{i+1})$ is given too. That is, for a certain substructure $f : F \hookrightarrow \Delta[n]$, a way to extend e_1 to a path along the 'freely added' variable a_{i+1} is given as:

$$e_F : s_i^* F \to E \tag{5.6}$$

This can be described as a lifting problem:

$$\begin{array}{ccc} s_i^* F \cup \Delta[n] & \xrightarrow{[e_F, e_1]} & E \\ {\scriptstyle s_i^* f \cup d_i}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta[n+1] & \xrightarrow{b} & B \end{array} \tag{5.7}$$

The following observation justifies the intuition so far:

Proposition 5.1.1. *A morphism $p : E \to B$ of simplicial sets is a Kan fibration if and only if it admits fillers for all diagrams of the form* (5.7) *(and those with d_i replaced by d_{i+1}) for all n, i.*

Proof. "⇒": It is enough to see that every horn inclusion $\Lambda_i[n+1] \to \Delta[n+1]$ is of the form $s_i^* f \cup d_i$ or $s_i^* f \cup d_{i+1}$ for some f. Indeed, take $f : F = \partial\Delta[n] \to \Delta[n]$. Recall that $\partial\Delta[n]$ is the union of all faces $d_0, \ldots, d_{n+1}$ of $\Delta[n]$. When $i \leq n+1$:

$$s_i^* d_j = \begin{cases} d_j & \text{if } j < i \\ d_{j+1} d_j & \text{if } j = i \\ d_{j+1} & \text{if } i < j \end{cases} \tag{5.8}$$

So $s_i^* F$ contains all faces except the i-th and the $(i+1)$-th, so that

$$s_i^* f \cup d_{i+1} \,:\, s_i^* F \cup \Delta[n] \hookrightarrow \Delta[n+1] \tag{5.9}$$

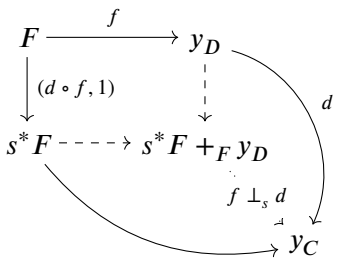
is the union of all faces except the i-th, in other words the i-th Horn inclusion. When $i = n+2$, one can take $s_{i-1}^* f \cup d_{i-1}$ instead.

"$\Leftarrow$" This is more work, but it is intuitively clear from the geometric realization of $s_i^* F \cup d_{i/i+1}$. A formal alternative is to fill $s_i^* F \cup d_{i/i+1}$ by induction on the dimension of subcomplexes. The idea is to complete every subcomplex to a Kan horn, using filling for the lower-dimensional ones, and then move up a dimension. □

5.1.1 Kan moulds

For a site $\mathcal{C}$ with a family of covers $\mathcal{E}$, one can define a class of arrows in $\mathrm{Sh}(\mathcal{C})$ which, for simplicial sets, defines the family of arrows appearing as the left vertical arrows in the diagrams of the form (5.7). It similarly generalizes the class of left arrows to fibrations for the cubical sets in [BCH14] and the generating trivial cofibrations of the evolved cubical sets in [Coh+18]. The definition is as follows. Suppose y_C is a sheaf and $s \,:\, C \twoheadrightarrow D$ is a cover with a section $d \,:\, D \to C$. Note that since y_D is a retract of a sheaf, it is also a sheaf. Further let $f \,:\, F \rightarrowtail y_D$ be an arbitrary subobject. Since $s \circ d = 1$, there is an induced morphism $(d \circ f, 1) \,:\, F \rightarrowtail s^* F$ into the pullback of f along y_s. Alternatively, this arrow is the pullback of d along $s^* f \,:\, s^* F \rightarrowtail y_C$, so it is a monomorphism. Abstracting from (5.7), the following defines the generating class of left arrows, or *trivial cofibrations*:

Definition 5.1.2. A *Kan mould* is a morphism $f \perp_s d \,:\, s^* F + y_D \rightarrowtail y_C$ induced by the pushout (5.10) defined from the above data and a subobject $f \,:\, F \hookrightarrow y_D$:



(5.10)

Note that since the outer square in the diagram (5.10) is a pullback, the pushout is given by the union of $s^* F$ and y_D as subobjects of y_C. Indeed, this is a property of a *coherent* category, of which every topos is an example (see [Joh02b], proposition A1.4.3). It follows that a Kan mould is always a monomorphism.

The idea of a Kan mould is just like for simplicial sets above. It defines a shape inside y_C consisting of a base y_D and a 'perpendicular' part s^*F, in the sense that $s^*F \cap y_D$ is contained in y_D. For cubical sets, the resulting arrows are essentially the 'Leibniz products' of Gambino-Sattler ([GS17], § 3) which are instances of a more general 'Leibniz construction' which appears in categorical homotopy theory and is due to Riehl and Verity [RV14]. Note that for simplicial sets, the Leibniz products of Gambino and Sattler are not the same as the Kan moulds defined here as Kan moulds always have representable codomain.

Definition 5.1.3. A *fibration* is a morphism $p : Y \to X$ that has the *right lifting property* with respect to every Kan mould. That is, there is a diagonal filler for every square:

$$\begin{array}{ccc} s^*F +_F y_D & \longrightarrow & Y \\ {\scriptstyle f \perp_s d}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle p} \\ y_C & \longrightarrow & X \end{array} \tag{5.11}$$

If a square (5.12) is a pullback square and d is a section of s, then there is an induced section of s' and a pullback square of Kan moulds (5.13) whenever $F \hookrightarrow y_D$ is a subobject:

$$\begin{array}{ccc} B & \xrightarrow{g'} & C \\ {\scriptstyle s'}\downarrow\uparrow{\scriptstyle d'} & & {\scriptstyle s}\downarrow\uparrow{\scriptstyle d} \\ E & \xrightarrow{g} & D \end{array} \tag{5.12}$$

$$\begin{array}{ccc} s'^*g^*F +_{g^*S} y_E & \longrightarrow & s^*F +_S y_D \\ \downarrow{\scriptstyle g^*f \perp_{s'} d'} & & \downarrow{\scriptstyle f \perp_s d} \\ y_B & \xrightarrow{g'} & y_C \end{array} \tag{5.13}$$

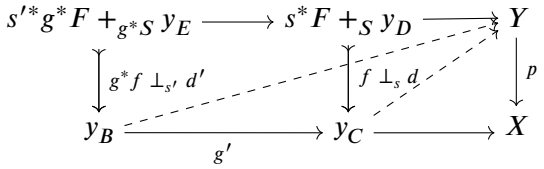
A pullback square as in (5.13) is called a *morphism of Kan moulds*. Kan moulds and pullback squares (5.13) define a small subcategory of the arrow category in $\mathrm{Sh}(\mathcal{C})$:

$$\boldsymbol{\Sigma} \hookrightarrow \mathrm{Sh}(\mathcal{C})^{\to} \tag{5.14}$$

For those in the know, I state the following fact as a direct application Theorem 4.4 in Garner [Gar09], under the assumption that $\mathcal{S} = \mathrm{Set}$ under the axioms and rules of ZFC set theory. A new proof of this result was given in Proposition 16 of Bourke and Garner [BG16]. The consequence for the theory above does not have much to do with the rest of this thesis, but forms an important ingredient when exploring the direction set out in this chapter in more depth.

Fact 5.1.4. *The inclusion* $\boldsymbol{\Sigma} \hookrightarrow \mathrm{Sh}(\mathcal{C})^{\to}$ *cofibrantly generates an algebraic weak factorization system* (L, R) *on* $\mathrm{Sh}(\mathcal{C})$*. The algebras for the mere pointed endofunctor* R *consist of morphisms* $p : Y \to X$ *equipped with a family of filling operations with respect to every square* (5.11)*, satisfying the following property:*

(i) For a morphism of Kan moulds as in (5.13)*, the chosen fillers in* (5.15) *below are* compatible*, i.e. they make the middle triangle commute.*



(5.15)

Such morphisms $p : Y \to X$ are called uniform fibrations*, and the morphisms which are co-algebras for the mere co-pointed endofunctor L are called (uniform)* trivial cofibrations. □

5.2 Kan dependent products

In a simplicial model of homotopy type theory (e.g. [KL16]) one would like to interpret the syntax of intensional Martin-Löf type theory in simplicial sets by interpreting a dependent type by a Kan fibration $p : E \to B$. Under this interpretation, one can use type-theoretic notation to talk about fibrations, e.g.:

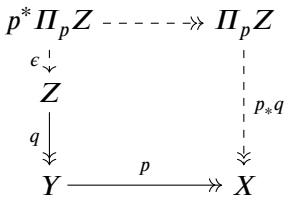
$$b : B \vdash E(b) : \text{Type} \tag{5.16}$$

The path lifting property of Kan fibrations discussed in the previous section can then also be formulated in the form of several type-theoretic rules. In the simplicial model of homotopy type theory, one wants to define identity types by a fibration in such a way that these rules are precisely the rules of the identity type in intensional Martin-Löf type theory.

In this section, I am however not concerned with identity types, so I will omit an introduction and refer the reader to [Pro13] for more on that topic. Rather, I just adopt the working assumption that one wants to interpret types as fibrations. If one is willing to accept Fact 5.1.4 above, one obtains an interpretation of the identity type by factorizing diagonals. The first question then becomes what other type constructors are permissible in this interpretation. First, one can show that the interpretation admits Σ-types, or *dependent sums*, when the composition of two fibrations is again a fibration. This can again be taken as a consequence of Fact 5.1.4 and is otherwise not very difficult to verify.

A second important type constructor is the *dependent product*. Categorically speaking, the issue here is whether dependent types (hence, Kan fibrations) are preserved under push-forward

along another dependent type (a Kan fibration):



(5.17)

If this is the case, then the interpretation of type theory is closed under taking dependent products. For simplicial sets, this property of Kan fibrations has long been known, but the proof is tedious and non-constructive. See for instance Theorem 7.8 in May[May67], which relies on heavy combinatorics. Indeed, it was shown by Bezem, Coquand and Parmann [BCP15] that this property of simplicial Kan fibrations (under a sufficiently 'naive' interpretation) does not admit a constructive proof. This has been one of the driving forces behind the subsequent cubical approaches to homotopy type theory.

In this section, I show how the notion of fibration suggested by Proposition 5.1.1 admits a constructive proof of Kan dependent products for certain conditions on a factorizing site $\mathcal{C}$. In the proof presented here, these conditions do not hold for the simplex category. For simplicial sets, Van den Berg and I have instead used path objects to deal with the issue, which requires a substantial amount of additional theory on Kan fibrations based on Kan moulds. As announced, this is forthcoming [BF], but a taste is given in Section 5.3. Still, the below result is of interest because it shows, for instance, that Pitts' cubical sets with diagonals [Pit15] also admit fibrant dependent products, and it attributes this property to essentially the same structure as that of the more complicated cubical sets in [Coh+18]. It further suggests a hosts of new examples where a 'fibrant type theory' may be considered.

Of course, this also leaves aside the many more questions that surround an interpretation of type theory. I had already mentioned identity types, but another essential ingredient to models of type theory is the interpretation of the type-theoretic universe. This becomes a difficult problem when this universe has to be fibrant. For simplicial sets, this has been addressed in the first model by Kapulkin and Lumsdaine [KL16] using non-constructive reasoning. Constructive methods have been based on *cubical sets* [BCH14] [Coh+18]. By abstracting on the notion of an interval, this method has a significant range [Sat17], although the case of simplicial sets remains notoriously difficult. In the announced work with Van den Berg, this is another issue that we aim to address.

5.2.1 A new class of models

In order to investigate the algebraic weak factorization system of Proposition 5.1.4 further, I make the following additional assumption on $\mathcal{C}$:

Assumption 5.2.1. There is a *dense* subcategory $\mathcal{F} \hookrightarrow \mathcal{C}$ such that:

(i) Every object of $\mathcal{F}$ is prime and the class $\mathcal{E}$ restricts to a coverage on $\mathcal{F}$
(ii) The arrows in $\mathcal{E}$ with domain and codomain in $\mathcal{F}$ are closed under pullback along arrows in $\mathcal{F}$
(iii) Every arrow in $\mathcal{E}$ between objects of $\mathcal{F}$ splits

As a consequence, $\mathrm{Sh}(\mathcal{C}) \cong [\mathcal{F}^{\mathrm{op}}, \mathcal{S}]$ is a presheaf topos.

Due to condition (ii), the simplex category $\mathcal{C} = \Delta$ is no longer an example. As announced, I return to simplicial sets in Section 5.3, where I point out that the omission of property (ii) is by no means a great loss, for simplicial sets admit a certain path object which in many cases actually works better than in the cubical setting. In this section, I show nevertheless that the following class of examples admits fibrant dependent products by virtue of property (ii) and thus forms a potentially interesting new class of models of (fibrant) dependent type theory.

Example 5.2.2. The reader may have already spotted that the assumptions (i)-(iii) hint at the class of examples discussed in Section 4.4.2, namely (non-empty) *flat* models of finitary algebraic theories. In that section, I had shown that these are classified by a site $\mathcal{C}$ of orbits of which the *free models* of the form $F\{\mathbf{x}\}$ form a dense subcategory. The restricted coverage is generated by *weakening*:

$$F\{\mathbf{x}\} \hookrightarrow F\{\mathbf{x}, y\} \tag{5.18}$$

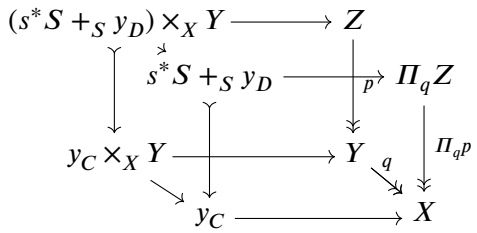
which defines a cover $F\{\mathbf{x}, y\} \twoheadrightarrow F\{\mathbf{x}\}$ in $\mathrm{FF}\mathbb{T}\text{-}\mathrm{Mod}(\mathcal{S})^{\mathrm{op}}$. Clearly the assumptions (i)-(iii) above are satisfied. This means that the definition of a fibration for cubical sets in [Coh+18] extends to all classifying toposes of flat models of (non-empty) algebraic theories. By Theorem 4.4.11, these are all of the form $[\mathbf{M}, \mathcal{S}]_{\mathrm{ev}}$ for $\mathbf{M}$ a monoid of endomorphisms on the free infinite algebra.

Example 5.2.3. As an example, take the category of *cubical sets with diagonals* as presented by Pitts [Pit15]. From Pitts' presentation and Theorem 4.4.11, it follows that this category classifies flat algebras for the finitary algebraic theory which consists of just two constants, 0 and 1. Pitts' relates the interpretation of the identity type in this category to the concept of *name abstraction*. From the theorem below it follows that this topos admits fibrant dependent products.

The goal of the below is to prove the following theorem:

Theorem 5.2.4. *If $p : Z \twoheadrightarrow Y$, $q : Y \twoheadrightarrow X$ are uniform fibrations, then $\Pi_q p : \Pi_q Z \to X$ can be equipped with the structure of a uniform fibration.*

To this end, the procedure begins with the usual observation that it is enough to prove that *generating* trivial cofibrations (the Kan moulds) are closed under pullback along fibrations. Indeed, consider a lifting problem as in (5.11) for $\Pi_q p$:



(5.19)

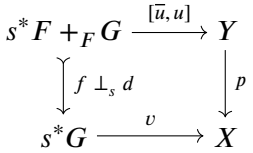
By pulling back q along $y_C \to X$, one obtains a fibration $y_C \times_X Y$. Pulling back the Kan mould along this fibration results in the arrow on the left, which by the universal property of dependent products admits an arrow into Z, making the diagram commute. By the universal property, a filler for the back square defines a filler for the front square. So it would be enough to show that pullbacks of Kan moulds along fibrations always lift against uniform fibrations. For this, the following lemma is needed first.

Lemma 5.2.5. *Suppose $f : F \hookrightarrow G$ is a monomorphism over y_D, i.e. for some $g : G \to y_D$, and let $s : C \twoheadrightarrow D$ and $d : D \rightarrowtail C$ be as in definition 5.1.2.*

Then if p is a uniform fibration, it can be equipped with a right lifting structure with respect to morphisms (5.20) *formed as the unique arrow from a pushout, cf. 5.10 above.*

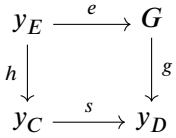
$$f \perp_s d : s^*F +_F G \rightarrowtail s^*G \tag{5.20}$$

Proof. Consider a square (5.21):



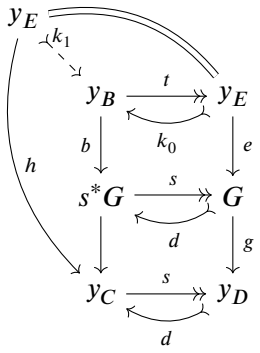
(5.21)

To define a filler, one can proceed pointwise by defining a natural transformation $\phi_E : s^*G(E) \to Y(E)$. An arbitrary element of $s^*G(E)$ corresponds to a pair (e, h) fitting in the square:



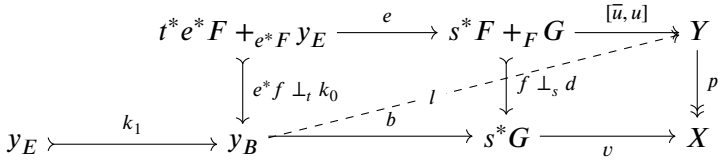
(5.22)

From this data, a Kan mould can be constructed as follows. Define $t : B \twoheadrightarrow E$ as the following pullback, with induced sections k_0, k_1 as drawn.



(5.23)

Observe that k_0,k_1 are typically distinct when h does not factor through d. The data (5.21)-(5.23) now fits into the following Kan mould:



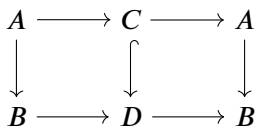
(5.24)

Now since p is a fibration, there exists a filler l as drawn. Because $b \circ k_1$ is the original element $(e, h) \in s^*G(E)$, the restriction $l \circ k_1$ is a suitable lift $\phi_E(e, h) \in Y(E)$.

Since p in fact comes with natural fillers with respect to pullback squares (5.13), and the above construction clearly commutes with pullback along arrows $E' \to E$, $\phi_E : s^*G(E) \to Y(E)$ defines a natural transformation, and therefore a filler $\phi : s^*G \to Y$ for the diagram (5.21). □

The following observation will also be useful:

Lemma 5.2.6 ('Swan's lemma' [Swa18], lemma 5.9)**.** *Suppose* (5.25) *is a retract diagram in some category. Then if the middle arrow $C \hookrightarrow D$ is a monomorphism, the left square is a pullback.*



(5.25)

□

Everything is now ready for the proof of the key ingredient of Theorem 5.2.4:

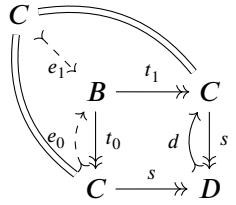
Proposition 5.2.7. *A pullback of a Kan mould along a fibration, as in* (5.26) *below, is a trivial cofibration.*

$$\begin{array}{ccc} p^*s^*S +_{p^*S} p^*y_D & \dashrightarrow & s^*S +_S y_D \\ {\scriptstyle g}\,\vdots & & \big\downarrow {\scriptstyle f \perp_s d} \\ G & \xrightarrow{\quad p \quad} & y_C \end{array} \tag{5.26}$$

Proof. The goal is to exhibit the arrow g in the diagram (5.26) as a retract of a trivial cofibration of the form $g_0 \perp_{t_0} e_0$ (5.20), where $t_0 : B \twoheadrightarrow C$, and g_0 is the restriction of g along (5.27):

$$i_0 : p^*s^*S \hookrightarrow p^*s^*S +_{p^*S} p^*y_D. \tag{5.27}$$

Define t_0,t_1 by the pullback of s along itself, together with the respective sections e_0, e_1 as in 5.28 (note that e_1 is a section of both t_0 and t_1).

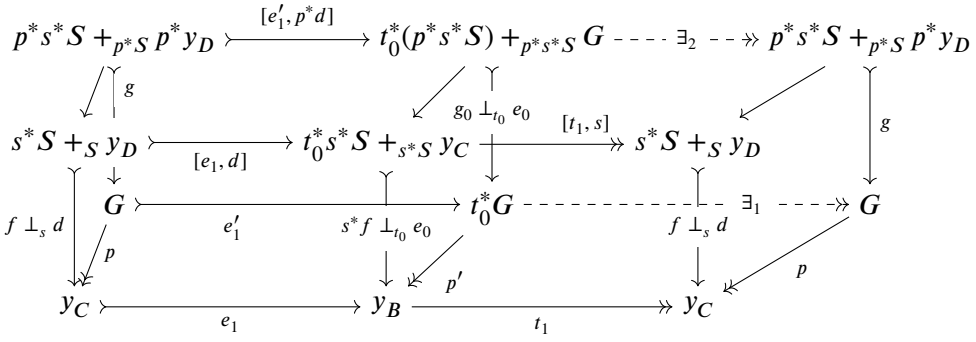


(5.28)

$$\begin{array}{ccc} D & \xrightarrow{\quad d \quad} & C \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle e_0} \\ C & \xrightarrow[\quad e_1 \quad]{} & B \end{array} \tag{5.29}$$

From (5.28), the square (5.29) commutes and is a pullback by Swan's lemma.

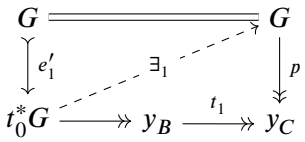
Using this, observe that the front part of the diagram (5.30) exhibits the Kan mould $i := f \perp_s d$ as a retract of the Kan mould $s^*f \perp_{t_0} e_0$. The arrow $[e_1, d]$ is well-defined since $e_0 d = e_1 d$, and similar for $[t_1, s]$ since $t_0^* s^* S \cong t_1^* s^* S$ (note that in the composition $[t_1, s] \circ [e_1, d]$, one passes through the isomorphisms $t_1^* s^* S \cong t_0^* s^* S \cong t_1^* s^* S$). It is verified at (5.32)-(5.34) below that both squares commute. By Swan's lemma, the left square is a pullback.



(5.30)

The left bottom square is defined by pulling back p along t_0 and taking the section induced by e_1. Next, the middle upright square is defined by pullback along p'. This defines, finally, the arrow $[e_1', p^*d]$ as the unique arrow factoring through this pullback. So the cube on the left is commutative and in fact a pullback. The diagram needs to be completed with the two missing morphisms.

Now, observe that since e_1 is a section of t_0, e_1' is a Kan mould of the form (5.20), namely $0 \perp_{t_0} e_1 : G \nrightarrow t_0^* G$. Since p is a fibration, there is a filler (5.31) fitting in (5.30).



(5.31)

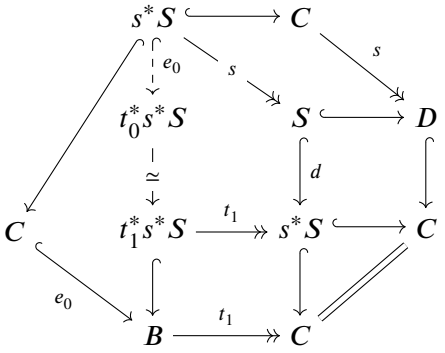
The diagram (5.28) can now be completed by the arrow $\exists_2$ induced by the pullback square on the right. This finally exhibits g as a retract of $g_0 \perp_{t_0} e_0$. Since cofibrations are stable under pullback, g_0 is a cofibration, and so $g \perp_{t_0} e_0$ is a trivial cofibration. So g is also a trivial cofibration. □

The following explicitly verifies the commutativity of the diagram (5.30):

Commutativity of the squares (5.30). The commutativity condition

$$[f \perp_s d] \circ [t_1, s] = t_1 \circ s^* f \perp_{t_0} e_0 \tag{5.32}$$

in the diagram 5.30 is expressed by commutativity of the following diagram, where I have repressed the $y_{(-)}$ notation for representable objects:

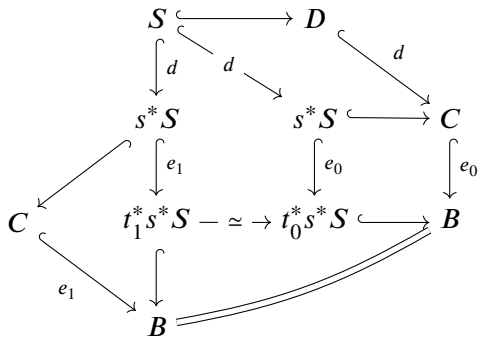


(5.33)

For the identity

$$s^* f \perp_{t_0} e_0 \circ [e_1, d] = e_1 \circ f \perp_s d \tag{5.34}$$

there is similarly a commutative diagram:



(5.35)

□

Remark 5.2.8. Note that the proof of Proposition 5.2.7 implies something stronger, because p only needs fillers for diagrams of the form (5.31). In such diagrams, the 'perpendicular part' is empty, so the lifting condition is much more relaxed.

Remark 5.2.9. The proof above looks very similar to the proof of lemma 4.7 in Gambino-Sattler [GS17], so it is important to stipulate why they are not the same. The equivalent of lemma 4.7 here would be the assertion that $f \perp_s d$ is a retract of $(f \perp_s d) \perp_{t_0} e_0$, where the inclusion is given by some natural transformation $1 \Rightarrow (-) \perp_{t_0} e_0$. In the setting of Proposition 5.2.7, it is not possible to prove this. This is due to the lack of a *connection*. Moreover, the above retract diagram does not exhibit $f \perp_s d$ as a *strong* homotopy equivalence: although the two morphisms $[t_1, s]$ and t_1 could be seen as 'homotopies' (5.36), there is no *strength* since the middle morphism is (manifestly) not $t_0^*(f \perp_s d)$.

$$(i_1 \circ s) \circ f \perp_s d \simeq 1_{s^*S +_S y_D} \quad f \perp_s d \circ (i_1 \circ s) \simeq 1_{y_C} \tag{5.36}$$

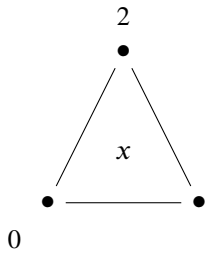
So proposition (5.2.7) is a different proof than that of Gambino-Sattler, and one which also works for the 'cubical sets with symmetries and diagonals' of Pitts [Pit15].

Actually, the proof closely resembles the original proof for the cubical sets model in [BCH14], although the cube category there does not satisfy the conditions of assumptions (i)-(iii) above. Yet it appears that the above proof still goes through - for the pullback condition is only used for very specific arrows and there is in each case an analogous construction in the cube category with the same properties (Swan's lemma still applies, for instance). Indeed, in theorem 5 of section 7.1 in [BCH14], the invocation of a 'fresh variable' z with respect to their context K corresponds to taking the pullback (5.28). The fact that, to define a lift, they lift 'downwards' first, and then up, corresponds to the intermediate lift denoted $\exists_1$ in the proof of proposition (5.28), and the subsequent application of Lemma 5.2.5.

Proof of Theorem 5.2.4. It is left to prove that the fillers for (5.19) provided by proposition (5.2.7) can be defined *uniformly*, i.e. they are compatible with morphisms of Kan moulds (5.13). Note that there are no dependent products at all without it, because Lemma 5.2.5 depends on uniformity.

Yet, all constructions in proposition 5.2.7 are pullbacks, so uniformity comes down to a straightforward diagram chase of a pullback (5.13) along all the above constructions, using that this time also p is uniform. This is left for the reader to convince themselves of. □

Example 5.2.10. For an example outside the 'cubical' paradigm where Theorem 5.2.4 contains an interesting result, take for instance the finite algebraic theory consisting of *three* constant symbols $0, 1, 2$. Finitely supported $\mathbf{M}$-sets for $\mathbf{M}$ the free infinite model have a very strange geometrical interpretation. The 1-dimensional representable is a 'line with three endpoints' $\{x\}$:



(5.37)

The 2-dimensional shape represented by $\{x, y\}$ has 7 faces of the form (5.37), corresponding to $x = i$, $y = i$ for $i = 0, 1, 2$ plus the face $x = y$, with each pair of faces sharing only one vertex. In another interpretation, one can view them as 'cubical sets with errors', where $0, 1$ have the usual geometric interpretation, but 2 represents an 'error'. Regardless of the intuition, by Theorem 5.2.4, these $\mathbf{M}$-sets interpret a version of Martin-Löf dependent type theory where dependent types are interpreted by fibrations.

5.3 The case of simplicial sets

This section picks up the thread from the beginning of Section 5.1, where I presented Kan moulds for simplicial sets as an alternative approach to Kan fibrations and the simplicial model of type theory. As in the case of cubical type theory [Coh+18], the biggest challenge when developing this further is working *constructively*, otherwise there would just be Proposition 5.1.1 and the work of Kapulkin and Lumsdaine [KL16]. Towards this, I was lucky to be able to join forces with Van den Berg [BF]. I will use this section to exhibit the basic ingredients of our work and present some new results. The main result that I will prove is that the simplicial Moore path functor, by Garner and Van den Berg [BG12], is a polynomial functor. This result reflects our key new insights into the Moore path functor that have led us on our quest. For the rest, however, I can but refer to our future work [BF]. The many other ingredients, such as cofibrations and trivial fibrations through

dominances, our constructive proof for dependent products and the local property of fibrations, go beyond the scope of this thesis. Yet I hope that what is said here may entice the reader to look out for this.

5.3.1 Paths

The motivating, and perhaps surprising, connection between the idea behind fibrations in Section 5.1 and type theory has its roots in the notion of simplicial *Moore path* due to Van den Berg and Garner [BG12]. This notion mimics Moore paths for topological spaces. Recall that a Moore path in a topological space X is a continuous map $[0, r] \to X$, where the domain denotes a real interval of length $r \in \mathbb{R}_{\geq 0}$. The salient feature of Moore paths is that they admit strictly associative composition (e.g. of a pair of paths of lengths r_0, r_1 to a path of length $r_0 + r_1$) so that the usual worry of composition being only associative 'up to homotopy' disappears. In addition, Moore paths on a space X can be equipped with a topology, so that composition of paths becomes a continuous operation. This can be extended to the structure of a *path object category* [BG12]. In simplicial sets, one can exploit a similar feature, which I introduce below. Let me note that cubical sets admit the structure of a path object category in a very similar way, which is due to Docherty [Doc14]. Further, the idea of using (a synthetic form of) *topological* Moore paths directly towards constructive models of type theory has been introduced by Orton and Pitts [OP19]. The following definition for simplicial sets is from [BG12].

Definition 5.3.1. An n-dimensional *traversal* is a function $\theta : \{1, \dots, k\} \to [n] \times \{+, -\}$, describing a zig-zag diagram of face maps as in the diagram below:

$$\begin{array}{ccccccccc} \Delta[n] & & & & \Delta[n] & & & & \Delta[n] \\ & \searrow^{d^{n+1}_{\theta^-(1)}} & & \swarrow_{d^{n+1}_{\theta^+(1)}} & & \searrow^{d^{n+1}_{\theta^-(2)}} & & \swarrow_{d^{n+1}_{\theta^+(k)}} & \\ & & \Delta[n+1] & & & & \Delta[n+1] \;\cdots\; \Delta[n+1] & & \end{array} \tag{5.38}$$

Here $\theta^-, \theta^+ : \{1, \dots, k\} \to [n+1]$ are defined as in (5.39):

$$\theta^-(j) = \begin{cases} i+1 \text{ if } \theta(j) = (i, +) \\ i \text{ if } \theta(j) = (i, -) \end{cases} \qquad \theta^+(j) = \begin{cases} i \text{ if } \theta(j) = (i, +) \\ i+1 \text{ if } \theta(j) = (i, -) \end{cases} \tag{5.39}$$

For n-dimensional traversals ϕ, ψ, their concatenation is denoted $\phi + \psi$ (note it is order-dependent). With concatenation, the set of n-dimensional traversals forms a monoid.

For every traversal θ, one can take the colimit of the above diagram, yielding its *realization* $\hat{\theta}$. For n-dimensional traversals θ, ψ, their concatenation $\theta + \psi$ is realized as a pushout:

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{\iota_1 \circ d_{\psi^-(1)}} & \widehat{\psi} \\ {\scriptstyle \iota_k \circ d_{\theta^+(k)}}\downarrow & & \downarrow \\ \hat{\theta} & \longrightarrow & \widehat{\theta + \psi} \end{array} \tag{5.40}$$

For every n-dimensional traversal, there is a canonical *retraction* onto $\Delta[n]$

$$r : \hat{\theta} \to \Delta[n] \tag{5.41}$$

given by the cocone $s_{\theta(i)_0} : \Delta[n+1] \to \Delta[n]$ on the diagram (5.38) defined by θ.

Example 5.3.2. For a typical example of Definition 5.3.1, take $k = 1$ and $\theta(1) = (i, +)$. Then the corresponding traversal is a descending pair of face maps:

$$\begin{array}{ccccc} \Delta[n] & & \xrightarrow{d_{i+1}^{n+1}} & & \Delta[n] \\ & & \Delta[n+1] & & \\ & & \xleftarrow{d_i^{n+1}} & & \end{array} \tag{5.42}$$

It may help the reader to observe that all traversals are concatenations of either ascending or descending pairs of face maps.

Just as the object of path lengths forms a topological space (that of non-negative real numbers), traversals can be equipped with a simplicial structure, that is, a right action by a simplicial map $\alpha : [m] \to [n]$ sending an n-dimensional traversal to an m-dimensional traversal.

Proposition 5.3.3. *For θ an n-dimensional traversal, and $\alpha : [m] \to [n]$ an arrow between simplices, there exists a unique traversal ψ such that the following square* (5.43) *is a pullback square of sets, $\hat{\alpha}$ is order-preserving, and the restriction of ψ to every fibre $\hat{\alpha}^{-1}(i)$ is order-reversing if $\theta(i) = (x, +)$ for some x and order-preserving in case $\theta(i) = (x, -)$.*

$$\begin{array}{ccc} \{1, \dots, l\} & \xrightarrow{\psi} & [m] \times \{+, -\} \\ {\scriptstyle \hat{\alpha}}\downarrow & & \downarrow{\scriptstyle \alpha \times \{+,-\}} \\ \{1, \dots, k\} & \xrightarrow{\theta} & [n] \times \{+, -\} \end{array} \tag{5.43}$$

Proof. This is a straightforward verification, see the discussion around Proposition 7.1.2 in [BG12] for more details. □

From this, one obtains a simplicial structure on traversals:

Corollary 5.3.4 (Van den Berg-Garner [BG12])**.** *Proposition 5.3.3 defines an action on traversals, sending an n-dimensional traversal θ to $\theta \cdot \alpha := \psi$ under $\alpha : [m] \to [n]$. Moreover, this action commutes with composition, or concatenation, of traversals.*

Proof. Equally straightforward by uniqueness. The last part follows from the fact that $\hat{\alpha}$ is order-preserving. □

Remark 5.3.5. Note that the action on traversals does not preserve their length, i.e. the number of segments it is composed of. It may increase through the action degeneracy maps, as shown in diagram A.9 in the appendix, and so it can decrease by the action of face maps.

Denote by $P1$ the simplicial set of traversals, with $P1[n]$ the set of n-dimensional traversals and the above action. This can be viewed as an object of path lengths. It has a category of elements $\mathbb{T}$ with objects (n, θ) and morphisms $\alpha : (m, \psi) \to (n, \phi)$ satisfying $\psi = \phi \cdot \alpha$.

Proposition 5.3.6. *The* realization *of a traversal extends to a functor*

$$C : \mathbb{T} \to \widehat{\Delta} \tag{5.44}$$

defined by $C\theta = \hat{\theta}$ on objects and on morphisms by the pullback squares:

$$\begin{array}{ccc} \widehat{\theta \cdot \alpha} & \xrightarrow{\hat{\alpha}} & \hat{\theta} \\ {\scriptstyle r_{\theta\cdot\alpha}}\downarrow & & \downarrow{\scriptstyle r_{\theta}} \\ \Delta[m] & \xrightarrow{\alpha} & \Delta[n] \end{array} \tag{5.45}$$

Proof. The proposition is that there is such a pullback square (5.45). This is non-trivial and since a full proof would break the continuity of this section, I have deferred this to Appendix A.2. □

The functor C has a left Kan extension $C : \hat{\mathbb{T}} \to \widehat{\Delta}$ along the Yoneda embedding, which has a right adjoint:

$$P_1(X)(n, \theta) := \mathrm{Hom}_{\widehat{\Delta}}(C\theta, X) \tag{5.46}$$

Since there is a natural equivalence $\hat{\mathbb{T}} \cong \widehat{\Delta}/P1$, the following definition makes sense:

Definition 5.3.7 ([BG12])**.** The *Moore path functor* on simplicial sets is defined as the functor P given by (5.47)

$$P := \Sigma_{P1 \to 1} \circ P_1 \tag{5.47}$$

The Moore path functor can be extended to the structure of a *path object category* due to van den Berg and Garner [BG12]. By showing this for simplicial sets, they are able to define a weak factorization system where the right class enjoys a path lifting property formulated for general path

object categories. These are different from Kan fibrations. Below, I begin to develop the new theory that also uses Moore paths, but towards a constructive theory of Kan fibrations instead. From this, the Kan moulds of Definition 5.1.2 will naturally appear as trivial cofibrations.

5.3.2 A polynomial path functor

Proposition 5.3.6 has a stronger consequence: namely that the canonical retraction (5.41) comes from a natural transformation:

$$\rho : C \Rightarrow y \circ U \tag{5.48}$$

between C and the composite of the Yoneda embedding with the forgetful functor $U : \mathbb{T} \to \Delta$, which satisfies colim $y \circ U = P1$. Since all naturality squares (5.45) are pullbacks, this natural transformation is further called *cartesian*.

The natural transformation (5.48) has two natural sections:

$$\sigma, \tau : y \circ U \Rightarrow C \tag{5.49}$$

defined on a traversal (n, θ) by the left- and rightmost face inclusions:

$$s_\theta := d_{\theta^-(1)} : \Delta[n] \to \hat{\theta} \quad t_\theta := d_{\theta^+(k)} : \Delta[n] \to \hat{\theta} \tag{5.50}$$

Define $C1 := \operatorname{colim} C$, the value of the left Kan extension of C on 1. The following proposition is key:

Proposition 5.3.8. *Let μ be σ or τ. Then there are morphisms $m : P1 \to C1$, $s : C1 \to P1$ such that for any n-dimensional traversal $\theta \in P1[n]$, the commutative diagram* (5.51) *factorizes the identity on θ as a composite of pullback squares.*

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{\theta} & P1 \\ {\scriptstyle \mu_\theta}\downarrow & & \downarrow{\scriptstyle m} \\ \hat{\theta} & \longrightarrow & C1 \\ {\scriptstyle \rho_\theta}\downarrow & & \downarrow{\scriptstyle r} \\ \Delta[n] & \xrightarrow[\theta]{} & P1 \end{array} \tag{5.51}$$

Proof. Because ρ, μ are natural transformations $C \Rightarrow y \circ U$ and $y \circ U \Rightarrow C$, taking colimits gives a pair of maps $r : C1 \to P1$, $m : P1 \to C1$ such that $r \circ m = 1_{P1}$.

It remains to prove that both squares in (5.51) are pullback. Since the vertical maps of the outer rectangle are identity arrows, it is enough to prove that the bottom square is a pullback. Since ρ is a *cartesian* natural transformation, this is a consequence of a general categorical result to which I have devoted Appendix A.3. □

Recall that an arrow $f : B \to A$ in a locally cartesian closed category $\mathbb{C}$ defines an endomorphism $P_f : \mathbb{C} \to \mathbb{C}$ called the *polynomial functor* defined by f:

$$P_f(X) := \Sigma_{A\to 1}\Pi_{f:B\to A}B^*X \cong \Sigma_{a:A}X^{B_a} \tag{5.52}$$

Here the last expression is the more type-theoretical interpretation of this object in the internal logic of $\mathbb{C}$, where B_a denotes the 'fibre' $f^{-1}(a)$. The name 'polynomial' comes from the fact that case A and B are finite sets, the result is an actual 'polynomial' in X. Polynomials functors have very interesting properties, for instance that their composition is also a polynomial functor. For more on them, see Gambino-Kock [GK13] or Von Glehn [Gle14]. The following theorem is a new result on simplicial Moore paths obtained by me and Van den Berg.

Theorem 5.3.9. *The Moore path functor $P : \widehat{\Delta} \to \widehat{\Delta}$ of definition 5.3.7 is isomorphic to the polynomial functor on $\widehat{\Delta}$ defined by the generic retraction $r : C1 \to P1$ of* (5.51).

Proof. By the previous proposition, one can extend $C : \mathbb{T} \to \widehat{\Delta}$ to a functor $\widehat{C} : \hat{\mathbb{T}} \cong \widehat{\Delta}/P1 \to \widehat{\Delta}$ simply by pullback:

$$\begin{array}{ccc} \widehat{C}X & \longrightarrow & C1 \\ \downarrow & & \downarrow \\ X & \xrightarrow[x]{} & P1 \end{array} \tag{5.53}$$

So $\widehat{C}(x) := \Sigma_{C1\to 1}r^*x$. Since pullback preserves colimits, it is easy to see that $\widehat{C}$ is the left Kan extension of C along $y : \mathbb{T} \to \hat{\mathbb{T}}$. Hence for the right adjoint one obtains the identity $P_1(X) := \Pi_{r:C1\to P1}C1 \times X$. With regards to definition 5.3.7, this yields:

$$P(X) = \Sigma_{P1\to 1}\Pi_{r:C1\to P1}C1 \times X \cong \Sigma_{\theta\in P1}X^{C1_\theta} \tag{5.54}$$

□

The previous theorem is very useful since the Moore path functor now has universal properties one can exploit, such as in the following. Suppose $\iota : F \hookrightarrow y_D$ is a subobject and one is given a 'path lifting problem' as follows:

$$\begin{array}{ccc} F & \xrightarrow{u} & PY \\ {\scriptstyle\iota}\big\downarrow & & \big\downarrow{\scriptstyle (t,Pf)} \\ y_D & \xrightarrow[(y_1,\gamma)]{} & Y \times_X PX \end{array} \tag{5.55}$$

Here the pullback $Y \times_X PX$ is of f along the target map $t : PX \to X$. So the above problem gives a path $\gamma : \hat{\theta} \to X$ in X with target x_0 and an element y_0 in the fibre $f^{-1}(x_0)$. In addition, it gives a

partial extension $u : F \to PY$ of this path to a path in Y lying above γ. Say the arrow $f : Y \to X$ is a *fibration* if every such lifting problem has a diagonal filler.

The problem can be cast into a familiar shape when the path γ is of length one, i.e. it corresponds to $\Delta[n+1] \to X$ for a traversal defined by a single degeneracy s_i. Note that the morphism (t, Pf) can be seen as a morphism in $\widehat{\Delta}/P1$ since there is always an arrow $P! : PX \to P1$ induced by the terminal arrow and this is compatible with $P! : PY \to P1$. So by Theorem 5.3.9 (u, γ) is really an arrow:

$$(u, \gamma) : \iota \to \Pi_{r:C1 \to P1} C1^* f \tag{5.56}$$

and thus has a unique transpose:

$$(\hat{u}, \hat{\gamma}) : \Sigma_{C1 \to 1} r^* \iota \to f \tag{5.57}$$

which, by (5.51), is commutative square:

$$\begin{array}{ccc} s_i^* F & \xrightarrow{\hat{u}} & Y \\ \downarrow & & \downarrow \\ y_C & \xrightarrow{\hat{\gamma}} & X \end{array} \tag{5.58}$$

In addition, there is the arrow $y_0 : y_D \to Y$ which satisfies $f \circ y_0 = \gamma \circ d_{i^+}$ where d_{i^+} is the face map corresponding to the target of γ, e.g. $i^+ = i$ or $i^+ = i + 1$ depending on the orientation of γ. Hence the data (5.55) is the same thing as a square:

$$\begin{array}{ccc} s_i^* F +_F \Delta[n] & \xrightarrow{[\hat{u}, y_0]} & Y \\ {\scriptstyle \iota \perp_{s_i} d_{i^+}} \downarrow & & \downarrow f \\ \Delta[n+1] & \xrightarrow{\hat{\gamma}} & X \end{array} \tag{5.59}$$

This is precisely a *Kan mould* as encountered in (5.7) and in the subsequent definition of Kan fibration in (5.1.3). For general paths $\gamma : \Delta[n] \to PX$ with an underlying traversal θ, the square (5.55) corresponds to a 'generalized' Kan mould:

$$\begin{array}{ccc} r_\theta^* F +_F \Delta[n] & \xrightarrow{[\hat{u}, y_1]} & Y \\ {\scriptstyle \iota \perp_{r_\theta} t_\theta} \downarrow & & \downarrow f \\ \hat{\theta} & \xrightarrow{\hat{\gamma}} & X \end{array} \tag{5.60}$$

It is easy to check that these squares have diagonal fillers precisely when the original (5.55) has one. Yet about this precise correspondence a lot more must be said in terms of structure, for in the case of a constructive theory this correspondence should certainly be natural in some sense. This needs a much more careful treatment than what is presented here, starting with a theory on *cofibrations*, essentially the left arrows in (5.55).

5.3.3 Uniform Kan fibrations in simplicial sets

But the generalized Kan moulds (5.60) raise more questions, such as whether a Kan fibration which lifts against the ordinary Kan moulds also lifts against generalized Kan moulds, in what sense there should be an *algebraic* weak factorization system induced by these Kan moulds, and whether it is possible to prove this constructively. After these urgent questions, there comes the problem of proving that fibrations are closed under dependent products along fibrations, turning fibrations into a model of Martin-Löf dependent type theory. Then one may ask what identity types could look like, whether they are the same as for Kapulkin-Lumsdaine [KL16] or the ones derived from a more 'cubical' approach by Gambino and Sattler [GS17][Sat17]. Lastly, one may wonder about univalent universes and the model structure on simplicial sets.

For these questions, I have to refer to the forthcoming [BF]. There, the above issues will be properly addressed. This theory makes fruitful use of the approach to algebraic weak factorization systems by Bourke and Garner [BG16], the ideas in Van den Berg-Garner [BG12] and what is adumbrated in the above. It would be an interesting further exercise to reflect on this theory after what has been said in this thesis about toposes of finitely supported $\mathbf{M}$-sets, of which many examples carry model structures and models of type theory, but about whose interrelation much remains to be said. More research is needed, as they say.

A Collected facts and results

A.1 The axioms of IZFA

Here I briefly introduce the formalism of intuitionistic Zermelo-Fraenkel (IZF) set theory with atoms (A). The idea, first studied by Myhill [Myh73] and Friedman [Fri73], is to take the axioms of ZF set theory but with the intuitionistic predicate calculus as underlying logic, that is, first-order logic without the law of excluded middle. When the logic is intuitionistic, some care has to be taken in the precise formulation of ZF. For example, the axiom of foundation implies the law of excluded middle (see Myhill [Myh73]) so it is replaced by an induction scheme. Friedman [Fri73] further uses the more general axiom of collection instead of replacement, and this has become the common formulation of IZF.

So IZF comes with the standard 2-ary relation symbol $\in$. In the formulation below, taken from Scott [Sco80], an additional 1-ary relation symbol $(-) \in V$ is added, to be read as 'x is a *set*'. This is to take into account the possibility of *individuals* or *atoms*, which are valid entities that are not sets and do not contain any elements. The additional rules of membership and decidability take care of this interpretation. So in fact, the below is precisely what is obtained when adapting Zermelo-Fraenkel set theory with atoms (ZFA) to the intuitionistic setting in the manner introduced above. Definition A.1.1 contains the precise formalization.

The semantics of IZF(A) was first studied using realizability models. These can now be understood as special cases of an interpretation in a topos, via the notion of a *realizability topos* [Oos08]. Since every topos interprets a *structural set theory*, where all quantifiers are bounded, the main challenge in interpreting set theory in a topos is the interpretation of unbounded quantifiers. For Grothendieck toposes, this is easily dealt with since these toposes have small colimits. One can thus construct a cumulative hierarchy V as demonstrated in Section 4.6, yielding a universe of sets, and unbounded quantification just becomes bounded quantification over the universe. This is the approach of Fourman [Fou80]. More general approaches can be developed using algebraic set

theory [JM95], or indeed Shulman's stack semantics [Shu10], but they are not necessary for what is presented in Section 4.6.

Definition A.1.1. The axioms of intuitionistic Zermelo-Fraenkel set theory with atoms (IZFA) are given by the following axiom scheme:

Decidability	$\forall x.x \in V \vee x \notin V$
Membership	$\forall x, y.x \in y \Rightarrow y \in V$
Extensionality	$\forall x, y \in V.(\forall z.(z \in x \leftrightarrow z \in y)) \rightarrow x = y$
Pairing	$\forall x, y.\exists z.(x \in z \wedge y \in z)$
Separation	$\forall x.\exists y \in V.\forall z.(z \in y \leftrightarrow z \in x \wedge \phi(z))$
Union	$\forall x.\exists y.\forall z.(\exists w \in x.z \in w \rightarrow z \in y)$
Powerset	$\forall x.\exists y.\forall z.(\forall w \in z.w \in x \rightarrow z \in y)$
Infinity	$\exists x.(\exists y.y \in x \wedge \forall y \in x \exists z \in x.y \in z)$
Induction	$\forall x.(\forall y \in x.\phi(y) \rightarrow \phi(x)) \rightarrow \forall x.\phi(x)$
Collection	$\forall x.(\forall y \in x.\exists z.\phi(y,z) \rightarrow \exists w.\forall y \in x.\exists z \in w.\phi(y,z))$

A.2 Simplicial Moore paths

The aim of this section is to establish Proposition 5.3.6. The first ingredient is the following lemma, which appears quite specific to the category of simplicial sets.

Lemma A.2.1. *Let $n \geq 0$ and $0 \leq i \leq n$ be arbitrary. Consider the commutative diagram* (A.1) *in $\widehat{\Delta}$, where the square (a) is a pullback. Then the square (b) is a pushout.*

$$\begin{array}{ccccc}
\Delta[n+1] & \xrightarrow{d_{i+1}} & \Delta[n+2] & & \\
{\scriptstyle d_{i+1}}\downarrow & (b) & \downarrow{\scriptstyle (s_{i+1}, s_i)} & & \\
\Delta[n+2] & \xrightarrow[(s_i, s_{i+1})]{} & \Delta[n+1] \times_{\Delta[n]} \Delta[n+1] & \dashrightarrow & \Delta[n+1] \\
 & & \vdots & (a) & \downarrow{\scriptstyle s_i} \\
 & & \Delta[n+1] & \xrightarrow[s_i]{} & \Delta[n]
\end{array} \tag{A.1}$$

Proof. This is a direct computation using the pointwise definition of pullbacks in $\widehat{\Delta}$. We want to prove that *(b)* satisfies the universal property of a pushout. So let X be an arbitrary simplicial set

fitting in a diagram (A.2).

$$\begin{array}{ccc} \Delta[n+1] & \xrightarrow{d_{i+1}} & \Delta[n+2] \\ {\scriptstyle d_{i+1}}\downarrow & & \downarrow{\scriptstyle x_1} \\ \Delta[n+2] & \xrightarrow{x_0} & X \end{array} \tag{A.2}$$

Such a pair x_0,x_1 is the same as a pair of $n+2$ simplices x_0,x_1 of X such that $d_{i+1}x_0 = d_{i+1}x_1$. To prove the statement, it is enough to prove that there exists a unique morphism $\overline{x}$ as in (A.3) with values x_0, x_1 on the respective $(n+2)$-simplices (s_i, s_{i+1}) and (s_{i+1}, s_i).

$$\overline{x} : \Delta[n+1] \times_{\Delta[n]} \Delta[n+1] \to X \tag{A.3}$$

Suppose (f, g) are arbitrary elements of $\Delta[n+1] \times_{\Delta[n]} \Delta[n+1](k)$, i.e. $f, g : [k] \to [n+1]$ and $s_i f = s_i g$. Let $h := s_i f$. Now f and g only differ on $h^{-1}(i)$, where they take values in $\{i, i+1\}$. Therefore we have either $\forall j : f(j) \leq g(j)$ or $\forall j : f(j) \geq g(j)$.

Assume we are in the first case. Then define $\alpha : [k] \to [n+2]$ as follows:

$$\alpha(j) = \begin{cases} f(j) \text{ if } g(j) \leq i \\ f(j) + 1 \text{ if } g(j) > i \end{cases} \tag{A.4}$$

Then one can verify that $s_i \circ \alpha = f$ and $s_{i+1} \circ \alpha = g$, in other words $(f, g) = \alpha \cdot (s_i, s_{i+1})$ so $\overline{x}_k(f, g)$ is fixed and uniquely defined by the equation (A.5).

$$\overline{x}_k(f, g) := \alpha \cdot x_0 \tag{A.5}$$

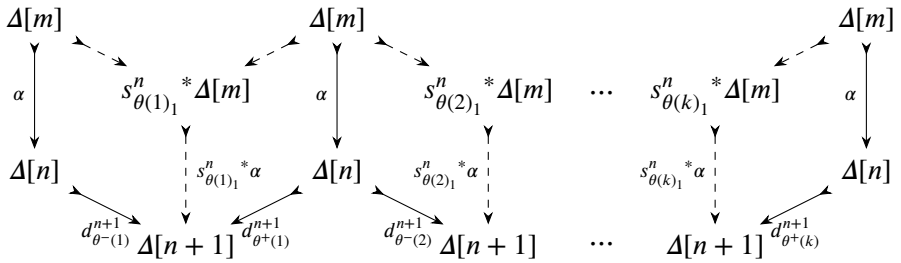
In the second case, the situation is of course completely symmetric, and the value of (f, g) is fixed by x_1. Observe that when $f = g$, α factors through d_{i+1} and both definitions agree. Clearly $\overline{x}$ is a morphism of simplicial sets, so this completes the proof. □

Now, recall that one wants to prove that there is a pullback square

$$\begin{array}{ccc} \widehat{\theta \cdot \alpha} & \xrightarrow{\hat{\alpha}} & \hat{\theta} \\ {\scriptstyle r_{\theta\cdot\alpha}}\downarrow & & \downarrow{\scriptstyle r_\theta} \\ \Delta[m] & \xrightarrow{\alpha} & \Delta[n] \end{array} \tag{A.6}$$

given by the simplicial action on traversals. When $\alpha : [m] \rightarrowtail [n]$ is a monomorphism, and θ is an n-dimensional traversal, the pullback of r_θ along α as in the above diagram can be readily seen to

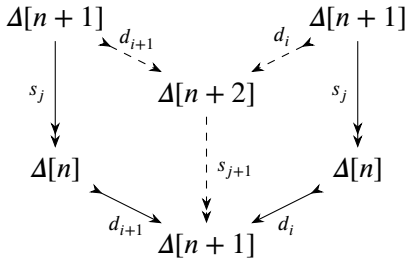
be given by a traversal, namely the top one in the diagram below, where all squares are pullback:



(A.7)

Indeed each ${s^n_{\theta(i)_1}}^*\Delta[m]$ is either $\Delta[m+1]$ (then the pullback square is a simplicial identity) or $\Delta[m]$, in which case the segment is trivial and dropped from the resulting traversal. One can verify that in this manner, it corresponds precisely to $\theta \cdot \alpha$.

The reason why this is easy is that the *simplicial identities involving face maps are pullbacks*. In fact, this goes further: since the simplicial identity $s_j s_i = s_i s_{j+1}$ is also a pullback when $i < j$, one can check the claim of the proposition for the action of s_j on the traversal given by s_i:



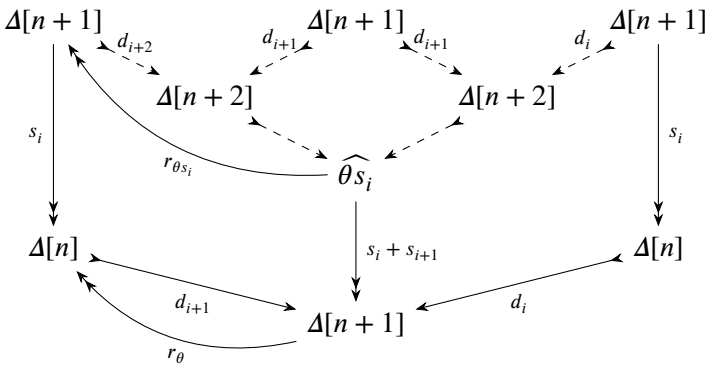
(A.8)

Indeed, the action is what it should be. By symmetry, this also works for the negatively oriented traversal given by s_i, and an analogous argument works when $j < i$.

However, it breaks when $i = j$. But since every map in the simplex category is composed of face and degeneracy maps, and the action commutes with composition (concatenation) of traversals, this leaves the only bit that still needs to verified in order to prove Proposition 5.3.6: namely for the traversal θ given by the single segment $\{i, +\}$ (the case for $\{i, -\}$ is symmetric) and $\alpha := s_i : [n+1] \to [n]$. This is done in the following lemma.

Lemma A.2.2. *For θ, α as above, consider the 'canonical' arrow $s_i + s_{i+1}$ between the traversals $\theta \cdot \alpha$ and θ induced by the action, as expanded in the diagram* (A.9) *below. Then the square of*

epimorphisms on the left-hand side of the diagram commutes and is a pullback.



(A.9)

Proof. Looking at the diagram, this can be deduced directly from lemma A.2.1 stated in the beginning. □

Hence:

Proof of Proposition 5.3.6. Since all cases have now been covered, it follows from the above discussion. □

A.3 Cartesian natural transformations

Recall the following definition:

Definition A.3.1. A natural transformation $\eta : F \Rightarrow G$ is *cartesian* if all naturality squares for η are pullbacks.

The aim of this section is to prove Lemma A.3.3 below, in order to derive from it the following fact: If $\mathbb{E}$ is the category of elements of some presheaf $P : \mathbb{C}^{\mathrm{op}} \to \mathrm{Set}$, $U : \mathbb{E} \to \mathbb{C}$ is the forgetful functor, and $F : \mathbb{E} \to \widehat{\mathbb{C}}$ is a functor with a *cartesian* natural transformation $\gamma : F \Rightarrow y \circ U$, where y is the Yoneda embedding, then the square:

$$\begin{array}{ccc} F(e) & \longrightarrow & \operatorname{colim} F \\ \downarrow & & \downarrow \\ y_C & \xrightarrow{e} & P \cong \operatorname{colim} y \circ U \end{array} \tag{A.10}$$

induced by the cocones (horizontal arrows) and γ (vertical arrows) is a pullback square.

To prove this, it is needed to go into some generalities. Suppose $\mathbb{C}$, $\mathbb{D}$ are (small) categories. Let $G : \mathbb{D} \to \widehat{\mathbb{C}}$ be a functor, and denote by $[\mathbb{D}, \widehat{\mathbb{C}}] \Downarrow G$ the full subcategory of $[\mathbb{D}, \widehat{\mathbb{C}}]/G$ on the

cartesian natural transformations. Write $G1 := \operatorname{colim} G$ and denote by $\iota : G \Rightarrow G1$ the colimiting cocone.

As a functor (A.11), taking colimits has a right adjoint ι^* given by pullback along ι.

$$\operatorname{colim} : [\mathbb{D}, \widehat{\mathbb{C}}] \Downarrow G \to \widehat{\mathbb{C}}/G1 \tag{A.11}$$

The below lemma defines a condition on the colimiting cocone ι. Colimits for which this condition holds are called *Van Kampen* in the literature.

Lemma A.3.2. *The adjunction* $\operatorname{colim} \dashv \iota^*$ *is an equivalence of categories if and only if whenever* (A.12) *is a commutative diagram with* α *cartesian, the statements (i),(ii) are equivalent.*

(i) The square (A.12) *is a pullback square*

(ii) The top arrow in (A.12) *is a colimiting cocone*

$$\begin{array}{ccc} F & \Longrightarrow & F1 \\ {\scriptstyle\alpha}\Big\Downarrow & & \Big\downarrow \\ G & \underset{\iota}{\Longrightarrow} & G1 \end{array} \tag{A.12}$$

Proof. If the top arrow is a colimiting cocone, then the unit of the adjunction at α is the induced morphism $\eta_\alpha : F \Rightarrow \iota^* F1$. If the square is a pullback, there is similarly a counit $\epsilon_{F1} : \operatorname{colim} F \to F1$ induced by the colimit.

It can now be readily seen that ϵ is an isomorphism precisely when (i)⇒(ii) and η is an isomorphism precisely when (ii)⇒(i), which implies the statement of the lemma. □

The goal is to prove the following lemma:

Lemma A.3.3. *Suppose* P *is an object of* $\widehat{\mathbb{C}}$ *and* $\mathbb{E}$ *is its category of elements. Write* $G := y \circ U : \mathbb{E} \to \widehat{\mathbb{C}}$. *Then the colimiting cocone* $\iota : G \Rightarrow P \cong G1$ *satisfies the condition of lemma A.3.2.*

Some more ingredients are needed in order to carry this out. Recall that a functor $u : \mathbb{D} \to \mathbb{C}$ is a *discrete fibration* if for all arrows $f : c \to ud$ in $\mathbb{C}$, there exists a *unique* lift $f' : f^*d \to d$ above f. It is well-known that, as a full subcategory of $\mathrm{Cat}/\mathbb{C}$, the category $\mathrm{DFib}(\mathbb{C})$ of discrete fibrations over $\mathbb{C}$ is equivalent to $\widehat{\mathbb{C}}$:

$$\mathrm{DFib}(\mathbb{C}) \cong [\mathbb{C}^{\mathrm{op}}, \mathrm{Set}] \tag{A.13}$$

For $u : \mathbb{D} \to \mathbb{C}$ a discrete fibration, define

$$Y(u) : \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathrm{Set} \tag{A.14}$$

as the transpose of the composition $y \circ u$ of u with the Yoneda embedding $y : \mathbb{C} \to \widehat{\mathbb{C}}$. Similar to the above, denote by

$$[\mathbb{C}^{\mathrm{op}} \times \mathbb{D}, \mathrm{Set}] \Downarrow Y(u) \tag{A.15}$$

the (full) subcategory of $[\mathbb{C}^{\mathrm{op}} \times \mathbb{D}, \mathrm{Set}]/Y(u)$ on the cartesian natural transformations.

Lemma A.3.4. *Suppose $u : \mathbb{D} \to \mathbb{C}$ is a discrete fibration. Then the functor*

$$T : \mathrm{DFib}\,(\mathbb{D}) \to [\mathbb{C}^{\mathrm{op}} \times \mathbb{D}, \mathrm{Set}] \Downarrow Y(u) \tag{A.16}$$

defined on discrete fibrations $v : \mathbb{F} \to \mathbb{D}$ by

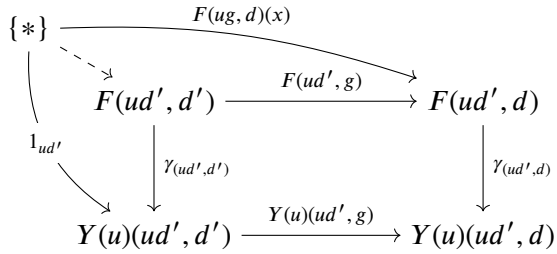
$$T(v)(c,d) := \sum_{f : c \to ud} v^{-1}(f^* d) \tag{A.17}$$

where f^*d denotes the domain of the lift of $f : c \to ud$ along u, with the obvious $\mathbb{C}^{\mathrm{op}} \times \mathbb{D}$ action and projection to $Y(u)$, is an equivalence of categories.

Proof. Given a functor $F : \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathrm{Set}$ and a cartesian natural transformation $\gamma : F \to Y(u)$, one can define a discrete fibration $v_\gamma : v(F) \to \mathbb{D}$ with fibres

$$v_\gamma(F)^{-1}(d) = \gamma^{-1}_{(ud,d)}(1_{ud}) \tag{A.18}$$

and the following action: for $g : d' \to d$ and $x \in \gamma^{-1}_{(ud,d)}(1_{ud})$, $g^*x \in \gamma^{-1}_{(ud',d')}(1_{ud'})$ is defined as the unique element induced by the following pullback diagram:



(A.19)

Then for $\gamma : F \Rightarrow Y(u)$ a cartesian natural transformation:

$$T(v_\gamma(F))(c,d) := \sum_{f : c \to ud} v_\gamma(F)^{-1}(f^*d) = \sum_{f : c \to ud} \gamma^{-1}_{(c, f^*d)}(1_c) \cong F(c,d) \tag{A.20}$$

where the latter identity is precisely the condition that γ is cartesian.

Conversely, if $\tau : T(v) \to Y(u)$ denotes the above projection, then:

$$v_\tau(T(v))^{-1}(d) = \tau^{-1}_{ud,d}(1_{ud}) = v^{-1}(1^*_{ud} d) = v^{-1}(d) \tag{A.21}$$

There are a few naturality checks left to the reader. □

Proof of Lemma A.3.3. Recall that the goal is to prove that

$$\imath^* : \widehat{\mathbb{C}}/\boldsymbol{P} \to [\mathbb{E}, \widehat{\mathbb{C}}] \Downarrow \boldsymbol{G} \tag{A.22}$$

is an equivalence of categories. I claim that $\imath^*$ factors as the following composite of equivalences:

$$\imath^* : \widehat{\mathbb{C}}/\boldsymbol{P} \cong \mathrm{DFib}\,(\mathbb{E}) \xrightarrow{T} [\mathbb{E}, \widehat{\mathbb{C}}] \Downarrow \boldsymbol{G} \tag{A.23}$$

where the second equivalence $\boldsymbol{T}$ is, up to a transpose, from Lemma A.3.4 and the first equivalence is standard. Note this also applies the well-known fact that the forgetful functor

$$\boldsymbol{U} : \mathbb{E} \to \mathbb{C} \tag{A.24}$$

from the category of elements $\mathbb{E}$ is a discrete fibration. I leave it to the reader to chase through (A.23) and observe that it is the same as $\imath^*$. □

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