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EXAMPLES OF FUNCTOR ADJUNCTIONS IN ELEMENTARY ANALYSIS

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Summary. Some basic concepts of real analysis, such as the "upper and lower limit", the "order of zero", the "upper and lower integral", are investigated by means of the language of categories and functors. It is shown that this approach provides a completely natural way for introducing these concepts and a convenient language for stating their properties. In this scheme, the most useful tools appear to be the notion of functor adjunctions and the technique of Kan extensions.

Prof. S. Mac Lane, in his book *Categories for the Working Mathematician*, has invited each reader "to find his own examples" of functor adjunctions, remembering also that "adjoint functors arise everywhere" **[1]**.

The principal and most attractive feature of the examples I am presenting here is that they are all concerned with elementary ideas and basic concepts of real analysis. In fact **[2, 3]**, the language of categories can be conveniently introduced even in the most elementary topics: the examples below will clearly confirm, in particular, that (i) this language provides a completely natural way for introducing some of the typical concepts of analysis, and (ii) the related results are often automatically self-contained in this approach and, in general, can be stated in a compact and expressive form which allows for a new insight—in a more general perspective—into these concepts.

For the convenience of the reader, I shall briefly recall here (in the form most suitable for our purposes; for details, see **[1]**) the two concepts of category theory which are being used in the following, namely the definition of functor adjunction and the construction of Kan extensions.

Given two categories A and B, the functor $F: B \to A$ is said to be a *left adjoint* for the functor $G: A \to B$ if for each object b in B, there is an arrow $\eta_b: b \to G \circ F(b)$ which is *universal* from b to G. This last statement means that for each arrow $\phi: b \to G(a)$, where a is an object of A, there exists one and only one arrow $\psi: F(b) \to a$, and ϕ factorizes $\phi = G(\psi) \circ \eta_b$.

Before giving the definition of Kan extension, we need the notion of ("projective") *limit* of a functor $F: J \to A$. It consists of an object r of A, together with a set ("cone") of arrows $\nu_j: r \to F(j)$, such that, for each object a which is vertex of another cone $\tau_j: a \to F(j)$, there is exactly one arrow $\phi: a \to r$ and $\tau_j = \nu_j \circ \phi$. Finally, instead of recalling its formal definition, we prefer to give a brief outline of the procedure used for constructing the Kan extensions. Let $F: A \to B$ and $G: A \to C$ be two given functors; for any fixed object c in C, consider the category of the arrows $\phi: c \to G(a)$, where a varies in (the class of objects of) A, and the functor sending each ϕ into $F(a)$. Call $R(c)$ the limit of this functor, if it exists (this is always true when B is complete). One then shows that the map $R: c \to R(c)$ of C in B defined in this way is really a functor, called the *right Kan extension* of F along G, and satisfying some peculiar "universal" properties.

Dualizing the above definitions, i.e., essentially reversing the arrows, one easily defines right adjoints, colimits, left Kan extensions.

A common feature of the examples we are considering below is that all the involved categories are preorders. A preorder in fact is simply a category in which there is at most one arrow between objects. We shall denote $\succeq$ the order relation, except for the standard relation $\geqq$ used between real quantities; we will also write $a \in P$ for "a is an object for the preorder (= category) P," and define the arrows $a_1 \to a_2$ in P with the meaning $a_1 \succeq a_2$. Note, of course, that the notion of limit and colimit of a functor whose codomain is a preorder ultimately coincides, in the above conventions, with that of least upper bound and of greatest lower bound, respectively.

In the following, R and N will denote respectively the set of real numbers (extended with $\pm\infty$) and of natural numbers, regarded as categories with the standard order relation $\geqq$.

Example 1 (limits of numerical sequences). Let S be the set of all real sequences $(a_n) \in S$; we shall define $(a_n) \succeq (b_n)$ when the following condition is satisfied: for each $c, c' \in R$ such that, for all

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$\varepsilon > 0$, relations $b_n \leqq c - \varepsilon$ and $a_n \geqq c' + \varepsilon$ hold at most for a finite number of indices, then the same must be true for relations $a_n \leqq c - \varepsilon$ and $b_n \geqq c' + \varepsilon$.

Let K be the (trivial!) functor $K: R \to S$ defined by $K(a) = (a)$, where (a) stands for the constant sequence $a_n = a$ for all n; then the left adjoint and the right adjoint of K are defined on the whole S and are the functors l and $L: S \to R$ giving respectively the lower and the upper limit of sequences. To see this, observe e.g. that the universality of the arrow $(a_n) \geq (\lambda)$, where $\lambda = l((a_n)) =$ lower limit of (a_n), expresses the well-known property that if $(a_n) \geq (\lambda')$, $\lambda' \in R$, then also $\lambda \geqq \lambda'$.

Example 2 (infinitesimal real functions). Let $\mathcal{J}$ be the set of all real continuous positive functions $f = f(x)$, defined on an interval $0 \leqq x \leqq b$ of the real axis, vanishing for x tending to zero. Let us introduce in $\mathcal{J}$ the relation $f_1 \geq f_2$ if $\lim_{x \to 0}(f_1(x)/f_2(x))$ exists and is a finite number. Let R^+ be the subcategory of real positive numbers (included 0 and $+\infty$) and let us include in $\mathcal{J}$ also the constant functions 0 and 1. Define now the functor $E: R^+ \to \mathcal{J}$ by $E(a) = x^a$, with the obvious definition $E(0) = 1$ and $E(\infty) = 0$. The left and right Kan extensions of the identity $I: R^+ \to R^+$ along E define for all functions $f \in \mathcal{J}$ a sort of "upper" and "lower order of zero," as clearly shown by their construction. These functors, however, are *not* adjoint functors of E, due to the nonexistence of the characteristic universal arrow for some of the elements f.

Instead, let us consider the subcategory $\mathcal{J}_0$ of $\mathcal{J}$ whose objects are the functions f such that the above introduced upper and lower order coincide (call $O(f)$ this number) and, in addition, both relations $f \geq x^{O(f)}$ and $x^{O(f)} \geq f$ hold: then the functor $O: \mathcal{J}_0 \to R^+$ is simultaneously left and right adjoint of $E: R^+ \to \mathcal{J}_0$, as easily seen in a similar way as in Example 1, and so $\mathcal{J}_0$ is *equivalent* to R^+.

Finally, if one adopts in $\mathcal{J}_0$ a new definition of arrows (preorder relations) writing $f_1 \to f_2$ if $\lim_{x \to 0}(f_1(x)/f_2(x)) = l$ with $0 \leqq l \leqq 1$, one easily verifies the equivalence of this subcategory with $(R^+)_{\mathrm{op}} \times R^+$, by means of the map sending f to the pair $(l(f), O(f))$ where $l(f) = \lim_{x \to 0}(f(x)/x^{O(f)})$. (In this case, exclude from $\mathcal{J}_0$ the functions 0 and 1, and from R^+ the elements 0 and $+\infty$. The subscript "op" stands for "opposite" category, i.e., with all the arrows reversed.) To see this result more clearly, one can observe that it ultimately expresses the standard computational rule of substituting each f in $\mathcal{J}_0$ with $l(f)x^{O(f)}$ in the evaluation of limits such as $\lim_{x \to 0}(f_1(x)/f_2(x))$.

Example 3 (limits of real functions and semicontinuous functions). Let E be a topological space, and $F = R^E$ the class of real functions f defined on E. For any fixed point x in E, let us define on F the following relation: we shall say $f_1 \geq f_2$ in x if there is a neighbourhood of x where $f_1 \geqq f_2$. Let F_0 denote the subset of constant functions and $l_0: F_0 \to R$ the obvious isomorphism of F_0 with R; for any fixed point x, the left and the right Kan extensions of l_0 along the inclusion $K: F_0 \to F$ give respectively the upper limit and the lower limit of functions $f \in F$ at the point x. Alternatively, this scheme of functors can be viewed as an adjunction, exactly as in Example 1, but the above presentation may be preferable because of its constructive character, which clearly exhibits the peculiar property of, say, the lower limit of being the "greatest" of the real numbers λ' such that $f \geq \lambda'$.

Another interesting feature can be viewed varying now the point x in E, but taking $f \in F$ fixed: thus the above left and right extensions describe respectively an upper semicontinuous function f^* and a lower semicontinuous function f_*, which are called the upper and lower envelopes of f [4]. The map sending each f to f^* is an example of the abstract situation called a *comonad* in category theory, this map being the right adjoint of the inclusion $K: F^* \to F$, where F^* is the coreflective subcategory of all upper semicontinuous functions defined on E. This inclusion does not possess a left adjoint: this fact depends essentially on the non-completeness of F^*. Similar (dual) results hold for the monad defined by the map sending each f to f_*.

Example 4 (integration of real functions). Let $\mathcal{R}$ be a Riesz space of real functions defined on an arbitrary set E and $\mu: \mathcal{R} \to R$ a Daniell measure on $\mathcal{R}$ [4, 5]. Let us consider the class $F = R^E$ of all real functions f defined on E as a partial order with the obvious definition $f_1 \geqq f_2$ if $f_1(x) \geqq f_2(x)$ for all

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$x \in E$. Then μ is a functor $\mu : \mathcal{R} \to R$. Let us consider now the functor categories $\mathcal{R}^N$, R^N, F^N, as well as the functors $\mu^N : \mathcal{R}^N \to R^N$ and $K^N : \mathcal{R}^N \to F^N$ defined as follows

$$\mu^N(s) = \mu \circ s; \qquad K^N(s) = K \circ s,$$

where s is a functor in $\mathcal{R}^N$ (i.e., an increasing sequence of functions belonging to $\mathcal{R}$) and $K : \mathcal{R} \to F$ the inclusion of $\mathcal{R}$ in F. Finally, let us denote $L_F : F^N \to F$ and $L_R : R^N \to R$ the functors "limit" in F and respectively in R. Then one has: the left Kan extension of the functor $L_R \circ \mu^N : \mathcal{R}^N \to R$ along the functor $L_F \circ K^N : \mathcal{R}^N \to F$ defines the upper integral $\mu^*(f) = \int^* f d\mu$ of functions $f \in F$. In fact, for any fixed f, by the very construction of Kan extension, $\mu^*(f)$ is obtained as the greatest lower bound of the numbers of the type $\sup_n \mu(\phi_n)$ where (ϕ_n) is an increasing sequence of functions belonging to $\mathcal{R}$ and such that $\sup_n \phi_n(x) \geqq f(x)$ for all x, in agreement with the Daniell definition.

"Dualizing" this construction, i.e., taking the right Kan extension of $\tilde{L}_R \circ \mu^{N_{op}} : \mathcal{R}^{N_{op}} \to R$ along $\tilde{L}_F \circ K^{N_{op}} : \mathcal{R}^{N_{op}} \to F$ (where $\tilde{L}_R$, $\tilde{L}_F$ denote the functors "colimit" in R and F), we define the lower integral $\mu^*(f)$. The left extension of this second construction, and the right extension of the first one, define instead a different upper and lower integral $\mu^0(f)$ and $\mu_0(f)$ respectively; it can be shown that the natural transformations $\mu^0 \dot{\to} \mu^* \dot{\to} \mu_* \dot{\to} \mu_0$ hold; this means that for each f one has $\mu^0(f) \geqq \mu^*(f)$, etc. To understand more clearly this point, suppose that E is a locally compact topological space with a countable base and μ is a Radon measure on E, then, in the subset B of bounded functions f with compact support, one has that $\mu^0(f)$ is equal to $\mu^*(f^*) = \mu_*(f^*)$ and $\mu_0(f)$ to $\mu^*(f_*) = \mu_*(f_*)$, where f^* and f_* are the semicontinuous envelopes of f defined as in Ex. 3.

In the subset B just considered, one can, similarly, introduce very easily the usual concepts of upper and lower Riemann integral: they are simply the left and the right Kan extension of the obvious integral (i.e., the Euclidean measure) of "step functions" along the inclusion of these functions in B. Instead, finally, if one takes as Riesz space $\mathcal{R}$ the set of Riemann-integrable functions and as measure μ the Riemann integral, the above integrals μ^* and μ_* define the usual Lebesgue integrals.

Further details, especially about Examples 2 and 4, will be considered in subsequent papers [3] and [6].

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MISCELLANEA

7. The clouded crystal ball. "In the old days when people invented a new function they had some useful purpose in mind; now they invent them deliberately just to invalidate our ancestors' reasoning, and that is all they are ever going to get out of them."

H. Poincaré, *Science et Méthode*, 1909.

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