The Art of Sketching (1): Sketch of the Real Number Line

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Abstract

The series *The Art of Sketching* aims to enrich the traditional collection of examples of sketches that have been constructed so far and to illustrate recent theoretical results about sketches. In this first installment, we describe a sketch whose category of models is equivalent to the completed real line $\overline{\mathbb{R}} = [-\infty, +\infty]$; then we indicate how to modify it to obtain categories of equivalent models, respectively, to $\mathbb{R} \cup \{-\infty\}$, $\mathbb{R} \cup \{+\infty\}$, $\mathbb{R}$, and $\mathbb{R} \setminus \mathbb{Q}$.

1 Cuts and Sketches

The goal is to sketch the completed real line $\overline{\mathbb{R}} = [-\infty, +\infty]$, viewed (in the category-theoretic sense) as the set of Dedekind cuts that has $\mathbb{Q}$ as support and is equipped with a map from 1. Recall that a cut is a pair (A', A'') of parts of $\mathbb{Q}$ such that

$$A' \cap A'' = \varnothing, \qquad A' \cup A'' = \mathbb{Q},$$

and every element of A' is less than every element of A''.

To represent $-\infty$ or $+\infty$, one does not impose that A' and A'' are nonempty. To avoid having representations for irrational numbers (the item itself and its complement in A' or A''), we restrict to left cuts, verifying for all $p < r < q$ in $\mathbb{Q}$:

$$\text{if } p, q \in A' \text{ then } r \in A', \qquad \text{and if } p, q \in A'' \text{ then } r \in A''.$$

Cuts are in bijection with decreasing maps M from $\mathbb{Q}$ to $\{0, 1\}$:

$$q \in A' \iff M(q) = 1, \qquad q \in A'' \iff M(q) = 0.$$

Left cuts correspond to those maps satisfying, for all $q < q'$,

$$M(q) = \inf_{q<r<q'} M(r).$$

To represent elements of $\overline{\mathbb{R}}$ as set-valued functors, we note that the category $\{0, 1\}$ is equivalent to the full subcategory of **Set** whose objects are sub-singletons. Hence a model in **Set** can be seen as a cone. This yields the following projective sketch $E_{\overline{\mathbb{R}}}$.

Proposition 1. *The category of models of the following sketch is equivalent to the ordered set* $\overline{\mathbb{R}} = [-\infty, +\infty]$*:*

- *support: the ordered set* $(\mathbb{Q}, \geq)$*;*
- *for each* $q \in \mathbb{Q}$*, a distinguished discrete-indexing projective cone at* q*;*

- *for each $q \in \mathbb{Q}$, a distinguished projective cone expressing the interval condition above.*

Proof. A model M of $E_{\overline{\mathbb{R}}}$ in **Set** identifies with a map from $\mathbb{Q}$ to $\{0,1\}$, hence with an element of $\overline{\mathbb{R}}$. Among these maps, exactly those satisfying the second family of cones are retained, namely

$$\forall q \in \mathbb{Q}, \quad M(q) = 1 \Rightarrow M(q') = 1 \text{ for all } q' < q,$$

which is equivalent to the inequality condition on associated elements. Therefore

$$\mathrm{Mod}(E_{\overline{\mathbb{R}}}) \simeq \overline{\mathbb{R}}.$$

2 Some Non-Algebraic Variants

Let Γ be the ordered set $\mathbb{R} \cup \{-\infty\}$ (that is, $\mathbb{R}_0 \cup \{+\infty\}$ in the notation of the original text). The preceding analysis can be read as excluding from $\overline{\mathbb{R}}$ the constant functor $M(q) = 1$ for all q, equivalently requiring

$$\prod_{q \in \mathbb{Q}} M(q) = \varnothing.$$

To express this, one enlarges the support of the sketch by adding cones corresponding to

- the empty-indexing inductive cone (interpreted as $+\infty$),
- a non-discrete-indexing projective cone (for $q \in \mathbb{Q}$),

and equivalently enforces

$$\lim_{q \in \mathbb{Q}} M(q) = 1.$$

Similarly, to sketch $\mathbb{R} \cup \{-\infty\}$ one modifies the support in the opposite direction and distinguishes the corresponding supplementary cones. Taking support $(\widehat{\mathbb{Q}}, \geq)$ and distinguishing all projective and inductive cones from both constructions gives a sketch of the real line itself, denoted $E_{\mathbb{R}}$, with

$$\mathrm{Mod}(E_{\mathbb{R}}) \simeq \mathbb{R}.$$

Finally, by replacing one of the discrete projective cones by an appropriate non-discrete one, we obtain a sketch $E_{\mathbb{R}\setminus\mathbb{Q}}$ whose category of models is equivalent to the ordered set $\mathbb{R} \setminus \mathbb{Q}$. Indeed, a model corresponds to an irrational iff, for every rational q,

$$M(q) = \lim_{q' < q} M(q').$$

Thus one sketches the doubleton functor (inclusion) $\mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$. (And similarly, as left as an exercise in the original text, one can sketch the doubleton functor $\mathbb{Q} \to \mathbb{R}$.)

3 Remarks

The sketches above provide interesting illustrations of several recent classification theorems for sketches and categories accessible in nature. From the sketch $E_{[-\infty,+\infty]}$ we note in particular:

(1) all distinguished projective cones are countably indexed and nonempty;

(2) all distinguished inductive cones are of empty indexing.

A natural question is which properties of the associated category $[-\infty, +\infty]$ are reflected by (1) and (2) together with the results cited in [**?**, **?**]. The following statement is verified in the source text.

Proposition 2. *The categories of models of sketches satisfying (1) and (2) are exactly (up to equivalence) the accessible categories with directed colimits such that:*

(a) they have $\aleph_1$-filtered limits and contain a set of $\aleph_1$-presentable objects whose directed colimit is each object;

(b) they have strict initial object;

(c) they have nonempty projective limits.

In particular, the ordered set $[-\infty, +\infty]$, seen as a category, satisfies these conditions. All of its objects are $\aleph_1$-presentable, and $-\infty$ is initial but not strict.

For $E_{[-\infty,+\infty]}$, one can formulate a more precise characterization by separating two classes of model categories:

(1) all distinguished projective cones are of countable indexing and nonempty;

(2) all distinguished inductive cones are of empty indexing.

This aligns with known theorems on categories accessible with finite projective limits and the role of conditional projective limits. A modified sketch can also be built in which all cones are as above except one non-discrete inductive cone indexed by $(\mathbb{Q}, \geq)$. This yields an equivalent sketch that represents real numbers no longer as cuts in $\mathbb{Q}$, but as cuts in $\mathbb{R}$.

Bibliography

References

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