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A self-contained, brief and complete formulation of Voevodsky's Univalence Axiom

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Abstract

In introductions to the subject for a general audience of mathematicians or logicians, the univalence axiom is typically explained by handwaving. This gives rise to several misconceptions, which cannot be properly addressed in the absence of a precise definition. In this short set of notes we give a complete formulation of the univalence axiom from scratch. The underlying idea of these notes is that they should be as concise as possible (and not more). They are not meant to be an Encyclopedia of Univalence.

Keywords. Univalence Axiom, Martin-Löf's identity type, type universe.

1 Introduction

The univalence axiom [10, 9, 2] is not true or false in, say, ZFC or the internal language of an elementary topos. It cannot even be formulated. As the saying goes, it is not even wrong. This is because

> univalence is a property of Martin-Löf's *identity type* of a universe of types.

Nothing like Martin-Löf's identity type occurs in ZFC or topos logic as a *native* concept. Of course, we can create *models* of the identity type in these theories, which will make univalence hold or fail. But in these notes we try to understand the primitive concept of identity type, directly and independently of any such particular model, as in (intensional) Martin-Löf type theory [6, 7], and the univalence axiom for it. In particular, we don't use the equality sign "=" to denote the identity type Id, or think of it as a path space.

Univalence is a type, and the univalence axiom says that this type has some inhabitant. It takes a number of steps to construct this type, in addition to subtle decisions (e.g. to work with equivalences rather than isomorphisms, as discussed below).

We first need to briefly introduce Martin-Löf type theory (MLTT). We will not give a full definition of MLTT. Instead, we will mention which constructs of MLTT are needed to give a complete definition of the univalence type. This will

be enough to illustrate the important fact that in order to understand univalence we first need to understand Martin-Löf type theory well.

2 Martin-Löf type theory, briefly

2.1 Types and their elements

Types are the analogues of sets in ZFC and of objects in topos theory. Types are constructed together with their elements, and not by collecting some previously existing elements. When a type is constructed, we get freshly new elements for it. We write

$$x : X$$

to declare that the element x has type X. This is not something that is true or false (unlike a membership relation $x \in X$ in ZFC). For example, if $\mathbb{N}$ is the type of natural numbers, we may write

$$0 : \mathbb{N},$$
$$(0, 0) : \mathbb{N} \times \mathbb{N}.$$

However, the following statements are nonsensical and syntactically incorrect, rather than false:

$$0 : \mathbb{N} \times \mathbb{N} \quad \text{(nonsense)},$$
$$(0, 0) : \mathbb{N} \quad \text{(nonsense)}.$$

This is no different from the situation in the internal language of a topos.

2.2 Products and sums of type families

Given a family of types $A(x)$ indexed by elements x of a type X, we can form its product and sum:

$$\Pi(x : X), A(x),$$
$$\Sigma(x : X), A(x),$$

which we also write ΠA and ΣA. An element of the type ΠA is a function that maps elements $x : X$ to elements of $A(x)$. An element of the type ΣA is a pair (x, a) with $x : X$ and $a : A(x)$. (We adopt the convention that Π and Σ scope over the whole rest of the expression.)

We also have the type $X \to Y$ of functions from X to Y, which is the particular case of Π with the constant family $A(x) := Y$. The cartesian product $X \times Y$, whose elements are pairs, is the particular case of Σ with $A(x) := Y$ again. We also have the disjoint sum $X+Y$, the empty type and the one-element type, which will not be needed to formulate univalence.

2.3 Quantifiers and logic

There is no underlying logic in MLTT. Propositions are types, and Π and Σ play the role of universal and existential quantifiers, via the so-called Curry-Howard interpretation of logic. As for the connectives, implication is given by the function-type construction $\to$, conjunction by the binary cartesian product $\times$, disjunction by the binary disjoint sum $+$, and negation by the type of functions into the empty type.

When a type is understood as a proposition, its elements correspond to proofs. In this case, instead of saying that A has a given element, it is common practice to say that A holds. Then a type declaration $x : A$ is read as saying that x is a proof of A. But this is just a linguistic device, which is not reflected in the formalism.

We remark that in univalent mathematics the terminology *proposition* is reserved for subsingleton types (types whose elements are all identified). The propositions that arise in the construction of the univalence type are all subsingletons.

2.4 The identity type

Given a type X and elements $x, y : X$, we have the identity type

$$\mathrm{Id}_X(x, y),$$

with the subscript X often elided. The idea is that $\mathrm{Id}(x, y)$ collects the ways in which x and y are identified. We have a function

$$\mathrm{refl} : \Pi(x : X), \mathrm{Id}(x, x),$$

which identifies any element with itself. Without univalence, refl is the only given way to construct elements of the identity type.

In addition to refl, for any given type family $A(x, y, p)$ indexed by elements $x, y : X$ and $p : \mathrm{Id}(x, y)$ and any given function

$$f : \Pi(x : X), A(x, x, \mathrm{refl}(x)),$$

we have a function

$$\mathrm{J}(A, f) : \Pi(x, y : X), \Pi(p : \mathrm{Id}(x, y)), A(x, y, p)$$

with $\mathrm{J}(A, f)(x, x, \mathrm{refl}(x))$ stipulated to be $f(x)$. We will see examples of uses of J in the steps leading to the construction of the univalence type.

Then, in summary, the identity type is given by the data $\mathrm{Id}, \mathrm{refl}, \mathrm{J}$. With this, the exact nature of the type $\mathrm{Id}(x, y)$ is fairly under-specified. It is consistent that it is always a subsingleton in the sense that $\mathrm{K}(X)$ holds, where

$$\mathrm{K}(X) := \Pi(x, y : X), \Pi(p, q : \mathrm{Id}(x, y)), \mathrm{Id}(p, q).$$

The second identity type $\mathrm{Id}(p, q)$ is that of the type $\mathrm{Id}(x, y)$. This is possible because any type has an identity type, including the identity type itself, and

the identity type of the identity type, and so on, which is the basis for univalent mathematics (but this is not discussed here, as it is not needed in order to construct the univalence type).

The K axiom says that $\mathrm{K}(X)$ holds for every type X. In univalent mathematics, a type X that satisfies $\mathrm{K}(X)$ is called a set, and with this terminology, the K axiom says that all types are sets.

On the other hand, the univalence axiom provides a means of constructing elements other than $\mathrm{refl}(x)$, at least for some types, and hence the univalence axiom implies that some types are not sets. (Then they will instead be 1-groupoids, or 2-groupoids, ..., or even ∞-groupoids, with such notions defined within MLTT rather than via models, but we will not address this important aspect of univalent mathematics here).

2.5 Universes

Our final ingredient is a "large" type of "small" types, called a universe. It is common to assume a tower of universes $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots$ of "larger and larger" types, with

$$\begin{aligned} &\mathcal{U}_0 : \mathcal{U}_1, \\ &\mathcal{U}_1 : \mathcal{U}_2, \\ &\mathcal{U}_2 : \mathcal{U}_3, \\ &\quad\vdots \end{aligned}$$

When we have universes, a type family A indexed by a type $X : \mathcal{U}$ may be considered to be a function $A : X \to \mathcal{V}$ for some universe $\mathcal{V}$. Universes are also used to construct types of mathematical structures, such as the type of groups, whose definition starts like this:

$$\mathrm{Grp} := \Sigma(G : \mathcal{U}), \mathrm{isSet}(G) \times \Sigma(e : G), \Sigma(- \cdot - : G \times G \to G), (\Pi(x : G), \mathrm{Id}(e \cdot x, x)) \times \cdots$$

Here $\mathrm{isSet}(G) := \Pi(x, y : G), \Pi(p, q : \mathrm{Id}(x, y)), \mathrm{Id}(p, q)$, as above. With univalence, Grp itself will not be a set, but a 1-groupoid instead, namely a type whose identity types are all sets. Moreover, if $\mathcal{U}$ satisfies the univalence axiom, then for $A, B : \mathrm{Grp}$, the identity type $\mathrm{Id}(A, B)$ can be shown to be in bijection with the group isomorphisms of A and B.

3 Univalence

Univalence is a property of the identity type $\mathrm{Id}_{\mathcal{U}}$ of a universe $\mathcal{U}$. It takes a number of steps to define the univalence type.

3.1 Construction of the univalence type

We say that a type X is a *singleton* if we have an element $c : X$ with $\mathrm{Id}(c, x)$ for all $x : X$. In Curry-Howard logic, this is

$$\mathrm{isSingleton}(X) := \Sigma(c : X), \Pi(x : X), \mathrm{Id}(c, x).$$

For a function $f : X \to Y$ and an element $y : Y$, its fiber is the type of points $x : X$ that are mapped to (a point identified with) y:

$$f^{-1}(y) := \Sigma(x : X), \mathrm{Id}(f(x), y).$$

The function f is called an equivalence if its fibers are all singletons:

$$\mathrm{isEquiv}(f) := \Pi(y : Y), \mathrm{isSingleton}(f^{-1}(y)).$$

The type of equivalences from $X : \mathcal{U}$ to $Y : \mathcal{U}$ is

$$\mathrm{Eq}(X, Y) := \Sigma(f : X \to Y), \mathrm{isEquiv}(f).$$

Given $x : X$, we have the singleton type consisting of the elements $y : X$ identified with x:

$$\mathrm{singletonType}(x) := \Sigma(y : X), \mathrm{Id}(y, x).$$

We also have the element $\eta(x)$ of this type:

$$\eta(x) := (x, \mathrm{refl}(x)).$$

We now need to *prove* that singleton types are singletons:

$$\Pi(x : X), \mathrm{isSingleton}(\mathrm{singletonType}(x)).$$

In order to do that, we use J with the type family

$$A(y, x, p) := \mathrm{Id}(\eta(x), (y, p)),$$

and the function

$$\begin{aligned} f &: \Pi(x : X), A(x, x, \mathrm{refl}(x)) \\ f(x) &:= \mathrm{refl}(\eta(x)). \end{aligned}$$

With this we get a function

$$\begin{aligned} \phi &: \Pi(y, x : X), \Pi(p : \mathrm{Id}(y, x)), \mathrm{Id}(\eta(x), (y, p)) \\ \phi &:= \mathrm{J}(A, f). \end{aligned}$$

Notice the reversal of y and x. With this, we can in turn define a function

$$\begin{aligned} g &: \Pi(x : X), \Pi(\sigma : \mathrm{singletonType}(x)), \mathrm{Id}(\eta(x), \sigma) \\ g(x, (y, p)) &:= \phi(y, x, p). \end{aligned}$$

Finally, using the function g, we get our desired result, that singleton types are singletons:

$$\begin{array}{rcl} h & : & \Pi(x : X), \Sigma(c : \mathrm{singletonType}(x)), \Pi(\sigma : \mathrm{singletonType}(x)), \mathrm{Id}(c, \sigma) \\ h(x) & := & (\eta(x), g(x)). \end{array}$$

Now, for any type X, its identity function id_X, defined by $\mathrm{id}(x) := x$, is an equivalence. This is because the fiber $\mathrm{id}^{-1}(x)$ is simply the singleton type defined above, which we proved to be a singleton. We need to name this function:

$$\mathrm{idIsEquiv} : \Pi(X : \mathcal{U}), \mathrm{isEquiv}(\mathrm{id}_X).$$

The identity function id_X should not be confused with the identity type Id_X. Now we use J a second time to define a function

$$\mathrm{IdToEq} : \Pi(X, Y : \mathcal{U}), \mathrm{Id}(X, Y) \to \mathrm{Eq}(X, Y).$$

For $X, Y : \mathcal{U}$ and $p : \mathrm{Id}(X, Y)$, we set

$$A(X, Y, p) := \mathrm{Eq}(X, Y)$$

and

$$f(X) := (\mathrm{id}_X, \mathrm{idIsEquiv}(X))$$

and

$$\mathrm{IdToEq} := \mathrm{J}(A, f).$$

Finally, we say that the universe $\mathcal{U}$ is univalent if the map $\mathrm{IdToEq}(X, Y)$ is itself an equivalence:

$$\mathrm{isUnivalent}(\mathcal{U}) := \Pi(X, Y : \mathcal{U}), \mathrm{isEquiv}(\mathrm{IdToEq}(X, Y)).$$

3.2 The univalence axiom

The type $\mathrm{isUnivalent}(\mathcal{U})$ may or may not have an inhabitant. The univalence axiom says that it does. The K axiom implies that it doesn't. Because both univalence and the K axiom are consistent, it follows that univalence is undecided in MLTT.

3.3 Notes

1. The minimal Martin-Löf type theory needed to formulate univalence has

$$\Pi, \Sigma, \mathrm{Id}, \mathcal{U}, \mathcal{U}'.$$

Two universes $\mathcal{U} : \mathcal{U}'$ suffice, where univalence talks about $\mathcal{U}$.

2. It can be shown, by a very complicated and interesting argument, that

$$\Pi(u, v : \mathrm{isUnivalent}(\mathcal{U})), \mathrm{Id}(u, v).$$

This says that univalence is a subsingleton type (any two of its elements are identified). In the first step we use u (or v) to get function extensionality (any two pointwise identified functions are identified), which is *not* provable in MLTT, but is provable from the assumption that $\mathcal{U}$ is univalent. Then, using this, one shows that being an equivalence is a subsingleton type. Finally, again using function extensionality, we get that a product of subsingletons is a subsingleton. But then $\mathrm{Id}(u, v)$ holds, which is what we wanted to show. But this of course omits the proof that univalence implies function extensionality (originally due to Voevodsky), which is fairly elaborate.

3. For a function $f : X \to Y$, consider the type

$$\mathrm{Iso}(f) := \Sigma(g : Y \to X), (\Pi(x : X), \mathrm{Id}(g(f(x)), x)) \times (\Pi(y : Y), \mathrm{Id}(f(g(y)), y)).$$

We have functions $r : \mathrm{Iso}(f) \to \mathrm{isEquiv}(f)$ and $s : \mathrm{isEquiv}(f) \to \mathrm{Iso}(f)$. However, the type $\mathrm{isEquiv}(f)$ is always a subsingleton, assuming function extensionality, whereas the type $\mathrm{Iso}(f)$ need not be. What we do have is that the function r is a retraction with section s.

Moreover, the univalence type formulated as above, but using $\mathrm{Iso}(f)$ rather than $\mathrm{isEquiv}(f)$, is provably empty, e.g. for MLTT with Π, Σ, Id, the empty and two-point types, and three universes, as shown by Shulman [8]. With only one universe, the formulation with $\mathrm{Iso}(f)$ is consistent, as shown by Hofmann and Streicher's groupoid model [3], but in this case all elements of the universe are sets and $\mathrm{Iso}(f)$ is a subsingleton, and hence equivalent to $\mathrm{isEquiv}(f)$.

So, to have a consistent axiom in general, it is crucial to use the type $\mathrm{isEquiv}(f)$. It was Voevodsky's insight not only that a subsingleton version of $\mathrm{Iso}(f)$ is needed, but also how to construct it. The construction of $\mathrm{isEquiv}(f)$ is very simple and elegant, and motivated by homotopical models of the theory, where it corresponds to the concept with the same name. But the univalence axiom can be understood without reference to homotopy theory.

4. Voevodsky gave a model of univalence for MLTT with Π, Σ, empty type, one-point type, two-point type, natural numbers, and an infinite tower of universes in simplicial sets [5, 4], thus establishing the consistency of the univalence axiom.

The consistency of the univalence axiom shows that, before we postulate it, MLTT is "proto-univalent" in the sense that it cannot distinguish concrete isomorphic types such as $X := \mathbb{N}$ and $Y := \mathbb{N} \times \mathbb{N}$ by a property $P : \mathcal{U} \to \mathcal{U}$ such that $P(X)$ holds but $P(Y)$ doesn't. This is because,

being isomorphic, X and Y are equivalent. But then univalence implies $\mathrm{Id}(X, Y)$, which in turn implies $P(X) \iff P(Y)$ using J. Because univalence is consistent, it follows that for any given concrete $P : \mathcal{U} \to \mathcal{U}$, it is impossible to prove that $P(X)$ holds but $P(Y)$ doesn't.

So MLTT is invariant under isomorphism in this doubly negative, meta-mathematical sense. With univalence, it becomes invariant under isomorphism in a positive, mathematical sense.

5. Thus, we see that the formulation of univalence is far from direct, and has much more to it than the (in our opinion, misleading) slogan "isomorphic types are equal".

   What the consistency of the univalence axiom says is that one possible understanding of Martin-Löf's identity type $\mathrm{Id}(X, Y)$ for $X, Y : \mathcal{U}$ is as precisely the type $\mathrm{Eq}(X, Y)$ of equivalences, in the sense of being in one-to-one correspondence with it. Without univalence, the nature of the identity type of the universe in MLTT is fairly under-specified. It is a remarkable property of MLTT that it is consistent with this understanding of the identity type of the universe, discovered by Vladimir Voevodsky (and foreseen by Martin Hofmann and Thomas Streicher [3] in a particular case).

This paper only explains what the *univalence axiom* is. A brief and reasonably complete introduction to *univalent mathematics* is given by Grayson [2].

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