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A Generalization of the Functorial Calculus

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The notions of category, functor and natural transformation lead to a calculus whose rules have been explicitly formulated by Godement [2; appendice] under the name of "five rules of the functorial calculus." A generalization that includes contravariant functors and functors of several variables is obtained trivially by using dual categories and products of categories.

There are, however, useful situations which are not covered by the existing notions. For example, let $\mathscr{A}$ be the category of Abelian groups and consider the family of morphisms

$$a(AB) : A \otimes \mathscr{A}(AB) \to B$$

(we write $\mathscr{A}(AB)$ instead of $\mathrm{Hom}(A, B)$) given by the correspondence $a \otimes f \to fa$ for $f \in \mathscr{A}(AB)$.

With respect to the variable B the transformation a is natural in the usual sense. With respect to the variable A, the expression $A \otimes \mathscr{A}(AB)$ is not even a functor. However, $A \otimes \mathscr{A}(A'B)$ is a functor $T : \mathscr{A} \times \mathscr{A}^* \times \mathscr{A} \to \mathscr{A}$ and in some sense a should be a natural transformation $a : T \to 1$ where $1 : \mathscr{A} \to \mathscr{A}$ is the identity functor.

The generalization of the notion of a natural transformation given in this paper has been designed to include the above example and various similar situations.

The definitions were essentially given by one of the authors in [3], but the possibility of composing two such extraordinary natural transformations to get a third, upon which the usefulness of the idea depends, was there

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considered only in two special cases. The general question of composability turns out to be quite subtle, and the bulk of this paper is devoted to it.

Consider functors

$$T : \mathscr{A} \times \mathscr{B}^* \times \mathscr{B} \to \mathscr{E}$$
$$S : \mathscr{A} \times \mathscr{C}^* \times \mathscr{C} \to \mathscr{E},$$

where $\mathscr{B}^*$ is the dual category of $\mathscr{B}$. Let

$$a(ABC) : T(ABB) \to S(ACC)$$

be morphisms in $\mathscr{E}$ defined for every triple $A \in \mathscr{A}$, $B \in \mathscr{B}$, $C \in \mathscr{C}$. We shall say that the family $a = \{a(ABC)\}$ is *natural* if the following three diagrams commute:

$$\begin{array}{ccc} \mathscr{A}(AA') & \xrightarrow{T(-,B,B)} & \mathscr{E}(T(ABB), T(A'BB)) \\ {\scriptstyle S(-,C,C)}\downarrow & & \downarrow{\scriptstyle \mathscr{E}(1,a)} \\ \mathscr{E}(S(ACC), S(A'CC)) & \xrightarrow[\mathscr{E}(a,1)]{} & \mathscr{E}(T(ABB), S(A'CC)) \end{array} \tag{1}$$

$$\begin{array}{ccc} \mathscr{B}(BB') & \xrightarrow{T(A,B',-)} & \mathscr{E}(T(AB'B), T(AB'B')) \\ {\scriptstyle T(A,-,B)}\downarrow & & \downarrow{\scriptstyle \mathscr{E}(1,a)} \\ \mathscr{E}(T(AB'B), T(ABB)) & \xrightarrow[\mathscr{E}(1,a)]{} & \mathscr{E}(T(AB'B),(S(ACC)) \end{array} \tag{2}$$

$$\begin{array}{ccc} \mathscr{C}(CC') & \xrightarrow{S(A,C-,)} & \mathscr{E}(S(ACC), S(ACC')) \\ {\scriptstyle S(A,-,C')}\downarrow & & \downarrow{\scriptstyle \mathscr{E}(a,1)} \\ \mathscr{E}(S(AC'C'), S(ACC')) & \xrightarrow[\mathscr{E}(a,1)]{} & \mathscr{E}(T(ABB), S(ACC')) \end{array} \tag{3}$$

Applied to $f : A \to A'$, $g : B \to B'$, $h : C \to C'$ the commutativity of the above diagrams asserts that the diagrams

$$\begin{array}{ccc} T(ABB) & \xrightarrow{T(f11)} & T(A'BB) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ S(ACC) & \xrightarrow[S(f11)]{} & S(A'CC) \end{array} \tag{1'}$$

$$\begin{array}{ccc} T(AB'B) & \xrightarrow{T(11g)} & T(AB'B') \\ {\scriptstyle T(1g1)}\downarrow & & \downarrow{\scriptstyle a} \\ T(ABB) & \xrightarrow[a]{} & S(ACC) \end{array} \qquad (2')$$

$$\begin{array}{ccc} T(ABB) & \xrightarrow{a} & S(ACC) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle S(11h)} \\ S(AC'C') & \xrightarrow[S(1h1)]{} & S(ACC') \end{array} \qquad (3')$$

commute. These three can be condensed into a single commuting diagram

$$\begin{array}{ccccc} T(AB'B) & \xrightarrow{T(f1g)} & T(A'B'B') & \xrightarrow{a} & S(A'CC) \\ {\scriptstyle T(1g1)}\downarrow & & & & \downarrow{\scriptstyle S(11h)} \\ T(ABB) & \xrightarrow[a]{} & S(AC'C') & \xrightarrow[S(fh1)]{} & S(A'CC') \end{array} \qquad (4)$$

Because of some generalizations of the notions of category and functor that will be mentioned later, we prefer to use the more complicated form (1)–(3) of the naturality conditions, which do not, however, incolve the individual morphisms f, g, h.

Let $\mathscr{I}$ be the category with one object, $*$, and one morphism. If $\mathscr{B} = \mathscr{I}$ we can identify $\mathscr{A} \times \mathscr{B}^* \times \mathscr{B}$ with $\mathscr{A}$, writing TA for $T(A**)$; if also $\mathscr{A} = \mathscr{I}$, T is given by the object $T(***)$ of $\mathscr{E}$, which we shall also denote by T. By taking two at a time of $\mathscr{A}, \mathscr{B}, \mathscr{C}$ to be $\mathscr{I}$, we obtain definitions of naturality for families of morphisms:

$$aA : TA \to SA \quad \text{when} \quad T, S : \mathscr{A} \to \mathscr{E}, \qquad (1'')$$

$$aB : T(BB) \to S \quad \text{when} \quad T : \mathscr{B}^* \times \mathscr{B} \to \mathscr{E} \quad \text{and} \quad S \in \mathscr{E}, \qquad (2'')$$

$$aC : T \to S(CC) \quad \text{when} \quad S : \mathscr{C}^* \times \mathscr{C} \to \mathscr{E} \quad \text{and} \quad T \in \mathscr{E}. \qquad (3'')$$

In case (1″) of course we regain the usual definition of naturality. The special case of (3″) in which $\mathscr{E}$ is the category of sets and T a set consisting of one point coincides with MacLane's notion of *diagonal spread* [4, p. 54].

If we replace $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ by products $\mathscr{A}_1 \times \cdots \times \mathscr{A}_q$, $\mathscr{B}_1 \times \cdots \times \mathscr{B}_t$, $\mathscr{C}_1 \times \cdots \times \mathscr{C}_s$, we obtain a definition of naturality for a family of morphisms

$$a(A_1 \cdots A_q B_1 \cdots B_t C_1 \cdots C_s) : T(A_1 \cdots A_q B_1 \cdots B_t B_1 \cdots B_t) \to \\ \to S(A_1 \cdots A_q C_1 \cdots C_s C_1 \cdots C_s),$$

where

$$T : \mathscr{A}_1 \times \cdots \times \mathscr{A}_q \times \mathscr{B}_1^* \times \cdots \times \mathscr{B}_t^* \times \mathscr{B}_1 \times \cdots \times \mathscr{B}_t \to \mathscr{E}$$

and

$$S : \mathscr{A}_1 \times \cdots \times \mathscr{A}_q \times \mathscr{C}_1^* \times \cdots \times \mathscr{C}_s^* \times \mathscr{C}_1 \times \cdots \times \mathscr{C}_s \to \mathscr{E}.$$

The reader will easily verify that $a(A_1 \cdots C_s)$ is natural if and only if it is natural in each of the $q + t + s$ variables $A_1 \cdots C_s$ *separately*, in the sense of (1″), (2″), or (3″), as is appropriate, when the other variables are held fixed.

It is clear that if $T^* : \mathscr{A}^* \times \mathscr{B} \times \mathscr{B}^* \to \mathscr{E}^*$ and $S^* : \mathscr{A}^* \times \mathscr{C} \times \mathscr{C}^* \to \mathscr{E}^*$ are the duals of T and S then $a^* : S^* \to T^*$ is natural.

It is also clear that if

$$P : \mathscr{A}' \to \mathscr{A}, \quad Q : \mathscr{B}' \to \mathscr{B}, \quad R : \mathscr{C}' \to \mathscr{C}, \quad V : \mathscr{E} \to \mathscr{E}' \tag{5}$$

are functors and if a is natural then

$$Va(PA', QB', RC') : VT(PA', Q^*B', QB') \to VS(PA', R^*C', RC')$$

is natural. Thus composition of natural transformations with functors poses no problems.

A problem does arise when we attempt to compose two natural transformations. Indeed consider a functor

$$R : \mathscr{A} \times \mathscr{D}^* \times \mathscr{D} \to \mathscr{E}$$

and let

$$b(ACD) : S(ACC) \to R(ADD)$$

be natural. The composition

$$T(ABB) \xrightarrow{a(ABC)} S(ACC) \xrightarrow{b(ACD)} R(ADD)$$

will in general depend on A, B, C, D and therefore will not yield a well-defined natural transformation. However, if S is a functor of A only (i.e., if $\mathscr{C} = \mathscr{I}$), then a depends only on A and B and b depends only on A and D and the composition $c(ABD) = b(AD)a(AB)$ is a natural transformation

$$c(ABD) : T(ABB) \to R(ADD).$$

In order to gain a better understanding of the composition of natural transformations suppose that the categories $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are products

$$\mathscr{A} = \mathscr{A}_1 \times \cdots \times \mathscr{A}_q, \qquad \mathscr{B} = \mathscr{B}_1 \times \cdots \times \mathscr{B}_t, \qquad \mathscr{C} = \mathscr{C}_1 \times \cdots \times \mathscr{C}_s.$$

Then T becomes a functor of $q + 2t$ variables, S becomes a functor of $q + 2s$ variables and $a : T \to S$ depends on $q + t + s$ variables. Now the $q + 2s$ variables of S may be regrouped in a different way to form $\mathscr{A} \times \mathscr{C}^* \times \mathscr{C}$ and this different grouping may be used for $b : S \to R$.

We shall show that this phenomenon is the key to the understanding of the composition of natural transformations. The three propositions that follow will give three cases when the composition ba is natural. Then an abstract theory will be given from which it will follow that these three cases are essentially all the ones in which composition is defined.

PROPOSITION 1. *Given functors*

$$T : \mathscr{A} \to \mathscr{E}, \qquad R : \mathscr{A} \to \mathscr{E}$$
$$S : \mathscr{A} \times \mathscr{A}^* \times \mathscr{A} \times \cdots \times \mathscr{A}^* \times \mathscr{A} \to \mathscr{E}, \qquad (2n + 1 \text{ variables})$$

and given natural transformations

$$a(AB_1 \cdots B_n) : TA \to S(AB_1B_1 \cdots B_nB_n)$$
$$b(B_1 \cdots B_nC) : S(B_1B_1 \cdots B_nB_nC) \to RC,$$

the composition

$$cA = b(A \cdots A)\, a(A \cdots A) : TA \to RA$$

is natural.

PROPOSITION 2. *Given functors*

$$T : \mathscr{A} \times \mathscr{A}^* \to \mathscr{E}, \qquad R : \mathscr{I} \to \mathscr{E}$$
$$S : \mathscr{A} \times \mathscr{A}^* \times \mathscr{A} \times \cdots \times \mathscr{A}^* \to \mathscr{E}, \qquad (2n \text{ variables}, n > 0)$$

and given natural transformations

$$a(AB_1 \cdots B_{n-1}C) : T(AC) \to S(AB_1B_1 \cdots B_{n-1}B_{n-1}C)$$
$$b(B_1 \cdots B_n) : S(B_1B_1 \cdots B_nB_n) \to R,$$

the composition

$$cA = b(AA \cdots A)\, a(AA \cdots A) : T(AA) \to R$$

is natural.

PROPOSITION 2*. *Given functors*

$$T : \mathscr{I} \to \mathscr{E}, \qquad R : \mathscr{A} \times \mathscr{A}^* \to \mathscr{E}$$
$$S : \mathscr{A} \times \mathscr{A}^* \times \mathscr{A} \times \cdots \times \mathscr{A}^* \to \mathscr{E}, \qquad (2n \text{ variables}, n > 0)$$

and given natural transformations

$$a(B_1 \cdots B_n) : T \to S(B_1B_1 \cdots B_nB_n)$$
$$b(AB_1 \cdots B_{n-1}C) : S(AB_1B_1 \cdots B_{n-1}B_{n-1}C) \to R(AC),$$

the composition

$$cA = b(A \cdots A)\, a(A \cdots A) : T \to R(AA)$$

is natural.

Proof of Proposition 1. We define for $0 \leqslant k \leqslant n$

$$D_k = S(\underbrace{A \cdots A}_{k}\, C \cdots C)$$

$$U_k = S(\underbrace{A \cdots A}_{2k}, -, C \cdots C) \colon \mathscr{A}(A, C) \to \mathscr{E}(D_{2k+1}, D_{2k})$$

$$V_k = S(\underbrace{A \cdots A}_{2k+1}, -, C \cdots C) \colon \mathscr{A}(A, C) \to \mathscr{E}(D_{2k+1}, D_{2k+2})$$

$$a_k = a(\underbrace{A \cdots A}_{k+1}\, C \cdots C) \colon TA \to D_{2k+1}$$

$$b_k = b(\underbrace{A \cdots A}_{k}\, C \cdots C) \colon D_{2k} \to RC.$$

For convenience we shall write $(\ ,\ )$ instead of $\mathscr{E}(\ ,\ )$. From the commuting diagram

$$\begin{array}{ccccc}
(D_{2k+1}, D_{2k}) & \xleftarrow{U_k} & \mathscr{A}(A, C) & \xrightarrow{U_{k+1}} & (D_{2k+3}, D_{2k+2}) \\
\downarrow{\scriptstyle (1, b_k)} & & \downarrow{\scriptstyle V_k} & & \downarrow{\scriptstyle (a_{k+1}, 1)} \\
(D_{2k+1}, RC) & \xleftarrow[(1, b_{k+1})]{} & (D_{2k+1}, D_{2k+2}) & \xrightarrow[(a_k, 1)]{} & (TA, D_{2k+2}) \\
& {\scriptstyle (a_k, 1)} \searrow & & \swarrow {\scriptstyle (1, b_{k+1})} & \\
& & (TA, RC) & &
\end{array}$$

we deduce

$$(a_k, b_k)U_k = (a_k, b_{k+1})V_k = (a_{k+1}, b_{k+1})U_{k+1}. \qquad (*)$$

From the commuting diagrams

$$\begin{array}{ccccc}
\mathscr{A}(A, C) & \xrightarrow{T} & (TA, TC) & & \\
\downarrow{\scriptstyle U_0} & & \downarrow{\scriptstyle (1, a)} & \searrow {\scriptstyle (1, c)} & \\
(D_1, D_0) & \xrightarrow[(a_0, 1)]{} & (TA, D_0) & \xrightarrow[(1, b_0)]{} & (TA, RC)
\end{array}$$

$$\begin{array}{ccccc}
\mathscr{A}(A, C) & \xrightarrow{R} & (RA, RC) & & \\
\downarrow{\scriptstyle U_n} & & \downarrow{\scriptstyle (b, 1)} & \searrow {\scriptstyle (c, 1)} & \\
(D_{2n+1}, D_{2n}) & \xrightarrow[(1, b_n)]{} & (D_{2n+1}, RC) & \xrightarrow[(a_n, 1)]{} & (TA, RC)
\end{array}$$

we deduce that $(1, c)T = (a_0, b_0)U_0$, $(c, 1)R = (a_n, b_n)U_n$. Thus by (*), $(1, c)T = (c, 1)R$ as required.

Proof of Proposition 2. We define D_k, U_k and V_k as in the previous proof. Then

$$a_k = a(\underbrace{A \cdots A}_{k+1} C \cdots C) \colon T(AC) \to D_{2k+1}$$

$$b_k = b(\underbrace{A \cdots A}_{k} C \cdots C) \colon D_{2k} \to R,$$

and as before we prove (*). From the commuting diagrams

$$\begin{array}{ccccc}
\mathscr{A}(A, C) & \xrightarrow{T(-,C)} & (T(AC), T(CC)) & & \\
\Big\downarrow{\scriptstyle U_0} & & \Big\downarrow{\scriptstyle (1,a)} & \searrow^{(1,c)} & \\
(D_1, D_0) & \xrightarrow[(a_0,1)]{} & (T(AC), D_0) & \xrightarrow[(1,b_0)]{} & (T(AC), R)
\end{array}$$

$$\begin{array}{ccccc}
\mathscr{A}(A, C) & \xrightarrow{T(A,-)} & (T(AC), T(AA)) & & \\
\Big\downarrow{\scriptstyle V_{n-1}} & & \Big\downarrow{\scriptstyle (1,a)} & \searrow^{(1,c)} & \\
(D_{2n-1}, D_{2n}) & \xrightarrow[(a_{n-1},1)]{} & (T(AC), D_{2n}) & \xrightarrow[(1,b_n)]{} & (T(AC), R)
\end{array}$$

we deduce that

$$(1, c)T(-, C) = (a_0, b_0)U_0, \qquad (1, c)T(A, -) = (a_{n-1}, b_n)V_{n-1}.$$

Thus by (*), $(1, c)T(-, C) = (1, c)T(A, -)$ as required.

Proposition 2* is dual of Proposition 2.

The language needed to formulate the general problem of composability will now be introduced.

Let X be a finite set. We shall consider graphs γ with vertices in X subject to the following conventions: the edges of γ are not oriented, and each edge of γ has two distinct vertices, but two edges may have the same vertices. A graph γ is *simple* if each point of X is a vertex of exactly one edge of γ. Thus a simple graph is a union of disjoint edges and X has an even number of points.

We now define the category $\mathscr{X}$ as follows. The objects of $\mathscr{X}$ are finite sets X, X', Y, etc. Given X, Y we consider the disjoint union (i.e., the *sum*) $X + Y$ and define a morphism $\alpha : X \to Y$ to be a simple graph on $X + Y$. Given $\beta : Y \to Z$ we have in $X + Y + Z$ the graphs α and β which we consider as disjoint except for the set of vertices Y, and we also have their union $\alpha + \beta$. This graph is no longer simple, but has no points of ramification.

Its connected components fall into the following four types:

(i) an arc L starting at a point of X and terminating at a point of Z with all intermediate vertices in Y; L consists of an even number of edges alternately from α and from β.

(ii) an arc L with both endpoints in X and all intermediate vertices in Y; L has an odd number of edges.

(iii) an arc L with both endpoints in Z and all intermediate vertices in Y; L has an odd number of edges.

(iv) a simple closed curve C with all vertices in Y; C has an even number of edges.

If we disregard the closed curves C and consider the arcs L of type (i), (ii) or (iii) as edges of a graph γ, we obtain a simple graph on $X + Z$ which we define as the composition $\gamma = \beta\alpha : X \to Z$. The product of X with an interval is a simple graph on $X + X$ and clearly is an identity morphism for X. To verify the associative law consider $\alpha : X \to Y$, $\beta : Y \to Y'$ and $\gamma : Y' \to Z$. The graph $\alpha + \beta + \gamma$ on $X + Y + Y' + Z$ again has no ramification. The components of $\alpha + \beta + \gamma$ which are arcs again yield a simple graph on $X + Z$ and thus yield a morphism $\gamma\beta\alpha : X \to Z$. Each such arc can be decomposed into edges of γ alternating with arcs made up of edges of α and β. These arcs must be components of $\alpha + \beta$, since $\alpha + \beta + \gamma$ has no ramification. In this way we see that $\gamma\beta\alpha = \gamma(\beta\alpha)$. Similarly $\gamma\beta\alpha = (\gamma\beta)\alpha$.

The morphisms $\alpha : X \to Y$, $\beta : Y \to Z$ will be called *compatible* if the graph $\alpha + \beta$ on $X + Y + Z$ has no closed curves. Similarly $\alpha : X \to Y$, $\beta : Y \to Y'$, $\gamma : Y' \to Z$ are said to be compatible if the graph $\alpha + \beta + \gamma$ on $X + Y + Y' + Z$ has no closed curves. One easily shows the equivalence of the following conditions:

(a) γ, β, α are compatible.

(b) β, α are compatible and $\gamma, \beta\alpha$ are compatible.

(c) γ, β are compatible and $\gamma\beta, \alpha$ are compatible.

One can easily construct examples where γ, β are compatible, β, α are compatible but γ, β, α are not.

We now define a category $\mathscr{H}$ as follows. An object $\mathbf{B}$ of $\mathscr{H}$ is a family of categories $\{\mathscr{B}_x\}$, $x \in X$, indexed by an object X of $\mathscr{K}$. If $\mathbf{C} = \{\mathscr{C}_y\}$, $y \in Y$, a morphism $\mathbf{B} \to \mathbf{C}$ is a triple $(\mathbf{B}, \alpha, \mathbf{C})$ where $\alpha : X \to Y$ is a morphism in $\mathscr{K}$ satisfying the following conditions:

(I) If $x \in X$, $y \in Y$ are vertices of an edge in α then $\mathscr{B}_x = \mathscr{C}_y$.

(II) If $x, x' \in X$ are vertices of an edge in α then $\mathscr{B}_x = \mathscr{B}_{x'}$*.

(III) If $y, y' \in Y$ are vertices of an edge in α then $\mathscr{C}_y = \mathscr{C}_{y'}$*.

If $(\mathbf{C}, \beta, \mathbf{D})$ is another morphism in $\mathscr{Y}$ then it is easy to see that

$$(\mathbf{C}, \beta, \mathbf{D})(\mathbf{B}, \alpha, \mathbf{C}) = (\mathbf{B}, \beta\alpha, \mathbf{D})$$

is again a morphism in $\mathscr{Y}$. This shows that $\mathscr{Y}$ is a category. Further we have a faithful functor $U : \mathscr{Y} \to \mathscr{X}$ given by $U\mathbf{B} = X$, $U(\mathbf{B}, \alpha, \mathbf{C}) = \alpha$.

By a functor $T : \mathbf{B} \to \mathscr{E}$ we shall understand a functor on the product category $\times \mathscr{B}_x$, $x \in X$. Let $(\mathbf{B}, \alpha, \mathbf{C})$ be a morphism in $\mathscr{Y}$ and let

$$T : \mathbf{B} \to \mathscr{E}, \qquad S : \mathbf{C} \to \mathscr{E}$$

be functors.

Using α we can arrange the variables of T and S so that

$$T : \mathscr{A}_1 \times \cdots \times \mathscr{A}_q \times \mathscr{B}_1^* \times \mathscr{B}_1 \times \cdots \times \mathscr{B}_t^* \times \mathscr{B}_t \to \mathscr{E}$$
$$S : \mathscr{A}_1 \times \cdots \times \mathscr{A}_q \times \mathscr{C}_1^* \times \mathscr{C}_1 \times \cdots \times \mathscr{C}_s^* \times \mathscr{C}_s \to \mathscr{E},$$

where the $\mathscr{A}$'s correspond to edges of α of type I, the pairs $\mathscr{B}_i^*, \mathscr{B}_i$ to edges of type II, and pairs $\mathscr{C}_i^*, \mathscr{C}_i$ to edges of type III. If we define

$$\mathscr{A} = \mathscr{A}_1 \times \cdots \times \mathscr{A}_q, \qquad \mathscr{B} = \mathscr{B}_1 \times \cdots \times \mathscr{B}_t, \qquad \mathscr{C} = \mathscr{C}_1 \times \cdots \times \mathscr{C}_s$$

then

$$T : \mathscr{A} \times \mathscr{B}^* \times \mathscr{B} \to \mathscr{E}, \qquad S : \mathscr{A} \times \mathscr{C}^* \times \mathscr{C} \to \mathscr{E},$$

and the definition of a natural transformation

$$a : T \to S$$

applies. Such a natural transformation will be said to be of *type* α. In practice, we do not introduce $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, but write

$$a(\cdots A_i \cdots B_j \cdots C_q \cdots) : T(\cdots A_i \cdots B_j B_j \cdots) \to S(\cdots A_i \cdots C_q C_q \cdots).$$

We then have r commuting diagrams (1), t commuting diagrams (2), and s commuting diagrams (3). The advantage of this notation is that the type α of a is implicitly given by the notation.

Now consider morphisms $(\mathbf{B}, \alpha, \mathbf{C})$ and $(\mathbf{C}, \beta, \mathbf{D})$ in $\mathscr{Y}$, functors

$$T : \mathbf{B} \to \mathscr{E}, \qquad S : \mathbf{C} \to \mathscr{E}, \qquad R : \mathbf{D} \to \mathscr{E},$$

and natural transformations

$$a : T \to S, \qquad b : S \to R$$

of type α and β, respectively.

Assume that the graph $\alpha + \beta$ is connected. If $\alpha + \beta$ is an arc, then by a suitable labeling of the categories we arrive at one of the cases discussed in Propositions 1, 2, and 2*. The composition $ba : T \to R$ is then a natural

transformation of type $\beta\alpha$. If $\alpha + \beta$ is a closed curve then we are in the following situation

$$T: \mathscr{I} \to \mathscr{E},\ S: \underbrace{\mathscr{A} \times \mathscr{A}^* \times \cdots \times \mathscr{A} \times \mathscr{A}^*}_{2s} \to \mathscr{E},\ R: \mathscr{I} \to \mathscr{E}$$

$$a(A_1 \cdots A_s) : T \to S(A_1 A_1 \cdots A_s A_s)$$
$$b(A_1 \cdots A_s) : S(A_1 A_2 A_2 \cdots A_s A_s A_1) \to R.$$

The composition $b(A \cdots A)\, a(A \cdots A) : T \to R$ will then in general depend upon $A \in \mathscr{A}$ and will not be a well defined transformation $T \to R$.

The case when $\alpha + \beta$ has several components is reduced to the case when $\alpha + \beta$ is connected by fixing all the variables except those occuring in one component of $\alpha + \beta$. In this way we obtain the conclusion that if α and β are compatible then the composition $ba : T \to R$ is a natural transformation of type $\beta\alpha$.

In a forthcoming paper [*1*], dealing with enriched categories, we shall have occasion to use the above theory in a generalized setting. This generalization will be formally identical with the above provided we use diagrams (1), (2), (3) rather than diagrams (1′), (2′), (3′), which in the generalized setting cannot be written.

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