DEDEKIND'S INVENTION OF IDEALS

HAROLD M. EDWARDS

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The present-day formulation of ideal theory is quite standardized. A subset of a commutative ring with unit is called an *ideal* if it is closed under addition and under multiplication by ring elements. The *product* of two ideals of a ring is the ideal consisting of all elements of the ring that can be written as sums of products $a \cdot b$ in which a is in the first ideal and b is in the second. The ring itself is an ideal and every ideal A of a ring R has the trivial factorization $A = RA$. An ideal A is called *prime* if it has no nontrivial factorizations, that is, if $A = BC$ implies $B = R$ or $C = R$. Under suitable conditions on R, *every ideal can be written as a product of prime ideals* and this representation is unique, up to the order of the factors.

This is almost exactly the way that Dedekind himself formulated the theory. He was dealing specifically with the ring of integers of an algebraic number field and did not use the term "ring", which was introduced later by Hilbert, and instead of speaking of a "subset" of the ring of integers he spoke of a "system" (*System* in German) of integers. Except for these minor differences, Dedekind's presentation of the theory in the supplements to the 3rd and 4th editions of Dirichlet's lectures on number theory (in 1879 and 1894) exactly fits this description.

It should be seen as a great triumph for Dedekind that his formulation of the basic concepts has won such universal acceptance. For him, the way in which one formulated the basic concepts of a mathematical theory was of the utmost importance, and his version of ideal theory was the result of the most careful deliberation, much in the way that Leibniz's philosophy assigned great importance to mathematical notation and the Leibniz notation for the calculus was the result of years of effort. And just as the success of Leibniz's notation is evident in today's notation for differential forms, the success of Dedekind's set-theoretical approach to the fundamental concepts is evident in today's approach to all of modern algebra and, indeed, today's approach to the foundations of mathematics in general.

The success of Dedekind's ideas has been so great that modern mathematicians unaware of the history have no idea how innovative they were. Pierre Dugac has rightly called Dedekind's first publication of his theory of ideals the "birthplace" of the modern set-theoretic approach to the foundations of mathematics. In order to appreciate how new it was, one must, of course, investigate the context in which it was created. This will also shed light on the considerations which led Dedekind to formulate the theory in the way that he did, and hence will give some insight into the reasons for the success of his ideas.

First, it should be said that there is no necessity whatsoever to formulate the theory this way. Kronecker's theory of divisors does exactly what Dedekind set out to do—that is, it generalizes Kummer's theory of ideal factorization from cyclotomic fields to arbitrary algebraic number fields—and it uses no "ideals" at all. Moreover, Dedekind's ideas aren't even sufficient. As I tried to show in my paper [3] on the

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genesis of ideal theory, the really crucial idea is that of an "integer" in an algebraic number field. Without that—that is, without the correct determination of the *ring* to which the ideal theory is to apply—Dedekind's ideas are of no help. With that, the theory can be developed in at least three quite distinct ways, one of them Dedekind's, one of them Kronecker's, and the third one a fairly automatic generalization of Kummer's original approach.

In fact—and this shows how far Dedekind's formulation is from being the obvious one—this third approach following Kummer is essentially the one Dedekind himself took in his original publication of the theory in 1871 in the 2nd edition of Dirichlet–Dedekind. In this version, the now-familiar definition of ideals is used, to be sure, but the notion of the *product* of two ideals occurs only at the very end of the theory. In its stead, the basic role is played, as it was in Kummer's theory, by the notion of the *divisibility* of one ideal by another, which is defined by certain congruence conditions; the theorem on unique factorization states that if two ideals are divisible by the same prime ideals with the same multiplicities, then they are equal. (The definition of the multiplicities involves, from Dedekind's point of view, technical difficulties which he was eager to remove.) Thus, far from being the most obvious and direct approach to the subject, Dedekind's final version was one he himself arrived at only after years of revision and reconsideration.

This last fact also shows that while Kummer's theory of ideal factorization for cyclotomic integers was, of course, the inspiration for Dedekind's work, his final result represented in some ways a *rejection* of Kummer's ideas. There are three specific objections to Kummer's theory which Dedekind expresses and which shed light on the philosophy of mathematics from which his ideas sprang.

The fundamental concept of Kummer's theory is that of an *ideal prime factor* of cyclotomic integers. Kummer defines what it means to say that a given cyclotomic integer is "divisible μ times" by a given ideal prime factor, but he never says what an ideal prime factor *is*. Moreover, he occasionally speaks of *ideal numbers*, rather than ideal prime factors, and he never says at all what an ideal number is. (One can infer from his use of the notion that by an ideal number he meant essentially a formal product of ideal prime factors.) This state of affairs was extremely unsatisfactory to Dedekind, who thought of *identifying ideal numbers* (and, in particular, ideal prime factors) *with the set* (system) *of integers that they divide*. He then found—not without considerable difficulty, for it is a difficult theorem—that such subsets are characterized by the fact that they are closed under addition and multiplication by ring elements. "This situation naturally led me to found the entire theory of numbers of the ring $\mathfrak{O}$ [of integers in a number field] on this simple definition [of "ideal"] *delivered of all obscurity and the admission of ideal numbers*" (emphasis added).* Thus Dedekind's formulation of the theory was a reaction to what he regarded as inadmissable obscurity and vagueness in Kummer's definition of ideal numbers.

Another aspect of Kummer's formulation that Dedekind rejected was that it was too specific. Kummer gave explicit rules for determining the ideal prime factors of a given cyclotomic integer. Dedekind saw the weakness of an approach that used specific properties of the cyclotomic integers and could not be applied to other number fields. "I am of course far from wanting to compare the value of [the development of ideal theory] with that of Kummer's original creative idea, but, apart from the aesthetic satisfaction of knowing that the same simple law according to which numbers are composed of primes applies in the form of the theory of ideals to

* Quotations are given in their original languages, together with citations, at the end of the paper.

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any field, the knowledge of the general theory has the great practical advantage that one is not obliged to return in each special case to the first principles and that the working out of the special laws is simplified to an extraordinary degree".

The third way in which Dedekind's work represents a negative reaction to Kummer's is less subtle. Kummer's work contained mistakes. They are rare, and they do not affect the main conclusions of his papers, but naturally any doubt about the correctness of one's conclusions is the antithesis of good mathematics. And the source of Kummer's problem was his suggestive terminology. Dedekind politely put it this way: "Kummer never defined the ideal numbers themselves but only divisibility by them.... Although this introduction of new numbers is entirely legitimate, one may well fear that the chosen mode of expression in which one speaks of determined ideal numbers and their products, and the presumed analogy with the theory of rational numbers, may lead one to premature conclusions, and then on to inadequate proofs—and, in fact, this outcome has not always been avoided".

So the inspiration for Dedekind's set-theoretic approach did not come from Kummer. Nor, in my opinion, did it come from Gauss, under whose direction Dedekind wrote his dissertation (on Eulerian integrals). Of course Gauss's appreciation of the role of generalization in the development of mathematics, and his unerring judgement as to what topics would prove fruitful, had a profound influence on the mathematics of that period and therefore on Dedekind. However, the set-theoretic approach is far removed from anything we know of Gauss's work. Anyone familiar with Dedekind's biography will be well aware that the decisive influence on his mathematical outlook was rather his association with the great G. Lejeune-Dirichlet.

When Gauss died in early 1855, just as Dedekind was beginning his period as a Privatdozent in Göttingen, Dirichlet came from Berlin to replace Gauss in Göttingen. His presence made of Dedekind, in his own words, a "new man". Paul Bachmann, who was a student in Göttingen at the time, recalled in later years that he only knew Dedekind by sight because Dedekind always arrived and left with Dirichlet and was completely eclipsed by him. Dedekind himself, in a letter of July 1856 that was recently published, describes how great a difference Dirichlet's presence made. "What is most useful to me is the almost daily association with Dirichlet, with whom I am for the first time beginning to learn properly; he is always completely amiable toward me, and he tells me without beating about the bush what gaps I need to fill and at the same time gives me the instructions and the means to do it. I thank him already for infinitely many things, and no doubt there will be many more".

Hermann Minkowski, in a memorial lecture about Dirichlet spoke of the "other Dirichlet Principle" (*the* Dirichlet Principle being, of course, the notion in function theory to which Riemann gave that name) which he described as being the principle of "overcoming problems with a minimum of blind calculation and a maximum of perceptive thought". It is this other Dirichlet principle that I see as having been the primary influence on Dedekind's philosophy of mathematics and on the development of Dedekind's mathematical style.

Of course Dedekind's own personality determined the way in which this influence would act. Kummer, too, was strongly influenced by Dirichlet, but, since his personality was so different, the influence acted in a different way. Basically, Kummer was a *computer* and Dedekind was an *expositor*. I don't mean, of course, that Kummer was *only* a computer or that Dedekind was *only* an expositor—both were mathematicians of wide-ranging abilities and great accomplishments. I only

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mean to give an indication of what seems to me a clear difference in their attitudes and ways of thinking about problems.

That Dedekind took his teaching very seriously, even at this early stage of his career, is clear. Dirichlet, in a letter recommending appointment of Dedekind to the Polytechnikum in Zürich, described him as "an exceptional pedagogue", and once Dedekind arrived in Zürich (he got the job) his teaching was an integral part of his mathematical thinking, as I will have occasion to mention in a few minutes. Another proof of Dedekind's calling as an expositor is his publication in 1863 of the first edition of "Dirichlet's" lectures on number theory. (Although the book is assuredly *based* on Dirichlet's lectures, and although Dedekind himself referred to the book throughout his life as Dirichlet's, the book itself was entirely written by Dedekind, for the most part after Dirichlet's death.) The importance of Dedekind's "supplements" to the later editions of this book should not be allowed to obscure the importance of the book itself, which was surely the most important book on number theory after Gauss's *Disquisitiones arithmeticae*. Shortly after it appeared, Kronecker mentioned it in the following way in a paper presented to the Berlin Academy:

> I should remark, in addition, that a similar derivation is also to be found in one of the very worthwhile supplements which Herr Dedekind added to his extremely commendable edition of Dirichlet's lectures on number theory, which is so skilfully and carefully executed.

This demonstrates, I think, the importance of Dirichlet's influence on Dedekind's mathematical formation, and on his attention to matters of exposition and "perceptive thought" as opposed to "blind calculation". Another major influence was the work of Riemann. Dedekind himself speaks of this influence in a letter he wrote in 1876 to R. Lipschitz:

> My efforts in number theory have been directed toward basing the work not on arbitrary representations or expressions but on simple foundational concepts and thereby—although the comparison may sound a bit grandiose—to achieve in number theory something analogous to what Riemann achieved in function theory, in which connection I cannot suppress the passing remark that Riemann's principles are not being adhered to in a significant way by most writers—for example, even in the newest works on elliptic functions. Almost always they mar the purity of the theory by unnecessarily bringing in forms of representation which should be results, not tools, of the theory.

Dedekind's aversion to basing anything in mathematics on forms of representation, which is so clear in this quotation, is expressed in another place in the correspondence with Lipschitz, where he says that in the exposition of ideal theory he himself published in the *Bulletin des Sciences Mathématiques*, in the interest of brevity, he marred (*ich verunziere*) the theory by using explicit representations of fields in the form $\mathbb{Q}(\alpha)$ where α satisfies an algebraic relation over $\mathbb{Q}$. To Dedekind, the field is what is important, and the arbitrarily chosen representation $\mathbb{Q}(\alpha)$ is to be avoided.

This Riemannian rejection of forms of representation and the use instead of characteristic properties as foundational concepts is closely related to that "other Dirichlet principle" and can in fact be viewed as another influence of Dirichlet on Dedekind, this time through the medium of Riemann. As Siegel has observed, Riemann was a computer of astonishing power and skill, but his published work always emphasized conceptual thinking and avoided formulas and "blind calculation".

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In considering possible influences on the development of Dedekind's set-theoretic approach, mention should be made of the work of Galois. Galois has been widely praised for the modernity and abstractness of his thinking, and it is known that Dedekind studied Galois's works very carefully during his formative years 1856–1858. Nonetheless, I am not able to see an influence here. Certainly the effort to understand and to give clear expositions of Galois's theory (Dedekind was the first to teach Galois theory at a German university) was a formative experience for Dedekind, but Galois showed no reluctance to base his formulation of Galois theory on the arbitrary choice of a Galois resolvent to represent the splitting field, and there appears to be no hint of the set-theoretic approach in his work.

But there is more to the set-theoretic approach than conceptual thinking. Dirichlet, Riemann, and Galois were all conceptual thinkers—and so, no doubt, were Gauss and many others—but this does not imply that they described the basic concepts in terms of sets. In fact, they most certainly would *not* have described basic concepts in terms of sets because of the classical doctrine which excluded any use of a completed infinite. As Gauss said in an oft-quoted letter to Schumacher, "But concerning your proof, I protest above all against the use of an infinite quantity as a *completed* one, which in mathematics is never allowed. The infinite is only a *façon de parler*, where one would properly speak of limits".

It must be asked what motivated Dedekind to defy this doctrine to the extent of *defining* his ideals as infinite sets, and to regard these infinite sets as completed entities. In this he was not following Dirichlet, and it seems unlikely that Dirichlet, had he been alive, would have followed him. The best indication I know of Dirichlet's views in such matters, comes, ironically, from Dedekind's introduction to *Was sind und was sollen die Zahlen*?: " ... Seen in this way it seems perfectly obvious and by no means new that every theorem of algebra and higher analysis, no matter how remote, can be expressed as a theorem about the natural numbers, an assertion that I heard repeatedly from the lips of Dirichlet". One can be certain that Dirichlet did not regard the natural numbers as a completed infinite and that his assertion here is much akin to Gauss's about the infinite—in Dirichlet's case, algebra and higher analysis—being a *façon de parler.*

Dedekind's contrary view—his use of completed infinites as basic concepts—goes back to his earliest writings. As Dugac has observed, already in his 1857 memoir *Abriss einer Theorie der höheren Kongruenzen* Dedekind described addition modulo a prime as addition of equivalence classes, that is, as addition of infinite sets regarded as entities. However, in all of elementary algebra, as in this example, the infinite sets involved are of a very tame type; they can be described explicitly in a way that is acceptable to even the most finitistic philosophy of mathematics. What prompted Dedekind to fly in the face of the doctrine against completed infinites could not have been anything in algebra; it must have been something from analysis.

In fact, there had been trouble brewing for quite a while in analysis. Consider, for example, the function concept. One of Dirichlet's greatest achievements was his proof in 1829 that the Fourier series of a function converges to the function for an extremely general class of functions, namely, all functions "which have a finite number of maxima and minima" as it was expressed at the time or which are "piecewise monotone" as we would express it today. What *is* such a function—how is it to be described? Surely not by a formula. The whole point of the theorem was to show that the functions to which Fourier analysis applies far transcend the functions given by simple formulas. "Functions that are given by a graph" was the way general functions were described in those days. But what is a graph but a subset of $\mathbb{R} \times \mathbb{R}$?

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And what is a subset of $\mathbb{R} \times \mathbb{R}$ but a completed infinite? Set theory is what is left after you reject all formulas.

It was just such considerations of very general functions and the convergence of Fourier series that led Cantor in the 1870's and 1880's to his bold advancement of the theory of infinite sets. For Dedekind, the impulse to overthrow the doctrine against the completed infinite came from a different direction—the need to elucidate the idea of a real number. Dedekind tells us the exact date—24 November 1858—on which the idea occurred to him to describe real numbers in terms of what we know today as *Dedekind cuts*. What prompted his thoughts in this direction was, characteristically, his teaching. This was his first year in Zürich and it was the first time he had taught differential and integral calculus. In his attempts to formulate an exposition of the fundamental concepts of the calculus for his students which he himself found acceptable, he found the existing expositions to be of no help and decided to do it on his own. The result was his description of real numbers in terms of the "cuts" of the rational numbers that they generate: A real number—whatever it is—effects a division of the rationals into two sets, those less than it and those greater than it. Dedekind turned the tables and took such a division of the rationals to *be*, or at any rate to *represent*, a real number.

The analogy between cuts and ideals is obvious. An ideal number in a ring is "concretely" represented by the set of all *actual* numbers of the ring that it divides. A real number is "concretely" represented by the set of all *rational* numbers that it is greater than (or by those it is less than). Here the use of the term "concrete" would scarcely be acceptable to anyone—Gauss, for example—who rejected the completed infinite. Still, it was "concrete" to Dedekind, and he used this term without apology, unless his delay of the publication of the idea until 1872 can be regarded as an acknowledgement that there might be strong objections.

Kronecker did object:

> These considerations oppose, it seems to me, the introduction of Dedekind's notions of modules, ideals, etc., as well as the introduction of various new concepts that have been used lately (in the first instance by Heine) in the attempt to understand and to give foundations to the "irrational" in full generality. The *general* concept of an infinite series itself, for example a power series, is in my judgement permissible only with the reservation that in each particular case the arithmetical rule by which the terms are given satisfies, as above, conditions which make it possible to deal with the series as though it were finite, and thus to make it unnecessary, strictly speaking, to go beyond the notion of a finite series.

This objection was in a footnote to a paper published in 1886.

Dedekind replied with a footnote of his own, published in *Was sind und was sollen die Zahlen*? (1888):

> In what way the determination [whether a given element is in a given set] comes about, or whether we know a way to decide it, is a matter of no consequence in what follows. The general laws that are to be developed do not depend on this at all. They hold under all circumstances. I mention this expressly because Herr Kronecker a short while ago sought to impose certain restrictions on the free introduction of concepts into mathematics, restrictions which I do not regard as justified. To discuss this in detail will be called for, however, only after that eminent mathematician publishes his reasons for asserting the necessity—or even simply the usefulness—of these restrictions.

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I award that round to Dedekind.

On the whole, the objections to Dedekind's ideas were surprisingly few, and the victory of his point of view was surprisingly swift and complete. One of the few indications of resistance other than Kronecker's is to be found in Dedekind's correspondence with Lipschitz. It seems that Lipschitz advised Dedekind *against* citing the analogy between ideals and Dedekind cuts in his exposition of ideal theory for the *Bulletin des Sciences Mathématiques*. Evidently Lipschitz felt that the Dedekind cut notion was not well accepted (not, however, because of its use of a completed infinite) and that this would not be seen as a point in favour of ideals. Dedekind's reply shows that Lipschitz had struck a nerve. The normally patient, mild, and altogether genial Dedekind suddenly became dogmatic, insistent, and almost out of control (only a *portion* of this letter is published in Dedekind's *Werke*—the whole letter is in the Göttingen archives) as he argues for the importance of the cut concept and for the necessity of some such concept for the proper foundation of analysis. This also demonstrates, if a demonstration is needed, the close relationship between ideals and Dedekind cuts. As I said above, the infinite sets needed in algebra are very tame; Dedekind would not have had the slightest difficulty in defining ideals in a way that met the most stringent restrictions on the use of the infinite, but he was not willing to take any steps in this direction because he knew full well that the same could not be done for his cuts.

There are several reasons for the success of Dedekind's ideas among his contemporaries. One was surely the publication in 1882 of his epoch-making paper with Heinrich Weber, which applied his ideal theory with such stunning success to the theory of Riemann surfaces (algebraic function fields of one variable). Here for the first time the Riemann–Roch theorem was proved rigorously under the most general assumptions using purely algebraic means—Dedekind's ideals and other ideas adapted from his work on number theory.

Another reason for the success of Dedekind's ideas was the marvellous clarity with which he presented them. He always assigned great importance to good exposition, and his writing shows it. By contrast, his rival, Kronecker, published a major work, his Kummer *Festschrift*, at exactly the same time that the paper of Dedekind–Weber appeared. No one doubted the importance of Kronecker's ideas, but it seems that no one could read them. The contrast could hardly have been more damaging from the point of view of winning allies for Kronecker's ideas on the foundations. With Kronecker, even his objectives were obscure, much less the question of whether or to what extent he achieved them. With Dedekind–Weber, the objective was one whose importance was universally recognized, and it was achieved with absolutely Euclidean economy and clarity.

The appearance of Cantor's revolutionary works on set theory in the 1880's may also have contributed to the acceptance of Dedekind's formulations. Cantor's use of the infinite was so much more aggressive than Dedekind's that mathematicians with scruples in such matters concentrated their fire on Cantor, not on Dedekind, and certainly not on Dedekind's number-theoretical works.

Dedekind's ideas also succeeded through a chance of circumstance that deserves to be mentioned. At the time Dedekind–Weber appeared, Weber happened to be teaching number theory at the University of Königsberg. Without doubt, Weber, who was the junior partner in the collaboration, would have done number theory at that time in an entirely Dedekindian way. One of his students was the young David Hilbert. Hilbert's early exposure to Dedekind's approach most probably was a help to him in his development into the foremost mathematician of his generation, and to

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this extent it is not mere chance that Dedekind's viewpoint came into the ascendancy via Hilbert. On the other hand, talents like Hilbert's are rare, and without him things might have been very different.

The extent to which objections to the completed infinite were forgotten is astounding. Hilbert's proof of the existence of an integral basis in the *Zahlbericht* (1897) begins "We imagine all the integers of the field in a list". He then proceeds to pick out of his imaginary list an entry in which a particular term has the smallest possible positive value! I have no way of proving it, but I believe even Dedekind would have balked at such a sweeping use of the completed infinite, especially to prove a theorem which can so easily be proved constructively, as Dedekind always did.

Dedekind once wrote of Kronecker that "I have always been far from wanting to compare my *aurea mediocritas*, the strength of which lies in obstinate perseverance, with Kronecker's outstanding talent". In this lecture I have tried to conceal my personal prejudice against Dedekind and for Kronecker, when it comes to questions about the foundations of mathematics. Even with this personal prejudice, however, I do not accept Dedekind's modest estimate of himself in comparison to Kronecker in the sentence just quoted. Kronecker's brilliance cannot be doubted. Had he had a tenth of Dedekind's ability to formulate and express his ideas clearly, his contribution to mathematics might have been even greater than Dedekind's. As it is, however, his brilliance for the most part died with him. Dedekind's legacy, on the other hand, consisted not only of important theorems, examples, and concepts, but of a whole *style* of mathematics that has been an inspiration to each succeeding generation.

Quotations

Cette constatation m'a conduit naturellement à fonder toute la théorie des nombres du domaine $\mathfrak{O}$ sur cette définition simple, entièrement délivrée de toute obscurité et de l'admission des nombres idéaux. [**1**, pp. 271–272]

Ich bin zwar weit davon entfernt, ihren Wert mit demjenigen des ersten schöpferischen Gedankens von Kummer vergleichen zu wollen; aber abgesehen von der ästhetischen Befriedigung, welche die Erkenntnis gewährt, daß dieselben einfachen Gesetze, nach welchen in der rationalen Zahlentheorie die Zahlen aus Primzahlen zusammengesetzt werden, in der Theorie der Ideals jedes Körpers wiederkehren, hat die Gewißheit der allgemeinen Theorie den großen praktischen Vorteil zur Folge, daß man nicht in jedem speziellen Falle immer wieder die ersten Grundlagen aufzusuchen braucht, und daß die Auffindung der speziellen Gesetze ganz außerordentlich erleichtert wird. [**1**, p. 419]

Kummer n'a pas défini les nombres idéaux eux-mêmes, mais seulement la divisibilité par ces nombres. ... Bien que cette introduction de nouveaux nombres soit tout à fait légitime, il est toutefois à craindre d'abord que, par le mode d'expression que l'on a choisi, dans lequel on parle de nombres idéaux déterminés et de leurs produits, et aussi par l'analogie présumée avec la théorie des nombres rationnels, on ne soit entrainé à des conclusions précipitées et par là à des démonstrations insuffisantes, et en effet cet écueil n'est pas toujours complétement évité. [**1**, p. 268]

Am nützlichsten ist mir wohl der fast tägliche Umgang mit Dirichlet, bei dem ich eigentlich erst recht zu lernen anfange; er ist immerfort gleich liebenswürdig gegen

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mich, sagt mir ohne Umschweife, was für Lücken ich auszufüllen habe, und giebt mir unmittelbar die Anweisung und die Mittel dazu. So verdanke ich ihm schon unendlich viel, und es wird wohl noch viel mehr werden. [**6**, p. 35]

... von dem anderen Dirichletschen Prinzipe, mit einem Minimum an blinder Rechnung, einem Maximum an sehenden Gedanken die Probleme zu zwingen, datiert die Neuzeit in der Geschichte der Mathematik. [**5**, pp. 460–461]

Ich bemerke dabei, daß eine ähnliche Herleitung sich auch in einem der verdienstvollen Supplemente findet, welche Hr. *Dedekind* seiner überaus dankenswerthen, mit geschickter und sorgsamer Hand veranstalteten Herausgabe der *Dirichlet*schen Vorlesungen über Zahlentheorie beigefügt hat. [**4**, Vol. 4, p. 232]

Mein Streben in der Zahlentheorie geht dahin, die Forschung nicht auf zufällige Darstellungsformen oder Ausdrücke sondern auf einfache Grundbegriffe zu stützen und hierdurch—wenn diese Vergleichung auch vielleicht anmaßend klingen mag—auf diesem Gebiete etwas Ähnliches zu erreichen, wie Riemann auf dem Gebiet der Functionentheorie, wobei ich die beiläufige Bemerkung nicht unterdrücken kann, daß die Riemannschen Principien von den meisten Schriftstellern, z. B. auch in den neuesten Werken über elliptische Functionen, nach meiner Ansicht nicht in consequenter Weise zur Anwendung gebracht werden; fast immer wird die einfache Theorie verunziert durch unnöthige Einmischung der Darstellungsformen, welche doch eigentlich nur Resultat, nicht Hülfsmittel der Theorie sein sollten. [**1**, pp. 468–469]

Was nun aber Ihren Beweis für 1) betrifft, so protestire ich zuvörderst gegen den Gebrauch einer unendlichen Grösse als einer *Vollendeten*, welcher in der Mathematik niemals erlaubt ist. Das Unendliche ist nur eine Façon de parler, indem man eigentlich von Grenzen spricht, denen gewisse Verhältnisse so nahe kommen als man will, während anderen ohne Einschränkung zu wachsen verstattet ist. [Gauss to Schumacher, 12 July 1831]

Gerade bei dieser Auffassung erscheint es als etwas Selbstverständliches und durchaus nicht Neues, daß jeder auch noch so fern liegende Satz der Algebra und höheren Analysis sich als ein Satz über die natürlichen Zahlen aussprechen läßt, eine Behauptung, die ich auch wiederholt aus dem Munde von Dirichlet gehört habe. [**1**, p. 338]

Die obigen Erwägungen stehen, wie mir scheint, der Einführung jener *Dedekind*'schen Begriffsbildungen wie „Modul", „Ideal" u. s. w. entgegen; ebenso auch der Einführung der verschiedenen Begriffsbildungen, mit Hülfe deren in neuerer Zeit vielfach (zuerst wohl von *Heine*) versucht worden ist, das „Irrationale" ganz allgemein zu fassen und zu begründen. Selbst der *allgemeine* Begriff einer unendlichen Reihe, z. B. einer solchen, die nach bestimmten Potenzen von Variabeln fortschreitet, ist meines Erachtens nur mit dem Vorbehalte zulässig, dass in jedem speciellen Falle auf Grund des arithmetischen Bildungsgesetzes der Glieder (oder der Coefficienten), ähnlich wie oben, gewisse Voraussetzungen als erfüllt nachgewiesen werden, welche die Reihen wie endliche Ausdrücke anzuwenden gestatten, und welche also das Hinausgehen über den Begriff einer *endlichen* Reihe eigentlich unnöthig machen. [**4**, Vol. 3, p. 156]

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Auf welche Weise diese Bestimmtheit zustande kommt, und ob wir einen Weg kennen, um hierüber zu entscheiden, ist für alles Folgende gänzlich gleichgültig; die zu entwickelnden allgemeinen Gesetze hängen davon gar nicht ab, sie gelten unter allen Umständen. Ich erwähne dies ausdrücklich, weil Herr Kronecker vor kurzem (im Band 99 des Journals für Mathematik, S. 334 bis 336) der freien Begriffsbildung in der Mathematik gewisse Beschränkungen hat auferlegen wollen, die ich nicht als berechtigt anerkenne; näher hierauf einsugehen, erscheint aber erst dann geboten, wenn der ausgezeichnete Mathematiker seine Gründe für die Notwendigkeit oder auch nur die Zweckmäßigkeit dieser Beschränkungen veröffentlicht haben wird. [**1**, p. 345]

Ich bin stets weit davon entfernt gewesen, meine aurea mediocritas, deren Stärke nur in hartnäckiger Ausdauer liegt, mit Kronecker's hervorragender Begabung vergleichen zu wollen ... . [**2**, p. 282]

... wir denken uns alle ganzen Zahlen des Körpers von der Gestalt ... berechnet ... ; wir können annehmen, daß ... [the first item in the list is one in which a certain term is the greatest common divisor of the corresponding terms in the other items in the list]. [Hilbert, *Zahlbericht*, §3]

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This list of works is merely a key to the above quotations, not a bibliography. For a bibliography, see my paper [**3**]. A much more complete bibliography appears with the edition of Dedekind's "Bunte Bemerkungen zu Kronecker's *Grundzüge*", edited by O. Neumann, W. Purkert, and myself, in the *Archive for History of Exact Sciences*, 27 (1982), 49–85.

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Courant Institute of Mathematical Sciences,
New York University,
New York, N.Y. 10012,
U.S.A.

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