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Dynamical Systems in Categories

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Abstract In this article we establish a bridge between dynamical systems, including topological and measurable dynamical systems as well as continuous skew product flows and nonautonomous dynamical systems; and coalgebras in categories having all finite products. We introduce a straightforward unifying definition of abstract dynamical system on finite product categories. Furthermore, we prove that such systems are in a unique correspondence with monadic algebras whose signature functor takes products with the time space. We discuss that the categories of topological spaces, metrisable and uniformisable spaces have exponential objects w.r.t. locally compact Hausdorff, strongly σ-compact or arbitrary time spaces as exponents, respectively. Exploiting the adjunction between taking products and exponential objects, we demonstrate a one-to-one correspondence between monadic

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algebras (given by dynamical systems) for the left-adjoint functor and comonadic coalgebras for the other. This, finally, provides a new, alternative perspective on dynamical systems.

Keywords Dynamical system · Theory of systems · Finite product category · Coalgebra

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1 Introduction

Dynamical behaviour of state based systems is a central research topic common to both computer science and dynamical systems theory. The former discipline studies such structures in a multitude of manifestations, e.g. as infinite state transition systems [30, 34, 41–43], Kripke structures [13, 26], Petri nets [20], event systems [14, 44], finite state machines, various kinds of automata [16, 17, 25, 28, 35, 39], Turing machines [29], etc.; the latter one devotes its main attention to the understanding and the description of different facets, such as, for instance, long-time behaviour, of complex dynamical systems on topological, metric, measurable or probability spaces (see e.g. [2, 3, 7, 23, 40]).

A problem common to both fields is to describe the transition of states, which may or may not be deterministic. For that the two fields have developed specific concepts and methods. Dynamical systems theory, for example, applies ideas from topology, functional analysis, measure theory or differential equations. Computer science exploits, inter alia, concepts from the theory of formal languages or various modal and timed logics to describe dynamical behaviour.

Already during the 1970ies the relationship between automata theory and linear control systems has been studied [5, 6] in category theoretic language. Later, Rutten in [36] proposed the theory of coalgebra, a branch of applied category theory already used for a uniform treatment of different sorts of state based systems in programming semantics, to investigate techniques and concepts of computer science, as well as dynamical systems theory, in a common setting.

In this paper we follow Rutten's suggestion and explore which types of dynamical systems are suitable to be modelled as coalgebras in appropriate categories. For those we give a detailed account of the translation process, involving certain monadic algebras as an intermediate step. Finally, we find out that our construction can be regarded as an instance of a general category theoretic result on the relationship of algebras and coalgebras. In the end the coalgebraic point of view offers one free parameter: the signature. With only small modifications dynamical behaviour containing observations or non-determinism can be described. Making such variability available for the study of classical dynamical systems constitutes one motivation for the origination of this paper, which is largely based on a more elaborate version [9].

The structure of the text is as follows: In Section 2 we gather basic concepts from topology, measure theory and category theory, as well as a number of variants of dynamical systems appearing in the literature. In Section 3 we pursue a straightforward modelling of these existing notions in categories with finite products. In this context we observe a general connection between nonautonomous dynamical systems and dynamical systems on product spaces. In Section 4, finally, we translate the coined definitions into the language of coalgebra.

In this respect we first exhibit a connection to monadic algebras w.r.t. an endo-functor that takes products with a time space (Sections 4.1 and 4.2).

Subsequently, we verify for our main examples that the mentioned endo-functor fulfils the property of being a left-adjoint. Based on this general assumption, we demonstrate how to transform monadic algebras in a one-to-one fashion into comonadic coalgebras. Thereby we exhibit the particular transformation of monadic algebras arising from dynamical systems into coalgebras as a special case of a well-understood abstract result in category theory: if F, G is a pair of endo-functors where F is left- and G is right-adjoint, then monads for F and comonads for G, as well as monadic F-algebras and comonadic G-coalgebras, are in bijective correspondence (cf. Propositions 4.10, 4.11 and 4.12).

Due to the presence of adjointness, for the specific structures arising from dynamical systems it makes no difference if they are considered as algebras or as coalgebras. However, from the coalgebraic understanding of transition systems in computer science, there are coalgebras known that look quite similar to those stemming from our construction, yet which may fail to satisfy the adjointness condition. Thus, they lack a corresponding equivalent on the side of algebras, i.e. a definition in the standard sense of dynamical systems, but they still represent dynamical behaviour. We pose it as a problem for future investigations to discover further all benefits coming from the realm of coalgebra in the context of, e.g., topological or symbolic dynamics.

2 Preliminaries and Notation

We start with recalling standard notions from topology and measure theory, then we introduce fundamental concepts from category theory needed for abstractly modelling dynamical systems. To enrich our approach with examples, we finally present key definitions from the wide-spread field of dynamical systems.

To summarise some basic notation, we write $\emptyset$ for the *empty set*, and $\mathfrak{P}(X)$ for the *powerset* of some set X. Moreover, we use $X \subseteq Y$ to express *set inclusion*, as opposed to $X \subset Y$ for *proper set inclusion*. If $f : X \to Y$ is a function from X to Y and $U \subseteq X$ is a subset, we write $f[U]$ for the *image* of U under f. Furthermore, we use $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ to denote the sets of *natural numbers including zero*, *integers* and *reals*, respectively. For $\mathbb{T} \in \{\mathbb{Z}, \mathbb{R}\}$ we abbreviate by $\mathbb{T}_{\geq 0}$ and $\mathbb{T}_{\leq 0}$ the respective non-negative and non-positive numbers.

2.1 Preliminaries Related to Topology and Measure Theory

Dynamical systems in topological spaces constitute an example of special importance in the following section. We therefore begin by recollecting some basic notions from topology.

We use the following notions as it is standard in topology and without further explanation: *topological space*, *base*, *subbase* and *neighbourhood base* of a topology, *product space*, *neighbourhood*, *interior*, *continuity*, *homeomorphism*, separation axioms such as the Hausdorff property, and *compactness*. Moreover, we shall also speak of *metric*, *uniform*, *metrisable* and *uniformisable* spaces without explicit introduction. For a topological space $\boldsymbol{X}$, we write $\boldsymbol{K}(\boldsymbol{X})$ for the *set of all compact subsets of* $\boldsymbol{X}$ and $\boldsymbol{C}(\boldsymbol{X}, \boldsymbol{Y})$ for the *set of all continuous mappings between* $\boldsymbol{X}$ and another topological space $\boldsymbol{Y}$. Further, we denote the *interior* of a subset $V \subseteq X$ by $\mathrm{int}_{\mathbf{x}}(V)$. Moreover, in this article we call a space $\boldsymbol{X}$ *locally compact* if for each of its points, it possesses a neighbourhood base consisting of compact

sets. A topological space $\boldsymbol{X}$ is said to be *σ-compact* if it can be written as $X = \bigcup_{n\in\mathbb{N}} K_n$ where $K_n \in K(\boldsymbol{X})$ for $n \in \mathbb{N}$ (cf. [46, 17.I, p. 126]). We say that $\boldsymbol{X}$ is *strongly σ-compact* (cp. [15, Definition 6.11.14, p. 346]) if it admits a *countable exhaustion by compact subsets*. That is, there exists a sequence $(K_n)_{n\in\mathbb{N}}$ in $K(X)^{\mathbb{N}}$ such that $K_n \subseteq \mathrm{int}_{\mathbf{x}}(K_{n+1})$ for each $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} K_n$.

A strongly σ-compact Hausdorff space is necessarily locally compact. In fact, the following lemma holds (we refer to [33, Satz 8.19(b), p. 111] or [4, Corollary 2.77, p. 58] for a proof):

Lemma 2.1 *If $\boldsymbol{X}$ is a Hausdorff space, it is strongly σ-compact if and only if it is locally compact and σ-compact.*

Another important class of structures for this article are *measurable spaces*, i.e. pairs (X, Σ) where X is a set and Σ is a *σ-algebra* on X. In this context we shall assume the following basic measure theoretic concepts to be clear: *product space*, *generated σ-algebra*, *Borel σ-algebra* generated by the topology of a topological space, *measurable map*, *measure*, *measure preserving map*, *push-forward measure*, *probability measure*, *Dirac measure* (i.e. measure centred in one point), *measure space*, and *probability space*.

2.2 Basic Notions from Category Theory

Since category theory may be less familiar from a dynamical systems perspective, we give a slightly more exhaustive introduction to this field. Nevertheless, we assume the reader to be familiar with the following notions: *category*, (*full*) *subcategory*, *monomorphism*, *epimorphism*, *isomorphism*, *functor*, *endo-functor*, *natural transformation* and *natural equivalence*. Further information may be found in [1, 8, 27] or Section 2.2 of [9].

To begin, we summarise some notation: if $\mathcal{C}$ is a category and A, B are objects, the set of morphisms from A to B is denoted by $\mathcal{C}(A, B)$ or $\mathrm{Hom}(A, B)$ if $\mathcal{C}$ is clear from the context. Morphisms $f \in \mathcal{C}(A, B)$ are also written as $A \xrightarrow{f} B$, *identical morphisms* as 1_A for A in $\mathcal{C}$, and *composition* by juxtaposition, i.e. $fg \in \mathcal{C}(A, C)$ for $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$ and A, B, C in $\mathcal{C}$. We notate application of functors to objects and morphisms from the left and usually omit brackets if only one symbol follows. This is to say, we use FGA and FGf to mean $F(G(A))$ and $F(G(f))$, respectively, and FA, Ff instead of $F(A)$, $F(f)$ for functors F, G, objects A and morphisms f. Moreover, for a category $\mathcal{C}$ we collect all its endo-functors in the class $\mathrm{End}\,\mathcal{C}$.

From every category $\mathcal{C}$ one can naturally derive the *opposite category* $\mathcal{C}^\partial$ by keeping the objects, reversing the direction of morphisms and swapping the order of composition. This allows for a strong duality principle via instantiation of definitions or statements in $\mathcal{C}^\partial$ and reinterpreting the meaning in $\mathcal{C}$.

All categories we use as examples in this article are *concrete* over $\mathcal{S}et$, the *category of sets*, having sets as objects, functions as morphisms, identical functions as identical morphisms and ordinary function composition as composition. Concreteness of the other examples means that the objects are sets with some additional structure, the morphisms are functions being in a certain sense structure preserving, and the identical morphisms and the composition are inherited from $\mathcal{S}et$.

For example, in the category $\mathcal{T}op$ of topological spaces, the additional structure is a topology, and structure preserving means continuity. Likewise, the category $\mathcal{M}eas\mathrm{rbl}$ of measurable spaces has σ-algebras as additional structure and measurability as requirement

for the morphisms. Instead of topological spaces we may also use the category $\mathcal{M}et_0$ of metric spaces with continuous mappings or that of metrisable spaces $\mathcal{M}et$, which is the corresponding full subcategory of $\mathcal{T}op$. Similarly, we can restrict the structure of $\mathcal{T}op$ to all locally compact Hausdorff spaces, yielding the category $\mathcal{LC}omp$. Another example are uniform spaces, which, together with continuous maps, form the category $\mathcal{U}nif_0$. As with metric spaces, we may also look at the underlying topological spaces of these, which yields the category $\mathcal{U}nif$ of uniformisable topological spaces, being again a full subcategory of $\mathcal{T}op$.

It is not hard to see that in the categories $\mathcal{S}et$, $\mathcal{T}op$ and $\mathcal{M}easrbl$ the monomorphisms and epimorphisms are exactly those morphisms having underlying injective and surjective maps, respectively.

An object I in a category $\mathcal{C}$ is said to be *terminal* if for every other object X of $\mathcal{C}$ there exists *exactly* one morphism from X to I. Assuming that the terminal object is fixed, we denote this unique morphism here by $X \xrightarrow{!_X} I$.

In $\mathcal{S}et$, $\mathcal{T}op$ and $\mathcal{M}easrbl$ terminal objects are given by equipping the one-element set with the obvious extra structure.

Terminal objects can also be seen as products with no factors. As usual an object P in a category $\mathcal{C}$ together with a family of morphisms $(P \xrightarrow{p_i} X_i)_{i\in I}$ is called a *product* of a set-indexed family $(X_i)_{i\in I}$ of objects, if for any object Q of $\mathcal{C}$ with morphisms $(Q \xrightarrow{q_i} X_i)_{i\in I}$ there exists *exactly* one morphism $Q \xrightarrow{h} P$ such that $q_i = hp_i$ holds for all $i \in I$. This unique morphism h is called $tupling$ of the morphisms $(q_i)_{i\in I}$ and denoted here by $\langle q_i\rangle_{i\in I}$. The members of the family $(P \xrightarrow{p_i} X_i)_{i\in I}$ are the *projection morphisms* or simply *projections*. In the case of finite index sets $I = \{\nu_1, \ldots, \nu_n\}$, we also write $X_{\nu_1} \times \cdots \times X_{\nu_n}$ instead of $\prod_{i=1}^{n} X_{\nu_i}$.

In the familiar categories we mentioned earlier, products exist and are given by the constructions that one expects. In $\mathcal{S}et$, $\mathcal{T}op$ and $\mathcal{M}easrbl$ the carrier set of a product can be given by the Cartesian product $\prod_{i\in I} X_i$ of the underlying sets $(X_i)_{i\in I}$ of the factor spaces, and the projections are $p_i\colon \prod_{j\in I} X_j \to X_i$, $(x_j)_{j\in I} \mapsto x_i$. The tupling of maps $q_i\colon Q \to X_i$, $i \in I$ underlying certain morphisms, is then given by $h(q) := (q_i(q))_{i\in I}$ for $q \in Q$. Moreover, in $\mathcal{T}op$ and $\mathcal{M}easrbl$ the Cartesian product of the carrier sets is equipped with the product topology and the product σ-algebra, respectively. The following remark about products will be helpful in later proofs.

Remark 2.2 Suppose that I is any index set, A and $(P_i)_{i\in I}$ are objects in a category $\mathcal{C}$ such that a product $\prod_{i\in I} P_i$ with projections $\left(\prod_{j\in I} P_j \xrightarrow{p_i} P_i\right)_{i\in I}$ exists. Then for any two morphisms $A \xrightarrow{f,g} \prod_{i\in I} P_i$, checking the equality $f = g$ is equivalent to verifying $fp_i = gp_i$ for all $i \in I$.

We say that a category $\mathcal{C}$ *has binary products* if for any two objects X, Y from $\mathcal{C}$ a product $X \times Y$ (with corresponding projection morphisms) exists. Moreover, we speak of a category *having finite products* (or of a *finite product category*) if it has binary products and a terminal object.

It is clear that in a finite product category, indeed, whenever $X_1, \ldots, X_n$ are $n \geq 0$ objects, their product $(\cdots((X_1 \times X_2) \times X_3) \times \cdots) \times X_n$ exists. We evidently have $\mathcal{S}et$, $\mathcal{T}op$ and $\mathcal{M}easrbl$ as examples of such categories.

Next, we present three functors that become relevant in Section 3 et seq.

Example 2.3 (a) Let $\mathcal{C}$ have binary products. Since products are unique only up to isomorphism, we consider now one particular choice for $X \times Y$ (together with corresponding projection morphisms) as fixed for any X, Y in $\mathcal{C}$. Furthermore, for objects X, Y, U, V and $\mathcal{C}$-morphisms $X \xrightarrow{f} U$ and $Y \xrightarrow{g} V$, we define $X \times Y \xrightarrow{f \times g} U \times V$ by $f \times g := \langle \mathrm{pr}_X\, f, \mathrm{pr}_Y\, g \rangle$ where $X \times Y \xrightarrow{\mathrm{pr}_X} X$ and $X \times Y \xrightarrow{\mathrm{pr}_Y} Y$ are the projection morphisms coming with $X \times Y$. It is easy to check that $\mathcal{C} \times \mathcal{C} \xrightarrow{-_1 \times -_2} \mathcal{C}$ is functorial in both arguments, i.e. a *bifunctor*.

(b) In a category $\mathcal{C}$ having binary products, one may substitute the constant endo-functor yielding some object T into the first coordinate of the bifunctor $\times$ given in (a). This yields an endo-functor $\mathcal{C} \xrightarrow{T \times -} \mathcal{C}$.

(c) The last example is also a bifunctor: if $\mathcal{C}$ is any category, the *hom-functor* is a bifunctor from $\mathcal{C}^\partial \times \mathcal{C}$ to $\mathcal{S}et$. For every pair of objects A, B from $\mathcal{C}$, it associates the set of morphisms $\mathcal{C}(A, B)$. Moreover, if C, D are further objects of $\mathcal{C}$ and $A \xrightarrow{f} C$ is a morphism in $\mathcal{C}^\partial$ and $B \xrightarrow{g} D$ is one in $\mathcal{C}$, then $\mathcal{C}(f, g) : \mathcal{C}(A, B) \to \mathcal{C}(C, D)$ is given by $\mathcal{C}(f, g)(h) := fhg$ for any morphism $h \in \mathcal{C}(A, B)$. Sometimes, also $\mathrm{Hom}(-, -)$ is written instead of $\mathcal{C}(-, -)$.

Remark 2.4 Assume that $\mathcal{C}$ is a finite product category and I a fixed terminal object. As I is the zeroth power of any object X from $\mathcal{C}$, we write $\mathcal{C} \xrightarrow{-^0} \mathcal{C}$ for the constant endo-functor with value $I = X^0$. It maps any morphism $X \xrightarrow{f} Y$ to $f^0 = 1_I = !_I$. The unique morphisms $X \xrightarrow{!_X} I$ can then be grouped together in a natural transformation $1_\mathcal{C} \xrightarrow{!} -^0$.

Besides, whenever we write $X \times Y$ in a finite product category $\mathcal{C}$, we agree to mean by this the result of the bifunctor $\mathcal{C} \times \mathcal{C} \xrightarrow{-_1 \times -_2} \mathcal{C}$ given by one particular (implicit or explicit) choice of the product (see Example 2.3(a)).[1]

Next, we recollect the notion of adjoint functors, which is central to Section 4.3 et seqq.

Definition 2.5 Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathcal{C} \xrightarrow{F} \mathcal{D}$, $\mathcal{D} \xrightarrow{G} \mathcal{C}$ be functors. One says that F is *left-adjoint* to G (and that G is *right-adjoint* to F), written as $F \dashv G$, if there exist natural transformations $1_\mathcal{C} \xrightarrow{\vartheta} GF$ (called *unit* of the adjunction) and $FG \xrightarrow{\varepsilon} 1_\mathcal{D}$ (called *co-unit* of the adjunction) such that for all objects X from $\mathcal{C}$ and Y from $\mathcal{D}$ the following two axioms, known as *co-unit-unit equations*, hold:

$$1_{FX} = F\vartheta_X \varepsilon_{FX} \qquad\qquad 1_{GY} = \vartheta_{GY} G\varepsilon_Y.$$

The following characterisation of adjointness is well-known.

Proposition 2.6 *For categories $\mathcal{C}$, $\mathcal{D}$ and functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} \mathcal{C}$ the following are equivalent:*

(a) $F \dashv G$

[1] This choice is naturally accompanied by a choice of projections for each product. However, instead of capturing these by two additional natural transformations, we leave them implicit and use ad-hoc notation as needed.

(b) *There exists a natural equivalence between the hom-bifunctors* $\mathcal{D}(F,-)$ *and* $\mathcal{C}(-,G)$.

More precisely, if $1_{\mathcal{C}} \overset{\vartheta}{\to} GF$ *and* $FG \overset{\varepsilon}{\to} 1_{\mathcal{D}}$ *are the unit and co-unit of the adjunction* $F \dashv G$, *then one defines the natural equivalence* $\mathcal{D}(F,-) \overset{\nu}{\to} \mathcal{C}(-,G)$ *by* $\nu_{X,Y}\left(FX \overset{g}{\to} Y\right) := \vartheta_X Gg$ *for* X *in* $\mathcal{C}$, Y *in* $\mathcal{D}$ *and* $g \in \mathcal{D}(FX,Y)$. *Its inverse* $\nu^{-1}_{X,Y}$ *is given by* $\nu^{-1}_{X,Y}\left(X \overset{f}{\to} GY\right) := Ff\varepsilon_Y$ *for* X *in* $\mathcal{C}$, Y *in* $\mathcal{D}$ *and* $f \in \mathcal{C}(X,GY)$.

Conversely, if the natural equivalence $\mathcal{D}(F,-) \overset{\nu}{\to} \mathcal{C}(-,G)$ *is given, then one puts* $\vartheta_X := \nu_{X,FX}(1_{FX})$ *for* X *in* $\mathcal{C}$ *and* $\varepsilon_Y := \nu^{-1}_{GY,Y}(1_{GY})$ *for* Y *in* $\mathcal{D}$.

If $\mathcal{C}$ is a category and $F \in \mathrm{End}\mathcal{C}$, then, as usual, an *algebra for the endo-functor* F (or an F-algebra) is a pair (A,φ), where A is an object and $\varphi \in \mathcal{C}(FA,A)$ a morphism. The dual concept is that of an F-coalgebra, i.e. any pair (A,ψ) where $\psi \in \mathcal{C}(A,FA)$.

To add some more structure to algebras and coalgebras, one can use monads and comonads, respectively.

Definition 2.7 (1) A triple (T,δ,η), in which $\mathcal{C} \overset{T}{\to} \mathcal{C}$ is an endo-functor, and $TT \overset{\delta}{\to} T$ and $1_{\mathcal{C}} \overset{\eta}{\to} T$ are natural transformations, is called a *monad* (originally called *standard construction*, [22], later also *triple*, see e.g. [19]) if the following two diagrams commute

$$\begin{array}{ccc} T(T(TX)) & \xrightarrow{T(\delta_X)} & T(TX) \\ {\scriptstyle \delta_{TX}}\downarrow & & \downarrow{\scriptstyle \delta_X} \\ T(TX) & \xrightarrow{\delta_X} & TX \end{array} \tag{2.1a}$$

$$\begin{array}{ccccc} TX & & \xrightarrow{\eta_{TX}} & & TTX \\ & {\scriptstyle T\eta_X}\searrow & & \swarrow{\scriptstyle 1_{TX}} & \downarrow{\scriptstyle \delta_X} \\ & & TTX & \xrightarrow{\delta_X} & TX \end{array} \tag{2.1b}$$

for every object X in $\mathcal{C}$.

(2) The dual notion is that of a *comonad*, i.e. a triple (T,δ,η), where $\mathcal{C} \overset{T}{\to} \mathcal{C}$ is an endo-functor, and $T \overset{\delta}{\to} TT$ and $T \overset{\eta}{\to} 1_T$ are natural transformations satisfying that the diagrams

$$\begin{array}{ccc} T(T(TX)) & \xleftarrow{T(\delta_X)} & T(TX) \\ {\scriptstyle \delta_{TX}}\uparrow & & \uparrow{\scriptstyle \delta_X} \\ T(TX) & \xleftarrow{\delta_X} & TX \end{array} \tag{2.2a}$$

$$\begin{array}{ccccc} TX & & \xleftarrow{\eta_{TX}} & & TTX \\ & {\scriptstyle T\eta_X}\nwarrow & & \nearrow{\scriptstyle 1_{TX}} & \uparrow{\scriptstyle \delta_X} \\ & & TTX & \xleftarrow{\delta_X} & TX \end{array} \tag{2.2b}$$

commute for every X from $\mathcal{C}$.

Using two additional commutative diagrams, monads can encode extra information about algebras, such as, for instance, identities.

Definition 2.8 Let C be a category and $C \xrightarrow{T} C$ be an endo-functor.

(1) If (T, δ, η) is a monad, then a T-algebra $\left(A, TA \xrightarrow{\varphi} A\right)$ is said to be *monadic w.r.t.* (T, δ, η) (or a *T-algebra for the monad* (T, δ, η)) if the following two diagrams

$$\begin{array}{ccc} TTX & \xrightarrow{T\varphi} & TX \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \varphi} \\ TX & \xrightarrow{\varphi} & X \end{array} \tag{2.3a}$$

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle \varphi} \\ & & X \end{array} \tag{2.3b}$$

commute for every X in C.

(2) If (T, δ, η) is a comonad, then a T-coalgebra $\left(A, TA \xrightarrow{\varphi} A\right)$ is said to be *comonadic w.r.t.* (T, δ, η) (or a *T-coalgebra for the comonad* (T, δ, η)) if the following two diagrams

$$\begin{array}{ccc} TTX & \xleftarrow{T\varphi} & TX \\ {\scriptstyle \delta_X}\uparrow & & \uparrow{\scriptstyle \varphi} \\ TX & \xleftarrow{\varphi} & X \end{array} \tag{2.4a}$$

$$\begin{array}{ccc} X & \xleftarrow{\eta_X} & TX \\ & {\scriptstyle 1_X}\nwarrow & \uparrow{\scriptstyle \varphi} \\ & & X \end{array} \tag{2.4b}$$

commute for every X in C.

2.3 Classical Dynamical Systems Theory

Dynamical systems, in the broader sense, can be interpreted as actions of a time monoid $\mathbf{T}$ (occasionally also stronger axioms such as group actions are demanded) on a state space X. Typical monoids occurring in this context are the additive monoids on the sets $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$ (*continuous time*), $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_{\leq 0}$ (*discrete time*). Often also the state space carries some extra structure such as a topology, a uniformity, a metric, a differentiable structure, a σ-algebra, a measure etc. The actions are then required to be structure preserving w.r.t. X, i.e. continuous (if X is a topological, uniform or metric space, e.g. a subspace of $\mathbb{R}^n$), differentiable (if X is a geometric manifold), measurable (if X is a measurable space), measure preserving (if X is a measure space, in particular a probability space) etc. Accordingly, there is a large variety of literature studying different types of dynamical systems depending on the setting that is assumed for X and T.

Therefore, one of the aims of this paper is to give a definition of dynamical system (see Definition 3.4) that encompasses many of the competing notions that can be found throughout the literature. This is possible using the language of category theory. In this formulation we see that dynamical systems are in fact a special instance of monadic algebras, occurring in algebra and theoretical computer science.

The following informal definition of a dynamical system seems to be the core of all the different variations that one encounters. Given a monoid $\mathbf{T} = \langle T; +, 0\rangle$ and a mapping $\varphi\colon T \times X \to X$, we say that $(\mathbf{T}, \varphi)$ is a dynamical system provided the following compatibility conditions hold and all involved mappings are structure preserving w.r.t. the framework assumed for X and T: for all $x \in X$ we have

(1) $\varphi(0, x) = x$, (initial condition)
(2) $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$ for all $s, t \in T$. (semigroup property)

We remark that in case X and T are just sets, conditions (1) and (2) jointly express the fact that $\varphi\colon T \times X \to X$ is a *monoid action* of $\mathbf{T}$ on X.

In the next step we put this into a formal definition for the setting of topological spaces, being the basis for a straightforward generalisation to arbitrary abstract categories. Altogether, the case of dynamical systems in a topological environment will receive the highest attention among all types of dynamical systems considered in this paper. After that we collect further variants from the literature to outline the scope of our general modelling: we either use them, with marginal modifications, as examples, or we discuss why they are not fitting into our framework. In this regard, we first briefly focus on special cases, e.g. dynamical systems in metric spaces. Subsequently, we introduce measurable dynamical systems and discuss measure preserving systems. Then we consider nonautonomous dynamical systems and their continuous and measurable variants, generalising, for example, skew product flows.

We recall that for a topological space T, a monoid $\langle T;\ +, 0\rangle$ is called *topological monoid* if the addition operation $+\colon T \times T \to T$ is continuous w.r.t. the product topology. The constant $0\colon T^0 \to T$ is automatically continuous w.r.t. the unique topology on the one-element (terminal) topological space.

Definition 2.9 Let T, X be topological spaces. A *topological dynamical system over a monoid* is a triple $(\mathbf{T} = \langle T; +, 0\rangle, X, \varphi\colon T \times X \to X)$ where

(1) $\mathbf{T}$ is a *topological monoid*,
(2) $\varphi\colon T \times X \to X$ is a *topological monoid action*, i.e. it is continuous w.r.t. the product topology and satisfies the equalities (1) and (2) from above.

Obviously, Definition 2.9 is a slight generalisation of the notion of topological dynamical system as defined e.g. in [21, 31], where groups are used instead of monoids. Our basic definition of topological dynamical system over a monoid subsumes both existing notions, and it will be the starting point in Section 3.

Of course, the previous definition can also be given in prominent full subcategories of $\mathcal{T}op$, e.g. Hausdorff spaces, compact Hausdorff spaces, metrisable spaces etc. Then one requires that T and X belong to this subcategory and that all morphisms are continuous. In this sense, we can, for example, define the notions of *metric dynamical system*, *(compact) Hausdorff topological dynamical system* etc. in complete analogy to Definition 2.9.

Another variant of dynamical system, which comes up via iterating measurable maps in the same way as explained earlier for continuous maps, is a *measurable dynamical system*,

see e.g. [7, p. 536]. Preparing the following definition, we call a monoid $\langle T; +, 0\rangle$ on a measurable space T a *measurable monoid* if addition $+\colon T \times T \to T$ is measurable w.r.t. the product σ-algebra. Note that the constant $0\colon T^0 \to T$ is automatically measurable w.r.t. the unique full σ-algebra on the one-element (terminal) measurable space.

Definition 2.10 Let T, X be measurable spaces. A *measurable dynamical system over a monoid* is a triple $(\mathbf{T} = \langle T; +, 0\rangle, X, \varphi\colon T \times X \to X)$ where

(1) $\mathbf{T}$ is a *measurable monoid*,
(2) $\varphi\colon T \times X \to X$ is a *measurable monoid action*, i.e. it is measurable w.r.t. the product σ-algebra and satisfies the equalities (1) and (2) from above.

Again, the prototype that motivated Definition 2.10 (cp. [7]) can clearly be seen as a special case where $\mathbf{T}$ is one of the additive monoids $\{\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{\leq 0}, \mathbb{Z}, \mathbb{N}, \mathbb{Z}_{\leq 0}\}$.

In [7, p. 537] the yet stronger notion of *measure preserving dynamical system* (or *metric dynamical system*, a historical term that we wish to avoid for clarity) is defined. We recall that a self-map $f\colon X \to X$ of a measure space X carrying a measure μ is measure preserving if $\mu \circ f^{-1} = \mu$.

Definition 2.11 Suppose that $\mathbf{T} = \langle T; +, 0\rangle$ is a measurable monoid and $\mathbb{X}$ is a measure space with measure μ and underlying measurable space X. A measurable dynamical system $(\mathbf{T}, X, \varphi\colon T \times X \to X)$ is called *measure preserving* if for every $t \in T$ the self-map $\varphi(t, \cdot)\colon \mathbb{X} \to \mathbb{X}$ is measure preserving.

Unfortunately, this definition is not suitable to be modelled within just one category: the requirement on the self-mappings $\varphi(t, \cdot)\colon \mathbb{X} \to \mathbb{X}$ to be measure preserving suggests to choose the category of measure spaces together with measure preserving mappings as morphisms. For a categorical modelling it would be desirable if not only the individual mappings $(\varphi(t, \cdot))_{t \in T}$ were measure preserving, but if the evolution rule $\varphi\colon T \times X \to X$ as a whole were measure preserving w.r.t. the product measure on $T \times X$ given by the measure μ on X and some measure m on T. Solving for simple cases (e.g. $\mathbf{T}$ being the reals with addition, and $\mathbb{X}$ being the reals with Lebesgue measure) the question if such a measure m exists at all, shows that it will often be uniquely determined by μ if it exists. On the other hand, for $\mathbf{T}$ to be in accordance with the category chosen for $\mathbb{X}$, the measure on T should also be such that $\mathbf{T}$ is a measure preserving monoid, i.e. addition and, moreover, $0\colon T^0 \to T$ have to be measure preserving, the latter w.r.t. the terminal measure space T^0. This means that the measure m on T is the Dirac measure centred in 0, prescribing a possibly different measure space on T than the requirement coming from φ. In view of these arguments it seems that a reasonable categorical modelling of measure preserving dynamical systems should be done with the option to choose X and T from different categories. As this contradicts our original intention, we shall not consider measure preserving dynamical systems further in this article.

A different sort of dynamical system arises when studying dynamical behaviour of a system under an external influence, which itself is modelled by a dynamical system. This kind of constellation entails a so-called *skew product*; its study is subject to the field of nonautonomous dynamics.

In this context we refer to the article [11], where the concepts of *nonautonomous dynamical system (NDS)* and *continuous skew product flow* are defined. While the notion of continuous skew product flow is suitable for categorical modelling (cf. Section 3.3), nonautonomous dynamical systems as defined in [11] do not fit into an easy generalisation using

just one category. With the following definition we remove the asymmetry between sets and metric spaces from [11, Definition 2.1] and emphasise its purely algebraic, non-topological aspect.

Definition 2.12 For monoids $\mathbf{S}$ and $\mathbf{T}$, such that $\langle S; +, 0\rangle = \mathbf{S} \leq \mathbf{T}$ is a submonoid, and sets X and Y, we call a pair $(\theta\colon T \times X \to X, \varphi\colon S \times X \times Y \to Y)$ *nonautonomous dynamical system with times* $\mathbf{S} \leq \mathbf{T}$ *on* Y *with base* X if

(1) $(\mathbf{T}, X, \theta)$ is a (non-topological) dynamical system over the monoid $\mathbf{T}$ (i.e. θ is a left-monoid action of $\mathbf{T}$ on X), called *driving system*, and
(2) φ is a *cocycle over* θ, i.e. for all $s, t \in S$, $x \in X$ and $y \in Y$ we have the equations $\varphi(0, x, y) = y$ and $\varphi(t + s, x, y) = \varphi(t, \theta(s, x), \varphi(s, x, y))$, known as *cocycle property*.

It is not only with respect to continuity of the cocycle, where our Definition 2.12 slightly differs from the one given in [11]. Also w.r.t. to the time spaces, we are marginally more general. The only time monoids $\mathbf{T}$ considered in [11] are $\mathbb{R}$ (continuous time) and $\mathbb{Z}$ (discrete time), and the submonoids $\mathbf{S}$ were always chosen as the non-negative points in time, i.e. $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$, respectively. Since we assume an arbitrary monoid $\mathbf{T}$ in our definition, which does not need to have a compatible order relation, we can in general not speak of 'non-negative' points in time. Hence, we replace the role of this special submonoid by allowing any submonoid $\mathbf{S}$.

The following definition is the companion to Definition 2.12, focussing on the continuous aspect of nonautonomous dynamics.

Definition 2.13 Suppose $\mathbf{S} = \langle S; +, 0\rangle$ and $\mathbf{T}$ are topological monoids, $\mathbf{S} \leq \mathbf{T}$ being a submonoid, and X and Y be topological spaces. A pair (θ, φ) of mappings $\theta\colon T \times X \to X$ and $\varphi\colon S \times X \times Y \to Y$ is called *continuous skew product system*[2] *with times* $\mathbf{S} \leq \mathbf{T}$ *on* Y *with base* X *if*

(1) $(\mathbf{T}, X, \theta)$ is a topological dynamical system over the monoid $\mathbf{T}$ (as in Definition 2.9), called *driving system*, and
(2) φ is a *continuous cocycle over* θ, that is, φ is continuous and fulfils the cocycle property as introduced in Definition 2.12.

Comparing again to [11], we have generalised the role of a continuous time monoid $\langle \mathbb{R}; +, 0\rangle$ with its submonoid $\langle \mathbb{R}_{\geq 0}; +, 0\rangle$ and that of a discrete time monoid $\langle \mathbb{Z}; +, 0\rangle$ with the submonoid $\langle \mathbb{N}; +, 0\rangle$, respectively, to any topological time monoid $\mathbf{T}$ with an arbitrary submonoid $\mathbf{S}$. Additionally, all spaces in Definition 2.6 of [11] were assumed to be metric spaces. However, since continuity is a purely topological concept, and we allow arbitrary continuous monoids, and groups in particular, for $\mathbf{T}$, it seems natural to formulate the whole definition within the setting of topological spaces. In this way we make sure not to lose interesting topological groups or monoids that are not metrisable. For instance, the topological monoid $\mathbf{T} := \langle \mathbb{R}; +, 0\rangle$ consisting of the real line with the usual addition operation and the Sorgenfrey topology, fails to be metrisable. This specific choice for $\mathbf{T}$ also motivates not

[2] We have replaced the word 'flow' by 'system' since 'flow' explicitly refers to $\mathbf{T}$ being the reals with addition, and we allow arbitrary topological monoids, instead.

to restrict our Definitions 2.9 and 2.13 to topological groups because it is an example of a topological monoid that is not a topological group (see e.g. [32, Example 3, p. 799]).

A third important notion in nonautonomous dynamics is that of *random dynamical system (RDS)*. A definition can, for instance, be found in [11, Definition 2.3] and in [7, Definition 1.1.1]. Rephrasing Arnold's definition in our language, an RDS is an NDS (θ, φ), where φ is measurable, and the driving system θ is measure preserving w.r.t. a probability space on the base space X as in Definition 2.11. Berger and Siegmund's definition additionally requires that θ be ergodic. Due to the concerns in connection with modelling measure preserving dynamical systems, we shall not consider RDS in this article.

However, relaxing the conditions on the driving system, we can introduce the following notion of *measurable nonautonomous dynamical system*.

Definition 2.14 Given measurable monoids $\mathbf{S}$ and $\mathbf{T}$ such that $\mathbf{S} \leq \mathbf{T}$ and measurable spaces X, Y, a pair $(\theta\colon T \times X \to X, \varphi\colon S \times X \times Y \to Y)$ is called *measurable nonautonomous dynamical system with times* $\mathbf{S} \leq \mathbf{T}$ *on* Y *with base* X if

(1) $(\mathbf{T}, X, \theta)$ is a measurable dynamical system over the monoid $\mathbf{T}$ (as in Definition 2.10), called *driving system*, and
(2) φ is a *measurable cocycle over* θ, that is, φ is measurable and fulfils the cocycle property as introduced in Definition 2.12.

3 Dynamical Systems in Abstract Categories

To model dynamical systems as coalgebras in abstract categories we are going to pursue the following strategy. We shall move all the additional conditions that mappings involved in the definition of a dynamical system have to satisfy (e.g. continuity) into the definition of a suitably chosen category $\mathcal{C}$. For the examples we shall primarily be interested in within this paper, this will mostly be one of the categories $\mathcal{Top}$, $\mathcal{Measrbl}$ and $\mathcal{Set}$ as introduced in Section 2.2.

Then we are going to explore what the two conditions (1) and (2) mentioned at the beginning of Section 2.3 mean in our abstract context. To this end we shall first define straightforward generalisations of monoids and monoid actions in abstract categories. This will allow us to state a general definition of dynamical system, which will comprise different variations of dynamical systems found in the literature. Further, we shall generalise the definition of nonautonomous dynamical system and show that on the abstract level these can be understood as a special instance of our general category theoretic formulation of dynamical system.

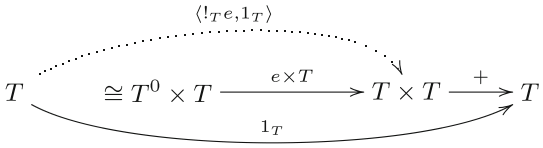
In our definitions and results we shall need certain requirements on the given category $\mathcal{C}$. For convenience we shall most frequently suppose that $\mathcal{C}$ is a finite product category. Regarding this assumption, we shall always consider a particular product construction to be fixed in advance as explained in Remark 2.4. Moreover, in some cases we are going to need that certain projection morphisms are epimorphisms. Concerning our main example categories $\mathcal{Set}$, $\mathcal{Top}$ and $\mathcal{Measrbl}$, this assumption is indeed a very mild condition.

3.1 Monoids and Monoid Actions in Abstract Categories

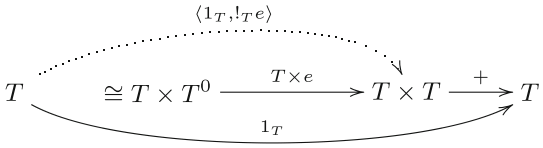
The purpose of this subsection is to lift the notions of monoid and monoid action to any abstract category.

Definition 3.1 In a finite product category C, we call every triple $(T, +, e)$ consisting of an object T of C and morphisms $T \times T \xrightarrow{+} T$ and $T^0 \xrightarrow{e} T$, a *C-monoid setting*.

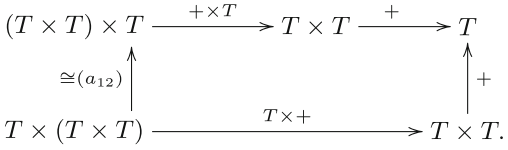
A C-monoid setting $(T, +, e)$ is called a *C-monoid* if the following three diagrams commute:



(3.1a)



(3.1b)



(3.1c)

The dotted arrows have just been added to make the morphism explicit and add nothing to the commutativity condition. The isomorphism a_{12} is the canonical isomorphism given by the appropriate tupling of projections.

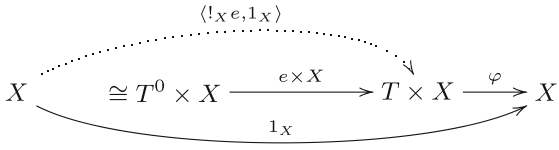
To model nonautonomous dynamics we need to generalise the concept of submonoid. The category theoretical answer to this task is, of course, to use embeddings, which are a certain kind of monic (homo)morphisms. However, category theory does not give a satisfactory one-and-only answer to the question what an embedding should be. There are various concrete notions of embedding occurring in specific categories, and most of them represent a category theoretic concept, in fact a certain type of monomorphism. However, not all of these specific concepts can be modelled by the same kind of monomorphism; sometimes just monomorphisms are the right choice (e.g. in the categories of sets, semigroups or rings, respectively), sometimes additional properties like extremality, strongness or regularity of the monomorphism (e.g. in the categories of topological, Hausdorff or metric spaces, respectively, see [1] for further details) need to be assumed. So each category comes with its own natural concept of embedding, which is why we do not define this term in abstract categories apart from requiring that it must be a monomorphism.

Definition 3.2 Suppose that $\mathbf{S} = \langle S; +^{\mathbf{S}}, e^{\mathbf{S}} \rangle$ and $\mathbf{T} = \langle T; +^{\mathbf{T}}, e^{\mathbf{T}} \rangle$ are C-monoid settings in a finite product category C. We call a morphism $S \xrightarrow{h} T$

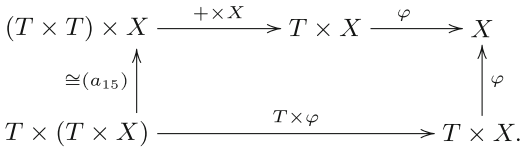
(1) a *homomorphism* between $\mathbf{S}$ and $\mathbf{T}$ if $(h \times h) +^{\mathbf{T}} = +^{\mathbf{S}} h$ and $h^0 e^{\mathbf{T}} = e^{\mathbf{S}} h$ hold.
(2) an *embedding* if it is a homomorphism and an embedding in C. We denote this by $\mathbf{S} \xhookrightarrow{h} \mathbf{T}$ or simply $\mathbf{S} \hookrightarrow \mathbf{T}$ if the particular embedding morphism is not interesting or is merely given by an existence condition. We then say that $\mathbf{S}$ *embeds as a submonoid setting* into $\mathbf{T}$.

Having dealt with the generalisation of monoids in abstract categories, we can now turn towards their actions.

Definition 3.3 Let $\mathcal{C}$ be a finite product category and $(T, +, e)$ be a $\mathcal{C}$-monoid. Furthermore, let X be an object in $\mathcal{C}$ and $T \times X \xrightarrow{\varphi} X$ be a morphism. We call the pair $\left(X, T \times X \xrightarrow{\varphi} X\right)$ a *$\mathcal{C}$-monoid action* (of $(T, +, e)$ on X) if the following two diagrams commute



(3.2a)



(3.2b)

Again the dotted arrow has been added for making the morphism explicit, and a_{15} denotes the canonical isomorphism given by the obvious tupling of projections.

3.2 Abstract Dynamical Systems

We shall define abstract dynamical systems as a straightforward generalisation of Definition 2.9 using the notions of $\mathcal{C}$-monoid and $\mathcal{C}$-monoid action from above.

In Section 4 we are going to establish a characterisation of such general dynamical systems as certain monadic algebras for the endo-functor $T \times -$, exploiting that products of the time space with every object in the considered category exist.

Definition 3.4 A *dynamical system on a finite product category* $\mathcal{C}$ is a triple $\left((T, +, e), X, T \times X \xrightarrow{\varphi} X\right)$, where $X \in \mathcal{C}$ is an object, $(T, +, e)$ is a $\mathcal{C}$-monoid and $\left(X, T \times X \xrightarrow{\varphi} X\right)$ is a $\mathcal{C}$-monoid action of $(T, +, e)$ on X.

To add some interpretative terminology to this definition, the object X will occasionally be called *state space*, T *time space*, $(T, +, e)$ *time structure* and the pair $\left(X, T \times X \xrightarrow{\varphi} X\right)$ or simply $T \times X \xrightarrow{\varphi} X$ *transition structure* of the dynamical system.

The following result makes sure that this is indeed a generalisation of the notions developed in Section 2.3.

Corollary 3.5 *The notions of topological dynamical system over a monoid (as in Definition 2.9) and dynamical system on $\mathcal{T}op$ (as in Definition 3.4) coincide.*

Likewise, the concepts of measurable dynamical system over a monoid (as in Definition 2.10) and of dynamical system on the category of measurable spaces $\mathcal{M}eas\mathcal{B}l$ (as in Definition 3.4) are the same.

Of course, in the same way we can use the categories $\mathcal{M}et_0$ of metric spaces or $\mathcal{U}nif_0$ of uniform spaces, each with continuous mappings as morphisms, instead of $\mathcal{T}op$

to study metric dynamics or uniform dynamics, respectively. With regard to basic aspects this is essentially the same as equipping the space with its underlying topology and forgetting about the metric or uniform structure, i.e. studying dynamical systems on the full subcategories $\mathcal{M}et$ and $\mathcal{U}nif$ of $\mathcal{T}op$, given by metrisable and uniformisable spaces, respectively.

One advantage of our abstract view on dynamical systems is that we now have a very simple way to translate a given system into others (in possibly different categories). Indeed, whenever we have a finite product preserving functor between two categories and a dynamical system on one of them, we also get one on the other category.

Remark 3.6 For a finite product category $\mathcal{C}$ and a finite product preserving functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$, each dynamical system $\left((T, +, e), X, T \times X \xrightarrow{\varphi} X\right)$ on $\mathcal{C}$ gives rise to a dynamical system $\left((FT, F+, Fe), FX, FT \times FX \xrightarrow{F\varphi} FX\right)$ on $\mathcal{D}$.

Proof It is clear that the functor F transforms the defining commutative diagrams for the dynamical system $\left((T, +, e), X, T \times X \xrightarrow{\varphi} X\right)$ into commutative diagrams in the category $\mathcal{D}$. Since it preserves finite products, the resulting diagrams also have the correct form to describe a dynamical system on $\mathcal{D}$. □

3.3 Nonautonomous Dynamics

Here we shall give an example how the notion of skew product system can also be lifted to our abstract setting. In fact, continuous skew product systems as given in Definition 2.13 have a straightforward generalisation in abstract categories, called *abstract NDS*, and Definition 2.13 is the specialisation of the general concept for the category of topological spaces. Subsequently, we shall see that for coinciding time monoids, abstract NDS are in turn a special instance of our abstract dynamical systems as given in Definition 3.4, namely in the case where the state space is a product of two spaces. This is a purely algebraic fact being true in the abstract categorical setting, no matter what category we choose.

The defining equations for continuous skew product systems clearly translate into commutative diagrams as we can see in the following definition.

Definition 3.7 Suppose $\mathbf{S} = \langle S; +, e\rangle$ and $\mathbf{T}$ are $\mathcal{C}$-monoids in a finite product category $\mathcal{C}$ such that $\mathbf{S}$ embeds as a submonoid into $\mathbf{T}$. Let X and Y be objects in $\mathcal{C}$ and $T \times X \xrightarrow{\theta} X$ and $S \times X \times Y \xrightarrow{\varphi} Y$ be morphisms. The pair (θ, φ) of morphisms is called *abstract nonautonomous dynamical system (abstract NDS) in $\mathcal{C}$ with times* $\mathbf{S} \hookrightarrow \mathbf{T}$ *on Y with base X* if

(1) $(\mathbf{T}, X, \theta)$ is a dynamical system on $\mathcal{C}$ over the monoid $\mathbf{T}$ (as in Definition 3.4), called *driving system*, and
(2) φ is an *abstract cocycle over* θ, that is, the following two diagrams, called *abstract cocycle property*,

$$X \times Y \cong S^0 \times (X \times Y) \xrightarrow{e \times (X \times Y)} S \times (X \times Y) \xrightarrow{\varphi} Y, \qquad X \times Y \overset{\langle !_{X \times Y} e, 1_{X \times Y}\rangle}{\dashrightarrow} S \times (X \times Y), \qquad X \times Y \xrightarrow{\mathrm{pr}'_Y} Y \tag{3.3a}$$

$$\begin{array}{ccccc}
(S \times S) \times (X \times Y) & \xrightarrow{+\times(X\times Y)} & S \times (X \times Y) & \xrightarrow{\varphi} & Y \\
\cong \uparrow & & & & \uparrow \varphi \\
S \times (S \times (X \times Y)) & & \xrightarrow{S\times\langle\langle \mathrm{pr}_S, \mathrm{pr}_X\rangle\theta,\varphi\rangle} & & S \times (X \times Y)
\end{array} \tag{3.3b}$$

commute. Here pr_S and pr_X denote the first and second projection morphism of the product $S \times (X \times Y)$, and pr'_X and pr'_Y are the projections belonging to $X \times Y$.

Evidently, we have the following corollary, which shows that our definition is sound, i.e. that it indeed entails the special case of continuous skew product system we started from.

Corollary 3.8 *Every continuous skew product system on a topological space with times* $\mathbf{S} \leq \mathbf{T}$ *(as in Definition 2.13) is an abstract NDS on* $\mathcal{T}op$ *with times* $\mathbf{S} \hookrightarrow \mathbf{T}$ *(as in the preceding Definition 3.7).*

Furthermore, every abstract NDS on $\mathcal{T}op$ *with times* $\mathbf{S} \overset{\varepsilon}{\hookrightarrow} \mathbf{T}$ *is a continuous skew product system with times* $\varepsilon(\mathbf{S}) \leq \mathbf{T}$*, where* $\varepsilon(\mathbf{S})$ *denotes the image of the topological monoid* $\mathbf{S}$ *under the embedding* ε*, which is isomorphic to* $\mathbf{S}$*.*

Similarly, interpreting Definition 3.7 in the category of sets, we obtain nonautonomous dynamical systems.

Corollary 3.9 *Every nonautonomous dynamical system on just a set with times* $\mathbf{S} \leq \mathbf{T}$ *(as in Definition 2.12) is an abstract NDS on* $\mathcal{S}et$ *with times* $\mathbf{S} \hookrightarrow \mathbf{T}$ *(as in Definition 3.7).*

Furthermore, every abstract NDS on $\mathcal{S}et$ *with times* $\mathbf{S} \overset{\varepsilon}{\hookrightarrow} \mathbf{T}$ *is a nonautonomous dynamical system with times* $\varepsilon(\mathbf{S}) \leq \mathbf{T}$*, where* $\varepsilon(\mathbf{S})$ *denotes the image of the monoid* $\mathbf{S}$ *under the embedding* ε*, which is isomorphic to* $\mathbf{S}$*.*

Likewise, abstract NDS on $\mathcal{M}easrbl$ correspond to measurable NDS as in Definition 2.14. For brevity an explicit corollary is omitted.

Next we prove that for two equal time monoids abstract NDS can be understood as a special kind of abstract dynamical system. This fact has been known for concrete cases of dynamical systems, e.g. continuous flows (topological dynamical systems as in Definition 2.9 where time is given by the real numbers with addition) arising as solutions of nonautonomous ordinary differential equations, cf. Chapter IV of [40], especially IV.A, IV.F and Theorem IV.11. Our lemma shows that this result only depends on the algebraic structure behind dynamical systems, not on the analytic or measure theoretic framework in which it is placed.

Since nonautonomous dynamics does not lie in the main focus of this article, we keep the proof sketchy and leave some details for the reader to work out.

Lemma 3.10 *Let* X*,* Y *belong to a finite product category* $\mathcal{C}$*, let* $\mathbf{T} = \langle T; +, e\rangle$ *be a* $\mathcal{C}$*-monoid setting and* $T \times (X \times Y) \overset{\Phi}{\to} X \times Y$*,* $T \times X \overset{\theta}{\to} X$ *be morphisms in* $\mathcal{C}$ *satisfying the condition*[3] $\Phi\,\mathrm{pr}'_X = \langle \mathrm{pr}_T, \mathrm{pr}_X\rangle\theta$*, i.e. the* X*-component of* Φ *does*

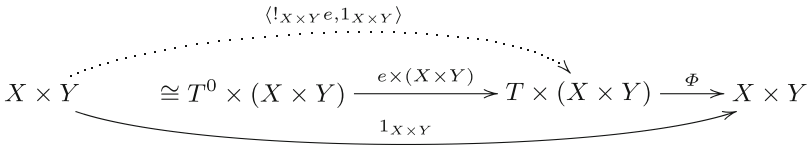
[3] As in Definition 3.7, pr'_X and pr'_Y denote the projection morphisms belonging to the product $X \times Y$.

not depend on Y and is given by θ. Furthermore, we require that the morphisms[4] $T \times (T \times (X \times Y)) \xrightarrow{\left\langle \mathrm{pr}_{T_1}, \left\langle \mathrm{pr}_{T_2}, \mathrm{pr}_X \right\rangle \right\rangle} T \times (T \times X)$ *and* $X \times Y \xrightarrow{\mathrm{pr}'_X} X$ *be epi. In these circumstances the following statements are equivalent:*

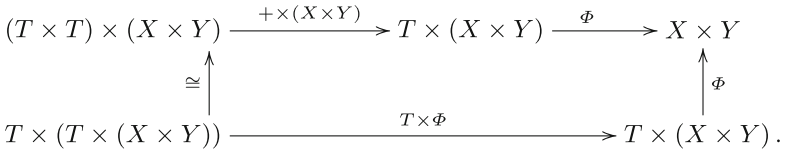
(a) $\left(\mathbf{T}, X \times Y, T \times (X \times Y) \xrightarrow{\Phi} X \times Y\right)$ *is a dynamical system on* $\mathcal{C}$.

(b) $\left(\theta, \Phi\, \mathrm{pr}'_Y\right)$ *is an abstract NDS in* $\mathcal{C}$ *with times* $\mathbf{S} = \mathbf{T}$ *on* Y *with base* X.

Proof The argument is based on transforming the defining condition for the dynamical system $\left((T, +, e), X \times Y, T \times (X \times Y) \xrightarrow{\Phi} X \times Y\right)$. According to Definition 3.4, the notion of dynamical system is built upon $(T, +, e)$ being a $\mathcal{C}$-monoid and $\left(X \times Y, T \times (X \times Y) \xrightarrow{\Phi} X \times Y\right)$ being a $\mathcal{C}$-monoid action. The latter fact is a conjunction of two commuting diagrams, (3.2a) and (3.2b). These have the form



and



Both diagrams express that two certain morphisms f, g, starting in the same object Z (either $X \times Y$ or $T \times (T \times (X \times Y))$) and ending in the product $X \times Y$, are identical. By definition of the product this is equivalent to the fact that the equalities $f\, \mathrm{pr}'_X = g\, \mathrm{pr}'_X$ and $f\, \mathrm{pr}'_Y = g\, \mathrm{pr}'_Y$ hold. This means that we can equivalently replace each of the two diagrams by a conjunction of two commutative diagrams.

Taking into account the assumption that $\Phi\, \mathrm{pr}'_X = \langle \mathrm{pr}_T, \mathrm{pr}_X \rangle \theta$, we obtain the equalities $T \times \langle \langle \mathrm{pr}_T, \mathrm{pr}_X \rangle \theta, \Phi\, \mathrm{pr}'_Y \rangle = T \times \langle \Phi\, \mathrm{pr}'_X, \Phi\, \mathrm{pr}'_Y \rangle = T \times \Phi$, and thus we see that the two diagrams arising from composition with pr'_Y are precisely the ones occurring in Definition 3.7. The other two ones, coming from composition with pr'_X, are equivalent to the two diagrams defining $\left((T, +, e), X, T \times X \xrightarrow{\theta} X\right)$. This can be seen from a short calculation using again the condition $\Phi\, \mathrm{pr}'_X = \langle \mathrm{pr}_T, \mathrm{pr}_X \rangle \theta$ and the assumption that the two projection morphisms mentioned in the statement of the lemma are epi. For further details we refer to the complete proof in [9]. □

[4] Here pr_{T_1}, pr_{T_2} and pr_X denote the projection morphisms of $T \times (T \times (X \times Y))$ onto the first factor T, onto the second factor T and on the X-component of the product.

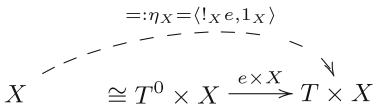
4 Dynamical Systems as Algebras and Coalgebras

4.1 From Monoids to Monads

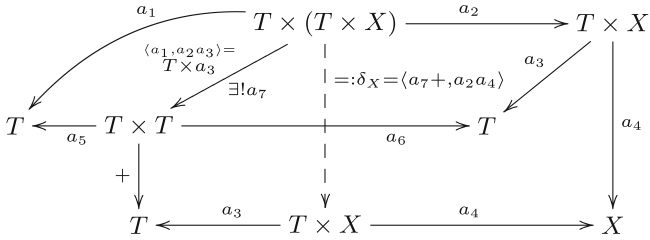
Here we are going to explore the connection of $\mathcal{C}$-monoids and their actions to monads and monadic algebras. For this we need to assume that for some object T from $\mathcal{C}$ all products $T \times X$ exist for X in $\mathcal{C}$. This allows us to define the endo-functor $T \times -$ on $\mathcal{C}$. Of course in finite product categories, this assumption is certainly valid.

The main and only result of this subsection is a lemma connecting the commutativity conditions from the definitions of a $\mathcal{C}$-monoid and a $\mathcal{C}$-monoid action with certain commutative diagrams for two derived natural transformations. In the next subsection we shall use this lemma to translate abstract $\mathcal{C}$-monoids into monads and $\mathcal{C}$-monoid actions into monadic algebras for the monad associated with the $\mathcal{C}$-monoid.

Lemma 4.1 *Suppose $\mathcal{C}$ is a finite product category with the endo-functor $T \times -$ for some object T and let $(T, +, e)$ be a $\mathcal{C}$-monoid setting. For every X from $\mathcal{C}$ let the morphisms η_X and δ_X be defined by the commutativity of the following diagrams*



and



(where the morphisms a_i, $i \neq 7$, are projection morphisms of the respective products). Then

(a) $1_C \xrightarrow{\eta} T \times -$ *and* $T \times (T \times -) \xrightarrow{\delta} T \times -$ *are natural transformations.*

(b) *For an object $X \in \mathcal{C}$ and the projection morphism $T \times X \xrightarrow{a_3} T$, the conditions $a_3 \langle !_T e, 1_T\rangle + = a_3 \cong (e \times T) + = a_3$ and $\eta_{T\times X}\delta_X = 1_{T\times X}$ are equivalent.*

(c) *If $(T, +, e)$ satisfies condition (3.1a), then we have $\eta_{T\times X}\delta_X = 1_{T\times X}$ for all $X \in \mathcal{C}$. If the projection morphism $T \times X \xrightarrow{a_3} T$ is epi for one $X \in \mathcal{C}$, then the converse implication is true, as well.*

(d) *For an object $X \in \mathcal{C}$ and the projection morphism $T \times X \xrightarrow{a_3} T$, the conditions $a_3 \langle 1_T, !_T e\rangle + = a_3 \cong (T \times e) + = a_3$ and $(T \times \eta_X)\delta_X = 1_{T\times X}$ are equivalent.*

(e) *If $(T, +, e)$ satisfies condition (3.1b), then $(T \times \eta_X)\,\delta_X = 1_{T\times X}$ holds for all $X \in \mathcal{C}$. If the projection morphism $T \times X \xrightarrow{a_3} T$ is epi for one $X \in \mathcal{C}$, then the converse implication is true, as well.*

(f) *For an object* $X \in \mathcal{C}$ *and the isomorphism* $T \times (T \times T) \xrightarrow{a_{12}} (T \times T) \times T$ *the following equalities hold*

$$\delta_{T\times X} a_7 + = (T \times a_7) a_{12} (+ \times T) + \tag{4.1}$$

$$(T \times \delta_X) a_7 + = (T \times a_7)(T \times +) + . \tag{4.2}$$

Furthermore, the equality $\delta_{T\times X}\delta_X = (T \times \delta_X)\delta_X$ *is equivalent to the condition* $\delta_{T\times X} a_7 + = (T \times \delta_X) a_7 +$ *and to* $(T \times a_7) a_{12} (+ \times T) + = (T \times a_7)(T \times +) +$.

(g) *If* $(T, +, e)$ *satisfies condition* (3.1c)*, then we have* $\delta_{T\times X}\delta_X = (T \times \delta_X)\,\delta_X$ *for all* $X \in \mathcal{C}$*. If* $T \times (T \times (T \times X)) \xrightarrow{T\times a_7} T \times (T \times T)$ *is an epimorphism for one* $X \in \mathcal{C}$*, then the converse implication is true, as well.*

(h) *If* $(T, +, e)$ *is a* $\mathcal{C}$*-monoid, then* $(T \times -, \delta, \eta)$ *is a monad. If the projection morphisms* $T \times X \xrightarrow{a_3} T$ *and* $T \times (T \times (T \times Y)) \xrightarrow{T\times a_7} T \times (T \times T)$ *are epi for some objects* $X, Y \in \mathcal{C}$*, then the converse also holds.*

(i) *Let an object* $X \in \mathcal{C}$ *and a morphism* $T \times X \xrightarrow{\varphi} X$ *be given. For the canonical isomorphism* $T \times (T \times X) \xrightarrow{a_{15}} (T \times T) \times X$ *from condition* (3.2b) *we have* $a_{15}\,(+ \times X) = \delta_X$*, whence diagram* (3.2b) *commutes if and only if the equality* $(T \times \varphi)\,\varphi = \delta_X \varphi$ *holds, and diagram* (3.2a) *commutes if and only if* $\eta_X \varphi = 1_X$ *is true.*

This lemma can be proven by mainly straightforward, but slightly technical calculations, the details can be found in [9, Lemma 4.1.1]. It enables us now to characterise abstract dynamical systems in terms of monadic algebras for the endo-functor $T \times -$ on $\mathcal{C}$.

4.2 From Abstract Dynamical Systems to Monadic Algebras

Here we finally relate our definition of abstract dynamical system on finite product categories to the well-known algebraic concept of monadic algebra.

Proposition 4.2 *Let* $\mathcal{C}$ *be a finite product category and* T *one of its objects. Suppose* $(T, +, e)$ *is a* $\mathcal{C}$*-monoid setting and* $1_{\mathcal{C}} \xrightarrow{\eta} T \times -$ *and* $T \times (T \times -) \xrightarrow{\delta} T \times -$ *are the associated natural transformations as in Lemma 4.1 (a). Furthermore, let* X *be an object of* $\mathcal{C}$ *with a morphism* $T \times X \xrightarrow{\varphi} X$*. If* $((T, +, e), X, \varphi)$ *is a dynamical system on* $\mathcal{C}$ *then* $(T \times -, \delta, \eta)$ *is a monad and* (X, φ) *is a monadic* $(T \times -)$*-algebra for this monad.*

If, for certain objects Y, Z *of* $\mathcal{C}$*, the morphism* $T \times Z \xrightarrow{a_3} T$ *and the morphism* $T \times (T \times (T \times Y)) \xrightarrow{T\times a_7} T \times (T \times T)$ *mentioned in Lemma 4.1(h) are epi, then also the converse implication holds.*

Proof If $((T, +, e), X, \varphi)$ is a dynamical system on $\mathcal{C}$, then $(T, +, e)$ is a $\mathcal{C}$-monoid, so by Lemma 4.1 (h), $(T \times -, \delta, \eta)$ is a monad. Furthermore, (X, φ) is a $\mathcal{C}$-monoid action, so diagrams (3.2a) and (3.2b) commute, which is equivalent to the two equalities stated in Lemma 4.1(i). This, however, means that (X, φ) is a monadic $(T \times -)$-algebra w.r.t. the monad $(T \times -, \delta, \eta)$.

Under the additional assumptions, the implication in Lemma 4.1(h) can be reversed, which shows the second part of the proposition. □

A much more concise formulation of this result is achieved if one starts with a monoid instead of a monoid setting as in the following corollary:

Corollary 4.3 *Let $\mathcal{C}$ be a finite product category and T, X be objects of $\mathcal{C}$ with a morphism $T \times X \xrightarrow{\varphi} X$. Furthermore, let $(T, +, e)$ be a $\mathcal{C}$-monoid and $(T \times -, \delta, \eta)$ the associated monad as in Lemma 4.1(h). Then*

$((T, +, e), X, \varphi)$ *is a dynamical system on* $\mathcal{C}$	*if and only if*	(X, φ) *is a monadic* $(T \times -)$*-algebra for* $(T \times -, \delta, \eta)$.

Proof Note that the additional assumptions in Proposition 4.2 have only been needed to show that $(T, +, e)$ is a $\mathcal{C}$-monoid provided that $(T \times -, \delta, \eta)$ is a monad. As the conclusion of this implication is already contained in the assumptions of the corollary, the same proof as for the proposition works, just using the part involving Lemma 4.1(i). □

It is now easy to see that the connection exhibited in the previous corollary can be formalised as an isomorphism between categories.

Remark 4.4 For a finite product category $\mathcal{C}$, any two objects T, X, a morphism $T \times X \xrightarrow{\varphi} X$ and a $\mathcal{C}$-monoid $(T, +, e)$ with associated monad $(T \times -, \delta, \eta)$ as in Lemma 4.1(h), mapping $((T, +, e), X, \varphi) \mapsto (X, \varphi)$ induces a categorical equivalence (even an isomorphism) between the category of abstract dynamical systems on $\mathcal{C}$ w.r.t. the $\mathcal{C}$-monoid $(T, +, e)$ and that of monadic $(T \times -)$-algebras for the associated monad $(T \times -, \delta, \eta)$. This is so because the condition for a morphism $X \xrightarrow{h} Y$ of $\mathcal{C}$ to be a morphism of dynamical systems $((T, +, e), X, \varphi)$ and $((T, +, e), Y, \psi)$ is precisely the same as for being a morphism of $(T \times -)$-algebras, namely that $(T \times h)\,\psi = \varphi h$. Therefore, the assignment above extends to a functor that maps morphisms identically and has the obvious inverse functor.

In particular, if we combine the latter observation with Corollary 3.5, we obtain that the category of topological dynamical systems over a fixed topological monoid $\mathbf{T} = \langle T; +, 0\rangle$, which are the dynamical systems on $\mathcal{T}op$ for this particular $\mathcal{T}op$-monoid, is isomorphic to the category of $(T \times -)$-algebras for the associated monad $(T \times -, \delta, \eta)$ as given in Lemma 4.1(h).

Observe that in this context the canonically given notion of isomorphism in the category of $(T \times -)$-algebras translates to the well-known concept of *topological conjugacy* from the world of dynamical systems.

For example, in Section 2.3.2 of [10] a prototypical example of a chaotic dynamical system is studied. It is a discrete time system as introduced at the beginning of Section 2.3, induced by iterating the logistic map on a certain Cantor set Λ within the real unit interval, viewed as a topological subspace X of the real numbers with the topology being given by the absolute value metric. In Theorem 2.20, Berger examines the topological dynamical system over the discrete topological monoid $\langle \mathbb{N}; +, 0\rangle$ given by $\varphi\colon \mathbb{N} \times X \to X$, where $\varphi(n, x) := f^n(x)$, and $f\colon X \to X$ is defined by $f(x) := \mu x(1 - x)$ for $x \in X$ and the special choice $\mu := 3.839$. The space X is partitioned into two disjoint parts and each state $x \in X$ is mapped to an ω-sequence $h(x) \in 2^\omega$ of indices zero and one, indicating which of the two parts the respective n-th iterate $\varphi(n, x) = f^n(x)$ belongs to. Thereby, Berger establishes that the particular discrete time dynamical system is isomorphic to a so-called *subshift of finite type*. The latter one is readily seen to fulfil the criteria of a chaotic system.

The condition that needs to be shown in the proof of the mentioned theorem is precisely that the two associated monadic algebras are isomorphic: the mapping h has to be a homeomorphism (an isomorphism in the category $\mathcal{T}op$) satisfying the condition that for every $x \in X$ shifting the sequence assigned to x to the left yields the sequence assigned to $f(x)$.

In a similar way, other category theoretic concepts and constructions, for instance existing limits for monadic algebras, can be shifted both ways between the algebraic world of $(T \times -)$-algebras and the analytic world of dynamical systems.

4.3 Connections to Coalgebras

The aim of this part is to establish a link between abstract dynamical systems that have now been understood as monadic algebras for the endo-functor $T \times -$ and coalgebras for another signature functor. It will turn out that these coalgebras will also carry a comonadic structure in a natural way.

The motivation for the rest of this section comes from regarding Corollary 4.3 in the special case of $\mathcal{C} = \mathcal{S}et$ and at first forgetting about monadicity conditions. What remains is an algebra $(X, \varphi\colon T \times X \to X)$ of signature $T \times -$ on the state space X. There is an easy construction, well-known from computer science as *currying*, transforming every map $\varphi\colon T \times X \to X$ into a map $\psi\colon X \to X^T$, where X^T denotes the set of all maps from T to X and $\psi(x)\colon T \to X$ is defined by $(\psi(x))(t) := \varphi(t, x)$ for $t \in T$ and all $x \in X$. Evidently, the mapping ψ suffices to encode all the information about state transitions that is contained in φ. That is, the currying operation can be reversed by assigning $\psi(x)(t)$ to every pair $(t, x) \in T \times X$, thus re-obtaining φ from ψ.

Consequently, in $\mathcal{S}et$ there is a one-to-one correspondence between mappings of the form $\varphi\colon T \times X \to X$ and $\psi\colon X \to X^T$ or, in other words, between algebras $(X, \varphi\colon T \times X \to X)$ and coalgebras $\left(X, \psi\colon X \to X^T\right)$ for the hom-functor $-^T = \mathrm{Hom}(T, -)$.

If $\mathcal{C} = \mathcal{S}et$, then $\mathrm{Hom}(T, -)$ is an endo-functor, but this does not hold in arbitrary categories $\mathcal{C}$. Searching for an appropriate generalisation of the construction above leads us back to the original idea of currying. In fact the one-to-one correspondence between mappings as described in the case of dynamical systems on $\mathcal{S}et$ is a bit more general: every mapping $\varphi\colon T \times X \to Y$ in $\mathcal{S}et$ can be translated into a mapping $\psi\colon X \to Y^T$ and vice versa. However, this is the defining property of an adjunction between the endo-functors $T \times -$ and $-^T$. It turns out that this is the right point of view for a generalisation to arbitrary categories, which as a side effect ensures that algebras for $T \times -$ and coalgebras for the adjoint functor are uniquely related.

In every finite product category $\mathcal{C}$ any object T gives rise to an endo-functor $T \times -$ on $\mathcal{C}$. We say that $\mathcal{C}$ *has exponential objects w.r.t.* T if the endo-functor $\mathcal{C} \xrightarrow{T \times -} \mathcal{C}$ has a right adjoint, called $\mathcal{C} \xrightarrow{-^T} \mathcal{C}$. Moreover, $\mathcal{C}$ *has exponential objects*, if it has exponential objects w.r.t. to any object T of $\mathcal{C}$.

These notions enable us to study the connections between dynamical systems as monadic algebras and a possible formalisation as coalgebras on the more general level of adjoint functors. In fact, this discussion can be done independently of the particular functor $T \times -$ and a possible adjoint $-^T$, see Section 4.5.

Since adjoint functors (if they exist at all) are unique up to isomorphism this method also yields a reasonable definition of the object X^T for our algebras. If $\mathcal{C}$ is concrete over $\mathcal{S}et$, then X^T often can be given by adding some extra structure to the set $\mathrm{Hom}(T, X)$. For

example in the category $\mathcal{Top}$ the set $\mathrm{Hom}(T, X)$ equipped with the compact open topology serves as an exponential object provided that the time space T is locally compact Hausdorff.

4.4 Exponential Objects in $\mathcal{Top}$ for Locally Compact Hausdorff Spaces

For topological spaces $\boldsymbol{X} = (X, \rho)$ and $\boldsymbol{Y} = (Y, \sigma)$, a compact set $K \in K(\boldsymbol{X})$ and $U \in \sigma$ we let $[K, U] := \{f \in C(\boldsymbol{X}, \boldsymbol{Y}) \mid f[K] \subseteq U\}$ and define $\kappa(\boldsymbol{X}, \boldsymbol{Y})$ to be the *compact-open topology* on $C(\boldsymbol{X}, \boldsymbol{Y})$, i.e. the topology generated by the subbase $\{[K, U] \mid K \in K(\boldsymbol{X}), U \in \sigma\}$. Moreover, we put $\boldsymbol{Y}^{\boldsymbol{X}} := (C(\boldsymbol{X}, \boldsymbol{Y}), \kappa(\boldsymbol{X}, \boldsymbol{Y}))$.

Note that the set $C(\boldsymbol{X}, \boldsymbol{Y})$ was called $\mathcal{Top}(\boldsymbol{X}, \boldsymbol{Y})$ in the general category theoretic setting introduced in Section 2.2. The next proposition clarifies that, indeed, the category of locally compact Hausdorff spaces has exponential objects. This fact is well-known and may e.g. be found in [24, Satz 8.1.19, p. 222], [45, p. 269], or [18, Chapter XII, 2.1 (for (a)), and Theorem 3.1 (for (b) and (c))].

Proposition 4.5 *Let $\boldsymbol{T} = (T, \tau) \in \mathcal{Top}$.*

(a) *The assignment $-^{\boldsymbol{T}} : \mathcal{Top} \to \mathcal{Top} : \boldsymbol{X} \mapsto \boldsymbol{X}^{\boldsymbol{T}}$ defines a functor, operating on morphisms $\boldsymbol{X} \xrightarrow{f} \boldsymbol{Y}$ via $\boldsymbol{X}^{\boldsymbol{T}} \xrightarrow{f^{\boldsymbol{T}}} \boldsymbol{Y}^{\boldsymbol{T}} : g \mapsto f \circ g$.*

(b) *The family of morphisms given by*

$$\begin{aligned} \Phi^{\boldsymbol{T}}_{\boldsymbol{X},\boldsymbol{Y}} : \mathcal{Top}(\boldsymbol{T} \times \boldsymbol{X}, \boldsymbol{Y}) &\to \mathcal{Top}(\boldsymbol{X}, \boldsymbol{Y}^{\boldsymbol{T}}) \\ \boldsymbol{T} \times \boldsymbol{X} \xrightarrow{f} \boldsymbol{Y} &\mapsto \boldsymbol{X} \xrightarrow{\Phi^{\boldsymbol{T}}_{\boldsymbol{X},\boldsymbol{Y}}(f)} \boldsymbol{Y}^{\boldsymbol{T}} : [x \mapsto [t \mapsto f(t, x)]], \end{aligned}$$

constitutes a natural transformation.

(c) *If $\boldsymbol{T}$ is locally compact Hausdorff, then $\Phi^{\boldsymbol{T}} : \mathcal{Top}(\boldsymbol{T} \times -_1, -_2) \to \mathcal{Top}\left(-_1, -_2^{\boldsymbol{T}}\right)$ is a natural equivalence.*

In the theory of dynamical systems, state spaces are often chosen as metric spaces. This motivates the search for those topological spaces $\boldsymbol{T}$ for which the space $\boldsymbol{X}^{\boldsymbol{T}}$ is metrisable whenever $\boldsymbol{X}$ is metrisable.

We recall that a topological space is strongly σ-compact if it has a countable exhaustion by compact subsets. Note, furthermore, that strongly σ-compact Hausdorff spaces are necessarily locally compact (see Lemma 2.1). The following proposition now answers the previously stated question.

Proposition 4.6 *If $\boldsymbol{T}$ is a strongly σ-compact topological space and $\boldsymbol{X}$ a metrisable topological space, then $\boldsymbol{X}^{\boldsymbol{T}}$ is metrisable, too.*

Proof Let $(K_n)_{n\in\mathbb{N}}$ be a countable exhaustion of $\boldsymbol{T}$ by compact subsets, and let d be a metric generating the topology of $\boldsymbol{X}$. Then it is not difficult to see that

$$\begin{aligned} d^* : C(\boldsymbol{T}, \boldsymbol{X})^2 &\to \mathbb{R}, \\ (f, g) &\mapsto \textstyle\sum_{n=0}^{\infty} \frac{1}{2^n} \min\left\{\sup_{x\in K_n} d(f(x), g(x)), 1\right\} \end{aligned}$$

is a metric on $C(\boldsymbol{T}, \boldsymbol{X})$ generating $\kappa(\boldsymbol{T}, \boldsymbol{X})$ (cp. also [46, 43G.1., p. 289]). □

Many well-known topological spaces are strongly σ-compact, such as all finite powers of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$. However, as it turns out, a slight generalization of metrisable spaces allows us to use a notably larger class of time spaces. Namely, if $\boldsymbol{X}$ is uniformisable, we shall see that $\boldsymbol{T}$ may indeed be an arbitrary topological space.

To give an example of uniform spaces that properly generalise metric spaces, let $\boldsymbol{X}$ be a topological space and consider the space $C(\boldsymbol{X}) := C(\boldsymbol{X}, \mathbb{R})$ of continuous real-valued functions on $\boldsymbol{X}$, equipped with the topology of compact convergence. Such function spaces $C(\boldsymbol{X})$ as state spaces promise a wide variety of dynamic behaviour, much more than just $\mathbb{R}^n$. It goes beyond the scope of this article to study them in more detail, but in [38] and [37], we examine topological dynamical systems on function spaces over topological groups and semigroups more closely. In particular, we study and characterise faithful strongly chaotic continuous actions of locally compact Hausdorff topological groups on such spaces.

Even though function spaces $C(\boldsymbol{X})$ sometimes lack metrisability and thus Proposition 4.6 fails to be applicable, these spaces are certainly uniformisable as said before. Hence, one may instead rely on the following well-known variant of Proposition 4.6, which, as a side-effect, allows us to drop the assumption of strong σ-compactness w.r.t. the time space $\boldsymbol{T}$. For a proof of this fact we refer to [46, Theorem 43.7].

Proposition 4.7 *If $\boldsymbol{T}$ is a topological space and $\boldsymbol{X}$ a uniformisable space, then the function space $\boldsymbol{X}^{\boldsymbol{T}}$ is uniformisable, too.*

According to Proposition 4.6, if $\boldsymbol{X}$ is a metrisable space and the time space $\boldsymbol{T}$ is strongly σ-compact, then also $\boldsymbol{X}^{\boldsymbol{T}}$ is metrisable, i.e. the compact-open topology on $\boldsymbol{X}^{\boldsymbol{T}}$ is induced by some metric. Such a metric in a natural way defines a uniform structure Θ on $\boldsymbol{X}^{\boldsymbol{T}}$, which is indeed the same as the uniform structure constructed by Proposition 4.7 applied to $\boldsymbol{T}$ and the uniform space on X derived from the metric on X.

With the previous result we have established the existence of exponential objects w.r.t. any time space in the full subcategory $\mathcal{U}nif$ of uniformisable spaces. With Proposition 4.6 we have done the same for the subcategory $\mathcal{M}et$ of metrisable spaces and strongly σ-compact times spaces. Furthermore, Proposition 4.5 solves this question for the category of locally compact Hausdorff spaces in general and for the category of topological spaces w.r.t. locally compact Hausdorff time spaces.

Thus, in many familiar situations, one can ensure that the functor taking products with the time space $\boldsymbol{T}$ has a right adjoint endo-functor $-^{\boldsymbol{T}}$. It is on the level of adjoint endo-functors that we shall now explore, how to understand dynamical systems in abstract categories in a different manner than as monadic algebras.

4.5 (Co)Monadic (Co)Algebras and Adjoint Functors

In this part we shall show that monadic algebras correspond closely to comonadic coalgebras if the respective signature functors are adjoint. Since under fairly weak assumptions on the considered category $\mathcal{C}$, general dynamical systems have been modelled as monadic algebras for the signature functor $T \times -$ (cf. Corollary 4.3), this result will in particular apply to an adjoint functor $-^T$ on $\mathcal{C}$ provided it exists. However, our treatment of this topic allows the functors $T \times -$ and $-^T$ to be replaced by any other adjoint pair of endo-functors $F \dashv G$.

Our first aim is to show how a monad (F, δ, η) for an endo-functor $F \in \mathrm{End}\mathcal{C}$ can be transformed into a comonad $(G, \tilde{\delta}, \tilde{\eta})$ for a right adjoint endo-functor $G \in \mathrm{End}\mathcal{C}$. We shall put the technical part of the construction into the following lemma:

Lemma 4.8 *Let $\mathcal{C}$ be a category and $F, G \in End\mathcal{C}$ be two adjoint endo-functors ($F \dashv G$). We denote the corresponding natural equivalence between the hom-set functors by $\mathrm{Hom}(F, -) \xrightarrow{\nu} \mathrm{Hom}(-, G)$, the unit by $1_{\mathcal{C}} \xrightarrow{\vartheta} GF$ and the co-unit by $FG \xrightarrow{\varepsilon} 1_{\mathcal{C}}$.*

Furthermore, let two natural transformations $FF \xrightarrow{\delta} F$ and $1_{\mathcal{C}} \xrightarrow{\eta} F$, and an arbitrary object $X \in \mathcal{C}$ be given.

For every object $Y \in \mathcal{C}$ we define the following morphisms:

$$FFGY \xrightarrow{\zeta_Y := \delta_{GY}\varepsilon_Y} Y \qquad GY \xrightarrow{\tilde{\delta}_Y := \nu_{GY,GY}(\mu_Y)} GGY$$
$$FGY \xrightarrow{\mu_Y := \nu_{FGY,Y}(\zeta_Y)} GY \qquad GY \xrightarrow{\tilde{\eta}_Y := \eta_{GY}\varepsilon_Y} Y.$$

Then the following assertions are true:

(a) $FFG \xrightarrow{\zeta} 1_{\mathcal{C}}$, $FG \xrightarrow{\mu} G$, $G \xrightarrow{\tilde{\delta}} GG$ *and* $G \xrightarrow{\tilde{\eta}} 1_{\mathcal{C}}$ *are natural transformations.*

(b) $\tilde{\delta}_X G\tilde{\eta}_X = \nu_{GX,X}(\eta_{FGX}\delta_{GX}\varepsilon_X) = \nu_{GX,X}(\eta_{FGX}\zeta_X)$ *and*

$\eta_{FX}\delta_X = F\vartheta_X\eta_{FGFX}\zeta_{FX}$.

(c) $\tilde{\delta}_X\tilde{\eta}_{GX} = \nu_{GX,X}(F\eta_{GX}\delta_{GX}\varepsilon_X) = \nu_{GX,X}(F\eta_{GX}\zeta_X)$ *and*

$F\eta_X\delta_X = F\vartheta_X F\eta_{GFX}\zeta_{FX}$.

(d) $\tilde{\delta}_X G\tilde{\delta}_X = \nu_{GX,G^2X}(\nu_{FGX,GX}(\nu_{F^2GX,X}(\delta_{FGX}\zeta_X)))$,

$\tilde{\delta}_X\tilde{\delta}_{GX} = \nu_{GX,G^2X}(\nu_{FGX,GX}(\nu_{F^2GX,X}(F\delta_{GX}\zeta_X)))$,

$F\delta_X\delta_X = F^3\vartheta_X F\delta_{GFX}\zeta_{FX}$ *and*

$\delta_{FX}\delta_X = F^3\vartheta_X\delta_{FGFX}\zeta_{FX}$.

Proof The statement of this lemma follows by skilfully intertwining the naturality conditions for ϑ, ε, δ, η, ζ and μ with the equations

$$1_{FX} = F\vartheta_X\varepsilon_{FX} \tag{4.3}$$
$$1_{GY} = \vartheta_{GY}G\varepsilon_Y \tag{4.4}$$
$$\vartheta_X Gg = \nu_{X,Y}(g) \tag{4.5}$$
$$F\nu_{X,Y}(g)\varepsilon_Y = g \tag{4.6}$$

that hold for all objects X, Y from $\mathcal{C}$ and morphisms $FX \xrightarrow{g} Y$. For more technical details the esteemed reader is referred to [9]. □

Proposition 4.9 *With the notation introduced in Lemma 4.8, the following equivalences hold:*

(a) $\forall X \in \mathcal{C}\colon\ \eta_{FX}\delta_X = 1_{FX} \iff \forall X \in \mathcal{C}\colon\ \tilde{\delta}_X G\tilde{\eta}_X = 1_{GX}$
$\iff \forall X \in \mathcal{C}\colon\ \eta_{FGX}\zeta_X = \varepsilon_X$.

(b) $\forall X \in \mathcal{C}\colon\ F\eta_X\delta_X = 1_{FX} \iff \forall X \in \mathcal{C}\colon\ F\eta_{GX}\zeta_X = \varepsilon_X$
$\iff \forall X \in \mathcal{C}\colon\ \tilde{\delta}_X\tilde{\eta}_{GX} = 1_{GX}$.

(c) $\forall X \in \mathcal{C}\colon\ F\delta_X\delta_X = \delta_{FX}\delta_X \iff \forall X \in \mathcal{C}\colon\ \tilde{\delta}_X\tilde{\delta}_{GX} = \tilde{\delta}_X G\tilde{\delta}_X$.

(d) (F, δ, η) *is a monad if and only if* $\left(G, \tilde{\delta}, \tilde{\eta}\right)$ *is a comonad.*

Proof In this proof underlined parts of formulae denote subformulae that are replaced using assumed or previously proven equalities. To demonstrate the stated equivalences, we rely upon Lemma 4.8 and the following equality holding for every object X in C:

$$\nu_{GX,X}(\varepsilon_X) \overset{(4.5)}{=} \vartheta_{GX} G\varepsilon_X \overset{(4.4)}{=} 1_{GX}. \tag{4.7}$$

(a) First, for a fixed object $X \in C$ it is easy to see, using Eq. 4.7 and Lemma 4.8 (b) that the equalities

$$\tilde{\delta}_X G\tilde{\eta}_X = 1_{GX} \quad \text{and} \quad \nu_{GX,X}\left(\eta_{FGX}\delta_{GX}\varepsilon_X\right) = \nu_{GX,X}(\varepsilon_X)$$

are equivalent. As $\nu_{GX,X}$ is a bijection, the latter equality is equivalent to $\eta_{FGX}\delta_{GX}\varepsilon_X = \varepsilon_X$. Taking into account that $\zeta_X = \delta_{GX}\varepsilon_X$, the equivalence of the two statements on the right-hand side is proven.

Now assume that $\eta_{FX}\delta_X = 1_{FX}$ holds for all $X \in C$. For every $Y \in C$, substituting $X = GY$ in this equality and composition with ε_Y yields

$$\eta_{FGY}\delta_{GY}\varepsilon_Y = 1_{FGY}\varepsilon_Y = \varepsilon_Y.$$

Conversely, suppose that this holds for all $Y \in C$. Considering any $X \in C$ and substituting $Y = FX$ yields the equality $\eta_{FGFX}\delta_{GFX}\varepsilon_{FX} = \varepsilon_{FX}$. From this observation and Lemma 4.8(b) one obtains

$$\eta_{FX}\delta_X \overset{4.8(b)}{=} F\vartheta_X \underline{\eta_{FGFX}\delta_{GFX}\varepsilon_{FX}} \overset{\text{v.s.}}{=} F\vartheta_X\varepsilon_{FX} \overset{(4.3)}{=} 1_{FX}.$$

(b) One may almost verbatim copy the argument of (a), just replacing the application of Lemma 4.8 (b) by Lemma 4.8 (c).

(c) Assume that $F\delta_X\delta_X = \delta_{FX}\delta_X$ holds for all $X \in C$ and consider any $Y \in C$. Substituting $X = GY$ in the given equality yields that $F\delta_{GY}\delta_{GY} = \delta_{FGY}\delta_{GY}$, whence one obtains $F\delta_{GY}\zeta_Y = \delta_{FGY}\zeta_Y$ by composition with ε_Y. From the latter equality one has

$$\begin{aligned}\tilde{\delta}_Y\tilde{\delta}_{GY} &\overset{4.8(d)}{=} \nu_{GY,G^2Y}(\nu_{FGY,GY}(\nu_{F^2GY,Y}(F\delta_{GY}\zeta_Y))) \\ &\overset{\text{v.s.}}{=} \nu_{GY,G^2Y}(\nu_{FGY,GY}(\nu_{F^2GY,Y}(\delta_{FGY}\zeta_Y))) \overset{4.8(d)}{=} \tilde{\delta}_Y G\tilde{\delta}_Y.\end{aligned}$$

Conversely, assume that this equality holds for all $Y \in C$. Substituting $Y = GX$ with an arbitrary $X \in C$ yields

$$\begin{aligned}&\nu_{GFX,G^2FX}(\nu_{FGFX,GFX}(\nu_{F^2GFX,FX}(F\delta_{GFX}\zeta_{FX}))) \overset{4.8\,(d)}{=} \tilde{\delta}_{FX}\tilde{\delta}_{GFX} \\ &= \tilde{\delta}_{FX}G\tilde{\delta}_{FX} \overset{4.8(d)}{=} \nu_{GFX,G^2FX}(\nu_{FGFX,GFX}(\nu_{F^2GFX,FX}(\delta_{FGFX}\zeta_{FX}))),\end{aligned}$$

whence $F\delta_{GFX}\zeta_{FX} = \delta_{FGFX}\zeta_{FX}$ as ν is a natural equivalence, so all its mappings are bijective. Composing the result with $F^3\vartheta_X$ yields

$$F\delta_X\delta_X \overset{4.8(d)}{=} F^3\vartheta_X\underline{F\delta_{GFX}\zeta_{FX}} \overset{\text{v.s.}}{=} \underline{F^3\vartheta_X\delta_{FGFX}\zeta_{FX}} \overset{4.8\,(d)}{=} \delta_{FX}\delta_X.$$

(d) This statement is a combination of the equivalences just shown. □

In the previous result we have established a relationship between monads for F and comonads for an adjoint endo-functor G. This connection extends to monadic algebras and comonadic coalgebras:

Proposition 4.10 *In the notational setting provided in Lemma 4.8 assume that (F, δ, η) is a monad with its associated comonad $\left(G, \tilde{\delta}, \tilde{\eta}\right)$ as given in Proposition 4.9. Let $X \in C$ be an object and $FX \xrightarrow{\varphi} X$ be a morphism. Defining the morphism $X \xrightarrow{\psi := \nu_{X,X}(\varphi) = \vartheta_X G\varphi} GX$, we have*

(a) $\psi\tilde{\delta}_X = \nu_{X,GX}(\nu_{FX,X}(\delta_X\varphi))$ *and* $\psi G\psi = \nu_{X,GX}(\nu_{FX,X}(F\varphi\varphi))$.
(b) $\psi\tilde{\eta}_X = \eta_X\varphi$.
(c) (X, φ) *is a monadic algebra w.r.t.* (F, δ, η) *if and only if* (X, ψ) *is a comonadic coalgebra w.r.t.* $\left(G, \tilde{\delta}, \tilde{\eta}\right)$.
(d) *Defining* $\Phi((X, \varphi)) := (X, \psi)$ *on objects and* $\Phi\left((X, \varphi) \xrightarrow{h} (X', \varphi')\right)$ *on homomorphisms as* $\Phi((X, \varphi)) \xrightarrow{h} \Phi((X', \varphi'))$ *yields a well-defined functor between the category of monadic F-algebras and comonadic G-coalgebras making both categories isomorphic.*

At this point it should be noted that the previous result is not entirely new. It seems to be the case that it has passed into common knowledge, yet we found it hard to give a specific reference, e.g. to one of the common text books on category theory. In [12, 2.6(1)] the authors collect the proof from different references, one of which is the original paper by Eilenberg and Moore, [19, Theorem 3.1], showing that every monad arises from a naturally given adjunction. An explicit proof of Proposition 4.10 can also be found in [9, Proposition 4.5.3].

It follows from the proof, that Φ and its inverse can be seen as inverse functors between arbitrary F-algebras and G-coalgebras that restrict docilely to monadic algebras w.r.t. (F, δ, η) and comonadic coalgebras w.r.t. $\left(G, \tilde{\delta}, \tilde{\eta}\right)$, respectively.

Dualising the two previous results yields the converse implication:

Proposition 4.11 *Let C be a category with a pair of adjoint endo-functors $F \dashv G$, the adjunction being specified by the hom-set equivalence* $\mathrm{Hom}(F, -) \xrightarrow{\nu} \mathrm{Hom}(-, G)$, *the unit* $1_C \xrightarrow{\vartheta} GF$ *and the co-unit* $FG \xrightarrow{\varepsilon} 1_C$.

Furthermore, let two natural transformations $G \xrightarrow{\tilde{\delta}} GG$ and $G \xrightarrow{\tilde{\eta}} 1_C$ be given. Define families of morphisms $FF \xrightarrow{\hat{\delta}} F$ and $1_C \xrightarrow{\hat{\eta}} F$ dually as in Lemma 4.8, i.e. $\hat{\delta}_Y := \nu^{-1}_{FY,FY}(\nu^{-1}_{Y,GFY}(\vartheta_Y\tilde{\delta}_{FY}))$ and $\hat{\eta}_Y := \vartheta_Y\tilde{\eta}_{FY}$. Let $X \in C$ be an object and $X \xrightarrow{\psi} GX$ be a morphism. Declaring the morphism $FX \xrightarrow{\varphi := \nu^{-1}_{X,X}(\psi)} X$, we have

(a) $FF \xrightarrow{\hat{\delta}} F$ *and* $1_C \xrightarrow{\hat{\eta}} F$ *are indeed natural transformations.*
(b) $\left(F, \hat{\delta}, \hat{\eta}\right)$ *is a monad if and only if* $\left(G, \tilde{\delta}, \tilde{\eta}\right)$ *is a comonad.*

(c) (X, φ) *is a monadic algebra w.r.t.* $\left(F, \hat{\tilde{\delta}}, \hat{\tilde{\eta}}\right)$ *if and only if* (X, ψ) *is a comonadic coalgebra w.r.t.* $\left(G, \tilde{\delta}, \tilde{\eta}\right)$.

Proposition 4.12 *Let $\mathcal{C}$ be a category and $F, G \in \mathrm{End}\mathcal{C}$ be two adjoint endo-functors $(F \dashv G)$. The constructions of comonads out of monads and vice versa, presented in the two previous propositions are mutually inverse, i.e.*

(a) $(F, \delta, \eta) \mapsto \left(G, \tilde{\delta}, \tilde{\eta}\right) \mapsto \left(F, \hat{\tilde{\delta}}, \hat{\tilde{\eta}}\right) = (F, \delta, \eta)$;

(b) $\left(G, \tilde{\delta}, \tilde{\eta}\right) \mapsto \left(F, \hat{\tilde{\delta}}, \hat{\tilde{\eta}}\right) \mapsto \left(G, \tilde{\tilde{\delta}}, \tilde{\tilde{\eta}}\right) = \left(G, \tilde{\delta}, \tilde{\eta}\right)$.

An analogous statement holds w.r.t. the monadic algebras and comonadic coalgebras.

Proof The statements (a) and (b) follow from the definitions, the naturality conditions for the unit ϑ, δ and η, and Eqs. 4.5 and 4.3, using similar manipulations as in the proof of Proposition 4.9. Details can again be found in [9]. □

As a consequence of Propositions 4.10, 4.11 and 4.12, it does not matter if we regard (abstract) dynamical systems as monadic algebras or comonadic coalgebras. The coalgebraic perspective however offers us some variability that is not necessarily available on the side of algebras. Slight modifications of the signature $-^T$ may result in a functor that fails to have a left-adjoint, and thus the corresponding coalgebras may lack a counterpart on the algebraic side.

From the applications of coalgebra to transition systems in computer science, a decent choice of related signature functors for the coalgebraic formulation suggests itself. For our convenience and since it is very common for transition systems, we state these functors for the case $\mathcal{C} = \mathcal{S}et$. Using $-^T$ every state x of the coalgebra is mapped to a T-sequence of successor states, the trajectory of $x \in X$. This closely corresponds to a deterministic automaton with state set X and possibly infinite alphabet T. Its behaviour can be extended by observations (or outputs) from a fixed set A. This is possible in at least two ways: one may add one observation per trajectory, resulting in the functor $X \mapsto A \times X^T$, or one per each successor state, yielding the functor $X \mapsto (A \times X)^T$. Instead of using a fixed set A as observables, one may also try the state space itself, giving rise to $X \mapsto X \times X^T$.

Besides, non-determinism can be represented without any difficulties: instead of assigning to each state a sequence of future states, one may assign a T-sequence of subsets of possible successor states. This is expressible using the endo-functor $X \mapsto \mathfrak{P}(X)^T$. Of course, all these different features may also be combined in one functor, such as $X \mapsto (A \times \mathfrak{P}(X))^T$.

We leave it as a task for future investigations to generalise these functors to categories like $\mathcal{T}op$ or $\mathcal{M}eas\mathcal{B}l$, and to explore the increased expressivity for particular examples of dynamical systems.

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