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GERM REPRESENTABILITY AND LOCAL INTEGRATION OF VECTOR FIELDS IN A WELL ADAPTED MODEL OF SDG

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Introduction

This paper deals with the topos $\mathscr{G}$ constructed in [3]. This topos is a Well Adapted Model of Synthetic Differential Geometry [3, Theorem 17] which satisfies furthermore two additional basic axioms concerning local (in the topological sense) aspects of the line R. The first asserts that germs of functions from R^n to R are representable, and the second that vector fields on R^n can be integrated up to all $t \in \Delta$, where Δ denotes the object that represents germs from R to R. Here we prove these two assertions; and include some minimal background on C^∞-rings, the topos $\mathscr{G}$ and Penon's results [8] concerning logical opens in the topos.

Some background on C^∞-rings

We consider the category $\mathscr{B}$ of finitely generated C^∞-rings of the form $A = C^\infty(\mathbb{R}^n)/I$, where I is an ideal of local nature (germ determined C^∞-rings). A function $f \in C^\infty(\mathbb{R}^n)$ is *locally* in I, denoted $f \mathring{\in} I$, when:

$$f \mathring{\in} I \quad \Leftrightarrow \quad \text{Each point } p \in Z(I) \text{ has a neighbourhood } U,\ p \in U, \text{ such that the restriction } f|_U \text{ is in the ideal } I/U. \text{ That is, there are finitely many } k_i \in I,\ \varphi_i \in C^\infty(U), \text{ such that}$$

$$f(x) = \sum_i \varphi_i(x) k_i(x) \quad (\forall x \in U).$$

(where $Z(I) \subset \mathbb{R}^n$ is the set of zeros of I). An ideal of local nature is an ideal I such that $f \in I$ iff $f \mathring{\in} I$.

To work with ideals of local nature is the same that to work with all ideals, but replacing the global membership relation '$\in$' by the local membership relation '$\mathring{\in}$'.

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More precisely, given any $I \subset C^\infty(\mathbb{R}^n)$, $A = C^\infty(\mathbb{R}^n)/I$, and $J \subset C^\infty(\mathbb{R}^s)$, $B = C^\infty(\mathbb{R}^s)/J$, a morphism $A \to B$ can be considered to be a morphism $C^\infty(\mathbb{R}^n) \xrightarrow{\varphi} C^\infty(\mathbb{R}^s)$ such that for all $h \in C^\infty(\mathbb{R}^n)$, if $h \mathring{\in} I$, then $\varphi(h) \mathring{\in} J$. Thus, the coproduct in the category $\mathscr{B}$ is given by the formula:

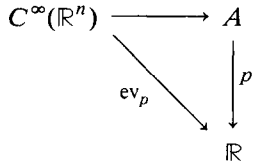
$$A \otimes_\infty B = C^\infty(\mathbb{R}^{n+s})/(I, J).$$

Given a function $f(x, y) \in C^\infty(\mathbb{R}^{n+s})$ we have

$f(x, y) \mathring{\in} (I, J) \quad \Leftrightarrow$ Each pair of points $p \in Z(I)$, $q \in Z(J)$ has a neighbourhood U, $(p, q) \in U$, such that there are finitely many $h_i \in I$, $l_j \in J$, and $\varphi_i, \psi_j \in C^\infty(U)$, such that

$$f(x, y) = \sum_i \varphi_i(x, y) h_i(x) + \sum_j \psi_j(x, y) l_j(y) \quad \text{(for all } (x, y) \text{ in } U).$$

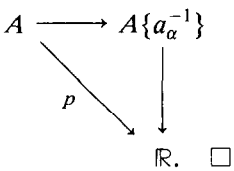
Given $A = C^\infty(\mathbb{R}^n)/I$, the set $Z(I) \subset \mathbb{R}^n$ is identified with the set of morphisms $A \to \mathbb{R}$ via the factorization



where ev_p denotes evaluation at the point p.

Given any open set $U \subset \mathbb{R}^n$ and any $f \in C^\infty(\mathbb{R}^n)$ such that $U = \{x \mid f(x) \neq 0\}$, then the restriction map $C^\infty(\mathbb{R}^n) \to C^\infty(U)$ is the solution in $\mathscr{B}$ to the universal problem of making f invertible [3]. Given $A \in \mathscr{B}$ and $a \in A$, we shall denote $A \to A\{a^{-1}\}$ the solution in $\mathscr{B}$ to the universal problem of making f invertible. We recall from [4]:

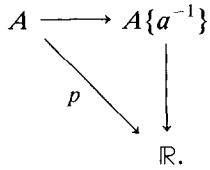
Proposition 1. *The open covers $U_a \subset U$ are universal effective epimorphic families $C^\infty(U) \to C^\infty(U_a)$ in $\mathscr{B}^{\mathrm{op}}$. Furthermore, given $A \in \mathscr{B}$ and a family $a_\alpha \in A$, $A \to A\{a_\alpha^{-1}\}$ covers in the open cover topology iff for all morphisms $A \xrightarrow{p} \mathbb{R}$ there exists α such that p factors,*



□

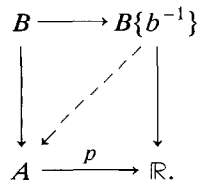
Corollary 2. *Given $A \in \mathscr{B}$ and $a \in A$, a is invertible in A iff for all morphisms $A \xrightarrow{p} \mathbb{R}$, $p(a) \neq 0$ in $\mathbb{R}$.*

Proof. For all $A \xrightarrow{p} \mathbb{R}$ there is a factorization



Therefore, the (one member) family $A \to A\{a^{-1}\}$ is effective epimorphic in $\mathscr{B}^{\mathrm{op}}$. Since it is clearly also a monomorphism in $\mathscr{B}^{\mathrm{op}}$, it follows that $A \approx A\{a^{-1}\}$. Thus a is invertible in A. □

Corollary 3. *Let $B \in \mathscr{B}$ and $b \in B$. Consider a map $B \to A$ into any other object A in $\mathscr{B}$, and the following diagram.*



If for every morphism $A \xrightarrow{p} \mathbb{R}$ there is a morphism $B\{b^{-1}\} \to \mathbb{R}$ making the square commutative, then the map $B \to A$ factors through $B\{b^{-1}\}$ as indicated in the diagram.

Proof. Immediate from Corollary 2. □

We finally remark that for any manifold M, the ring $C^\infty(M)$ is in $\mathscr{B}$ (cf. [3]) and that Proposition 1 also holds for coverings $U_a \subset M$. Similarly, for $U \subset M$, the restriction map $C^\infty(M) \to C^\infty(U)$ is the solution in $\mathscr{B}$ to the universal problem of making invertible any function $f \in C^\infty(M)$ such that $U = \{x \mid f(x) \neq 0\}$.

Some background on the topos $\mathscr{G}$

The topos $\mathscr{G}$ is defined to be the topos of sheaves on $\mathscr{B}^{\mathrm{op}}$ for the open cover topology. $\mathscr{G}$ is a *well adapted model* of S.D.G. There are full embeddings $\mathscr{B}^{\mathrm{op}} \to \mathscr{G}$ and $\mathscr{M} \xrightarrow{i} \mathscr{G}$, where $\mathscr{M}$ denotes the category of (paracompact) C^∞-manifolds. The embedding i preserves transversal pullbacks and open covers (cf. [3]).

If $A \in \mathscr{B}$, then we denote by $\bar{A} \in \mathscr{B}^{\mathrm{op}}$ the corresponding object in $\mathscr{B}^{\mathrm{op}}$ (and also in $\mathscr{G}$). Given $M \in \mathscr{M}$, by definition we have

$$iM = \overline{C^\infty(M)} \quad \text{in } \mathscr{G}.$$

We denote by R the image of $\mathscr{G}$ of the ring of real numbers $\mathbb{R}$ in $\mathscr{M}$. Thus,

$$R = i\mathbb{R} = \overline{C^\infty(\mathbb{R})} \quad \text{in } \mathscr{G}.$$

Since $1 = \bar{\mathbb{R}}$, given $A = C^\infty(\mathbb{R}^n)/I$, the set $Z(I)$ is identified with the set of global sections of $\bar{A}$ in $\mathcal{G}$. We shall denote by Γ the global sections functor. $\Gamma(\bar{A}) \approx Z(I)$. Clearly we have:

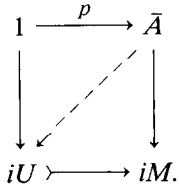
$$\Gamma iM = M \quad \text{for all } M \in \mathcal{M}.$$

The following is an immediate consequence of the definitions:

Proposition 4. *Given $A \in \mathcal{B}$, $A = C^\infty(\mathbb{R}^n)/I$, and any pair of subobjects $X \rightarrowtail \bar{A}$, $Y \rightarrowtail \bar{A}$, then $X \cup Y = \bar{A}$ in $\mathcal{G}$ iff there exists $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^n$ open, $iU \cap \bar{A} \subset X$, $iV \cap \bar{A} \subset Y$ and $Z(I) \subset U \cap V$. In particular, when $\bar{A} = R^n$, we have that $X \cup Y = R^n$ in $\mathcal{G}$ iff there are open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^n$ such that $iU \subset X$, $iV \subset Y$ and $U \cup V = \mathbb{R}^n$.*

Proof. Straightforward from Proposition 1. □

Proposition 5. *Let $M \in \mathcal{M}$ and $A \in \mathcal{B}$, and let $U \subset M$ be any open subset. Consider a map $\bar{A} \to iM$ and the following diagram:*



If for every global section $1 \xrightarrow{p} \bar{A}$ there is a map $1 \to iU$ making the square commutative, then the map $\bar{A} \to iM$ factors through iU as indicated in the diagram.

Proof. This is just Corollary 3 above. □

Corollary 6. *Let $U \subset M$ be any open set, and let $X \to iM$ be any subobject in $\mathcal{G}$. If $\Gamma(X) \subset U$, then $X \subset iU$. In particular, given $A \in \mathcal{B}$ of the form $A = C^\infty(\mathbb{R}^n)/I$, and $U \subset \mathbb{R}^n$, if $Z(I) \subset U$, $\bar{A} \subset iU$.*

Proof. Observe that objects in $\mathcal{B}^{\mathrm{op}}$ generate $\mathcal{G}$. Then utilize Proposition 5. □

1. Penon's characterization of open sets in $\mathcal{G}$

Recall that given an object X in a topos, a part $U \in \Omega^X$ is said to be *Penon open* iff the following condition holds:

$$\forall x \in U \ \forall y \in X \ y \neq x \vee y \in U.$$

For details, we refer the reader to [5, 7, 8]. We shall give here a direct proof of

Penon's characterization theorem of logical opens for the particular case of manifolds in the topos $\mathscr{G}$.

Theorem 7. *Given a manifold $M \in \mathscr{M}$, we have*:

(a) *If $U \subset M$ is any open subset, then $iU \subset iM$ is a Penon open subobject, and $U = \Gamma iU$.*

(b) *If $X \subset iM$ is any Penon open subobject, then $\Gamma(X) \subset M$ is an open subset, and $X = i\Gamma(X)$. Thus, Γ and i establish a bijection between open subsets of M and Penon open subobjects of iM.*

Proof. (a) Consider the diagram

$$\begin{array}{ccccc} & & iU \times iU & \xrightarrow{\pi} & iU \\ & & \downarrow & & \downarrow \\ \neg \Delta_{iU} & \rightarrowtail & iM \times iU & \xrightarrow{\pi} & iM \end{array}$$

where Δ_{iU} indicates the diagonal and $\neg$ the negation in $\mathscr{G}$. We have to show that $iU \times iU$ and $\neg \Delta_{iU}$ cover $iM \times iU$ in $\mathscr{G}$. Let $C\Delta_U$ be the set-theoretical complement of the diagonal Δ_U in $M \times U$. It is an open subset, thus we can consider the object $iC\Delta_U$ in $\mathscr{G}$. Since $i\Delta_U = \Delta_{iU}$, it follows that $iC\Delta_U \rightarrow \neg \Delta_{iU}$. (Recall that $i\emptyset = 0$ and that i preserves products.) With this, the claim follows from the fact that $U \times U$ and $C\Delta_U$ is an open cover of $M \times U$.

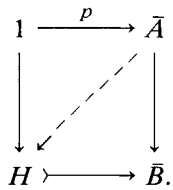
(b) Let $p \in \Gamma X$ be any point of ΓX, $1 \xrightarrow{p} X$. We have that $\neg\{p\} \cup X = iM$ in $\mathscr{G}$. Thus (Proposition 4) there exist open subsets $U \subset M$, $V \subset M$ such that $iU \subset \neg\{p\}$, $iV \subset X$ and $U \cup V = M$. Clearly, $V = \Gamma iV \subset \Gamma X$ and $p \in V$, which shows that ΓX is open. (If $p \notin V$, then $p \in U$. Thus $1 \xrightarrow{p} iU$ which implies $1 \xrightarrow{p} \neg\{p\}$. Then the empty family would cover 1.)

It remains to see that $X = i\Gamma X$. In the argument above, the open sets V (one for each p) cover the open set ΓX. This implies that $iV \subset i\Gamma X$ cover in $\mathscr{G}$. Since each $iV \subset X$, it follows that $i\Gamma X \subset X$. On the other hand, since $\Gamma X = \Gamma i \Gamma X$, it follows from Corollary 6 that $X \subset i\Gamma X$. □

Corollary 8. *Let $A \in \mathscr{B}$, $A = C^\infty(\mathbb{R}^n)/I$ and let $X \subset \bar{A}$ be any subobject in $\mathscr{G}$. Then X is Penon open iff it is of the form $X = iU \cap \bar{A}$ for some $U \subset \mathbb{R}^n$ open.*

Proof. Assume X is Penon open. Then for any $p \in Z(I)$ such that $1 \xrightarrow{p} X$, $\neg\{p\} \cup X = \bar{A}$. Then, by Proposition 4 and arguing exactly as in the proof of Theorem 7(b), it follows that there is an open set $V \subset \mathbb{R}^n$, $iV \cap \bar{A} \subset X$ and $p \in V$. Let U be the union of all the sets V (one for each p). Then $iU \cap \bar{A} \subset X$. As before, since $\Gamma(X) \subset U$, it follows from Corollary 6 that $X \subset iU$. It remains to see that any subobject of the form $X = iU \cap \bar{A}$ is Penon open. But this is immediate from Theorem 7(a). □

Proposition 9. *Let $A, B \in \mathscr{B}$, and let $H \subset \bar{B}$ be any Penon open subobject. Consider a map $\bar{A} \to \bar{B}$ and the following diagram:*



If for every global section $1 \xrightarrow{p} \bar{A}$ there is a map $1 \to H$ making the square commutative, then the map $\bar{A} \to \bar{B}$ factors through H as indicated in the diagram.

Proof. Immediate from Proposition 5 and Corollary 8. □

Corollary 10. *Let $B \in \mathscr{B}$, and let $H \subset \bar{B}$ be any Penon open subobject of $\bar{B}$. Given any subobject $X \subset \bar{B}$ of $\bar{B}$ in $\mathscr{G}$, if $\Gamma(X) \subset \Gamma(H)$, then $X \subset H$.* □

Proposition 11. *All objects of the form $\bar{A}, A \in \mathscr{B}$, satisfy the covering principle with respect to the intrinsic Penon topology in $\mathscr{G}$. That is, the following condition holds:*

$$\forall X, Y \in \Omega^{\bar{A}} (X \cup Y = \bar{A} \Rightarrow lX \cup lY = \bar{A})$$

where l indicates the interior (largest open inside).

Proof. Let $\bar{B} \xrightarrow{X} \Omega^{\bar{A}}$, $\bar{B} \xrightarrow{Y} \Omega^{\bar{A}}$. The corresponding subobjects $X \subset \bar{A} \times \bar{B}$, $Y \subset \bar{A} \times \bar{B}$ are such that $X \cup Y = \bar{A} \times \bar{B} = \overline{A \otimes_{\infty} B}$. By Propositions 4 and 8 there are Penon open $H \subset X$, $G \subset Y$ such that $H \cup G = \bar{A} \times \bar{B}$. But $\bar{B} \xrightarrow{H} \Omega^{\bar{A}}$, $\bar{B} \xrightarrow{G} \Omega^{\bar{A}}$ factor through $P(\bar{A}) \subset \Omega^{\bar{A}}$, where $P(\bar{A})$ indicates the object of Penon Opens of $\bar{A}$. (This is just sheaf semantics.) This proves the statement. □

Proposition 12. *All objects of the form $\bar{A}, A \in \mathscr{B}$ are separated for the intrinsic Penon topology in $\mathscr{G}$. That is, the following condition holds:*

$$\forall x \in \bar{A} \neg \{x\} \text{ is Penon open.}$$

Proof. It follows easily from Propositions 4 and 8 arguing as in Proposition 11. □

Proposition 13. *Let A, B in $\mathscr{B}$ and $1 \xrightarrow{p} \bar{B}$. Let G be any family of Penon Opens indexed by $\bar{A}$. That is, $G \subset \bar{B} \times \bar{A}$ is a Penon Open of $\bar{B}$ in $\mathscr{G}/\bar{A}$. Then, if $\{p\} \times \bar{A} \subset G$, there exists a Penon open X of $\bar{B} \times \bar{A}$ in $\mathscr{G}$ such that $\{p\} \times \bar{A} \subset X \subset G$.*

Proof. This is just [2, Lemma II.1.7]. Notice that the hypothesis in the lemma is precisely Proposition 11. We remark that this lemma has a simple synthetic proof which is valid in any topos. □

2. The axiom of germ representability

Let us recall that $\Delta(n) \subset R^n$ denotes the object $\Delta(n) = \neg\neg\{0\} \subset R^n$. It is the largest infinitesimal neighbourhood of $0 \in R^n$ in $\mathscr{G}$. It follows from Propositions 9 and 10 above and a simple synthetic argument (see [2, II.1.10]) that $\Delta(n) = P_0(R^n)$, where $P_0(R^n)$ denotes the intersection of all Penon open neighbourhoods of $0 \in R^n$. That is,

$$P_0(R^n) = [[x \in R^n \mid \forall U \in \Omega^{R^n}(U \in P(R^n) \wedge 0 \in U) \Rightarrow x \in U]]$$

where $P(R^n) \subset \Omega^{R^n}$ is the object (locale) of Penon open parts of R^n.

Consider now the C^∞-ring of germs of smooth functions on n variables, $C_0^\infty(\mathbb{R}^n)$. It is well known that $C_0^\infty(\mathbb{R}^n)$ is in $\mathscr{B}$ and equals the quotient $C^\infty(\mathbb{R}^n)/J$, where J is the ideal of functions with null germ at the origin.

Proposition 14. *$\Delta(n)$ is representable in $\mathscr{G}$ by the ring $C_0^\infty(\mathbb{R}^n)$.*

Proof. By definition $\overbrace{C_0^\infty(\mathbb{R}^n)} \to R^n$ is the (external) intersection of $C^\infty(U)$ for all open sets $0 \in U \subset \mathbb{R}^n$. Thus by the theorem of characterization of open sets (Theorem 7) we have $P_0(R^n) \subset \overline{C_0^\infty(\mathbb{R}^n)}$. It only remains to see the other inclusion. That is, we have to check the statement

$$\forall U \in \Omega^{R^n}(0 \in U \in P(R^n) \Rightarrow \overline{C_0^\infty(\mathbb{R}^n)} \subset U).$$

Let $A \in \mathscr{B}$, $A = C^\infty(\mathbb{R}^p)/I$, and let $\bar{A} \xrightarrow{U} P(R^n)$. Consider the corresponding subobject $U \subset R^n \times \bar{A}$. We have $\{0\} \times \bar{A} \subset U$. Then, by Proposition 13 there exists a Penon open H of $R^n \times \bar{A}$ such that $\{0\} \times \bar{A} \subset H \subset U$. In particular $\{0\} \times Z(I) \subset \Gamma(H)$. But $\{0\} \times Z(I) = Z(J) \times Z(I) = Z(J, I)$. Thus, by Corollary 10 it follows that $C_0^\infty(\mathbb{R}^n) \times \bar{A} \subset H$. This finishes the proof. □

Consider now the object $\mathrm{Partial}(R^n, R) \xrightarrow{\partial} \Omega^{R^n}$ of partial maps, with ∂ denoting the 'domain' map. A germ at 0 is an equivalence class of elements $f \in \mathrm{Partial}(R^n, R)$ such that $0 \in \partial(f) \in P(R^n)$. The equivalence relation is:

$$f \sim g \quad \Leftrightarrow \quad \exists U \in P(R^n),\ 0 \in U \subset \partial(f) \cap \partial(g) \colon f|_U = g|_U.$$

The object of germs is denoted $C_0^g(R^n, R)$. Since $0 \in \partial(f) \in P(R^n)$ clearly implies $\Delta(n) \subset \partial(f)$, there is a restriction map

$$C_0^g(R^n, R) \xrightarrow{j} R^{\Delta(n)}.$$

The axiom of germ representability, axiom III of S.D.G., says that this map is invertible. For more details the reader should consult [2]. We shall now prove that it holds in $\mathscr{G}$. (See also [6, 8] for earlier proofs of this.)

Theorem 15. *In $\mathscr{G}$, for each positive integer n, the restricton map j (defined above) is invertible. This statement is divided in two parts, the injectivity and the surjectivity*

of the map j. They mean the validity in the topos of the following two formulae:

(a) $\forall f, g \in \text{Partial}(R^n, R),\ 0 \in \partial(f) \in P(R^n),\ 0 \in \partial(g) \in P(R^n),$

$$f|_{\Delta(n)} = g|_{\Delta(n)} \quad \Rightarrow \quad \exists U \in P(R^n),\ 0 \in U \subset \partial(f) \cap \partial(g),\ f|_U = g|_U.$$

Remark that all this means is the purely logical statement:

$$\frac{\neg(x \neq 0) \Rightarrow (f(x) = g(x))}{(x \neq 0) \vee (f(x) = g(x))}.$$

(b) $\forall g \in R^{\Delta(n)}\ \exists f \in \text{Partial}(R^n, R),\ 0 \in \partial(f) \in P(R^n),\ f|_{\Delta(n)} = g.$

Proof. (a) Let $A \in \mathscr{B}$, $A = C^\infty(\mathbb{R}^S)/I$ and let $\bar{A} \xrightarrow{f} \text{Partial}(R^n, R)$, $\bar{A} \xrightarrow{g} \text{Partial}(R^n, R)$. Then f and g are partial maps from $R^n \times \bar{A}$ into R in $\mathscr{G}$, and this domain (which we can assume to be the same) is a subobject $H \to R^n \times \bar{A}$ such that the corresponding map $\bar{A} \to \Omega^{R^n}$ factors $\bar{A} \to P(R^n) \subset \Omega^{R^n}$, and the map $\bar{A} \xrightarrow{(0, \text{id})} R^n \times \bar{A}$ factors $\bar{A} \to U \to R^n$. Also, $\Delta(n) \times \bar{A} \to H$. In a diagram:

$$\begin{array}{ccccc} \{0\} \times \bar{A} & \rightarrowtail & \Delta(n) \times \bar{A} & \rightarrowtail & H \longrightarrow R^n \times \bar{A} \\ & & & & \Big\downarrow f \quad \Big\downarrow g \\ & & & & R \end{array}$$

The object H is a Penon Open in the topos $\mathscr{G}/\bar{A}$ but not necessarially in the topos $\mathscr{G}$ (since a family of opens is not (in general) open in the product). This problem does not arise in the proofs given in [6, 8], since these germs are supposed to be globally defined. Here we do as follows. By Proposition 13 there is a Penon open $G \subset R^n \times \bar{A}$, $\{0\} \times \bar{A} \subset G \subset H$. We take then by Corollary 8 an open subset $W \subset \mathbb{R}^n \times \mathbb{R}^s$ such that $G = iW \cap \bar{A}$. That is, $G = \overline{C^\infty(W)/I|_W}$. Recall that J denotes the ideal of functions on n-variables of germ null at the origin. Let us also denote by f, g representives in $C^\infty(W)$ for the maps $f, g : G \to R$. The fact that f and g are equal on $\Delta(n) \times \bar{A} \subset iW$ means that $(f-g) \overset{\circ}{\in} (J, I)|_W$. Thus, (see the introduction) each point $(0, p) \in Z(J, I) \subset W$ has a neighbourhood V_p where $f-g$ is of the form

$$f - g = \sum_i \varphi_i h_i, \quad \varphi_i \in C^\infty(V_p),\ h_i \in I$$

(since the part corresponding to J is zero). This shows that there is an open set $V \subset W$, $\{0\} \times Z(I) = Z(J, I) \subset V$, and such that $f - g \overset{\circ}{\in} I|_V$. That is, f and g are equal on $U = \overline{C^\infty(V)/I|_V}$, $U = iV \cap \bar{A} \subset R^n \times \bar{A}$. Since U is a Penon Open of $R^n \times \bar{A}$ in $\mathscr{G}$, it follows that the corresponding map $\bar{A} \to \Omega^{R^n}$ factors $\bar{A} \to P(R^n) \subset \Omega^{R^n}$.

(b) The proof of (b) follows easily from Proposition 14. □

Actually, in $\mathscr{G}$ a stronger statement is true.

Proposition 16. *In $\mathcal{G}$, germs are globally defined. That is, the restriction map $R^{R^n} \to R^{\Delta(n)}$ is an epimorphism. In fact, for every A in $\mathcal{B}$, the map $R^{\bar{A}\times R^n} \to R^{\bar{A}\times\Delta(n)}$ is an epimorphism.*

Proof. It follows easily from Proposition 14. □

Given any A in $\mathcal{B}$, a 'germ defined around the axis $\bar{A}\times\{0\}\subset\bar{A}\times R^n$, is an equivalence class of elements $f\in\text{Partial}(\bar{A}\times R^n, R)$, such that $\bar{A}\times\{0\}\subset\partial(f)\in P(\bar{A}\times R^n)$. The equivalence relation is:

$$f\sim g \quad\Leftrightarrow\quad \exists U\in P(\bar{A}\times R^n),\ \bar{A}\times\{0\}\subset U\subset\partial(f)\cap\partial(g)\colon f|_U = g|_U.$$

The object of germs so defined is denoted $C^g_{\bar{A}\times\{0\}}(\bar{A}\times R^n, R)$, and, as before, we have a restriction map:

$$C^g_{\bar{A}\times\{0\}}(\bar{A}\times R^n, R) \xrightarrow{j} R^{\bar{A}\times\Delta(n)}$$

Theorem 17. *In $\mathcal{G}$, for each A in $\mathcal{B}$ and each positive integer n, the restriction map j (defined above) is invertible.*

Proof. It follows arguing in exactly the same way that in the proof of Theorem 15. □

We remark that an alternative proof can be obtained synthetically from Proposition 16 together with Theorem 15. Indeed, clearly from Proposition 16 it follows that the map j is an epimorphism. On the other hand, the injectivity of j follows from Theorem 15 together with Proposition 13, by the simple argument of [2, II.3.1].

3. The postulate of Δ-infinitesimal integration

The following considerations can be done in any model of S.D.G. In what follows, M will denote the Euclidean space R^m of dimension m, considered as a manifold, $R^m = M$. However, when R^m plays the role of the tangent space at a point of M, we shall denote it R^m. Given a function $g\in R^{m^M}$ and a point $p\in M$, they determine a differential equation:

$$\begin{cases} y' = g(y), \\ y(0) = p. \end{cases} \tag{1}$$

A solution f to this equation is a map f on the variables $(p,t)\in M\times R$ such that:

$$\begin{cases} \dfrac{\mathrm{d}f}{\mathrm{d}t}(p,t) = g(f(p,t)), \\ f(p,0) = p. \end{cases} \tag{2}$$

It will have, as domain of definition, some 'neighbourhood' H of the axe, $M\times\{0\}\subset H\subset M\times R$. For example, there is always (tautologically) a solution on $H=M\times D$ defined by $f(p,t)=p+tg(p)$. (Recall that $D=[[t\in R \mid t^2=0]]$.)

Recall that an integral flow of g is a map f on the variables $(p,t)\in M\times R$ such that:

$$\begin{cases} f(p,d)=p+d\cdot g(p) \quad \forall d\in D, \\ f(p,s+t)=f(f(p,s),t), \end{cases} \tag{3}$$

Similarly, f will have, as domain of definition, some part $H\subset M\times R$, $M\times\{0\}\subset H\subset M\times R$.

Any map $f(p,t)$ which satisfies (3) also satisfies (2), since

$$f(p,s+d)=f(f(p,s),d)=f(p,s)+dg(f(p,s)).$$

However, to show that any map which satisfies (2) also satisfies the flow equation (3), *uniqueness* of solutions to differential equations is essential. Indeed, given $f(p,t)$ such that (2), for each p and each s (fixed), the functions $y_1(t)=f(f(p,s),t)$ and $y_2(t)=f(p,s+t)$ both satisfy the differential equation $y'=g(y)$ with initial condition $y(0)=f(p,s)$.

We shall consider now the infinitesimal neighbourhood of the axis $H=M\times\Delta$, $M\times\{0\}\subset M\times\Delta\subset M\times R$. The previous considerations show the following:

Proposition 18. *The following two statements are equivalent in any model of S.D.G.:*

(1) $$\forall g\in R^{m^M}\ \exists! f\in M^{M\times\Delta}\colon f(p,0)=p,\ \frac{\mathrm{d}f}{\mathrm{d}t}(p,t)=g(f(p,t)).$$

(2) $$\forall g\in R^{m^M}\ \exists! f\in M^{M\times\Delta}\colon$$
$$\forall d\in D[f(p,d)=p+d\cdot g(p)],\ f(p,s+t)=f(f(p,s),t). \qquad \square$$

We shall now prove that they are valid in the model $\mathscr{G}$. This is the Postulate WA 2 of [2, II, 3.1, 3.2].

Consider the rings of global sections of the objects

$$R^{m^M} \quad \text{and} \quad M^{M\times\Delta} \quad (\text{recall that } M=R^m).$$
$$\Gamma(R^{m^M})=C^\infty(M,\mathbb{R}^m)=C^\infty(\mathbb{R}^m)^m.$$
$$\Gamma(M^{M\times\Delta})=(C^\infty(M\times\mathbb{R})/(0,J))^m,$$

where, as before, $J\subset C^\infty(\mathbb{R})$ is the ideal of functions with null germ at zero, and where we have also denoted with the letters M the manifold $\mathbb{R}^m$, $M=\mathbb{R}^m$. Given an element g in $\Gamma(R^{mM})$, $g\colon M\to\mathbb{R}^m$, let f be the solution of the differential equation (2) given by the classical theory of ordinary differential equations. f will be defined in an open set U, $M\times\{0\}\subset U\subset M\times\mathbb{R}$, $f\colon U\to\mathbb{R}^m$. Since $Z(0,J)=M\times\{0\}\subset U$, it follows from Corollary 3 that the quotient map $C^\infty(M\times\mathbb{R})\to C^\infty(M\times\mathbb{R})/(0,J)$ factors through $C^\infty(U)$. Thus f determines an element (actually each coordinate of f) in this quotient ring. The uniqueness (actually local uniqueness would suffice)

of f with respect to g implies that this determines a map:

$$C^\infty(M, \mathbb{R}^m) \to (C^\infty(M \times \mathbb{R})/(0, J))^m.$$

Clearly, f considered as an element of the ring $C^\infty(M \times \mathbb{R})^m$, satisfies the differential equation (2) modulo the ideal $(0, J)$. We shall now show that this map lifts into a map in the topos.

Theorem 19. *There is a map in* $\mathscr{G}$,

$$R^{m^M} \xrightarrow{S} M^{M \times \Delta}$$

which sends $g \in R^{mM}$ *into a solution* $f \in M^{M \times \Delta}$ *of the differential equation* (2). *This actually means the following*:

Given any A *in* $\mathscr{B}$ *and* $\bar{A} \xrightarrow{g} R^{m^M}$ $(\bar{A} \times M \xrightarrow{g} R^m)$, *there is* $f \in M^{M \times \Delta}$ $(\bar{A} \times M \times \Delta \xrightarrow{f} R^m)$ *such that*

(i) $$\forall \lambda \in \bar{A}, \begin{cases} \dfrac{\mathrm{d}f}{\mathrm{d}t}(\lambda, p, t) = g(\lambda, f(\lambda, p, t)), \\ f(\lambda, p, 0) = p. \end{cases}$$

(ii) *The correspondence* $g \mapsto f$ *is natural in* $\bar{A}$.

Proof. It follows easily from Lemma 20 below.

Lemma 20. *Let* I *be any ideal,* $I \subset C^\infty(\mathbb{R}^n)$, $g, h : \mathbb{R}^n \times M \to \mathbb{R}^m$ *any two smooth functions,* U *be an open set,* $\mathbb{R}^n \times M \times \{0\} \subset U \subset \mathbb{R}^n \times M \times \mathbb{R}$, *and* $f, l : U \to \mathbb{R}^m$ *smooth functions such that*:

$$\frac{\mathrm{d}f}{\mathrm{d}t}(\lambda, p, t) = g(\lambda, f(\lambda, p, t)), \qquad f(\lambda, p, 0) = p,$$

$$\frac{\mathrm{d}l}{\mathrm{d}t}(\lambda, p, t) = h(\lambda, l(\lambda, p, t)), \qquad l(\lambda, p, 0) = p.$$

Then, the following implication holds (*for each one of the m coordinates, that by abuse of notation we write without subindices*)

$$(g - h) \mathring{\in} (I, 0) \Rightarrow (f - l) \mathring{\in} (I, 0, J)\,|_U.$$

Note. Given g, a function f as above always exists by the classical theory of ordinary differential equations.

Proof. We have $Z(I, 0, J) = Z(I) \times M \times \{0\}$. Thus, we have to show that given any $p_0 \in M$, $\lambda_0 \in Z(I)$, there exists an open set $W \in U$, $(\lambda_0, p_0, 0) \in W$, such that $(f - l)\,|_W \in (I, 0, J)_W$. We do this now. The point $(\lambda_0, p_0) \in Z(I, 0) = Z(I) \times M$. Thus

by assumption, there is an open set $H \subset \mathbb{R}^n \times M$, $(\lambda_0, p_0) \in H$, and such that $(g-h)|_H \in (I,0)|_H$. That is, there are finitely many functions $s_i \in I$, $h_i \in C^\infty(H)$, $i=1,\dots,r$, such that

$$h(\lambda, p) = g(\lambda, p) + \sum_{i=1}^{r} s_i(\lambda) h_i(\lambda, p), \quad \forall (\lambda, p) \in H.$$

We now utilize a technique introduced in [1] to prove a similar theorem [1, Theorem 1.8]. Consider a parameter space $\mathbb{R}^r$, $s \in \mathbb{R}^r$, $s = (s_1, \dots, s_r)$, and the function

$$\varphi : \mathbb{R}^r \times H \to \mathbb{R}^m$$

defined by

$$\varphi(s, \lambda, p) = g(\lambda, p) + \sum_{i=1}^{r} s_i h_i(\lambda, p).$$

Clearly $\varphi(0, \lambda, p) = g(\lambda, p)$ and $\varphi(s(\lambda), \lambda, p) = h(\lambda, p)$. (Notice the double meaning of the letter "s"!) Let now $G \subset \mathbb{R}^n \times M$ open, $(\lambda_0, p_0) \in G$, $V \subset \mathbb{R}^r$ open, $0 \in V$, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ and $\psi : V \times G \times (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ be such that for all $(s, \lambda, p, t) \in V \times G \times (-\varepsilon, \varepsilon)$, $(\lambda, \psi(s, \lambda, p, t)) \in H$, and

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(s, \lambda, p, t) = \varphi(s, \lambda, \psi(s, \lambda, p, t)), \quad \psi(s, \lambda, p, 0) = p.$$

Such a ψ is given by the classical theory of ordinary differential equations. Let now $l_i : V \times G \times (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ be such that

$$\psi(s, \lambda, p, t) = \psi(0, \lambda, p, t) + \sum_{i=1}^{r} s_i l_i(s, \lambda, p, t).$$

The functions l_i are given by a well known result often quoted as local Hadamard's lemma (remark that the open set V can be considered to be a product of intervals). Now, since $\lambda_0 \in Z(I)$ and the functions $s_i \in I$, we have that $s(\lambda_0) = 0$. Let $H' \subset H$ be small enough so that $s(\lambda) \in V$ for all $\lambda \in H'$. Then, we have

$$\psi(s(\lambda), \lambda, p, t) = \psi(0, \lambda, p, t) + \sum_{i=1}^{r} s_i(\lambda) l_i(s(\lambda), \lambda, p, t)$$

for all (λ, p, t) in the open set $W = (G \cap H') \times (-\varepsilon, \varepsilon)$. But by the uniqueness of the solution for classical ordinary differential equations, it follows that $f(\lambda, p, t) = \psi(0, \lambda, p, t)$ and $l(\lambda, p, t) = \psi(s(\lambda), \lambda, p, t)$. This finishes the proof. □

This theorem establishes the existence condition in statement (1) of Proposition 18 (i.e., the existence condition of Postulate WA 2). But it establishes much more, since it says that the map

$$M^{M \times \Delta} \xrightarrow{D} R^{m^M}, \qquad f(p,t) \mapsto \frac{\mathrm{d}f}{\mathrm{d}t}(p,t)|_{t=0}$$

has a section in the topos. We shall now prove the uniqueness of solutions to dif-

ferential equations, that is, the uniqueness condition in statement (2) of Proposition 18. We thank Bruno who indicated to us a proof, which actually follows the same lines as the proof here of Lemma 20.

Lemma 21. *Let I be any ideal, $I \subset C^\infty(\mathbb{R}^n)$, $g: \mathbb{R}^n \times M \to \mathbb{R}^m$ and $f, l: \mathbb{R}^n \times M \times \mathbb{R} \to \mathbb{R}^m$ be smooth functions such that (each one of the m coordinates, that as before we denote without subindices)*

$$\frac{\mathrm{d}f}{\mathrm{d}t}(\lambda, p, t) - g(\lambda, f(\lambda, p, t)) \overset{\circ}{\in} (I, 0, J), \qquad f(\lambda, p, 0) - p \overset{\circ}{\in} (I, 0, J),$$

$$\frac{\mathrm{d}l}{\mathrm{d}t}(\lambda, p, t) - g(\lambda, l(\lambda, p, t)) \overset{\circ}{\in} (I, 0, J), \qquad l(\lambda, p, 0) - p \overset{\circ}{\in} (I, 0, J).$$

Then,

$$f(\lambda, p, t) - l(\lambda, p, t) \overset{\circ}{\in} (I, 0, J).$$

Proof. As before (Lemma 20) we have to show that given any $p_0 \in M$, $\lambda_0 \in Z(I)$, there exists an open set W, $(\lambda_0, p_0, 0) \in W$, such that $(f-l)\big|_W \in (I, 0, J)\big|_W$. By assumption, there is an open set H, $(\lambda_0, p_0, 0) \in H$ and functions $a_1(\lambda, p, t)$, $a_2(\lambda, p, t)$, $b_1(\lambda, p, t)$, $b_2(\lambda, p, t)$ (defined on H) in the ideal $(I, 0, J)\big|_H$ such that

$$\frac{\mathrm{d}l}{\mathrm{d}t}(\lambda, p, t) = g(\lambda, l(\lambda, p, t)) + a_1(\lambda, p, t), \qquad l(\lambda, p, 0) = p + b_1(\lambda, p, t),$$

$$\frac{\mathrm{d}f}{\mathrm{d}t}(\lambda, p, t) = g(\lambda, f(\lambda, p, t)) + a_2(\lambda, p, t), \qquad f(\lambda, p, 0) = p + b_2(\lambda, p, t)$$

(the functions b_1, b_2 actually do not depend on t).

Consider now $\mathbb{R}^2$ as a parameter space, $(a, b) \in \mathbb{R}^2$, and let ψ be such that:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(a, b, \lambda, p, t) = g(\lambda, \psi(a, b, \lambda, p, t)) + a, \qquad \psi(a, b, \lambda, p, 0) = p + b.$$

Such a ψ can be taken defined in an open set of the form $V \times G \times (-\varepsilon, \varepsilon)$, where $V \in \mathbb{R}^2$, $0 \in V$, is open (take V as a product of two intervals), $(\lambda_0, p_0) \in G$, and $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. By the uniqueness of the solution for classical ordinal differential equations, it follows that $l(\lambda, p, t) = \psi(a_1(\lambda, p, t), b_1(\lambda, p, t), \lambda, p, t)$ and $f(\lambda, p, t) = \psi(a_2(\lambda, p, t), b_2(\lambda, p, t), \lambda, p, t)$. As in Lemma 20, let now $r, s: V \times V \times G \times (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ be such that

$$\psi(a_2, b_2 \lambda, p, t) - \psi(a_1, b_1, \lambda, p, t) = (a_2 - a_1)r + (b_2 - b_1)s.$$

Now, since $\lambda_0 \in Z(I)$, it follows that $a_1(\lambda_0, p_0, 0) = 0$, $a_2(\lambda_0, p_0, 0) = 0$, $b_1(\lambda_0, p_0, 0) = 0$ and $b_2(\lambda_0, p_0, 0) = 0$. Let $W \subset H$, $(\lambda_0, p_0, 0) \in W$, be small enough so that $(a_1(\lambda, p, t), b_1(\lambda, p, t)) \in V$ and $(a_2(\lambda, p, t), b_2(\lambda, p, t)) \in V$ for all $(\lambda, p, t) \in W$. This finishes the proof. □

Theorem 22. *In the topos $\mathscr{G}$, the following holds:*

$$\forall g \in R^{m^M}\ \exists! f \in M^{M\times\Delta}\colon f(p,0)=p,\ \frac{\mathrm{d}f}{\mathrm{d}t}(p,t)=g(f(p,t))$$

where $M=R^m$. (See Proposition 18.)

Proof. The existence is guaranteed by Theorem 19, and the uniqueness follows immediately from Lemma 21. □

We finish with a remark. From Lemma 21, that is, the uniqueness of the solution of differential equations on $M\times\Delta$, it follows that in order to prove the existence of a section $R^{m^M}\xrightarrow{s} M^{M\times\Delta}$ to the map

$$M^{M\times\Delta}\xrightarrow{D} R^{m^M},\qquad D(f(p,t))=\frac{\mathrm{d}f}{\mathrm{d}t}(p,t)\,|_{t=0},$$

it is enough to prove that D is an epimorphism. Now, to prove that D is an epimorphism is very simple and easy, and does not need Lemma 20. We did prove Theorem 19 independently in order to show that the lifting theorem of Bruno [1, Theorem 1.8] can be proved also for other maps that the ones considered there. Theorem 19 should be helpful in order to formulate and prove a general lifting theorem for the topos $\mathscr{G}$.

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