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Duals Invert

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Abstract Monoidal objects (or pseudomonoids) in monoidal bicategories share many of the properties of the paradigmatic example: monoidal categories. The existence of (say, left) duals in a monoidal category leads to a dualization operation which was abstracted to the context of monoidal objects by Day et al. (Appl Categ Struct 11:229–260, 2003). We define a relative version of this called exact pairing for two arrows in a monoidal bicategory; when one of the arrows is an identity, the other is a dualization. In this context we supplement results of Day et al. (Appl Categ Struct 11:229–260, 2003) (and even correct one of them) and only assume the existence of biduals in the bicategory where necessary. We also abstract recent work of Day and Pastro (New York J Math 14:733–742, 2008) on Frobenius monoidal functors to the monoidal bicategory context. Our work began by focusing on the invertibility of

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In memory of Max

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components at dual objects of monoidal natural transformations between Frobenius monoidal functors. As an application of the abstraction, we recover a theorem of Walters and Wood (Theory Appl Categ 3:25–47, 2008) asserting that, for objects A and X in a cartesian bicategory $\mathscr{C}$, if A is Frobenius then the category $\mathrm{Map}\mathscr{C}(X, A)$ of left adjoint arrows is a groupoid. Also, the characterization in Walters and Wood (Theory Appl Categ 3:25–47, 2008) of left adjoint arrows between Frobenius objects of a cartesian bicategory is put into our current setting. In the same spirit, we show that when a monoidal object admits a dualization, its lax centre coincides with the centre defined in Street (Theory Appl Categ 13:184–190, 2004). Finally we look at the relationship between lax duals for objects and adjoints for arrows in a monoidal bicategory.

Keywords Monoidal bicategory · Monoidal object · Dual · Frobenius condition · Cartesian bicategory · Centre construction

Mathematics Subject Classifications (2000) Primary 18D05 · Secondary 18D10 · 18D15 · 18D20 · 18D25 · 18D35

1 Introduction and Preliminaries

1.1

The theorem that τV is invertible, for $\tau : F \Rightarrow G : \mathscr{V} \to \mathscr{W}$ a monoidal natural transformation between strong monoidal functors and V part of a dual situation in $\mathscr{V}$, goes back at least to [20]. When the second author heard the third reporting, on joint work with Bob Walters, that $\mathrm{Map}\mathscr{C}(X, A)$ is a groupoid for Frobenius A in cartesian $\mathscr{C}$, he conjectured that it should follow from Saavedra Rivano's theorem internalized to a monoidal bicategory (and dualized). At about the same time, we heard of Day and Pastro's discovery of Frobenius monoidal functors and their consequent improvement of Saavedra's theorem [6]. In fact, Frobenius monoidal functors are a degenerate case of the linear functors introduced in [4] which later gave rise to the linear bicategories of [2]. Frobenius functors with an extra separability condition appear in the work of Szlachányi [24, 25] on weak bialgebras. However, it was [6] that provided us with the impetus to prove such theorems in a general monoidal bicategory.

Meanwhile, the first author had gone beyond [15] (part of which will become [16] and [17]) in extending the work of [5]. The second author is grateful to the first for finding omissions in the statement of Proposition 1.1 (b) and (c) of [5]: the correct statement is a special case of our Proposition 4.2.

1.2

In the remaining subsections of Section 1 we recall the notions of a bidual situation and of a monoidal object in a monoidal bicategory. In Section 2 we generalize, to a

monoidal bicategory $\mathscr{M}$, what it is for an object V in a monoidal category $\mathscr{V}$ to be part of a dual situation. Here $\mathscr{V}$ is replaced by a monoidal object X in $\mathscr{M}$ and V is replaced by an arrow $x : A \nrightarrow X$. Ultimately, we need to assume that such an A has a right bidual A° in $\mathscr{M}$. The left dual of $x : A \nrightarrow X$ is an arrow $x^+ : A^\circ \nrightarrow X$ together with two 2-cells that satisfy two equations. We refer to such a dual situation as an *exact pairing* and give an interesting perspective of it that was noticed by Steve Lack. We note that if x is 1_X (so that X has a right bidual) then 1_X^+ was called a left dualization $d : X^\circ \nrightarrow X$ in [5]. In Section 3 we internalize Day and Pastro's notion of Frobenius monoidal functor and prove their theorems that Frobenius monoidal arrows 'preserve exact pairings' and that a monoidal and comonoidal 2-cell $\tau : f \Rightarrow g : X \nrightarrow Y$ with f and g Frobenius, has τx invertible whenever x is part of an exact pairing. For these key theorems, it is convenient to employ the extraordinary natural 2-cells of [22].

Section 4 provides equivalent formulations of exact pairings. These enable many interesting isomorphisms. In particular, when $(y, x; \alpha, \beta) : A \nrightarrow X$ is an exact pairing we can 'solve to within isomorphism' for each of x and y in terms of the other, the monoidal data for X, and the counit for the bidual situation $(A, A^\circ; e, n)$. An interesting application of the 'duals invert' theme is contained in Section 5, wherein it is shown that lax braidings for autonomous monoidal objects are necessarily braidings.

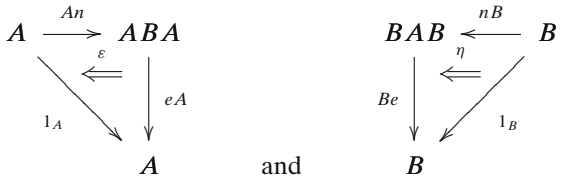
In Section 6 we show that what we call a *naturally Frobenius map monoidal object* A has itself as a right bidual, so that we can take $A^\circ = A$ and that the identity 1_A is part of an exact pairing, so that we can also take $1_A^+ = 1_A$. Some of our results in this section were anticipated by [15]. In Section 7 we make our promised application to cartesian bicategories by using the $(-)^{\mathrm{coop}}$ dual of our general results.

There is also in [26] a characterization of maps between Frobenius objects in a cartesian bicategory. In Section 8 we characterize *monoidal* maps between Frobenius map monoidal objects in a general monoidal bicategory. From the discussion in Section 7, it is clear how to recover the cartesian theorem of [26] from our general theorem. Finally, in Section 9 we take a brief look at *lax* duals in a monoidal bicategory. In particular, we describe lax duals in the bicategory $\mathrm{Sqr}\mathscr{M}$ of what might be called the lax squares of $\mathscr{M}$. The objects of $\mathrm{Sqr}\mathscr{M}$ are arrows of $\mathscr{M}$ and we show that certain lax right duals in $\mathrm{Sqr}\mathscr{M}$ have right adjoints in $\mathscr{M}$. This suggest that we can add to the slogan-like title of this paper, 'lax duals adjoin'. We conclude with two theorems providing justification for our declaration, 'lax transport exists'.

1.3

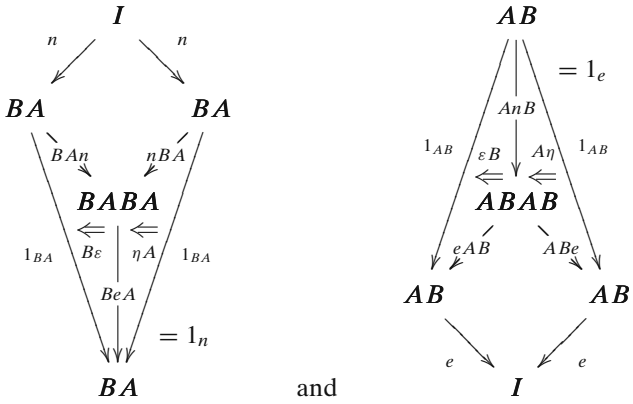
We work in a monoidal bicategory $\mathscr{M} = (\mathscr{M}, \otimes, I, \cdots)$, as in [8] for example. Especially for objects in $\mathscr{M}$, we usually collapse $\otimes$ to concatenation in displayed diagrams, so that $B \otimes C$ becomes BC, to save space. We take a few other liberties with some of our diagrams omitting, for example, structural isomorphisms of the form $(f \otimes 1)(1 \otimes g) \cong (1 \otimes g)(f \otimes 1)$. For A in $\mathscr{M}$, a right bidual for A consists of

an object B, and arrows $n : I \to B \otimes A$ and $e : A \otimes B \to I$, together with invertible 2-cells



(1)

subject to the coherence equations



(2)

We sometimes speak of the *bidual situation* $(A, B; e, n; \varepsilon, \eta)$ or, more simply $(A, B; e, n)$. We call n the unit, e the counit, and ε and η the constraints of the bidual situation. A right bidual for A is determined to within equivalence in $\mathscr{M}$ and we denote a choice of it by A°. When a bidual situation is given in advance we can treat ε and η as identities. However, in Section 6 we have occasion to construct a bidual situation and in this event we take care to verify the equations (Eq. 2). To the extent to which right biduals are available, we can use the extraordinary 2-cells and their composites defined and studied in [22].

1.4

We refer to a pseudomonoid $(X, p, j, \cdots)$ in $\mathscr{M}$ as a *monoidal object* in $\mathscr{M}$. For monoidal objects we often write $\bullet = p$ and use infix notation when it improves readability to do so. Thus, for $x : B \to X$ and $y : C \to X$, we write $x \bullet y = p(x \otimes y) : B \otimes C \to X$.

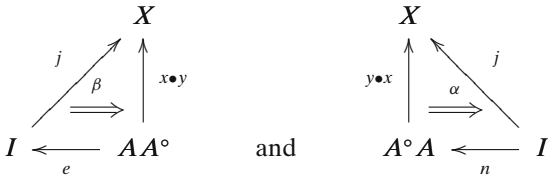
Since pseudofunctors send adjunctions to adjunctions and $A \otimes -$ and $- \otimes A$ are pseudofunctors we note that if $\eta, \varepsilon; f \dashv u : B \to X$ is an adjunction then, for example, in collapsed notation $A\eta, A\varepsilon; Af \dashv Au : AB \to AX$ is again an adjunction.

We refer to left adjoint 1-cells as *maps*. A *map monoidal object* is a monoidal object $(X, p, j, \cdots)$ where p, j are maps. By contrast a *monoidal map* is a left adjoint 1-cell in the bicategory of monoidal objects, lax monoidal arrows, and monoidal 2-cells.

2 Duals

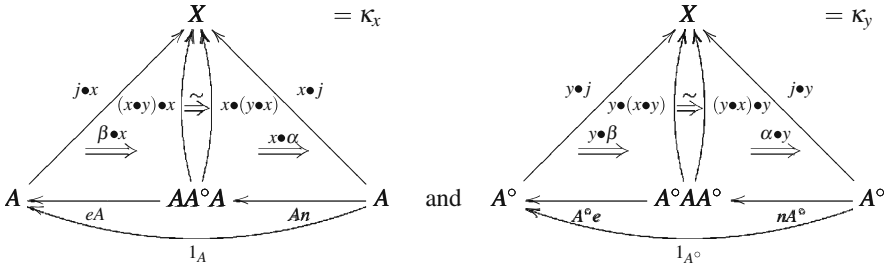
Our first definition is a generalization of *left dualization* as given in [5].

Definition 2.1 For A with a right bidual and X a monoidal object, an *exact pairing* for arrows $x : A \nrightarrow X$ and $y : A^\circ \nrightarrow X$ in $\mathscr{M}$ consists of the further data



(3)

subject to the equations



(4)

where the unlabelled 2-cells are instances of the associativity constraint for X and the κ's are instances of the obvious composites of the unitary constraints for X. In this situation we say that y is the *left dual* of x and that x is the *right dual* of y. We call α the *counit* and β the *unit* of the exact pairing and we also write $(y, x; \alpha, \beta) : A \nrightarrow X$.

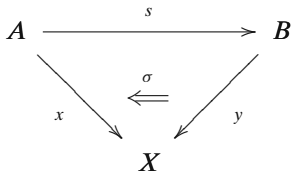
Remark 2.2 If A is also a monoidal object then a left dual for 1_A is precisely a left dualization for A in the sense of [5].

There are times when it is convenient to renormalize the equations (Eq. 4) so that the κ's become identities.

2.1

Steve Lack pointed out to us that to ask for a left dual in the sense of Definition 2.1, for an arrow $x : A \nrightarrow X$, is to ask for a *right bidual* for x seen as an object of a certain comma bicategory $\mathscr{M} /\!\!/ X$. Explicitly, the objects of $\mathscr{M} /\!\!/ X$ are the arrows

$x : A \nrightarrow X$ of $\mathscr{M}$ with codomain X. An arrow from $x : A \nrightarrow X$ to $y : B \nrightarrow X$ in $\mathscr{M} /\!\!/ X$ is a pair (s, σ), where $s : A \nrightarrow B$ is an arrow in $\mathscr{M}$ and σ is a 2-cell as in



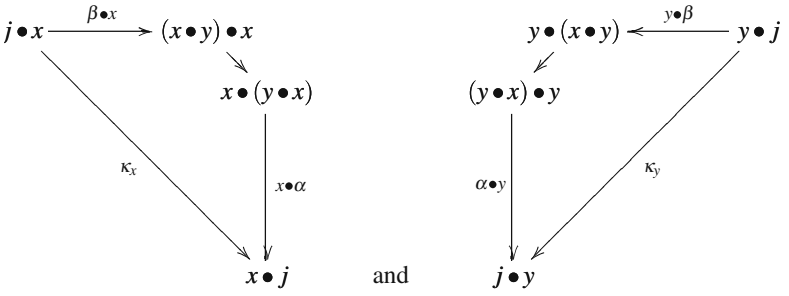
A 2-cell $\phi : (s, \sigma) \nrightarrow (t, \tau) : x \nrightarrow y$ in $\mathscr{M} /\!\!/ X$ is a 2-cell $\phi : s \nrightarrow t$ in $\mathscr{M}$ satisfying $\tau . y\phi = \sigma$. For $(X, p, j, \cdots)$ a monoidal object in $\mathscr{M}$, $\mathscr{M} /\!\!/ X$ becomes a monoidal bicategory with unit object $j : I \nrightarrow X$ and tensor product of $x : A \nrightarrow X$ and $y : B \nrightarrow X$ given by $x \bullet y : A \otimes B \nrightarrow X$. (In [5] the full subbicategory of $\mathscr{M} /\!\!/ X$ whose objects are maps is used for similar purposes but there it is assumed that $\mathscr{M}$ is autonomous and X is a left autonomous map monoidal object.) Of course there is, in the terminology of Cockett, Neifield, and Wood ('Variation and Enrichment', unpublished manuscript), a strict monoidal forgetful pseudofunctor $\mathscr{M} /\!\!/ X \nrightarrow \mathscr{M}$ which necessarily preserves right biduals. From this point of view, the data (Eq. 3) exhibit (n, α) as the unit and (e, β) as the counit for the bidual situation in $\mathscr{M} /\!\!/ X$. Also, the equations (Eq. 4) (renormalized) merely state that the invertible 2-cells $eA.An \nrightarrow 1_A$ and $1_{A^\circ} \nrightarrow A^\circ e.nA^\circ$ (which we have not even displayed, much less named in Eq. 4) are 2-cells in $\mathscr{M} /\!\!/ X$. The coherence conditions that these 2-cells are required to satisfy for a bidual situation in $\mathscr{M} /\!\!/ X$ are precisely the coherence conditions for a bidual situation $(A, A^\circ; e, n)$ in $\mathscr{M}$, which we gave in Eq. 2. As such they do not involve x, y, α, and β—which is why we have not fully embraced this point of view and continue to assume a bidual situation $(A, A^\circ; e, n)$ in $\mathscr{M}$.

Lack's observation does allow us to say immediately that a left dual for an arrow $x : A \nrightarrow X$ in $\mathscr{M}$ is determined to within equivalence in $\mathscr{M} /\!\!/ X$. However, observe that the typical arrow (s, σ) in $\mathscr{M} /\!\!/ X$ is an equivalence if and only if s is an equivalence in $\mathscr{M}$ and σ is an isomorphism. Thus if A° is indeed fixed we can take any two left duals $y : A^\circ \nrightarrow X$ and $z : A^\circ \nrightarrow X$ of x to be equivalent in $\mathscr{M} /\!\!/ X$ via an equivalence of the form $(1_{A^\circ}, \sigma)$ showing y and z to be isomorphic in $\mathscr{M}$. It follows that it makes sense to write $x^+ = y$ and, similarly, $y^- = x$ when y is a left dual of x.

2.2

We can see α and β as *extraordinary 2-cells* in the sense of [22]. Precisely, the (ordinary) 2-cell $\beta : j.e \nrightarrow x \bullet y$ becomes the extraordinary 2-cell $\beta : j \nrightarrow x \bullet y$ while the ordinary $\alpha : (y \bullet x)n \nrightarrow j$ becomes the extraordinary $\alpha : y \bullet x \nrightarrow j$. A calculus

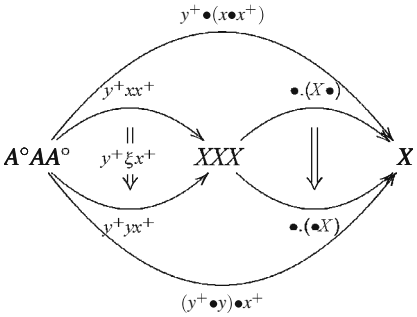
of composition of extraordinary 2-cells is given in [22], in terms of which the equations (Eq. 4) take the form



(5)

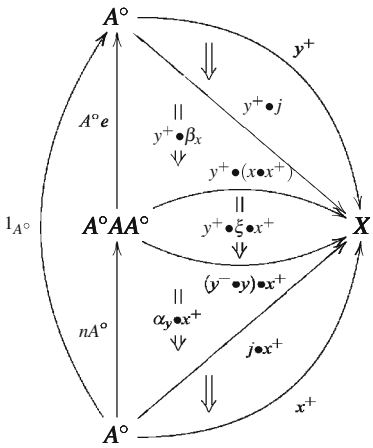
2.3

We can sharpen an application we wish to make by extending $(-)^+$ to 2-cells. Suppose that we have a 2-cell $\xi : x \nrightarrow y : A \nrightarrow X$ and exact pairings $(x^+, x; \alpha_x, \beta_x) : A \nrightarrow X$, $(y^+, y; \alpha_y, \beta_y) : A \nrightarrow X$. Firstly, we have the 2-cell obtained as the pasting composite



which could be called $y^+ \bullet \xi \bullet x^+$ (without parentheses). The unlabelled 2-cell is (the inverse of) the associativity constraint isomorphism for the monoidal object X. Next, the 2-cells $\beta_x : j.e \nrightarrow x \bullet x^+$ and $\alpha_y : (y^+ \bullet y)n \nrightarrow j$ give rise to 2-cells $y^+ \bullet \beta_x : (y^+ \bullet j)(A^\circ e) \nrightarrow y^+ \bullet (x \bullet x^+)$ and $\alpha_y \bullet x^+ : (y^+ \bullet y)(nA^\circ) \bullet x^+ \nrightarrow j \bullet x^+$, using the

monoidal structures both of $\mathscr{M}$ and of X as we did in Definition 2.1. These can be pasted with $y^+ \bullet \xi \bullet x^+$ and instances of the unit constraints for X as in



to give a 2-cell that we call $\xi^+ : y^+ \nrightarrow x^+ : A^\circ \nrightarrow X$. However, treating β_x and α_y as extraordinary 2-cells we can again follow [22], dispensing with explicit mention of n and e, to see ξ^+ as the composite

$$y^+ \longrightarrow y^+ \bullet j \xrightarrow{y^+ \bullet \beta_x} y^+ \bullet (x \bullet x^+) \xrightarrow{y^+ \bullet \xi \bullet x^+} (y^+ \bullet y) \bullet x^+ \xrightarrow{\alpha_y \bullet x^+} j \bullet x^+ \longrightarrow x^+$$

Similarly, if $\eta : y^+ \nrightarrow x^+$ is given, we can define $\eta^- : x \nrightarrow y$ as the composite

$$x \longrightarrow j \bullet x \xrightarrow{\beta_y \bullet x} (y \bullet y^+) \bullet x \xrightarrow{y \bullet \eta \bullet x} y \bullet (x^+ \bullet x) \xrightarrow{y \bullet \alpha_x} y \bullet j \longrightarrow y$$

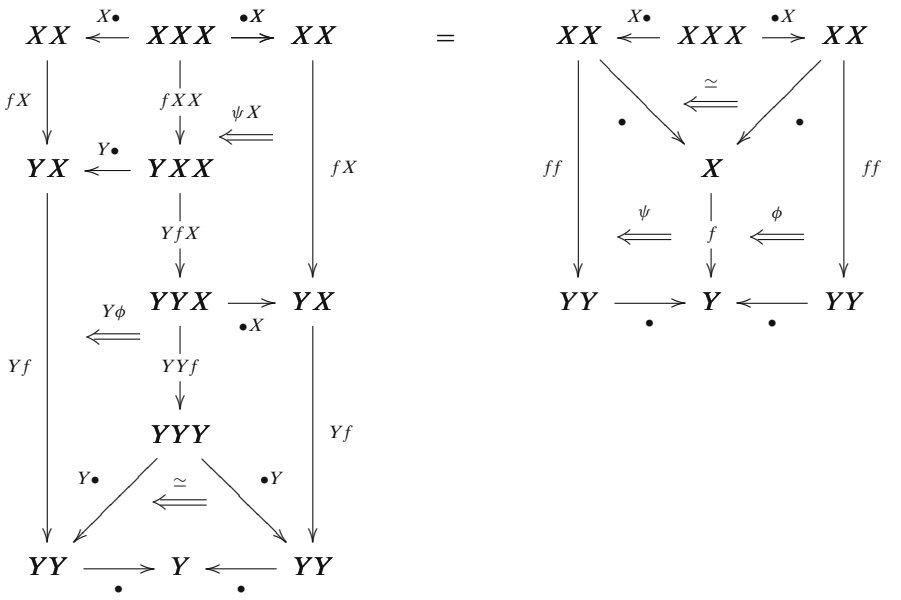
and it is then standard that $\xi^{+-} = \xi : x \nrightarrow y$ and $\eta^{-+} = \eta : y^+ \nrightarrow x^+$.

3 Frobenius Monoidal Arrows

3.1

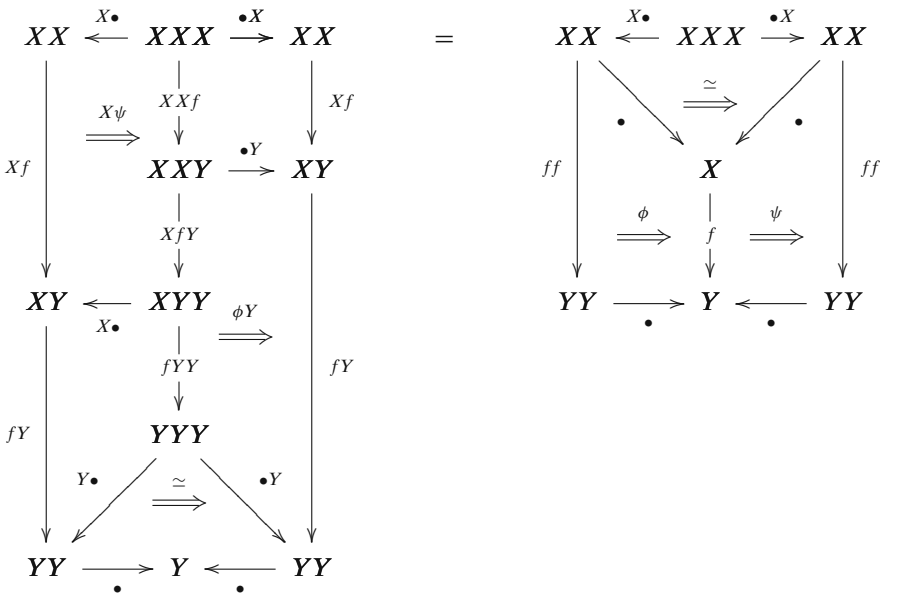
Suppose that an arrow $f : X \nrightarrow Y$ between monoidal objects is equipped with both lax monoidal structure (ϕ, ϕ_0) and colax monoidal structure (ψ, ψ_0). Following

[6] we say that $(f, \phi, \phi_0, \psi, \psi_0) : X \nrightarrow Y$ is a Frobenius monoidal arrow if it satisfies the Frobenius equations



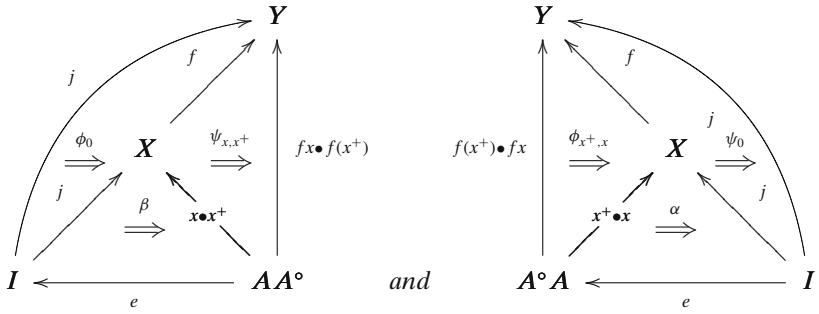
(6)

and



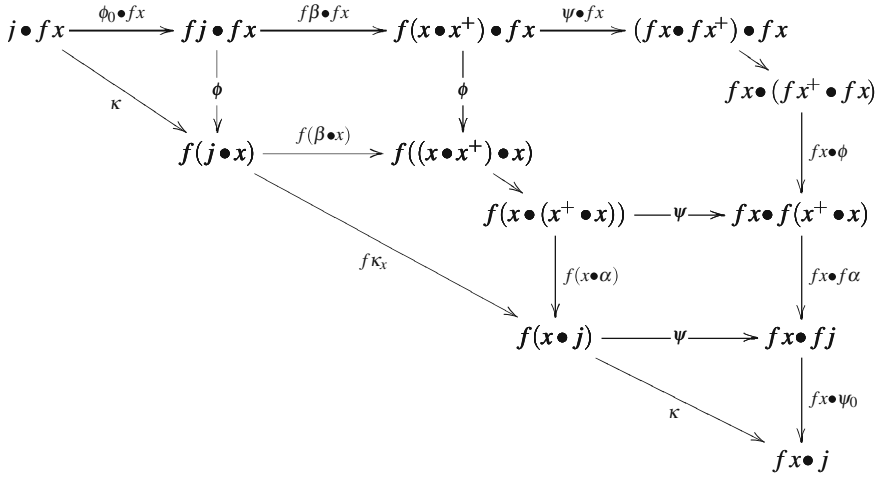
(7)

Theorem 3.1 (Day and Pastro) *If A has a right dual, X, Y are monoidal objects, $(x^+, x; \alpha, \beta) : A \nrightarrow X$ is an exact pairing, and $(f, \phi, \phi_0, \psi, \psi_0) : X \nrightarrow Y$ is Frobenius then $f(x^+)$ together with*



is a left dual for fx (so that we can write $(fx)^+ \cong f(x^+)$ and fx^+ is unambiguous).

Proof It is convenient to use commutative diagrams involving both (ordinary) 2-cells and extraordinary 2-cells, rather than pasting diagrams. We establish the first of the equations of Eq. 5. At this point, fx^+ is an abbreviation for $f(x^+)$.



The upper right hexagon is an instance of the first Eq. 6 of the Frobenius equations. The two unlabelled arrows arise from associativity constraints. The unadorned κ's are the obvious invertible 2-cells constructed via unitary constraints. The composite of the κ's and $f\kappa_x$ is clearly the 2-cell κ_{fx}. □

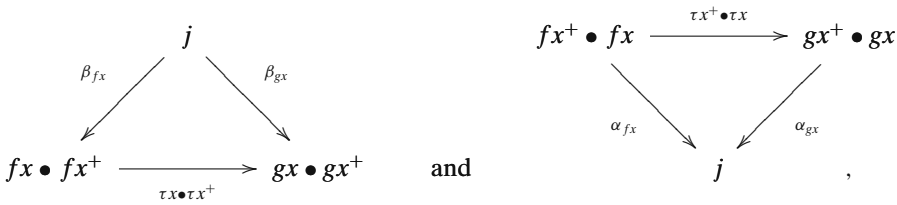
Theorem 3.2 (Day and Pastro) *If A has a right dual, X, Y are monoidal objects, $(x^+, x; \alpha, \beta) : A \nrightarrow X$ is an exact pairing, and $\tau : f \nrightarrow g : X \nrightarrow Y$ is a both monoidal and comonoidal 2-cell between Frobenius arrows then $\tau x : fx \nrightarrow gx$ is invertible with*

$$(\tau x)^{-1} = (\tau x^+)^-$$

and $\tau x^+ : fx^+ \to gx^+$ *is invertible with*

$$(\tau x^+)^{-1} = (\tau x)^+.$$

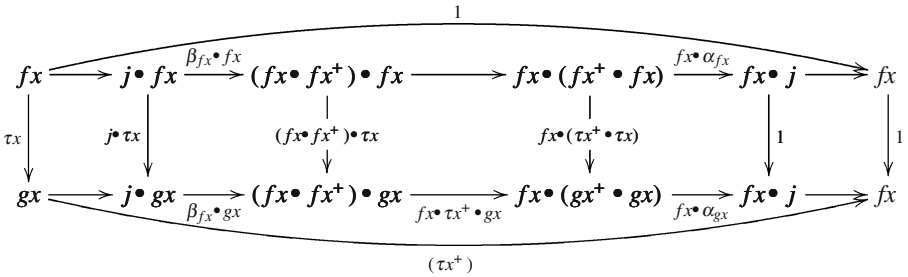
Proof We will prove the first equation, the second being similar. In the terminology of this paper, it is shown in [6] that, for $\tau : f \to g$ as in the hypothesis,



(8)

where $(fx^+, fx; \alpha_{fx}, \beta_{fx}) : A \to Y$ and $(gx^+, gx; \alpha_{gx}, \beta_{gx}) : A \to Y$ are the dual situations given by Theorem 3.1.

Consider $\tau x^+ : fx^+ \to gx^+$. By Theorem 3.1 it can be seen as $\tau x^+ : (fx)^+ \to (gx)^+$. From the discussion in 1.3 we have $(\tau x^+)^- : gx \to fx$ and



where commutativity of the centre rectangle is by naturality of the associativity constraint (which is a factor of $fx \bullet \tau x^+ \bullet gx$) and that of the next rectangle to the right is from the second diagram of Eq. 8. The equation $\tau x.(\tau x^+)^- = 1_g$ follows similarly using the first diagram of Eq. 8. □

Corollary 3.3 *If A has left dualization as a monoidal object of* $\mathscr{M}$*, X is a monoidal object, and* $\tau : f \to g : A \to X$ *is a monoidal and comonoidal 2-cell between Frobenius arrows then* τ *is invertible with*

$$\tau^{-1} = (\tau 1_A^+)^-$$

4 Equivalent Formulations of Exact Pairing

4.1

In this section we give equivalent conditions for an arrow $x : A \to X$ to have a left dual. The results are similar in spirit to some results in [5] and in [15]. Indeed, our Proposition 4.2 corrects and relativizes Proposition 1.1 of [5].

Recall the notion of pseudomonad from, for instance, [14] or [18]. In short, a pseudomonad on a bicategory $\mathscr{B}$ is a monoidal object in the monoidal bicategory $\mathbf{Hom}(\mathscr{B}, \mathscr{B})$ of endo-pseudofunctors, pseudonatural transformations and modifications. As observed in [9], there are two canonical choices of a tensor product of non-identity 1-cells in the bicategory $\mathbf{Hom}(\mathscr{B}, \mathscr{B})$, since there are two (isomorphic) canonical ways of horizontally composing two non-identity pseudonatural transformations. However, this kind of tensor product does not appear in the definition of a monoidal object, and hence the notion of pseudomonad we obtain is independent of which of the two monoidal structures we choose.

Let $T : \mathscr{B} \to \mathscr{B}$ be a pseudomonad, with unit $i : 1 \to T$ and multiplication $m : T^2 \to T$. A pseudoalgebra for T, or pseudo-T-algebra, is an object $A \in \mathscr{B}$ equipped with an "action" $a : TA \to A$ and an invertible 2-cell

$$\begin{array}{ccc} T^2A & \xrightarrow{Ta} & TA \\ {\scriptstyle m_A}\downarrow & \Downarrow \bar{a} & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array}$$

satisfying coherence conditions. If $X \in \mathscr{B}$ is an object, then TX has a "free" pseudoalgebra structure, with action given by $m_X : T^2X \to TX$ and an isomorphism $m_X(Tm_X) \cong m_X m_{TX}$ given by the pseudomonad structure of T. A pseudomorphism between pseudo-T-algebras A and B is an arrow $f : A \to B$ in $\mathscr{B}$ together with an invertible 2-cell

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & \Downarrow \bar{f} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

satisfying compatibility conditions with respect to the 2-cells $\bar{a}$, $\bar{b}$ and the pseudomonad structure of T. Pseudo-T-algebras, pseudomorphisms and the appropriate 2-cells form a bicategory Ps-T-Alg.

Given a monoidal object X in a monoidal bicategory $\mathscr{M}$, the pseudofunctors $(X \otimes -) : \mathscr{M} \to \mathscr{M}$ and $(- \otimes X) : \mathscr{M} \to \mathscr{M}$ have canonical pseudomonad structures. If A is an object of $\mathscr{M}$, we consider $X \otimes A$ as a free $(X \otimes -)$-algebra. The multiplication $p : X \otimes X \to X$ has a canonical structure of a 1-cell in the 2-category Ps-$(X \otimes -)$-Alg, and hence so does any arrow of the form $X \bullet x$ for $x : A \to X$. Similarly, the canonical pseudomorphism structure of $pA : XXA \to XA$ induces another on $(pA)(XyA)(Xn) : X \to XA$, for all $y : A^\circ \to X$. By the same argument, $z \bullet X : BX \to X$ and $(B^\circ p)(B^\circ zX)(nX) : X \to B^\circ X$ have a structure of pseudomorphism between pseudo-$(- \otimes X)$-algebras.

First, we state a very easy lemma that we will need later.

Lemma 4.1 *Let X be a monoidal object and regard $X \otimes A$, $X \otimes B$ as free pseudo-$(X \otimes -)$-algebras. Then, for any pseudomorphism $f : X \otimes A \to X \otimes B$, there exists a canonical isomorphism*

$$f \cong (X \otimes A \xrightarrow{X \otimes j \otimes A} X \otimes X \otimes A \xrightarrow{X \otimes f} X \otimes X \otimes B \xrightarrow{p \otimes B} X \otimes B)$$

in Ps-$(X \otimes -)$-Alg*, where the pseudomorphism structure of the second arrow is the composition of the free structure of $X \otimes (f(j \otimes A))$ and the canonical structure of $p \otimes B$.*

Proof The isomorphism is given by $f \cong f(p \otimes A)(X \otimes j \otimes A) \cong (p \otimes B)(X \otimes f)(X \otimes j \otimes A)$, where the last isomorphism is induced by the pseudomorphism structure of f. □

Proposition 4.2 *Let X be a monoidal object and A an object with a right bidual in $\mathscr{M}$. Given arrows $x : A \to X$, $y : A^\circ \to X$, there is a bijection between the following structures.*

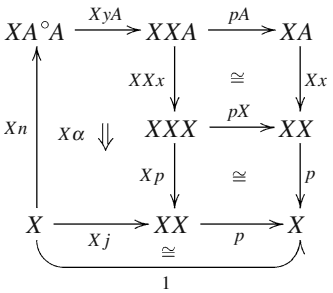
(i) *Pairs of 2-cells (α, β) such that $(y, x; \alpha, \beta)$ is an exact pairing.*
(ii) *Adjunctions in* Ps-$(X \otimes -)$-Alg

$$(XA \xrightarrow{X \bullet x} X) \dashv (X \xrightarrow{Xn} XA^\circ A \xrightarrow{(X \bullet y)A} XA)$$

(iii) *Adjunctions in* Ps-$(- \otimes X)$-Alg

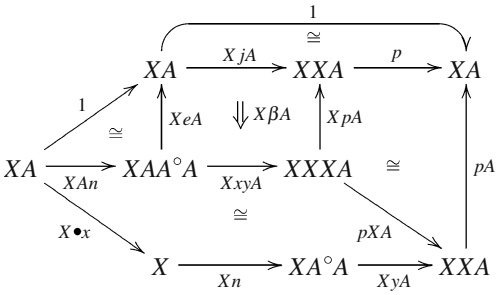
$$(A^\circ X \xrightarrow{y \bullet X} X) \dashv (X \xrightarrow{nX} A^\circ AX \xrightarrow{A^\circ(x \bullet X)} A^\circ X)$$

Proof First we exhibit a bijection between the structures in (i) and (ii). Given α, β, define ε as



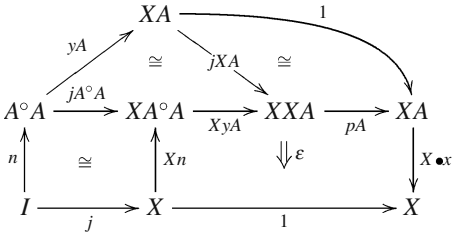
(9)

and η as

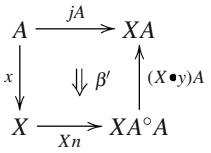


Both ε and η are clearly composition of 2-cells in Ps-$(X \otimes -)$-Alg.

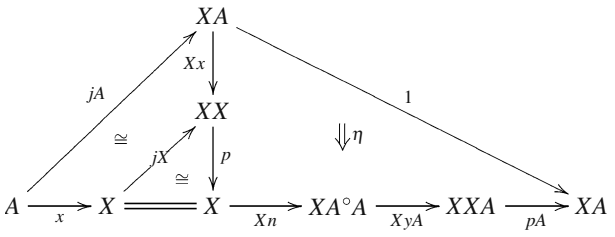
Conversely, given ε in Ps-$(X \otimes -)$-Alg, define α as



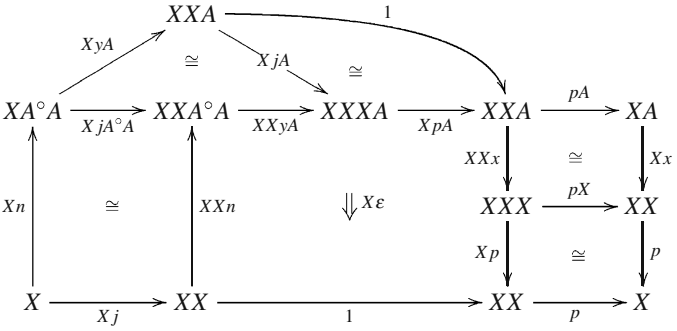
2-cells $\beta : je \to x \bullet y$ are in bijection with 2-cells



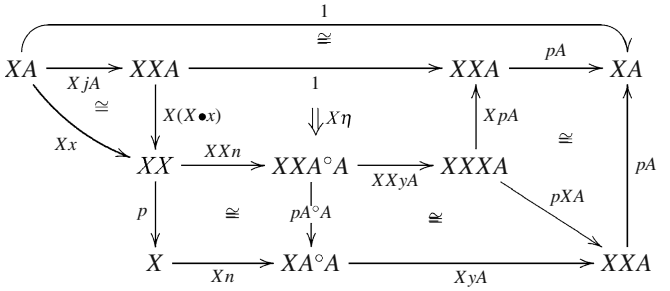
Hence, given η we define β' as



It is easy to see that the correspondence $\alpha \mapsto \varepsilon \mapsto \alpha$ is the identity. If we perform $\varepsilon \mapsto \alpha \mapsto \varepsilon$, we obtain the composition below



which is equal to ε because ε is a 2-cell in Ps-$(X \otimes -)$-Alg. Similarly, that the correspondence $\beta' \mapsto \eta \mapsto \beta'$ is the identity is easily shown. When we perform $\eta \mapsto \beta' \mapsto \eta$ we obtain the 2-cell



which is equal to η because η is a 2-cell in Ps-$(X \otimes -)$-Alg.

Each of the two axioms for α and β for an exact pairing is equivalent to one of the axioms for ε and η; we omit the long diagrammatic proof.

The equivalence of (i) and (iii) follows from the equivalence of (i) and (ii) and the following two observations. Firstly, $(y, x; \alpha, \beta)$ is an exact pairing in $\mathscr{M}$ if and only if $(x, y; \alpha, \beta)$ is an exact pairing in $\mathscr{M}^{\mathrm{rev}}$ ($\mathscr{M}$ equipped with the reverse tensor product). Secondly, there exists a bijection between adjunctions as in (iii) in $\mathscr{M}$ and adjunctions as in (ii) in $\mathscr{M}^{\mathrm{rev}}$ with the roles of x and y interchanged. Hence, by the equivalence between (i) and (ii), adjunctions in (iii) are in bijection with pairs of 2-cells (α, β) such that $(x, y; \alpha, \beta)$ is an exact pairing in $\mathscr{M}^{\mathrm{rev}}$, or equivalently, such that $(y, x; \alpha, \beta)$ is an exact pairing in $\mathscr{M}$. □

The corollary below gives yet another proof of the uniqueness of duals.

Corollary 4.3 *Let $(y, x; \alpha, \beta) : A \nrightarrow X$ be an exact pairing. Then:*

(i) – $x \cong (A \xrightarrow{Aj} AX \xrightarrow{A(y\bullet X)^*} AA^\circ X \xrightarrow{eX} X)$

– $y \cong (A^\circ \xrightarrow{jA^\circ} XA^\circ \xrightarrow{(X\bullet x)^* A^\circ} XAA^\circ \xrightarrow{Xe} X)$

(ii) *If the unit j is a map*

- *x has right adjoint* $X \xrightarrow{Xn} XA^\circ A \xrightarrow{(X\bullet y)A} XA \xrightarrow{j^*A} A$
- *y has right adjoint* $X \xrightarrow{nX} A^\circ AX \xrightarrow{A^\circ(x\bullet X)} A^\circ X \xrightarrow{A^\circ j^*} A^\circ$

Proof (i) We have

$$\begin{aligned} eX.A(y\bullet X)^*.Aj &\overset{(1)}{\cong} eX.A(A^\circ(x\bullet X).nX.j) \\ &\cong eX.A(A^\circ(x\bullet X).A^\circ Aj.n) \\ &\cong eX.A(A^\circ p.A^\circ xX.A^\circ Aj.n) \\ &\cong eX.A(A^\circ(p.Xj.x).n) \\ &\cong eX.AA^\circ x.An \\ &\cong x.eA.An \\ &\cong x \end{aligned}$$

where in (1) we have used $(y\bullet X)^* \cong A^\circ(x\bullet X).nX$ which is (iii) of Proposition 4.2. Derivation of the second isomorphism is similar and uses (ii) of Proposition 4.2.

(ii) For the arrow $x : A \nrightarrow X$ we have

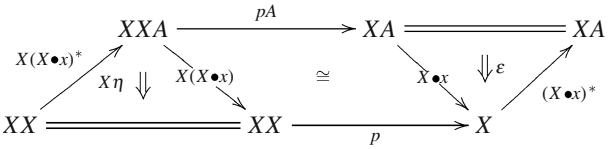
$$x \cong p.jX.x \cong p.Xx.jA \cong X\bullet x.jA$$

and by Proposition 4.2 (ii) we have

$$x^* \cong (X\bullet x.jA)^* \cong (jA)^*(X\bullet x)^* \cong j^*A.(X\bullet y)A.Xn.$$

Again, the proof for the case of $y : A^\circ \nrightarrow X$ is similar. □

Corollary 4.4 *Given an arrow $x : A \nrightarrow X$ with a left dual and any choice of an adjunction η, ε; $(X\bullet x) \dashv (X\bullet x)^*$, the following 2-cell is invertible.*



(10)

Proof The adjunction of Proposition 4.2(ii) induces an isomorphism

$$(X\bullet x)^* \cong ((X\bullet y)\otimes A)(X\otimes n)$$

compatible with the units and counits of both adjunctions. We equip $(X\bullet x)^*$ with a structure of a pseudomorphism of pseudo-$(X\otimes -)$-algebras by transporting through this isomorphism the corresponding structure of $((X\bullet y)\otimes A)(X\otimes n)$. This structure automatically makes η, ε 2-cells in Ps-$(X\otimes -)$-Alg. Using this latter fact, it is easy to see that Eq. 10 is the inverse of the pseudomorphism structure of $(X\bullet x)^*$. □

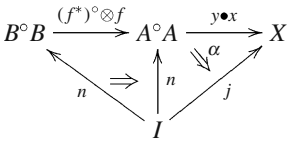
Lemma 4.5 *Suppose $x : A \nrightarrow X$ has a left dual $y : A^\circ \nrightarrow X$ and B is an object with a right bidual. Then, for every map $f : B \nrightarrow A$, the arrow $xf : B \nrightarrow X$ has a left dual given by $y(f^*)^\circ$.*

Proof The proof is the following simple calculation. The adjunction in Ps-$(X \otimes -)$-Alg, $X \bullet x \dashv (pA)(XyA)(Xn)$ of Proposition 4.2 induces another adjunction

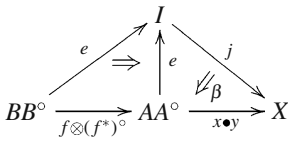
$$X \bullet (xf) \dashv (Xf^*)(pA)(XyA)(Xn)$$

in the same bicategory. This last arrow is isomorphic in Ps-$(X \otimes -)$-Alg to the composite $(pB)(X(y(f^*)^\circ)B)(Xn)$, yielding the result. □

In the result above, if $(y, x; \alpha, \beta) : A \nrightarrow X$ is an exact pairing, and $f : B \nrightarrow A$ is a map where B has a right bidual, then $(y(f^*)^\circ, xf, \hat{\alpha}, \hat{\beta}) : B \nrightarrow X$ is an exact pairing, where $\hat{\alpha}$ and $\hat{\beta}$ are respectively



and



The unlabelled 2-cells are the obvious ones induced by the counit and unit of the adjunction $f \dashv f^*$. In particular, if $d : X^\circ \nrightarrow X$ is a left dualization, every map $f : A \nrightarrow X$ from an object with a right bidual has a left dual given by

$$A^\circ \xrightarrow{(f^*)^\circ} X^\circ \xrightarrow{d} X.$$

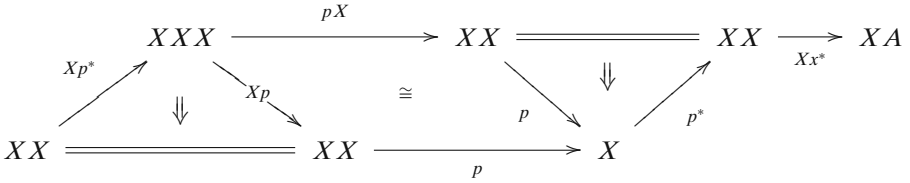
Proposition 4.6 *Let X be a monoidal object and A an object with a right bidual in $\mathscr{M}$. Then, an arrow $x : A \nrightarrow X$ has a left dual if and only if the arrow $X \bullet x : XA \nrightarrow X$ is a map and the 2-cell* Eq. 10 *is invertible.*

Proof The direct implication was proved in Corollary 4.4. To show the converse, observe that the 2-cell Eq. 10 is a structure of a lax morphism between pseudo-$(X \otimes -)$-algebras. This follows by "doctrinal adjunction" [12], since $p(X \otimes (X \bullet x)) \cong (X \bullet x)(p \otimes A)$ is a pseudomorphism structure for $X \bullet x$. Now, by hypothesis, Eq. 10 is invertible and hence a pseudomorphism structure. Then, by Lemma 4.1, there is an isomorphism

$$(X \bullet x)^* \cong (X \xrightarrow{X \otimes j} X \otimes X \xrightarrow{X \otimes (X \bullet x)^*} X \otimes X \otimes A \xrightarrow{p \otimes A} X \otimes A)$$

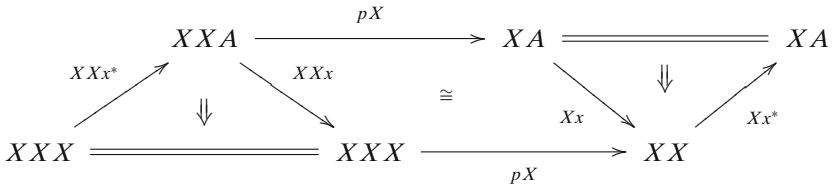
in Ps-$(X \otimes -)$-Alg. Denote by y the composition $(X \otimes e)(X \otimes (X \bullet x)^*)(X \otimes j) : A^\circ \nrightarrow X$. Applying Proposition 4.3 (ii), we deduce that y is a left dual of x. □

Corollary 4.7 *In the hypothesis of the proposition above, if X is a map monoidal object and $x : A \nrightarrow X$ is a map, then $x : A \nrightarrow X$ has a left dual if and only if the 2-cell below is invertible.*



(11)

Proof The result follows from Corollary 4.4, observing that the 2-cell Eq. 10 equals the pasting of Eq. 11 with the isomorphism



□

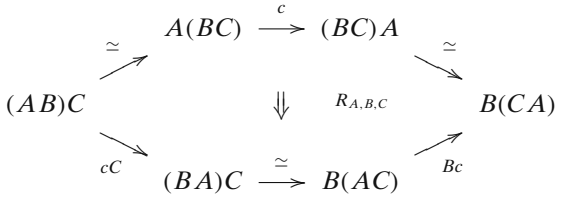
5 Lax Centres Become Centres

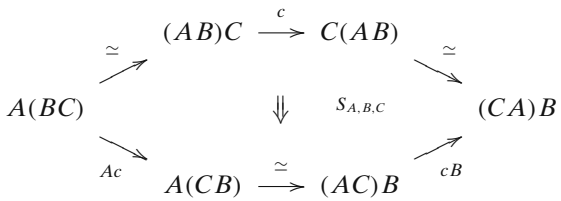
In this section we study the relationship between the existence of duals and the lax centre of a monoidal object. The centre of a monoidal category was defined in [10], while the lax variant appeared in [21] for ordinary monoidal categories (under the name 'weak center') and in [7] for enriched (pro)monoidal categories. The centre of a monoidal object was introduced in [23], and a lax variant of it was considered in [15, 17].

It was observed in [27] and [10] that, in a left autonomous monoidal category, the invertibility condition in the definition of a braiding is redundant. In other words, in this context every lax braiding is a braiding. The main result of this section is a version of this phenomenon for monoidal objects. A related result, less general than the one we provide here, was shown in [15, 17]: the lax centre of a (left and right) autonomous map monoidal object coincides with the centre.

We work in a braided monoidal bicategory $\mathscr{M}$, as defined in [19]. When $\mathscr{M}$ is a Gray monoid, this definition coincides with the definition of braided Gray monoid

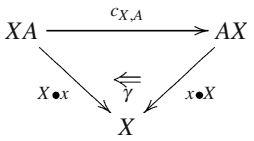
given in [8]. A braiding in the monoidal bicategory $\mathscr{M}$ is a pseudonatural equivalence $c_{A,B} : AB \to BA$ with two invertible modifications





satisfying coherence conditions that involve the monoidal structure of $\mathscr{M}$.

Definition 5.1 Let (X, j, p) be a monoidal object in a braided monoidal bicategory $\mathscr{M}$. A *left lax centre piece* of X with base A is a pair (x, γ) where $x : A \to X$ is an arrow and γ is a 2-cell

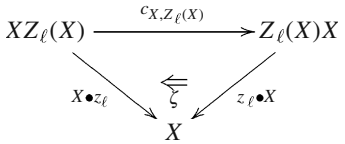


(12)

satisfying two axioms depicted in Fig. 1. A morphism of left lax centre pieces with base A from (x, γ) to (y, δ) is a 2-cell $\varepsilon : x \to y$, compatible with γ, δ in the obvious sense. Left lax centre pieces with base A and their morphisms form a category $CP_\ell(A, X)$. In fact this is the object part of a pseudofunctor $CP_\ell(-, X) : \mathscr{M}^{\mathrm{op}} \to \mathbf{Cat}$, induced by precomposition in $\mathscr{M}$.

Definition 5.2 A left lax centre for a monoidal object X is a birepresentation of the pseudofunctor $CP_\ell(-, X) : \mathscr{M}^{\mathrm{op}} \to \mathbf{Cat}$.

Thus a left lax centre for X is an arrow $z_\ell : Z_\ell(X) \to X$ together with a left lax centre piece structure



(13)

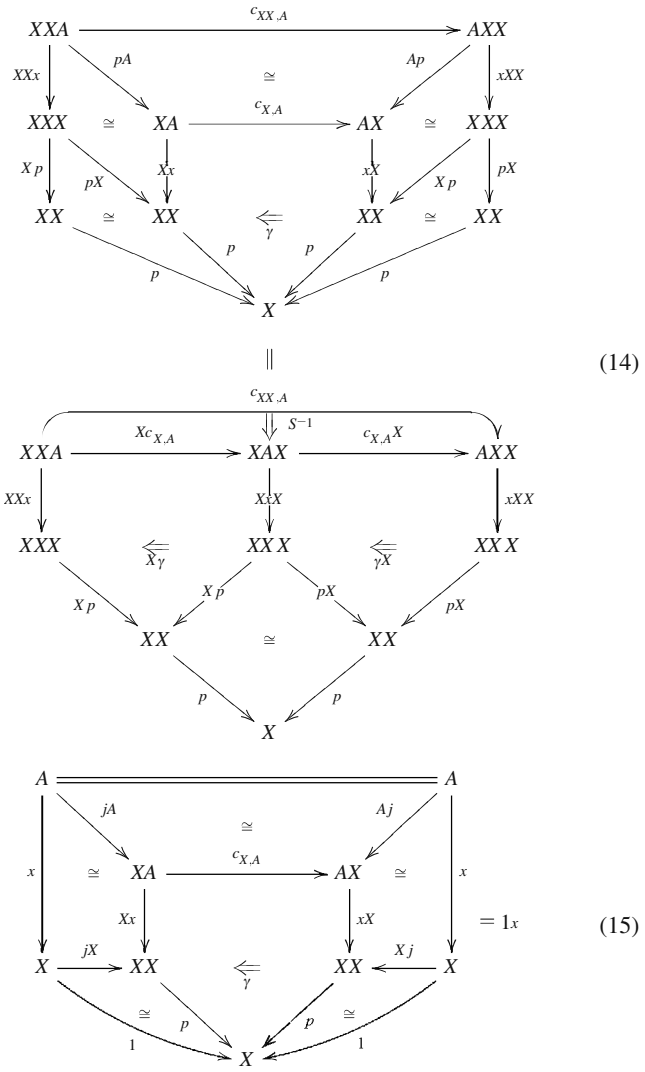
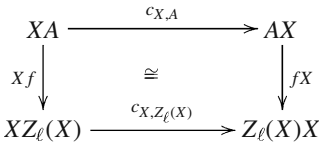


Fig. 1 Left lax centre piece axioms

satisfying the following universal property: For any left lax centre piece (x, γ) with base A, there exists an arrow $f : A \to Z_\ell(X)$ such that (x, γ) is isomorphic to the left lax centre piece obtained by pasting the following 2-cell on top of Eq. 13



(so (x, γ) is isomorphic to $CP_r(f, X)(z_\ell, \zeta)$) and, for all $f : A \to Z_\ell(X)$ and $g : A \to Z_\ell(X)$, any morphism of left lax centre pieces from $CP_\ell(f, X)(z_\ell, \zeta)$ to $CP_\ell(g, X)(z_\ell, \zeta)$ is of the form $z_\ell \mu : z_\ell f \Rightarrow z_\ell g$ for a unique 2-cell $\mu : f \Rightarrow g$. (The second conjunct of the universal property ensures that the f of the first conjunct is unique to within isomorphism.)

If we require the 2-cells Eq. 12 to be invertible, we recover the definition of centre piece and centre of a monoidal object introduced in [23]. We write $CP(A, X)$ for the full subcategory of $CP_\ell(A, X)$ with objects the centre pieces. We will often call left lax centre pieces and left lax centres simply lax centre pieces and lax centres.

The notion of a lax centre piece can be interpreted in terms of the monoidal bicategory $\mathscr{M} /\!\!/ X$ of Section 2.1. To give a 2-cell Eq. 12 is the same as giving $c_{X,A}$ a structure of a 1-cell $X \bullet x \to x \bullet X$ in $\mathscr{M} /\!\!/ X$. Giving this structure, we can also equip $c_{XX,A}$ with a structure of a 1-cell $(X \bullet X) \bullet x \to x \bullet (X \bullet X)$ in $\mathscr{M} /\!\!/ X$, given by the 2-cell (where this time we write the tensor product symbol explicitly, to avoid confusion)

$$p(x \otimes X)(A \otimes p)c_{XX,A} \xrightarrow{p(x \otimes X)c_{p,1}} p(x \otimes X)c_{X,A}(p \otimes A) \xrightarrow{\gamma(p \otimes A)} p(X \otimes x)(p \otimes A).$$

The axiom (14) of a lax centre piece says that there is an isomorphism in $\mathscr{M} /\!\!/ X$

$$\begin{array}{ccccc} 1_X \bullet (1_X \bullet x) & \xrightarrow{1_X \bullet c_{X,A}} & 1_X \bullet (x \bullet 1_X) & \xrightarrow{\simeq} & (1_X \bullet x) \bullet 1_X \\ \simeq \downarrow & & \Uparrow & & \downarrow c_{X,A} \bullet 1_X \\ (1_X \bullet 1_X) \bullet x & \xrightarrow{c_{XX,A}} & x \bullet (1_X \bullet 1_X) & \xrightarrow{\simeq} & (x \bullet 1_X) \bullet 1_X \end{array} \tag{16}$$

whose image under the forgetful pseudofunctor $\mathscr{M} /\!\!/ X \to \mathscr{M}$ is the isomorphism $S_{X,X,A}$. Analogously, $c_{I,A} : IA \longrightarrow AI$ is equipped with the following structure of a 2-cell $j \bullet x \to x \bullet j$ in $\mathscr{M} /\!\!/ X$:

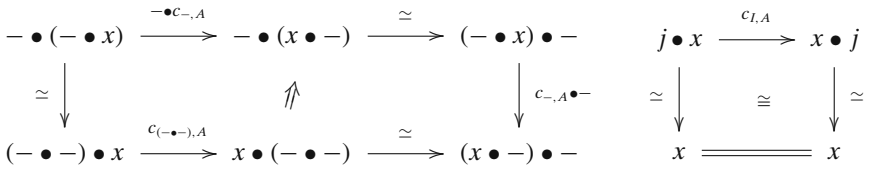
$$p(x \otimes X)(A \otimes j)c_{I,A} \xrightarrow{p(x \otimes X)c_{j,1}} p(x \otimes X)c_{X,A}(j \otimes A) \xrightarrow{\gamma(j \otimes A)} p(X \otimes x)(j \otimes A)$$

The axiom (15) requires the existence of an isomorphism in $\mathscr{M} /\!\!/ X$, depicted in the left hand side below, whose image under $\mathscr{M} /\!\!/ X \to \mathscr{M}$ is the canonical isomorphism on the right hand side.

$$\begin{array}{ccc} j \bullet x & \xrightarrow{c_{I,A}} & x \bullet j \\ \simeq \downarrow & \cong & \downarrow \simeq \\ x & = & x \end{array} \qquad \begin{array}{ccc} IX & \xrightarrow{c_{I,A}} & XI \\ \simeq \downarrow & \cong & \downarrow \simeq \\ X & = & X \end{array} \tag{17}$$

Lemma 5.3 *There is a bijection between the following structures on an arrow* $x : A \to X$.

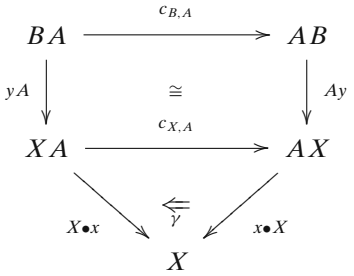
1. *Lax centre piece structures on x.*
2. *Pseudonatural transformations with isomorphisms* $c_{-,A} : (- \bullet x) \to (x \bullet -) : \mathscr{M} /\!\!/ X \to \mathscr{M} /\!\!/ X$



(18)

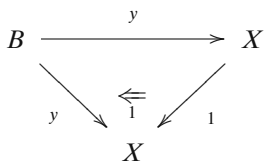
whose images under $\mathscr{M} /\!\!/ X \to \mathscr{M}$ *are respectively* $S_{-,-,A}$ *and the canonical isomorphism in the right hand side of* Eq. 17.

Proof If we evaluate the 2-cell on the left hand side of Eq. 18 on the object $(1_X, 1_X)$, we obtain the reinterpretation of the axioms of a right lax centre piece discussed above. So, it is clear that 2 gives 1. Conversely, given a structure as in 1, and hence isomorphims as in Eqs. 16 and 17, we have to construct data as in 2. If $y : B \to X$ is an object of $\mathscr{M} /\!\!/ X$, define the component of $c_{-,A}$ at y as the 1-cell of $\mathscr{M} /\!\!/ X$ below.



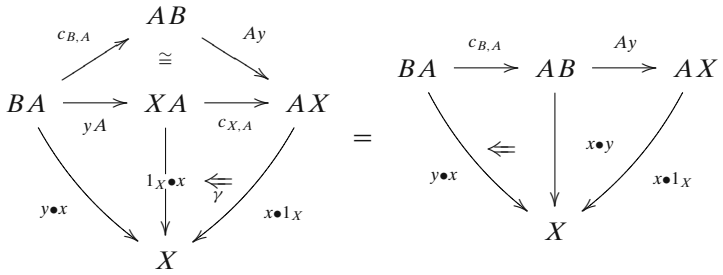
The pseudonaturality of $c_{-,A}$ follows from the pseudonaturality of the braiding c in $\mathscr{M}$.

To show that these constructions are inverses of one another, it is enough to show that a pseudonatural transformation $c_{-,A}$ as in 2 is determined by its component at the object $1_X : X \to X$. Consider the following arrow in $\mathscr{M} /\!\!/ X$.



(19)

Pseudonaturality of $c_{-,A}$ implies that the isomorphism $c_{y,1} : (A \otimes y)c_{B,A} \cong c_{X,A}(y \otimes A)$ is a 2-cell in $\mathscr{M} /\!\!/ X$. That means we have an equality



Hence, the component at y of $c_{-,A}$ is determined by the 2-cell γ. □

Recall from [1] [7.2] that, for any bicategory $\mathscr{B}$, we have the *classifying category* $\widehat{\mathscr{B}}$. The objects of $\widehat{\mathscr{B}}$ are those of $\mathscr{B}$ and the arrows $\widehat{f} : B \to A$ of $\widehat{\mathscr{B}}$ are isomorphism classes of 1-cells $f : B \to A$ of $\mathscr{B}$. Note that if $\widehat{f} : B \to A$ is an isomorphism in $\widehat{\mathscr{B}}$ then $f : B \to A$ is an equivalence in $\mathscr{B}$. If $\mathscr{M}$ is a monoidal bicategory, then $\widehat{\mathscr{M}}$ is a monoidal category.

Proposition 5.4 *Let the pseudomonoid X be left autonomous and $x : A \to X$ be an arrow. Then every lax centre piece structure on an arrow x is invertible, and hence a centre piece structure.*

Proof Let γ be a lax centre piece structure on x, as depicted in Eq. 12, and consider the pseudonatural transformation $c_{-,A}$ of Lemma 5.3. The conditions in the lemma imply that, in $\widehat{\mathscr{M}}$, $\widehat{c_{-,A}} : - \bullet x \to x \bullet -$ is an element of the right lax centre of the monoidal category $\widehat{\mathscr{M} /\!\!/ X}$. On the other hand, by Lack's observation in Section 2.1, the fact that 1_X has a left dual (a left dualization) means that 1_X has a right bidual in $\mathscr{M} /\!\!/ X$, and hence a right dual in $\widehat{\mathscr{M} /\!\!/ X}$. It follows from a result dual to Proposition 3.1 of [7] that the component $\widehat{c_{1_X,A}}$ is an isomorphism in $\widehat{\mathscr{M} /\!\!/ X}$, and hence an equivalence in $\mathscr{M} /\!\!/ X$. By the description of equivalences in $\mathscr{M} /\!\!/ X$ it follows that the 2-cell γ Eq. 12 is an isomorphism. □

Corollary 5.5 *If X is left autonomous, then a lax centre of X exists if and only if a centre of X does, and they coincide.*

Proof For any object A, we have an equality $CP(A, X) = CP_\ell(A, X)$ by Proposition 5.4. Now it is clear that the pseudofunctor $CP(-, A)$ is birepresentable if and only if $CP_\ell(-, X)$ is so, and the birepresentations coincide. □

Corollary 5.6 *If X is left autonomous then every lax braiding on X is a braiding.*

Proof A lax braiding $\gamma : pc_{X,X} \to p$ gives 1_X the structure of a right lax centre piece. Then, by the Proposition, we deduce that γ is invertible. □

Remark 5.7 The technique of using Bénabou's classifying category construction [1] [7.2] applied to the bicategory $\mathscr{M} /\!\!/ X$ in Proposition 5.4 can be used to give a proof of our Theorem 3.2 so that it becomes a direct corollary of the corresponding original Day and Pastro theorem, set in the context of monoidal categories.

6 Naturally Frobenius Map Monoidal Objects

6.1

Recall from the end of 1.4 that a monoidal object $(A, p, j, \alpha, \lambda, \rho)$ is map monoidal if p and j are maps (meaning that they are the left adjoint components of adjunctions $(p, p^*; \varepsilon_p, \eta_p)$ and $(j, j^*; \varepsilon_j, \eta_j)$). Here we are writing α for the associativity constraint and λ and ρ for the unitary constraints. We will continue to use α also for the counit of an exact pairing. There should be no confusion; the role of an α will be clear from the context.

For p a map, the associativity constraint isomorphism and its inverse have mates

$$\begin{array}{ccc} A^3 & \xleftarrow{Ap^*} & A^2 \\ {\scriptstyle pA}\downarrow & \overset{\phi}{\Longrightarrow} & \downarrow{\scriptstyle p} \\ A^2 & \xleftarrow[p^*]{} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} A^3 & \xleftarrow{p^*A} & A^2 \\ {\scriptstyle Ap}\downarrow & \overset{\psi}{\Longrightarrow} & \downarrow{\scriptstyle p} \\ A^2 & \xleftarrow[p^*]{} & A \end{array} \tag{20}$$

respectively. We say that a map monoidal object is *naturally Frobenius* if both ϕ and ψ are invertible. We are aware that there is considerable overlap between our treatment of naturally Frobenius map monoidal objects and the work of Section 8 in [15]. However, we provide more details, which we consider necessary, than given in [15].

Remark 6.1 The proof of Lemma 3.2 in [26], purportedly about a more specific context, actually shows that if A is a symmetric naturally Frobenius map monoidal object then invertibility of ϕ is equivalent to that of ψ.

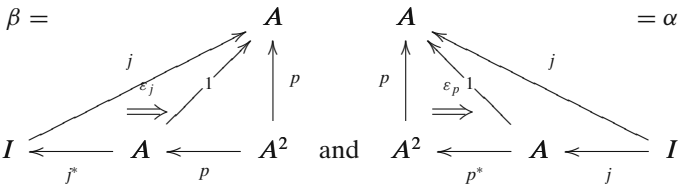
Remark 6.2 If (A, p, j) is part of the data for a map monoidal object in $\mathscr{M}$ then (A, p^*, j^*) is part of the data for a comonoidal object in $\mathscr{M}$. A general *Frobenius object* $(A, p, j, d, t, \phi, \psi)$ in $\mathscr{M}$ consists of a monoidal object (A, p, j) and a comonoidal object (A, d, t) (not necesssarily related by adjunction) together with isomorphisms ϕ and ψ, as in the diagrams Eq. 20 above but with p^* replaced by d. In this general case ϕ and ψ are required to satisfy coherence conditions, which are automatic for naturally Frobenius map monoidal objects.

Remark 6.3 In the (cartesian) monoidal bicategories **rel** of sets and relations and **spn** of sets and spans, both with $\otimes$ given by cartesian product, a map monoidal object is precisely a monoid. It is naturally Frobenius precisely when it is a group.

In the (cartesian) monoidal bicategory **pro** of categories and profunctors with $\otimes$ given by cartesian product, a map monoidal object A, for A in which idempotents split, is precisely a monoidal category. Such an A is naturally Frobenius precisely if it is autonomous.

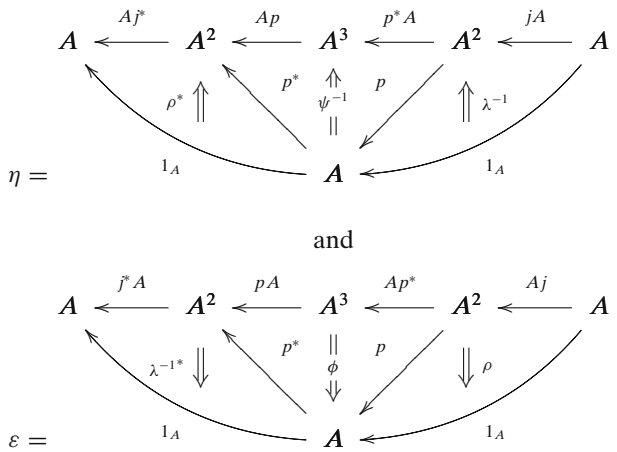
6.2

For a naturally Frobenius map monoidal object $(A, p, j, \varphi, \lambda, \rho)$ we define 2-cells by the following pasting composites



(21)

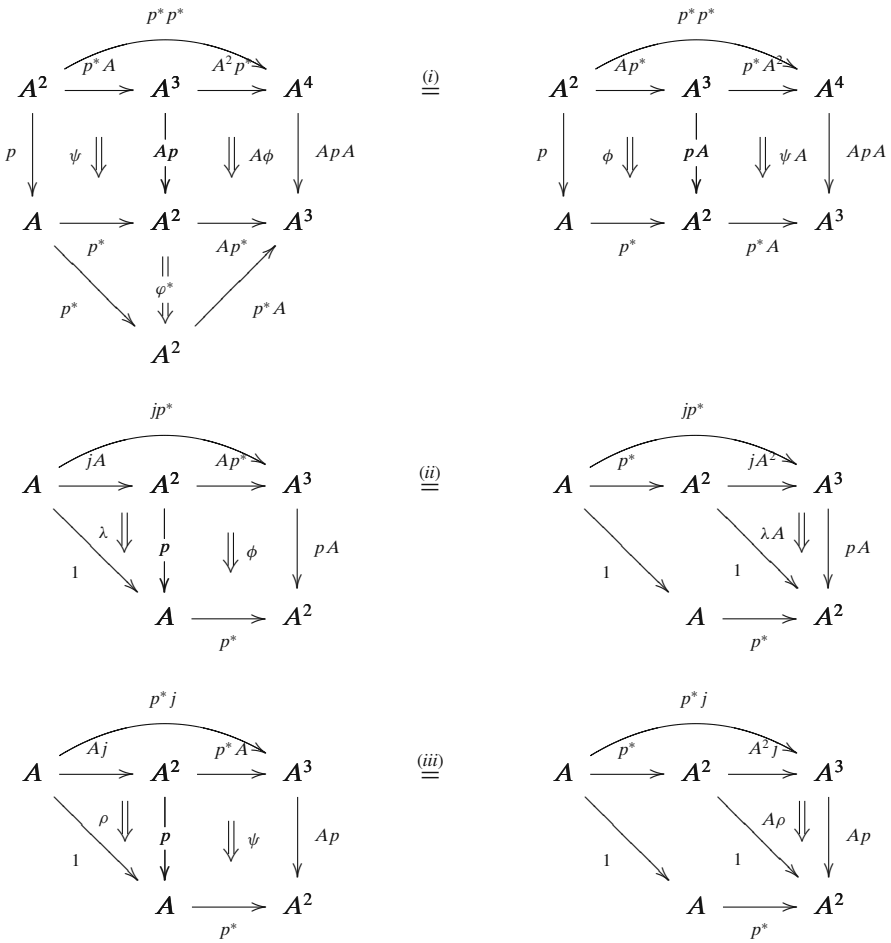
and



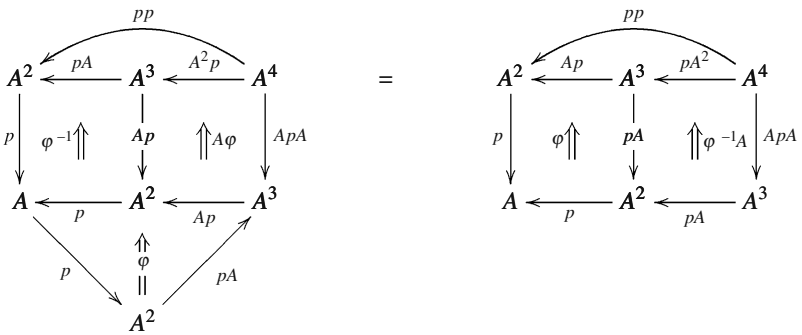
(22)

If we further define the composite arrows $n = p^*.j : I \nrightarrow A \otimes A$ and $e = j^*.p : A \otimes A \nrightarrow I$ then $(1_A, 1_A; \alpha, \beta)$ becomes a candidate for an exact pairing provided we can also show that $(A, A; e, n; \varepsilon, \eta)$ exhibits A as a right bidual of itself.

Proposition 6.4 *For* $(A, p, j, \varphi, \lambda, \rho)$ *a map monoidal object in a monoidal bicategory, the following pasting equalities hold.*

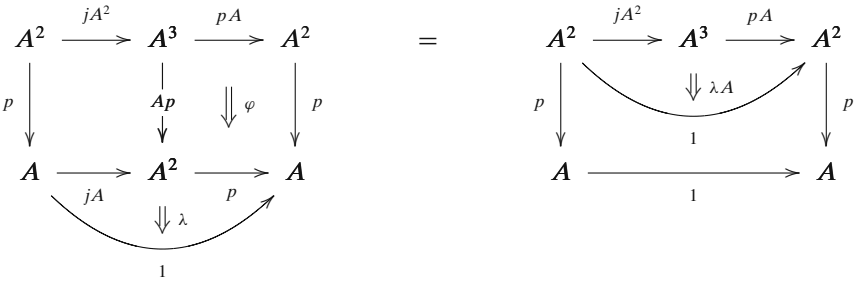


Proof For (i), taking mates with respect to the evident adjunctions, the equality is equivalent to that of

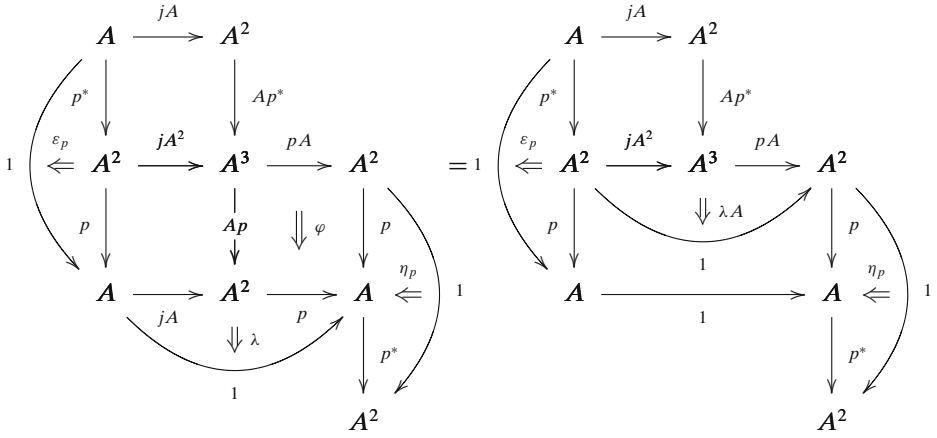


which is the classical pentagonal coherence of the associativity constraint.

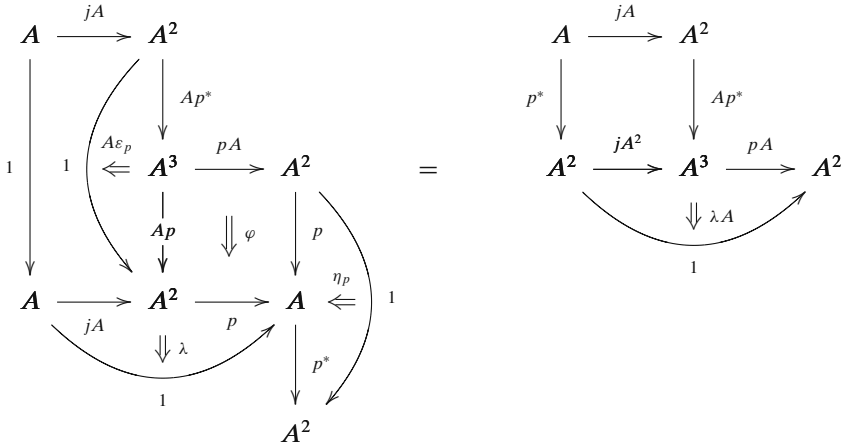
For (ii), begin with the equation for the coherence of associativity and left unit for A.



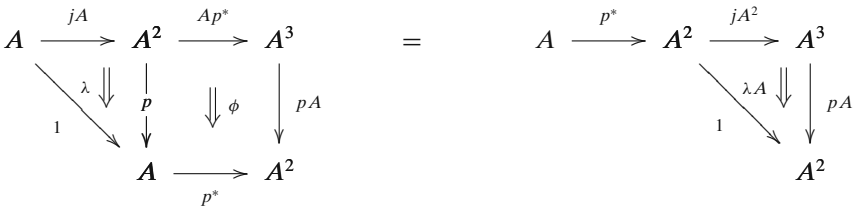
Pasting the same elements to both sides we have the equality



Using naturality on the left side and a triangle equation for the $p \dashv p^*$ adjunction on the right we have



Finally, noting the definition of ϕ as a mate of φ and simplifying the diagrams we have

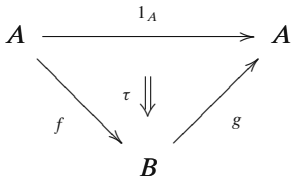


The equality (iii) is just the equality (ii) for the map monoidal object A in the reverse monoidal bicategory. □

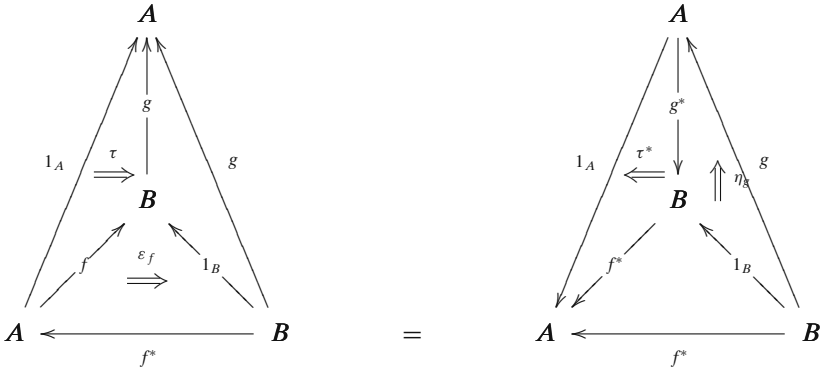
Remark 6.5 Note that Proposition 6.4 holds for a not-necessarily-Frobenius map monoidal object—there is no use of invertibility of ϕ and ψ.

The following simple pasting lemma (whose instances in large pasting composites can be deceptive) will be helpful.

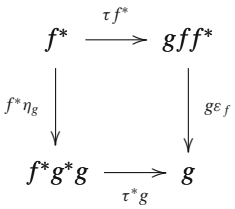
Lemma 6.6 *In any bicategory, for any 2-cell*



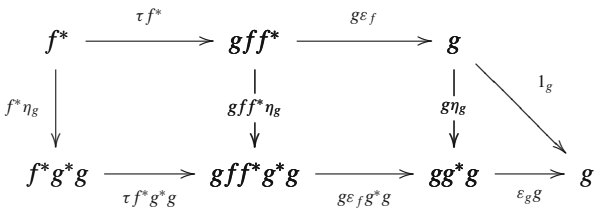
with both f and g maps,



Proof We offer a commutative diagram proof. We are to show



Expanding τ^*, more precisely $\tau^* g$, along the bottom edge we have

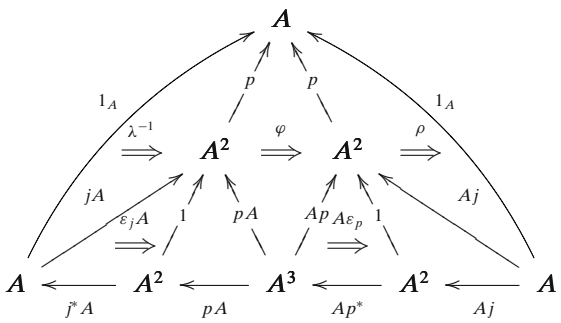


□

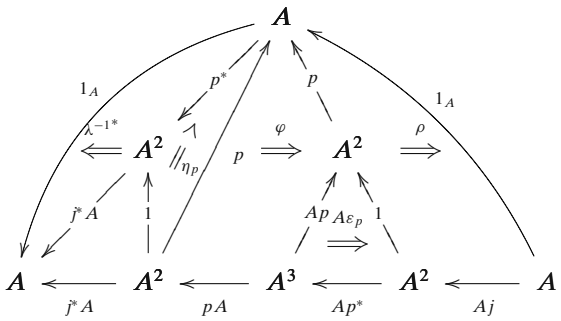
Lemma 6.7 *For* $(A, p, j, \varphi, \lambda, \rho)$ *a naturally Frobenius map monoidal object in a monoidal bicategory and* β *and* α *the 2-cells of* Eq. 21*,*

(i) *the 2-cell* ε *of* Eq. 22 *is equal to the pasting composite of* $\beta \bullet 1_A$ *and* $1_A \bullet \alpha$ *mediated by the constraints of A;*
(ii) *the 2-cell* η^{-1}*, for* η *the 2-cell of* Eq. 22*, is equal to the pasting composite of* $1_A \bullet \beta$ *and* $\alpha \bullet 1_A$ *mediated by the constraints of A.*

Proof For (i) the claim is that ε is equal to



However, if we reorient ε upwards and express ϕ in terms of the associativity constraint the result is

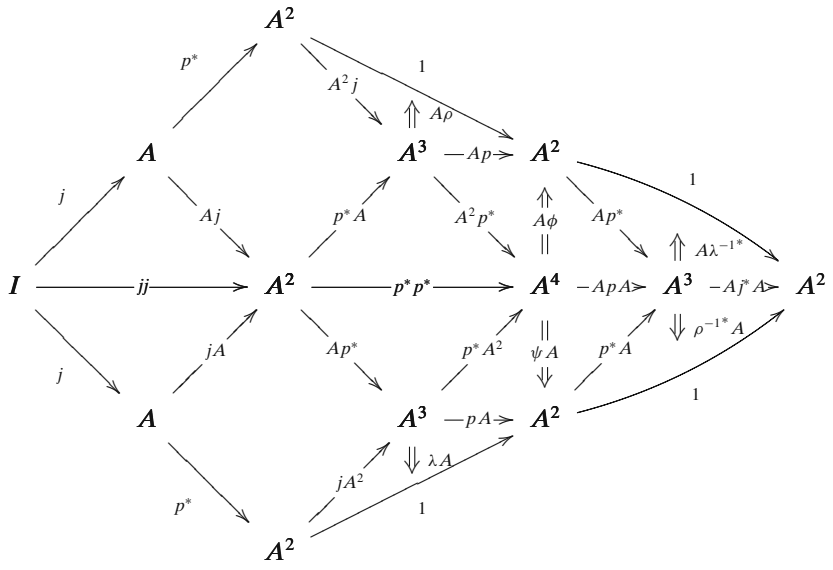


Applying Lemma 6.6 to the two leftmost 2-cells completes the proof. In a similar way we can obtain (ii). □

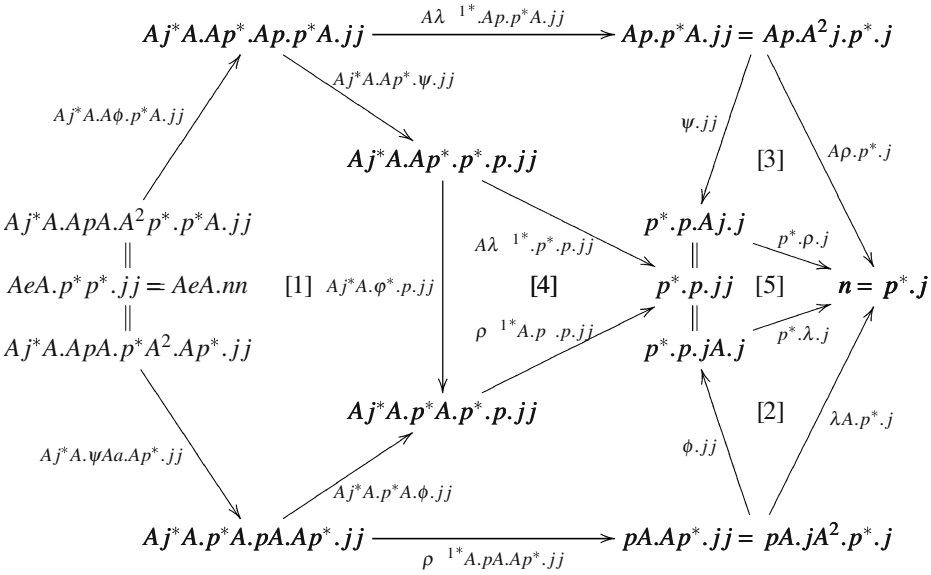
Theorem 6.8 *For $(A, p, j, \varphi, \lambda, \rho)$ a naturally Frobenius map monoidal object in a monoidal bicategory,*

(i) *the arrows $n = p^*.j : I \to A \otimes A$ and $e = j^*.p : A \otimes A \to I$, together with the 2-cells η and ε defined in* Eq. 22*, exhibit A as a right bidual for itself;*
(ii) *the 2-cells β and α defined in* Eq. 21 *exhibit $(1_A, 1_A; \alpha, \beta)$ as an exact pairing, so that A has left dualization, given by the identity, as a monoidal object of $\mathscr{M}$.*

Proof For (i), we have only to verify the coherence equations (Eq. 2). In this case the first of these, $(B\varepsilon.n)(\eta A.n) = 1_n$, is the same as $A\varepsilon.n = \eta^{-1}A.n$. In the following pasting diagram, $A\varepsilon.n$ is the pasting of the northern hemisphere and $\eta^{-1}A.n$ that of the southern.



The following commutative diagram shows the equality of the hemispheres. The clockwise path starting from the left is $A\varepsilon.n$ while the counter-clockwise path is $\eta^{-1}A.n$. The equalities shown, other than definitions such as $n = p^*.j$, can be read from the previous diagram. Most of the regions of the commutative diagram are numbered. (The two regions not numbered commute by functoriality of horizontal composition of the bicategory.) Regions [1], [2], and [3] commute by equations (i), (ii), and (iii) respectively of Proposition 6.4. Region [4] commutes by applying $(-)^*$ to the classical triangle coherence equation for a monoidal object. Region [5] is the classical coherence equation $\lambda_j = \rho_j$.



The other equation of Eq. 2 can be established using similar diagrams.

For (ii), we observe that the first equation of Eq. 4 (renormalized) says, in the language of Lemma 6.7, that ε^{-1} pasted to "the pasting composite of $\beta \bullet 1_A$ and $1_A \bullet \alpha$ mediated by the constraints of A" is 1_{1_A}. This is immediate from (i) of Lemma 6.7 and, similarly, the second equation of Eq. 4 is immediate from (ii) of Lemma 6.7. □

7 An Application to Cartesian Bicategories

7.1

For $\mathscr{C}$ a cartesian bicategory, it was shown in [26] that if A is a Frobenius object in $\mathscr{C}$ and X is any object in $\mathscr{C}$ then the category $\mathrm{Map}\mathscr{C}(X, A)$, of left adjoint arrows from X to A, is a groupoid. We now show that this result is a special case of the Day and Pastro Theorem that we set in the the context of a monoidal bicategory in Theorem 3.2.

As shown in [3], a cartesian bicategory $\mathscr{C}$ has a canonical monoidal bicategory structure that extends the (bicategorical) finite product structure of $\mathrm{Map}\mathscr{C}$. It follows that $\mathscr{C}^{\mathrm{coop}}$ is also a monoidal bicategory and it is $\mathscr{C}^{\mathrm{coop}}$ that will provide an instance

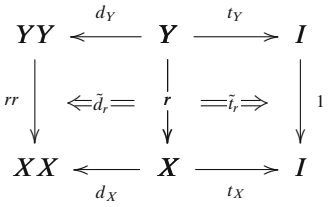
of the general $\mathscr{M}$ of this paper. It will be helpful to recall that adjunctions η, ε; $f \dashv u : B \nrightarrow X$ in $\mathscr{C}$ correspond to adjunctions ε, η; $f \dashv u : X \nrightarrow B$ in $\mathscr{C}^{\mathrm{coop}}$.

In a cartesian bicategory $\mathscr{C}$, every object X has, essentially uniquely, the structure of a comonoidal object

$$I \xleftarrow{t_X} X \xrightarrow{d_X} X \otimes X$$

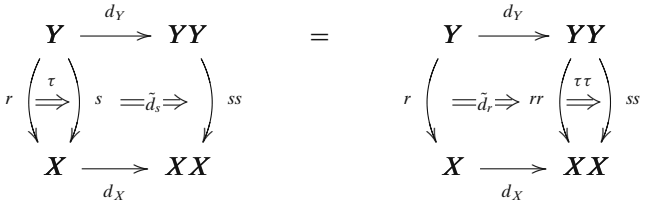
where t_X is the essentially unique *map* to the unit object I and d_X is the diagonal *map*, since $\mathrm{Map}\mathscr{C}$ has finite biproducts, given by I and $\otimes$. (The constraints can be taken to be identities.) It follows that in $\mathscr{C}^{\mathrm{coop}}$ every object is a (strict) map monoidal object. Moreover, an object A in $\mathscr{C}^{\mathrm{coop}}$ is a naturally Frobenius map monoidal object in the sense of this paper if and only if A is a Frobenius object in $\mathscr{C}$ in the sense of [26].

Any arrow $r : Y \nrightarrow X$ in $\mathscr{C}$ is equipped with a unique comonoidal structure

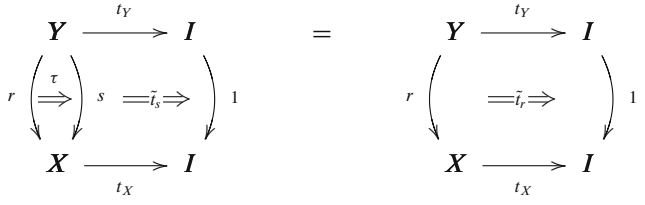


which was shown in [26] to provide a strong comonoidal arrow (meaning that both $\tilde{d}_r$ and $\tilde{t}_r$ are invertible) if and only if r is a map. (We will return to this point in greater detail in the next section.) It follows that in $\mathscr{C}^{\mathrm{coop}}$ every arrow r is monoidal. An arrow r in $\mathscr{C}^{\mathrm{coop}}$ is strong monoidal if and only if r is a map.

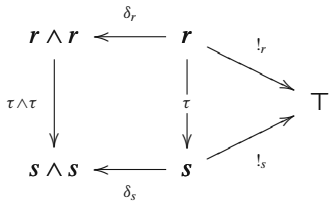
Any 2-cell $\tau : r \Rightarrow s : Y \nrightarrow X$ in $\mathscr{C}$ is comonoidal since we have



and



in virtue of the commutativities



in the hom-category with finite products $\mathscr{C}(Y, X)$. (We recall from [3] the definitions of $\tilde{d}_r$ and $\tilde{t}_r$ in terms of the local binary product $\wedge = \wedge_{Y,X}$ and the local terminal object $\top = \top_{Y,X}$.) It follows that if r and s are maps then in $\mathscr{C}^{\mathrm{coop}}$, τ is both monoidal and comonoidal.

Theorem 7.1 (Walters and Wood) *For a Frobenius object A and any object X in a cartesian bicategory $\mathscr{C}$, the category* $\mathrm{Map}\mathscr{C}(X, A)$ *is a groupoid.*

Proof From the discussion of Subsection 7.1, $\mathrm{Map}\mathscr{C}(X, A)^{\mathrm{op}} = \mathrm{Map}\mathscr{C}^{\mathrm{coop}}(A, X)$ is a full subcategory of the category of Frobenius arrows and monoidal and comonoidal 2-cells from A to X in $\mathscr{C}^{\mathrm{coop}}$. Since the Frobenius object A has left dualization by Theorem 6.8, it is an immediate consequence of Corollary 3.3 that $\mathrm{Map}\mathscr{C}(X, A)^{\mathrm{op}} = \mathrm{Map}\mathscr{C}^{\mathrm{coop}}(A, X)$ is a groupoid. Of course the ordinary categorical dual of a groupoid is a groupoid so that $\mathrm{Map}\mathscr{C}(X, A)$ is so. □

8 Arrows Between Naturally Frobenius Map Monoidal Objects

8.1

Another theorem from [26] that warrants consideration in the generality of this paper is the assertion that, for an arrow $r : X \nrightarrow A$ in a cartesian bicategory with X and A Frobenius, the following are equivalent:

(i) r is a map;
(ii) r is strong comonoidal;
(iii) $r \dashv r^{\circ}$.

Of course being a map is a property of an arrow but we have already remarked in Section 7 that any arrow $r : X \nrightarrow A$ has a unique comonoidal structure $(\tilde{d}_r, \tilde{t}_r)$ so that the point of (ii) is that $\tilde{d}_r$ and $\tilde{t}_r$ are invertible—also a property of an arrow r. By r° we mean the (right) mate of r with respect to the bidual situations $(X, X; e_X, n_X)$ and $(A, A; e_A, n_A)$. Thus r° is the composite

$$A \xrightarrow{n_X A} XXA \xrightarrow{XrA} XAA \xrightarrow{Xe_A} X \tag{23}$$

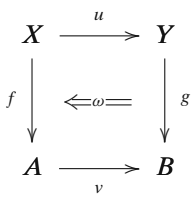
which in a symmetric monoidal bicategory, such as a cartesian bicategory, is isomorphic to the left mate which is the composite

$$A \xrightarrow{An_X} AXX \xrightarrow{ArX} AAX \xrightarrow{e_A X} X \tag{24}$$

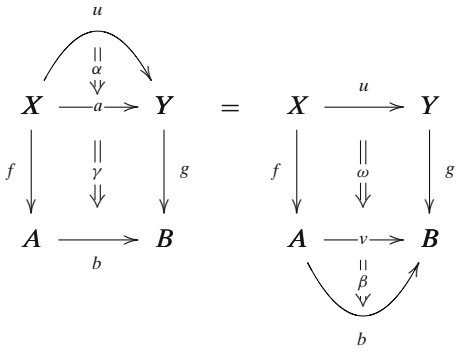
and called r^* in [11].

8.2

For a monoidal $f : X \nrightarrow A$ in $\mathscr{M}$ we recall from 1.4 that f is a *monoidal map* if f has a right adjoint in the bicategory Mon$\mathscr{M}$ of monoidal objects, (lax) monoidal arrows and monoidal 2-cells in $\mathscr{M}$. We will also need to consider the bicategory that we call Sqr$\mathscr{M}$ (but it has also been called Fun$\mathscr{M}$, Cyl$\mathscr{M}$, and Sq$\mathscr{M}$ by other authors). The objects of Sqr$\mathscr{M}$ are the arrows $f : X \nrightarrow A$ of $\mathscr{M}$. An arrow in Sqr$\mathscr{M}$ from $f : X \nrightarrow A$ to $g : Y \nrightarrow B$ is a square



A 2-cell from $(u, \omega, v) : f \nrightarrow g$ to $(a, \gamma, b) : f \nrightarrow g$ in Sqr$\mathscr{M}$ is a pair of 2-cells $\alpha : u \nrightarrow a, \beta : v \nrightarrow b$ in $\mathscr{M}$ such that



Monoidal structure on $\mathscr{M}$ imparts monoidal structure, pointwise, on Sqr$\mathscr{M}$.

8.3

We leave as exercises various calculations in Sqr$\mathscr{M}$. In the first case, we assert that $f : X \nrightarrow A$ has the structure of a monoidal object in Sqr$\mathscr{M}$ if and only if X and A have the structure of monoidal objects in $\mathscr{M}$ and f has comonoidal arrow structure. Next, we claim that $(u, \omega, v) : f \nrightarrow g$ is a map in Sqr$\mathscr{M}$ if and only if u and v are maps in $\mathscr{M}$ and ω is invertible. It is then easy to see that an object $f : X \nrightarrow A$ is

map monoidal in Sqr.$\mathscr{M}$ if and only if X and A are map monoidal and f is strong monoidal, while $f : X \nrightarrow A$ is then naturally Frobenius map monoidal if and only if X and A are so and f is strong monoidal. We recall too that in [11] it is shown that a general arrow f in $\mathscr{M}$ has a right bidual in Sqr.$\mathscr{M}$ if and only if X and A have right biduals in $\mathscr{M}$ and f has a right adjoint in $\mathscr{M}$—given by f^* as in Eq. 24.

Theorem 8.1 *For any monoidal $f : X \nrightarrow A$ in $\mathscr{M}$, with X and A naturally Frobenius map monoidal objects, the following are equivalent:*

(i) *f is a monoidal map;*
(ii) *f is strong monoidal;*
(iii) *$f \dashv f^*$ is a monoidal adjunction, where f^* is as defined in* Eq. 24.

Proof For monoidal f, that (i) implies (ii) follows from [12] (for *arbitrary* monoidal objects). Since (iii) implies (i) is trivial we have only to show that (ii) implies (iii). From the definitions in Subsection 8.2 and the exercises in Subsection 8.3 we see, by applying Theorem 6.8 to the object f in Sqr.$\mathscr{M}$, that f has a right bidual in Sqr.$\mathscr{M}$. From the theorem of [11] quoted in Subsection 8.3, there is an adjunction $f \dashv f^*$ in $\mathscr{M}$ and now, again by [12], the adjunction lifts to Mon.$\mathscr{M}$. □

Of course the cartesian theorem of [26] can be recovered from Theorem 8.1 via the observations made in Section 7.

9 Lax Duals

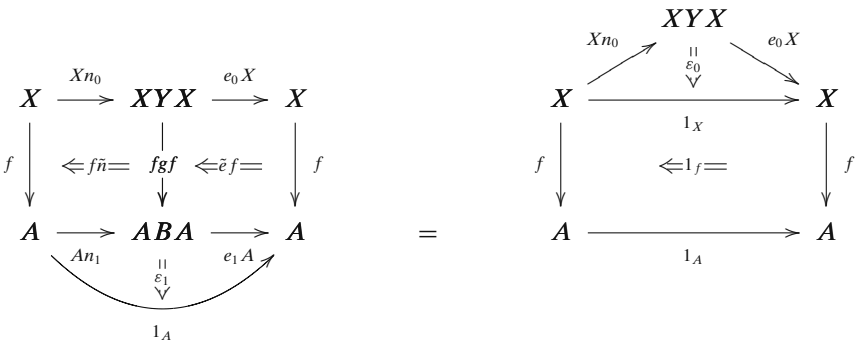
Definition 9.1 A lax dual situation $(A, B; e, n; \varepsilon, \eta)$ in a monoidal bicategory $\mathscr{M}$ consists of the data for a bidual situation as given in Eq. 1, but without requiring invertibility of ε and η, while still subject to the equations (Eq. 2). Then B is said to be a lax right dual for A.

9.1

It follows that in the bicategory Sqr.$\mathscr{M}$ a lax right dual for an $f : X \nrightarrow A$ consists of a $g : Y \nrightarrow B$, together with Sqr.$\mathscr{M}$ arrows

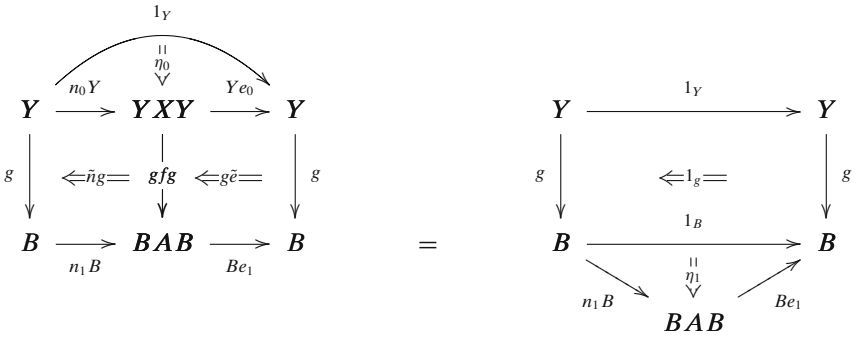
$$\begin{array}{ccc} XY & \xrightarrow{e_0} & I \\ {\scriptstyle fg}\downarrow & \Leftarrow\tilde{e}= & \downarrow{\scriptstyle 1_I} \\ AB & \xrightarrow[e_1]{} & I \end{array} \qquad \text{and} \qquad \begin{array}{ccc} I & \xrightarrow{n_0} & YX \\ {\scriptstyle 1_I}\downarrow & \Leftarrow\tilde{n}= & \downarrow{\scriptstyle gf} \\ I & \xrightarrow[n_1]{} & BA \end{array} \tag{25}$$

and 2-cells ε_0 and ε_1; and η_0, and η_1 as depicted below and satisfying the accompanying equations



(26)

and



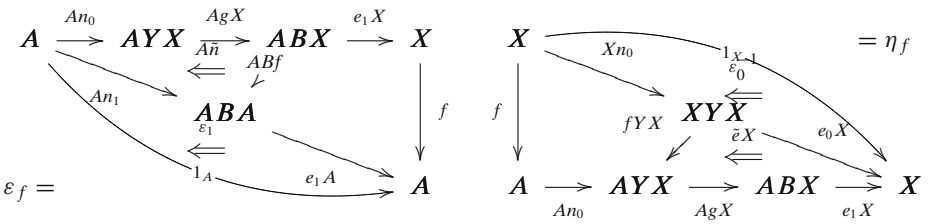
(27)

Equation 26 makes $(\varepsilon_0, \varepsilon_1)$ a 2-cell from $(e_0X, \tilde{e}f, e_1A)(Xn_0, f\tilde{n}, An_1)$ to $(1_X, 1_f, 1_A)$ in $\mathrm{Sqr}\mathscr{M}$ while Eq. 27 makes (η_0, η_1) a 2-cell from $(1_Y, 1_g, 1_B)$ to $(Ye_0, g\tilde{e}, Be_1)(n_0Y, \tilde{n}g, n_1B)$. Additionally, we require that the 2-cells $(\varepsilon_0, \varepsilon_1)$ and (η_0, η_1) in $\mathrm{Sqr}\mathscr{M}$ satisfy the two equations of Eq. 2; but it is trivial that this amounts to the pair (ε_0, η_0) and the pair (ε_1, η_1) each satisfying both equations of Eq. 2. Thus it follows that

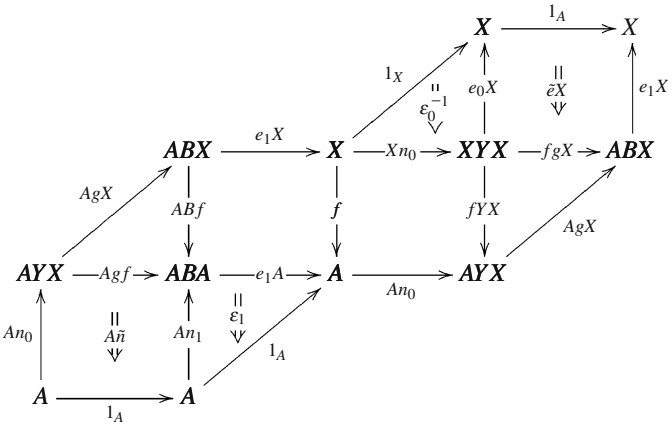
Proposition 9.2 *To give a lax right dual for an object $f : X \nrightarrow A$ in* $\mathrm{Sqr}\mathscr{M}$ *is to give, in $\mathscr{M}$, lax dual situations $(X, Y; e_0, n_0; \varepsilon_0, \eta_0)$ and $(A, B; e_1, n_1; \varepsilon_1, \eta_1)$, an arrow $g : Y \nrightarrow B$, and 2-cells $\tilde{e}$ and $\tilde{n}$ as in* Eq. 25 *satisfying the further equations* Eqs. 26 *and* 27.

Theorem 9.3 *If $f : X \nrightarrow A$ has lax right dual $g : Y \nrightarrow B$ in* Sqr$\mathscr{M}$ *with data as in Proposition 9.2 and ε_0 is invertible then f is a map with adjunction data provided by $(f, f^*; \varepsilon_f, \eta_f)$, where*

$$A \xrightarrow{f^*} X = A \xrightarrow{An_0} AYX \xrightarrow{AgX} ABX \xrightarrow{e_1X} X$$

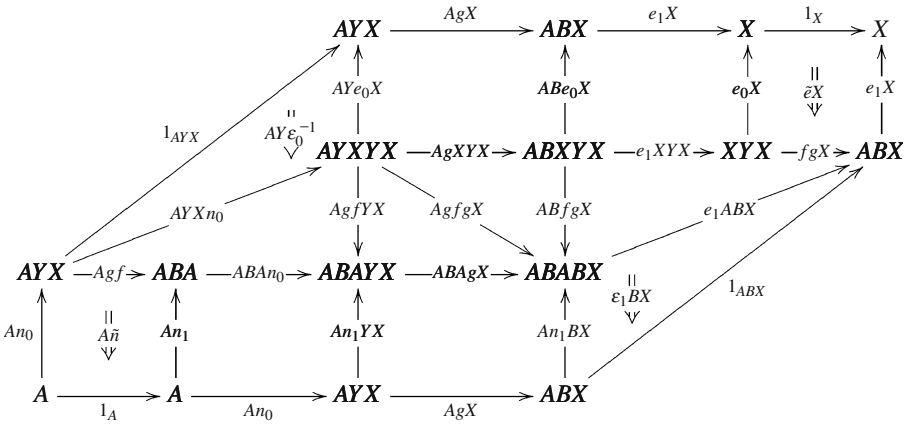


Proof When the 2-cells ε_f and η_f are pasted along $f^* = e_1X.AgX.An_0$, the result is easily seen to be the 2-cell on the left side of Eq. 26 pasted to ε_0^{-1} along $e_0X.Xn_0$. By Eq. 26 this is 1_f which establishes one of the equations required by the statement. For the other equation, begin by displaying the pasting composite of ε_f to η_f along f as shown below.

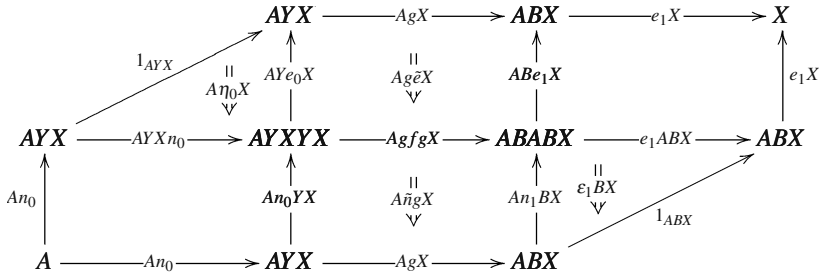


Replace the whisker $(AgX)(An_0)\varepsilon_1$ by the whisker $\varepsilon_1 BX(AgX)(An_0)$ using pseudo-functoriality of $\otimes$ (which as usual is best displayed, as we have it, with concatenation).

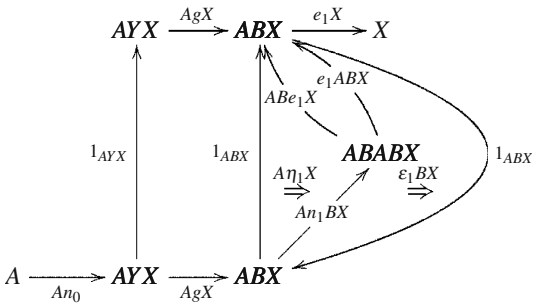
Similarly, replace $\varepsilon_0^{-1}(e_1X)(AgX)$ by $(e_1X)(AgX)AY\varepsilon_0^{-1}$ and fill in the commutativites shown.



By the first equation of Eq. 2 the whisker composite $AY\varepsilon_0^{-1}(An_0)$ can be replaced by the composite $A\eta_0X(An_0)$. In the next diagram we make this replacement. We also replace the pasting of the top three squares and the two triangles below them by the top two squares in the next diagram. We make a similar replacement of the pasting of the bottom three squares and the two triangles above them by the bottom two squares in the next diagram.



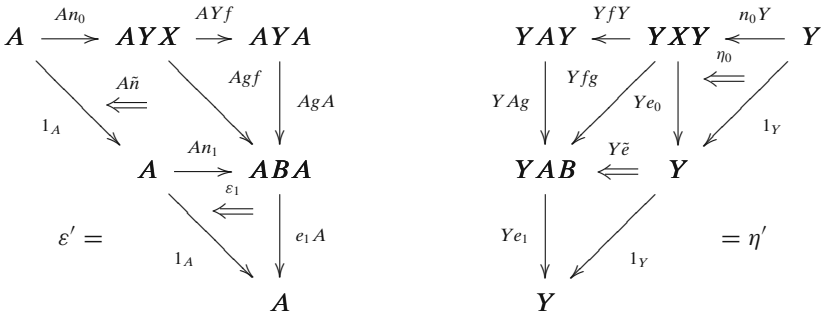
By Eq. 27 (tensored on the left by A and on the right by X) the diagram becomes



and now, by the second equation of Eq. 2 (tensored on the right by X), the pasting composite of the whiskers $(e_1X)A\eta_1X$ and $(e_1X)\varepsilon_1BX$ is 1_{e_1X}. Thus the pasting is equal to $1_{e_1X.AgX.An_0} = 1_{f^*}$ as required. □

Theorem 9.4 *If $g : Y \nrightarrow B$ is a lax right dual for an object $f : X \nrightarrow A$ in* Sqr.$\mathscr{M}$*, with data as in Proposition 9.2, then $(A, Y; e', n'; \varepsilon', \eta')$ is a lax dual situation in $\mathscr{M}$, where*

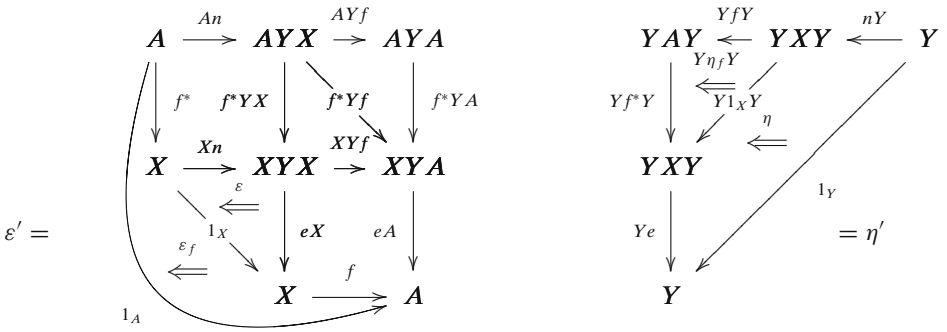
$$e' = (AY \xrightarrow{Ag} AB \xrightarrow{e_1} I) \qquad (I \xrightarrow{n_0} YX \xrightarrow{Yf} YA) = n'$$



Proof The proof is straightforward and we content ourselves with indicating to the reader how to proceed with it. To establish the first equation of Eq. 2 for the putative lax dual situation, construct the requisite pasting using the ingredients displayed above. Replace the whisker $(\eta_0 A)Yf$ by the whisker $Yf(\eta_0 X)$ using pseudofunctoriality of $\otimes$. In the resulting diagram, find the left side of Eq. 26 "tensored" on the left by Y. Using Eq. 26 replace it by the right side of the equation tensored on the left by Y. Now use the first equation of Eq. 2 for the lax dual situation $(X, Y; e_0, n_0, \varepsilon_0, \eta_0)$ that we have in virtue of Proposition 9.2. This will complete the derivation of the first equation and that of the second equation is similar. □

Theorem 9.5 *If $f : X \nrightarrow A$ is a map with adjunction data $(f, f^*; \varepsilon_f, \eta_f)$ and $(X, Y; e, n; \varepsilon, \eta)$ is a lax dual situation then $(A, Y; e', n'; \varepsilon', \eta')$ is a lax dual situation where*

$$e' = (AY \xrightarrow{f^*Y} XY \xrightarrow{e} I) \qquad (I \xrightarrow{n} YX \xrightarrow{Yf} YA) = n'$$



Proof As we did for Theorem 9.4, we just describe the steps of the proof. To establish the first equation of Eq. 2 for the putative lax dual situation, construct the requisite pasting using the ingredients displayed above. Using pseudofunctoriality replace the whisker $(Y\eta_f YA)(XXYf)YXn$ by the whisker $YXYf(YXn)(Y\eta_f)$. We have $(Yf, Yf^*; Y\varepsilon_f, Y\eta_f)$ an adjunction (again by pseudofunctoriality of $\otimes$). Use the equation $(Y\varepsilon_f Yf)(Yf Y\eta_f) = 1$ to replace the left hand side. Replace the whisker $(\eta A)Yf$ by $Yf(\eta X)$. Use the first equation of Eq. 2 for the lax dual situation $(X, Y; e, n, \varepsilon, \eta)$. This establishes the first equation. To establish the second equation of Eq. 2, construct the requisite pasting using the ingredients displayed above. Replace the whisker $eXY(f^*YXY)(AY\eta_f Y)$ by the whisker $f^*YXY(eXY)(\eta_f Y)$. We have $fY \dashv f^*Y$. Use an adjunction equation to eliminate $\varepsilon_f Y$ and $\eta_f Y$. Replace the whisker $f^*Y(A\eta)$ by $(\eta X)f^*Y$. Use the second equation of Eq. 2 for the lax dual situation $(X, Y; e, n, \varepsilon, \eta)$. This establishes the second equation. □

Corollary 9.6 *In* $\mathscr{V}$-mod , *for braided monoidal* $\mathscr{V}$, *any* $\mathscr{V}$*-functor* $f : X \nrightarrow A$ *provides data to exhibit* (A, X^{op}) *as part of a lax dual situation.*

Proof (X, X^{op}) is part of a bidual situation, which is in particular a lax dual situation, and any $\mathscr{V}$-functor $f : X \nrightarrow A$ gives a map $X \nrightarrow A$ in $\mathscr{V}$-mod . □

The Corollary shows that lax duality can be a very weak notion and that (unlike biduals) lax duals are not determined to within equivalence.

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