2) The NG-flow is ergodic on each of the components $SL(2,\mathbb{R})/\mathcal{D} \times \beta_\tau$.

Remark. The one-parameter subgroups $SL(2,\mathbb{R})/\mathcal{D} \times \gamma_i$ are everywhere dense in $SL(2,\mathbb{R})/\mathcal{D}$.

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QUANTUM GROUPS

V. G. Drinfel'd

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The paper is the expanded text of a report to the International Mathematical Congress in Berkeley (1986). In it a new algebraic formalism connected with the quantum method of the inverse problem is developed. Examples are constructed of noncommutative Hopf algebras and their connection with solutions of the Yang–Baxter quantum identity are discussed.

This paper* is devoted to recent work on Hopf algebras (or, as is more or less the same, on quantum groups), of M. Jimbo and the author. Our approach to Hopf algebras is motivated by the quantum method of the inverse problem (QMIP), by the method of constructing and studying integrable quantum systems developed largely by L. D. Faddeev and his collaborators. A large part of the definitions, constructions, examples, and theorems of the present paper arose under the influence of QMIP. Nevertheless, I begin with these definitions, constructions, etc., and only later explain their connection with the QMIP. This order of exposition is opposite to the history of the subject, but on the other hand, as it seems to the author, it permits the clarification of its logic.

The author dedicates this paper to Yurii Ivanovich Manin for his fiftieth birthday.

*The paper is an expanded version of a report to the International Mathematical Congress in 1986.

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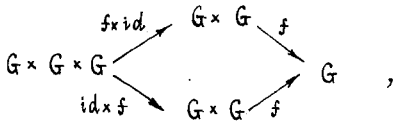
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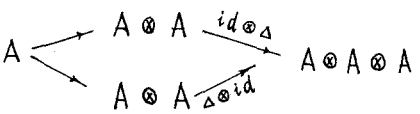
1. So What Is a Quantum Group?

We recall that both in classical and quantum mechanics there are two basic concepts, state and observable. In classical mechanics states are points of a manifold M, and observables are functions on M. In the quantum case, states are one-dimensional subspaces of a Hilbert space H, and observables are operators on H (we neglect the self-adjointness condition). It is easier to understand the connection between classical and quantum mechanics in the language of observables. Both in classical and quantum mechanics the observables form an associative algebra, which is commutative in the classical case and noncommutative in the quantum case. Thus quantizing is somewhat like replacing commutative algebras by noncommutative ones.

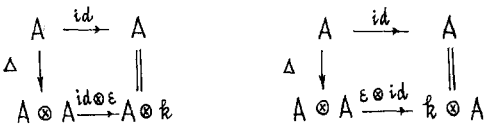
We shall now consider elements of the group G as states, and functions on G as observables. The concept of group is usually defined in the language of states. In order to prequantize, it is first necessary to translate it into the language of observables. This translation is well known. Nevertheless, I recall it. We consider the algebra $A = \mathrm{Fun}(G)$, consisting of functions on G, assumed to be smooth, if G is a Lie group, regular, if G is an algebraic group, etc. A is a commutative associative algebra with unit. It is clear that $\mathrm{Fun}(G \times G) = A \otimes A$, if we understand the symbol $\otimes$ in a suitable sense (for example, if G is a Lie group, then it is necessary to understand $\otimes$ as the topological tensor product). Hence the group operation, considered as a map $f: G \times G \to G$, induces a homomorphism of algebras $\Delta: A \to A \otimes A$ called comultiplication. In order to formulate the associativity of the group operation in terms of Δ, we express it as the condition of commutativity of the diagram



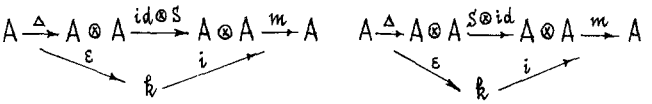
and then we apply the functor $X \mapsto \mathrm{Fun}(X)$. As a result, we get that Δ should be coassociative, i.e., the diagram



should be commutative. The translation of the concepts of group unit e and inverse map $g \mapsto g^{-1}$ is completely analogous. One considers maps $\varepsilon: A \to k$, $S: A \to A$, defined by the formulas $\varepsilon: \varphi \mapsto \varphi(e)$ and $S\varphi(g) = \varphi(g^{-1})$, respectively (here k is the ground field, i.e., $k = \mathbb{R}$, if G is a real Lie group, $k = \mathbb{C}$, if G is a complex Lie group, etc.). Then the identities $g \cdot e = e \cdot g = g$, $gg^{-1} = g^{-1}g = e$ are translated into the language of commutative diagrams and the functor $X \mapsto \mathrm{Fun}(X)$ is applied. As a result one gets commutative diagrams



(1)



(2)

Here m is the multiplication map (i.e., $m(a \otimes b) = ab$), and $i(c) = c \cdot 1_A$. The commutativity of (1) (respectively, (2)) is expressed by the words "ϵ is a counit" (respectively, "S is an antipode"). The properties of (A, m, Δ, i, ϵ, S) listed above mean that A is a commutative Hopf algebra.

There is a general principle: the functor X → Fun(X) from the category of "spaces" to the category of commutative associative algebras with unit (possibly with additional structures or properties) is an antiequivalence. This principle becomes a theorem if "space" is understood as "affine scheme," or if "space" is understood as "compact topological space," and "algebra" as "C^*-algebra." It follows from the principle indicated that the category of groups is antiequivalent to the category of commutative Hopf algebras.

Now we define the category of quantum spaces as the category dual to the category of (not necessarily commutative) associative algebras with unit. We denote by spec A (the spectrum of A) the quantum space corresponding to the algebra A. Finally, we define a quantum group as the spectrum of a (not necessarily commutative) Hopf algebra. Thus, the concepts of Hopf algebra and quantum group are actually equivalent, but the second concept has a geometric inflection.

We make some remarks in connection with the definition of a Hopf algebra. First of all it is known that for given A, m, and Δ, the counit $\epsilon: A \to k$ is unique and is a homomorphism. The antipode $S: A \to A$ is also unique and is an antihomomorphism (in relation both to the multiplication and the comultiplication). Further, in the noncommutative case one also requires the existence of a "twisted antipode" $S': A \to A$, which is actually an antipode for the opposite multiplication and the same comultiplication. S' is also an antipode for the opposite comultiplication and the same multiplication. It is known that $SS' = S'S = id$. We note that in the commutative or cocommutative case $S' = S$, but in general $S' \neq S$ and $S^2 \neq id$. The proofs of these assertions are contained in the standard texts on Hopf algebras [1-3]. In [4] there is a nonformal discussion of some aspects of the concept of Hopf algebra. One should keep in mind that our use of the term "Hopf algebra" is not the ordinary one: some do not require the existence of a unit, counit and antipode (Hopf algebras in this sense correspond to quantum semigroups).

It is important to note that a quantum group is not a group or even a group object of the category of quantum spaces. The fact is that for noncommutative algebras the tensor product is not the categorical coproduct.

The question now arises do there exist natural examples of noncommutative Hopf algebras. A simple way of constructing such algebras is to consider A^*, where A is a commutative, but not cocommutative algebra (we recall that the dual space of a Hopf algebra A has the structure of a Hopf algebra: the multiplication $A^* \otimes A^* \to A^*$ is induced by the comultiplication of the algebra A, and the comultiplication of the algebra A^* is induced by the multiplication of the algebra A). In this way one gets more or less all commutative noncocommutative Hopf algebras. The words "more or less" are connected with the fact that it is not true that for any vector space V one has $V^{**} = V$. But on the heuristic level $V^{**} = V$ and consequently any cocommutative Hopf algebra has the form $(\mathrm{Fun}(G))^*$ for some group G. It is clear that the commutativity of $(\mathrm{Fun}(G))^*$ is equivalent with the commutativity of G. We note that $(\mathrm{Fun}(G))^*$ is nothing but the group algebra of G. The universal enveloping algebras (the comultiplication $\mathfrak{U}\mathfrak{g} \to \mathfrak{U}\mathfrak{g} \otimes \mathfrak{U}\mathfrak{g}$ has the form $\Delta(X) = X \otimes 1 + 1 \otimes X, X \in \mathfrak{g}$) form an important class of cocommutative Hopf algebras. If $\mathfrak{g}$ is the Lie algebra of the Lie group G, then $\mathfrak{U}\mathfrak{g}$ can be considered as the subalgebra of $(C^\infty(G))^*$, consisting of distributions $\varphi \in C_0^{-\infty}(G)$ such that $\mathrm{supp}\ \varphi \subset \{e\}$. One can identify $(\mathfrak{U}\mathfrak{g})^*$ with the completion of the local ring of points $e \in G$ or with $\mathrm{Fun}(\hat{G})$, where $\hat{G}$ is the formal group corresponding to $\mathfrak{g}$ (the second realization of $(\mathfrak{U}\mathfrak{g})^*$ makes sense even if G does not exist, which may very well happen if $\dim \mathfrak{g} = \infty$).

The most interesting and enigmatic Hopf algebras are those which are neither commutative nor cocommutative. Although Hopf algebras have been studied intensively by "pure" algebraists [2-12], as well as by specialists in von Neumann algebras [13-26], a large part of the examples of noncommutative noncocommutative Hopf algebras, invented independently of the theory of quantum integrable systems, are, from my point of view, rather counterexamples, than "natural" examples (there are, however, notable exceptions: cf., e.g., [12, 16], and [2, pp. 89-90]. We discuss a general method for constructing noncommutative noncocommutative Hopf algebras given in [27, 28] under the influence of the QMIP. This method is based on the concepts of classical limit and quantization. It can be considered as the realization of ideas of G. I. Kats and V. G. Palyutkin (cf. end of [16]).

2. Quantization

Roughly speaking, the quantization of a commutative associative algebra A_0 over k is a not necessarily commutative deformation of A_0, depending on a parameter h (Planck's constant), i.e., an associative algebra A over k[[h]] such that $A/hA = A$ and A is a topologically free k[[h]]-module. If A is given then in A_0 one can define a new operation (Poisson bracket) by the formula

$$\{a \bmod h, b \bmod h\} = (h^{-1}[a, b]) \bmod h. \qquad (3)$$

Thus A_0 becomes a Poisson algebra (i.e., an algebra with respect to $\{,\}$ and a commutative associative algebra with respect to multiplication, where these two operations are compatible in the following sense: $\{a, bc\} = \{a,b\}c + b\{a,c\}$. Now we slightly change the point of view about quantization.

Definition. Quantization of the Poisson algebra A_0 is a deformation A of the algebra A_0 over $\mathbb{C}[[\hbar]]$ in the sense of associative algebras such that the Poisson bracket on A_0 defined by (3) is equal to the bracket previously defined.

Of course this approach to quantization has been known as long as quantum mechanics has existed. It was explained to mathematicians by F. A. Berezin, J. Vey, A. Lichnerowicz, M. Flato, D. Sternheimer, etc.

We need an analog of the definition given above for Hopf algebras. In this case A_0 is a Poisson Hopf algebra (i.e., on A_0 there are given Hopf algebra and Poisson algebra structures with the same multiplication, while the comultiplication $A_0 \to A_0 \otimes A_0$ should be a homomorphism of Poisson algebras; it is understood that the Poisson bracket on $A_0 \otimes A_0$ has the form $\{a \otimes b, c \otimes d\} = ac \otimes \{b,d\} + \{a,c\} \otimes bd$, and A is a deformation of A_0 as Hopf algebras. We shall also use the dual concept of quantization of copoisson Hopf algebras (a copoisson Hopf algebra is a cocommutative Hopf algebra B with a Poisson cobracket $B \to B \otimes B$, compatible with the Hopf algebra structure).

We discuss the structure of Poisson and copoisson Hopf algebras in Secs. 3 and 4. Afterwards we discuss the quantization problem.

3. Poisson Groups and Lie Bialgebras

A Poisson group is a group G together with a Poisson bracket defined on Fun(G), which turns Fun(G) into a Poisson Hopf algebra. In other words, the Poisson bracket should be compatible with the group operation, which means, by definition, that the map $\mu: G \times G \to G$, $\mu(g_1, g_2) = g_1 g_2$ should be a Poisson map in the sense of [33], i.e., $\mu^*: \mathrm{Fun}\, G \to \mathrm{Fun}(G \times G)$ should be a homomorphism of Lie algebras. By refining the meaning of the word "group" and the symbol Fun(G), we get the concept of Poisson–Lie group, Poisson formal group, Poisson algebraic group, etc. According to our principles, the concepts of Poisson group and Poisson Hopf algebra are equivalent.

There is a very simple description of Poisson–Lie group in terms of Lie bialgebras.

Definition. A Lie bialgebra is a vector space $\mathfrak{g}$, on which there are given Lie algebra and Lie coalgebra structures, which are compatible in the following sense: the cocommutator map $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ should be a 1-cocycle ($\mathfrak{g}$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ by means of the adjoint representation).

THEOREM 1. The category of connected simply connected Poisson–Lie groups is equivalent with the category of finite-dimensional Lie bialgebras.

Sketch of Proof. A Poisson structure on a Lie group G is defined by a formula of the form

$$\{\varphi, \psi\} = \eta^{\mu\nu} \partial_\mu \varphi \cdot \partial_\nu \psi, \qquad \varphi, \psi \in C^\infty(G), \tag{4}$$

where $\{\partial_\mu\}$ is a basis of right-invariant vector fields on G. We consider η as a function $G \to \overset{2}{\wedge} \mathfrak{g}$. The compatibility of the group operation with the bracket (4) means that η is a 1-cocycle, the 1-cocycles of G are in one-to-one correspondence with 1-cocycles of the $\mathfrak{g}$. The operation $\mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$, defined by the 1-cocycle $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ corresponding to η, is the infinitesimal part of the bracket (4). Hence, if the bracket (4) satisfies the Jacobi identity, then $\mathfrak{g}^*$ is a Lie algebra. To prove the converse assertion, we note that the expression $\{\{\varphi,\psi\},\chi\} + \{\{\psi,\chi\},\varphi\} + \{\{\chi,\varphi\},\psi\}$, $\varphi, \psi, \chi \in C^\infty(G)$ can be written in the form $\xi^{\lambda\mu\nu} \partial_\lambda \varphi \cdot \partial_\mu \psi \partial_\nu \chi$. One can show that $\xi: G \to \overset{3}{\wedge} \mathfrak{g}$ is a 1-cocycle. Hence $\xi = 0$ if the infinitesimal part of ξ is equal to zero.

One can prove the analogous theorem for Poisson formal groups over a field of characteristic zero in exactly the same way. Here is a different proof which is also instructive. The algebra of functions on the formal group corresponding to $\mathfrak{g}$ is $(U\mathfrak{g})^*$. A Poisson Hopf algebra structure on $(U\mathfrak{g})^*$ is equivalent with a Poisson Hopf algebra structure on $U\mathfrak{g}$. Hence it suffices to prove the following simple theorem.

THEOREM 2. Let $\delta: U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ be a Poisson cobracket, which turns $U\mathfrak{g}$ into a copoisson Hopf algebra (the Hopf algebra structure on $U\mathfrak{g}$ is the usual one). Then $\delta(\mathfrak{g}) \subset \mathfrak{g} \otimes \mathfrak{g}$ and $(\mathfrak{g}, \delta/\mathfrak{g})$ is a Lie bialgebra. In this way one gets a one-to-one correspondence between copoisson Hopf algebra structures on $U\mathfrak{g}$, inducing the usual Hopf algebra structure and Lie bialgebra structures on $\mathfrak{g}$ inducing a given Lie algebra structure.

Now we discuss the concept of Lie bialgebra. First of all, there is a one-to-one correspondence between Lie bialgebras and Manin triples. A Manin triple $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a Lie algebra with a nondegenerate invariant scalar product and isotropic Lie subalgebras $\mathfrak{p}_1$, $\mathfrak{p}_2$ such that $\mathfrak{p}$ as a vector space is the direct sum of $\mathfrak{p}_1$ and $\mathfrak{p}_2$. The correspondence mentioned above can be constructed as follows: if $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a Manin triple, then we set $\mathfrak{g} = \mathfrak{p}_1$ and we define the cocommutator $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ to be the map dual to the commutator map $\mathfrak{p}_2 \otimes \mathfrak{p}_2 \to \mathfrak{p}_2$ (we note that $\mathfrak{p}_2$ is naturally isomorphic to $\mathfrak{g}^*$). Conversely, if there is given a Lie bialgebra $\mathfrak{g}$, then we set $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{g}^*$, $\mathfrak{p}_1 = \mathfrak{g}$, $\mathfrak{p}_2 = \mathfrak{g}^*$ and we define the commutator $[x, \ell]$ for $x \in \mathfrak{g}$, $\ell \in \mathfrak{g}^*$ so that the natural scalar product in $\mathfrak{p}$ is invariant. We note that if $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a Manin triple, then $(\mathfrak{p}, \mathfrak{p}_2, \mathfrak{p}_1)$ is also a Manin triple. Hence, the concept of Lie bialgebra is self-dual.

Here are some examples of Lie bialgebras. Examples 2-4 are important for the method of the inverse problem.

Example 1. If $\dim \mathfrak{g} = 2$, then any linear maps $\wedge^2 \mathfrak{g} \to \mathfrak{g}$ and $\mathfrak{g} \to \wedge^2 \mathfrak{g}$ define a Lie bialgebra structure on $\mathfrak{g}$. A two-dimensional Lie bialgebra is called nondegenerate, if the composition $\wedge^2 \mathfrak{g} \to \mathfrak{g} \to \wedge^2 \mathfrak{g}$ is nonzero. In this case there exists in $\mathfrak{g}$ a basis $\{x_1, x_2\}$ such that $[x_1, x_2] = \alpha x_2$, and the cocommutator is given by the formula $x_1 \mapsto 0$, $x_2 \mapsto \beta x_2 \wedge x_1$. Here $\alpha\beta \neq 0$ and $\alpha\beta$ is independent of the choice of e_1, e_2.

Example 2. Let $\mathfrak{g}$ be a Kac-Moody algebra (in the sense of [55]) with fixed invariant scalar product $\langle , \rangle$, $\mathfrak{f}$ be a Cartan subalgebra, $\mathfrak{b}_+ \supset \mathfrak{f}$ be Borel subalgebras. We set $\mathfrak{p} = \mathfrak{g} \times \mathfrak{g}$, $\mathfrak{p}_1 = \{(x, y) \in \mathfrak{g} \times \mathfrak{g};\ x = y\} \simeq \mathfrak{g}$ $\mathfrak{p}_2 = \{(x, y) \in \mathfrak{b}_- \times \mathfrak{b}_+;\ x_{\mathfrak{f}} + y_{\mathfrak{f}} = 0\}$. As the scalar product of the elements $(x_1, y_1) \in \mathfrak{p}$ and $(x_2, y_2) \in \mathfrak{p}$ we take $\langle x_1, x_2 \rangle - \langle y_1, y_2 \rangle$. Since $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a Manin triple, $\mathfrak{g}$ has the structure of a Lie bialgebra. The cocommutator $\varphi: \mathfrak{g} \to \wedge^2 \mathfrak{g}$ can be described explicitly in terms of the canonical generators X_i^+, X_i^-, H_i (here $X_i^\pm \in [\mathfrak{b}_\pm, \mathfrak{b}_\pm]$, and H_i image of a simple root $\alpha_i \in \mathfrak{f}^*$ under the isomorphism $\mathfrak{f}^* \to \mathfrak{f}$): $\varphi(H_i) = 0$, $\varphi(X_i^\pm) = 1/2\, X_i^\pm \wedge H_i$ we note that $\mathfrak{b}_+$ and $\mathfrak{b}_-$ are subalgebras of $\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{sl}(2)$, then $\mathfrak{b}_+$ and $\mathfrak{b}_-$ are of the type described in Example 1. The Manin triple corresponding to $\mathfrak{b}_+$ is $(\mathfrak{g} \times \mathfrak{f}, \operatorname{Im} \psi_+, \operatorname{Im} \psi_-)$, where $\psi_\pm : \mathfrak{b}_\pm \hookrightarrow \mathfrak{g} \times \mathfrak{f}$ is defined by $\psi_\pm(a) = (a, \pm a_{\mathfrak{f}})$ and the scalar product on $\mathfrak{g} \times \mathfrak{f}$ is equal to $\langle , \rangle_{\mathfrak{g}} - \langle , \rangle_{\mathfrak{f}}$.

Example 3. We fix a simple Lie algebra $\mathfrak{A}$ ($\dim \mathfrak{A} < \infty$) and an invariant scalar product on it. We set $\mathfrak{p} = \mathfrak{A}((u^{-1}))$, $\mathfrak{p}_1 = \mathfrak{A}[u]$, $\mathfrak{p}_2 = u^{-1}\mathfrak{A}[[u^{-1}]]$ and we define a scalar product in $\mathfrak{p}$ by $(f, g) = \operatorname{res}_{u=\infty} (f(u), g(u))\, du$. The Manin triple defines a Lie bialgebra structure on $\mathfrak{g} = \mathfrak{A}[u]$. The cocommutator in $\mathfrak{g}$ is given by

$$a(u) \mapsto [a(u) \otimes 1 + 1 \otimes a(v),\ r(u, v)]\,. \tag{5}$$

Here $a \in \mathfrak{A}[u]$, $\mathfrak{g} \otimes \mathfrak{g}$ is identified with $(\mathfrak{A} \otimes \mathfrak{A})[u, v]$, and $r(u, v) = t/u - v$, where t is the element of $\mathfrak{A} \otimes \mathfrak{A}$ corresponding to our scalar product. The right side of (5) has no pole for $u = v$, since t is invariant.

Example 4 (cf. [29-31]). We fix a nonsingular irreducible projective algebraic curve X over $\mathbb{C}$. We denote by E (respectively A, $\mathcal{O}_x$) the field of rational functions on X (respectively, the ring of adeles, the completion of the local ring of the point $x \in X$). We fix an absolutely simple Lie algebra G over E ($\dim G < \infty$) and a rational differential ω on X, $\omega \neq 0$. We define an invariant $\mathbb{C}$-valued scalar product on $G \otimes_E A$ by the formula $(u, v) = \sum_{x \in X} \operatorname{res}_x \omega \operatorname{Tr}(\rho(u_x)\rho(v_x))$, where $\rho: G \to \mathfrak{gl}(n, E)$ is a faithful representation, $u = (u_x)_{x \in X}$, $v = (v_x)_{x \in X}$. Then G is a maximal isotropic subspace of $G \otimes_E A$. If there exists an open isotropic $\mathbb{C}$-subalgebra $\Lambda \subset G \otimes_E A$ such that $G \otimes_E A = G \oplus \Lambda$, then on G there arises a Lie bialgebra structure. To each subset $S \subset X$, $S \neq \emptyset$ corresponds a subbialgebra $G_S = \{a \in G;$ the image of a in $G \otimes A^S$ belongs to the image of Λ in $G \otimes A^S\}$, where A^S is the ring of adeles without x-component, $x \in S$. The bialgebra of Example 3 is actually G_S, where $X = \mathbb{P}^1$, $\omega = d\lambda$, $S = \{\infty\}$, $G = \mathfrak{A} \otimes E$, $\Lambda = \mathfrak{A} \otimes (\mathfrak{M}_\infty \times \prod_{x \in X \setminus S} \mathcal{O}_x)$, $\mathfrak{M}_\infty$ being a maximal ideal of $\mathcal{O}_\infty$. If $\mathfrak{g}$ is an affine Kac-Moody algebra with the bialgebra structure of Example 2, then $\mathfrak{g}' \overset{\text{def}}{=} [\mathfrak{g}, \mathfrak{g}] / (\text{the center of } \mathfrak{g})$ actually coincides with G_S for $X = \mathbb{P}^1$, $\omega = \lambda^{-1} d\lambda$, $S = \{0, \infty\}$, $G = \mathfrak{g}' \otimes_{\mathbb{C}[\lambda, \lambda^{-1}]} \mathbb{C}(\lambda)$ ($\mathfrak{g}'$

has a natural $\mathbb{C}[\lambda, \lambda^{-1}]$-algebra structure), $\Lambda = \mathbb{C} \times \prod_{x \in X \setminus S} (\mathfrak{g}' \otimes_{\mathbb{C}[\lambda,\lambda^{-1}]} \mathcal{O}_x)$, where $\mathbb{C} \subset (\mathfrak{g}' \otimes_{\mathbb{C}[\lambda,\lambda^{-1}]} \mathbb{C}((\lambda^{-1}))) \times \mathfrak{g}' \otimes_{\mathbb{C}[\lambda,\lambda^{-1}]} \mathbb{C}((\lambda)))$ is the closure of the algebra $\mathcal{P}'_x = \{(x,y) \in \mathfrak{g}' \times \mathfrak{g}' ;\ x \in \mathcal{b}'_-,\ y \in \mathcal{b}'_+,\ x_\mathfrak{g} + y_\mathfrak{g} = 0\}$.

4. Classical Yang–Baxter Equation

Let $\mathfrak{g}$ be a Lie algebra and $\varphi : \mathfrak{g} \to \Lambda^n \mathfrak{g}$ be the coboundary of the element $r \in \Lambda^2 \mathfrak{g}$. One can show that $(\mathfrak{g}, \varphi)$ is a Lie bialgebra if and only if

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \quad \mathfrak{g} \text{ being invariant} \tag{6}$$

Here for example $[r^{12}, r^{13}] \overset{def}{=} \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j$, where $r = \sum a_i \otimes b_i$ (one can suppose that $r^{12} = \sum a_i \otimes b_i \otimes 1 \in (U\mathfrak{g})^{\otimes 3}$, $r^{13} = \sum a_i \otimes 1 \otimes b_i \in (U\mathfrak{g})^{\otimes 3}$, $r^{23} = \sum 1 \otimes a_i \otimes b_i \in (U\mathfrak{g})^{\otimes 3}$. In particular, if

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0, \tag{7}$$

then $(\mathfrak{g}, \varphi)$ is a Lie bialgebra. (7) is nothing else than the classical Yang–Baxter Equation (CYBE), or the classical equation of triangles. There is an important case, intermediate between (6) and (7). Let us assume that $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies (7), but the skew-symmetry condition $r^{12} + r^{21} = 0$ is replaced by the condition of $\mathfrak{g}$-invariance of $r^{12} + r^{21}$ (then the coboundary $\varphi = \partial r$ is skew-symmetric). In this situation $(\mathfrak{g}, \varphi)$ is also a Lie bialgebra. If $r^{12} + r^{21} = P$, $\rho = r - 1/2\, P$, then $\rho \in \Lambda^2 \mathfrak{g}$, $\varphi = \partial \rho$ and $[\rho^{12}, \rho^{13}] + [\rho^{12}, \rho^{13}] + [\rho^{13}, \rho^{23}] = 1/4\, [P^{12}, P^{23}]$.

Definition. A cobounding Lie bialgebra is a pair $(\mathfrak{g}, r)$, where $\mathfrak{g}$ is a Lie bialgebra, $r \in \Lambda^2 \mathfrak{g}$, ∂r is equal to the commutator of $\mathfrak{g}$. A cobounding Lie bialgebra $(\mathfrak{g}, r)$ is called triangular if r satisfies (7). A quasitriangular Lie bialgebra is a pair $(\mathfrak{g}, r)$, where $\mathfrak{g}$ is a Lie bialgebra, $r \in \mathfrak{g} \otimes \mathfrak{g}$, ∂r is equal to the cocommutator of $\mathfrak{g}$, and r satisfies (7).

It was noted in [32, 27] that the expression $[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$, $r \in \Lambda^2 \mathfrak{g}$, is equal to $1/2\{r, r\}$, where $\{,\}$ is a bilinear operation in $\Lambda^* \mathfrak{g}$ such that $\{a, b\} = [a, b]$ for $a, b \in \mathfrak{g}$ and $\{a, b\} = (-1)^{(k+1)(\ell+1)+1}\{b, a\}$, $\{a, b \wedge c\} = \{a, b\} \wedge c + (-1)^{(k+1)\ell} b \wedge \{a, c\}$ for $a \in \Lambda^k \mathfrak{g}$, $b \in \Lambda^\ell \mathfrak{g}$, $c \in \Lambda^* \mathfrak{g}$. ($\Lambda^* \mathfrak{g}$ is a Poisson superalgebra with odd Poisson brackets; if the elements of $\Lambda^* \mathfrak{g}$ are considered as left-invariant polyvector fields on the Lie group corresponding to $\mathfrak{g}$, then the operation $\{,\}$ on $\Lambda^* \mathfrak{g}$ is nothing but the Schouten bracket [34]).

The Poisson bracket on the Poisson–Lie group G corresponding to the cobounding bialgebra $(\mathfrak{g}, r)$ has the form

$$\{\varphi, \psi\} = r^{\mu\nu} (\partial'_\mu \varphi\, \partial'_\nu \psi - \partial_\mu \varphi\, \partial_\nu \psi), \tag{8}$$

where ∂_μ (respectively ∂'_μ) are right-invariant (respectively left-invariant) vector fields on G corresponding to some basis of $\mathfrak{g}$, and $r^{\mu\nu}$ are the coordinates of r. The following property of the bracket (8) plays a key role in the Hamiltonian version of the classical method of the inverse problem (there is a systematic account of this method in [35]; cf. also [36-38]):

$$\varphi, \psi \in C^\infty(G),\ \varphi(gxg^{-1}) = \varphi(x),\ \psi(gxg^{-1}) = \psi(x) \Rightarrow \{\varphi, \psi\} = 0. \tag{9}$$

We return to Examples 1-4 of Sec. 3. In a nondegenerate two-dimensional bialgebra a cocommutator is not a coboundary. We consider now the cocommutator (5) in the algebra $\mathfrak{g} = \mathfrak{A}[u]$. First let us assume that r(u, v) is an $\mathfrak{A} \otimes \mathfrak{A}$-valued polynomial. Then one can consider r as an element of $\mathfrak{g} \otimes \mathfrak{g}$, and the cocommutator (5) as the coboundary of r. The CYBE and the condition $r \in \Lambda^2 \mathfrak{g}$ assume the form

$$[r^{12}(u_1, u_2),\ r^{13}(u_1, u_3)] + [r^{12}(u_1, u_2), r^{23}(u_2, u_3)] + [r^{13}(u_1, u_3),\ r^{23}(u_2, u_3)] = 0, \tag{10}$$

$$r^{12}(u_1, u_2) + r^{23}(u_2, u_1) = 0. \tag{11}$$

Thus, each polynomial solution of (10), (11) defines a triangular bialgebra structure on $\mathfrak{g}$. In Example 3 r(u, v) is a nonpolynomial solution of (10), (11). Hence the corresponding bialgebra is "pseudotriangular" (nevertheless, (9) holds).

The bialgebra $\mathfrak{g}$ of Example 2 has quasitriangular structure if $\dim \mathfrak{g} < \infty$ we consider the element $t \in \mathfrak{g} \otimes \mathfrak{g}$ corresponding to the scalar product in $\mathfrak{g}$; we represent t in the form $t_+ - t_-$

t_0+t_{-+}, $t_{+-}\in[\mathfrak{b}_+,\mathfrak{b}_+]\otimes[\mathfrak{b}_-,\mathfrak{b}_-]$, $t_{-+}\in[\mathfrak{b}_-,\mathfrak{b}_-]\otimes[\mathfrak{b}_+,\mathfrak{b}_+]$, $t_0\in\mathfrak{f}\otimes\mathfrak{f}$ and we set $r=t_{+-}+\frac{1}{2}t_0$. If $\dim\mathfrak{g}=\infty$ then r does not belong to the algebraic tensor product $\mathfrak{g}\otimes\mathfrak{g}$. Thus, $(\mathfrak{g},r)$ is a "pseudotriangular" bialgebra. We consider the nontwisted affine case in more detail (the twisted affine case is analogous). In this case $\mathfrak{g}'=[\mathfrak{g},\mathfrak{g}]$ / (the center of $\mathfrak{g}$) $=\mathfrak{A}[\lambda,\lambda^{-1}]$ for some simple algebra $\dim\mathfrak{A}<\infty$. We denote by $\bar{r}$ the analog of r for $\mathfrak{g}'$. We consider $\bar{r}$ as a formal series in $\lambda=\lambda\otimes 1$ and $\mu=1\otimes\lambda$ with coefficients in $\mathfrak{A}\otimes\mathfrak{A}$. Then $\bar{r}(\lambda,\mu)$ is the Laurent series of a rational function of λ/μ, having a simple pole for $\lambda/\mu=1$ with residue $t_{\mathfrak{A}}$ ($t_{\mathfrak{A}}\in\mathfrak{A}\otimes\mathfrak{A}$ corresponds to the scalar product in $\mathfrak{A}$). This rational function satisfies (10) and (11). Moral ([39, Sec. 3]): if the solution r(u, v) of Eqs. (10) and (11) has a pole for u = v, then the corresponding Lie bialgebra is "pseudoquasitriangular" rather than "pseudotriangular"; in other words, $r^{12}(u_1,u_2)+r^{21}(u_2,u_1)$ is equal to $t_{\mathfrak{A}}\cdot\delta(u_1-u_2)$, rather than to zero.

I. V. Cherednik proved [31] that the bialgebras of Example 4 also correspond to solutions of (10) and (11), having a pole on the diagonal. Moreover, it follows from [40-42] that there is a one-to-one correspondence between "nondegenerate" solutions of (10) and (11) up to a certain equivalence relation (such solutions always have a pole on the diagonal) and the quadruples (X, ω, G, Λ) from Example 4. Moreover, the classification of nondegenerate solutions of (10) and (11) is given in [40, 41]. In terms of (X, ω, G, Λ) the results of [40, 41] can be formulated as follows.

1) For the existence of Λ it is necessary that ω not have zeros, and consequently there are three possibilities: a) X is an elliptic curve and ω is a regular differential, b) $X=\mathbb{P}^1$, $\omega=\lambda^{-1}d\lambda$ c) $X=\mathbb{P}^1$, $\omega=d\lambda$. The corresponding solutions r(u, v) are elliptic, trigonometric, or rational functions of u − v.

2) In case a) for the existence of Λ it is necessary that G be isomorphic with $s\ell(n)$, and for each n there are a finite number of possibilities for Λ; all of them are enumerated. In case b) G is of the type described at the end of Example 4, and all the possibilities for Λ are enumerated (they are analogous to the algebra Λ described at the end of Example 4). In case c) little is known.

Unfortunately, inadequate space does not permit me to analyze the important ideas and results of M. A. Semenov-Tyan-Shanskii [39, 43], relating to Poisson–Lie groups and the CYBE.

5. Some Remarks of Historical Character

The Poisson bracket (8) and the CYBE were introduced by E. K. Sklyanin [44, 37] as the classical limits of the corresponding quantized objects. The compatibility of the bracket (8) with the group operation was formulated by him almost explicitly. The abstract concept of Poisson–Lie group appeared later [27]. The classification of solutions of the CYBE was based on extensive experimental material, due to many authors (cf. the appendix to [45]). In particular, the solution r(u, v) = t/u − v was found in [45, 46].

6. Quantization of Lie Bialgebras (Examples)

By definition, quantization of the Lie bialgebra $\mathfrak{g}$ is quantization of $U\mathfrak{g}$ in the sense of Sec. 2, where $U\mathfrak{g}$ is considered as a copoisson Hopf algebra according to Theorem 2. If A is a quantization of the Lie bialgebra $\mathfrak{g}$, then we shall call $\mathfrak{g}$ the classical limit of A.

Example 1 (cf. [2, pp. 89-90]). We consider the $\mathbb{C}[[\hbar]]$-algebra A, generated (in the h-adic sense, i.e., as an algebra completed in the h-adic topology) by elements x_1, x_2 with defining relation $[x_1,x_2]=\alpha x_2$, $\alpha\in\mathbb{C}^*$. We define $\Delta:A\to A\otimes A$ by the formulas $\Delta(x_1)=x_1\otimes 1+1\otimes x_1$, $\Delta(x_2)=x_2\otimes\exp(1/2\hbar\beta x_1)+\exp(-1/2\hbar\beta x_1)\otimes x_2$, $\beta\in\mathbb{C}^*$. Then A is the quantization of the bialgebra of Example 1 of Sec. 3. It follows from the results of Sec. 9 that this quantization is unique up to the substitution $\hbar\mapsto\hbar+\sum_{i=2}^{\infty}c_i\hbar^i$, $c_i\in\mathbb{C}$.

Example 2. Let $\mathfrak{g}$, $\mathfrak{f}$, α_i, $\langle,\rangle$, H_i mean the same things as in Example 2 of Sec. 3. We consider the $\mathbb{C}[[\hbar]]$-algebra $U_\hbar\mathfrak{g}$, generated (in the h-adic sense) by the space f and the elements X_i^+, X_i^- with the defining relations $[a_1, a_2] = 0$ for $a_1,a_2\in\mathfrak{f}$, $[a,X_i^\pm]=\pm\alpha_i(a)X_i^\pm$ for $a\in f$, and also

$$[X_i^+, X_j^-] = 2\delta_{ij} h^{-1} \operatorname{sh}(hH_i/2),$$

$$i=j,\ n = 1 - A_{ij},\ q = \exp\frac{h}{2}\langle H_i, H_i\rangle \Rightarrow \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{-k(n-k)/2} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{n-k} = 0.$$

Here A_{ij} are Cartan matrices and $\binom{n}{k}_q$ is the Gauss polynomial ([47, Sec. 3.3]), i.e., $\binom{n}{k}_q$ $(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)/(q^k-1)(q^{k-1}-1)\dots(q-1)$. One can show that $U_h\mathfrak{g}$ is a topologically free $\mathbb{C}[[h]]$-module and that there exists a homomorphism $\Delta: U_h\mathfrak{g} \to U_h\mathfrak{g} \otimes U_h\mathfrak{g}$ such that $\Delta(X_i^\pm) = X_i^\pm \otimes \exp(hH_i/4) + \exp(-hH_i/4) \otimes X_i^\pm$, $\Delta(a) = a \otimes 1 + 1 \otimes a$ for $a \in \mathfrak{f}$. $(U_h\mathfrak{g}, \Delta)$ is a quantization of $\mathfrak{g}$. Using the results of Sec. 9, one can show that $U_h\mathfrak{g}$ is the unique (up to the substitution $h \mapsto h + \sum_{i=2}^{\infty} c_i h^i$, $c_i \in \mathbb{C}$) quantization A of the bialgebra $\mathfrak{g}$, for which there exist a cocommutative Hopf subalgebra $C \subset A$ and a map $\theta: A \to A$ such that the map $C/hC \to U\mathfrak{g}$ is injective, and its image is equal to $U\mathfrak{f}$, $\theta^2 = id$, $\theta(C) = C$, θ being an automorphism of A as an algebra and an anti-automorphism as a coalgebra, $\theta \pmod h$ being the Cartan involution (if $A = U_h\mathfrak{g}$, then we set $C = U\mathfrak{f}[[h]]$, $\theta(X_i^\pm) = -X_i^\mp$, $\theta|_C = -id$). Hence it is natural to call $U_h\mathfrak{g}$ a quantized Kac-Moody algebra. The algebra $U_h\mathfrak{sl}(2)$ was introduced by P. P. Kulish and N. Yu. Reshetikhin [48], and also by E. K. Sklyanin [49]. For arbitrary $\mathfrak{g}$ the algebra $U_h\mathfrak{g}$ was introduced by M. Jimbo [50, 51] and the author [28]. All these papers were motivated by the QMIP. For affine and finite-dimensional $\mathfrak{g}$ the algebra $U_h\mathfrak{g}$ is closely connected with trigonometric solutions of the quantum Yang–Baxter equation (QYBE); cf. [50-52], and also Sec. 13.

Example 3 ([28, 54]). Let $\mathfrak{A}$ and $\mathfrak{g}$ mean the same things as in Example 3 of Sec. 4. Since $\mathfrak{g}$ is graded ($\deg au^k = k$ for $a \in \mathfrak{A}$, and the cocommutator has degree 1, it is natural to seek a quantization Q, which is a graded Hopf algebra over $\mathbb{C}[[h]]$, where the algebra $\mathbb{C}[[h]]$ is graded so that $\deg h = 1$. With the help of the results of Sec. 9 one can show that Q exists and is unique. As an h-adic algebra Q is generated by the Lie algebra $\mathfrak{A}$ ($\deg a = 0$ for $a \in \mathfrak{A}$) and elements $J(a)$ of degree 1, $a \in \mathfrak{A}$, with the defining relations

$$J(\lambda a + \mu b) = \lambda J(a) + \mu J(b),\quad J([a,b]) = [a, J(b)] \qquad \text{for } a, b \in \mathfrak{A},\ \lambda, \mu \in \mathbb{C}, \tag{12}$$

$$a_i, b_i \in \mathfrak{A},\ \sum_i [a_i, b_i] = 0 \Rightarrow \sum_i [J(a_i), J(b_i)] = \frac{h^2}{12} \sum_i ([[a_i, I_\alpha], [b_i, I_\beta]], I_\gamma)\{I_\alpha, I_\beta, I_\gamma\}, \tag{13}$$

$$a_i, b_i, c_i \in \mathfrak{A},\ \sum_i [[a_i, b_i], c_i] = 0 \Rightarrow \sum_i [[J(a_i), J(b_i)], J(c_i)] =$$

$$= \frac{h^2}{4} \sum_i \sum_{\alpha,\beta,\gamma} f(a_i, b_i, c_i, I_\alpha, I_\beta, I_\gamma)\{I_\alpha, I_\beta, J(I_\gamma)\}, \tag{14}$$

where $\{I_\alpha\}$ is an orthonormal basis of $\mathfrak{A}$, $f(a,b,c,x,y,z) = \underset{a,b}{Alt}\ \underset{a,c}{sym}([a[b,x]],[[c,y],z])$, $\{x_1,x_2,x_3\} = 1/6 \sum_{i\neq j\neq k} x_i x_j x_k$. $\Delta: Q \to Q \otimes Q$ is defined by the formulas

$$\Delta(a) = a \otimes 1 + 1 \otimes a,\ \Delta(J(a)) = J(a) \otimes 1 + 1 \otimes J(a) + \frac{h}{2}[a \otimes 1, t], \quad a \in \mathfrak{A}, \tag{15}$$

where $t = \sum I_\alpha \otimes I_\alpha$. We note that the classical limit of $J(a)$ is $au \in \mathfrak{g}$. Hence (12) and the left sides of (13) and (14) are completely natural. (15) is also natural, since $[a \otimes 1, t]$ is nothing but the image of au under the cocommutator map. The horrible right sides of (13) and (14) are obtained from (12) and (15). We also note that for $\mathfrak{A} = \mathfrak{sl}(2)$ the relations (13) are superfluous, and for $\mathfrak{A} \neq \mathfrak{sl}(2)$ (14) is superfluous. The Hopf algebra over $\mathbb{C}$ obtained from Q for $h = 1$, is denoted by $Y(\mathfrak{A})$ and is called the Yangian of $\mathfrak{A}$. $Y(\mathfrak{A})$ is connected with rational solutions of the QYBE (cf. Sec. 12), the simplest of which were found by C. N. Yang [53].

Fortunately, Q has another system of generators ξ_{ik}^+, ξ_{ik}^-, φ_{ik}, $i \in \Gamma$, $k = 0, 1, 2, \dots$, $\deg \xi_{ik}^\pm = \deg \varphi_{ik} = k$, where Γ is the set of simple roots of $\mathfrak{A}$. The classical limits of $\xi_{ik}^\pm$ and φ_{ik} are equal to $X_i^\pm u^k$ and $H_i u^k$, where $X_i^\pm$, H_i are the generators of $\mathfrak{A}$, used in Example 2 of Sec. 3. Here is a complete system of relations between the $\xi_{ik}^\pm$ and φ_{ik}:

$$[\varphi_{ik},\varphi_{j\ell}]=0,\ [\varphi_{i0},\xi^{\pm}_{j\ell}]=\pm 2B_{ij}\xi^{\pm}_{j\ell},\ [\xi^{+}_{ik},\xi^{-}_{j\ell}]=\delta_{ij}\varphi_{i,k+\ell},$$

$$[\varphi_{i,k+1},\xi^{\pm}_{j\ell}]-[\varphi_{ik},\xi^{\pm}_{j,\ell+1}]=\pm hB_{ij}(\varphi_{ik}\xi^{\pm}_{j\ell}+\xi^{\pm}_{j\ell}\varphi_{ik}),$$

$$[\xi^{\pm}_{i,k+1},\xi^{\pm}_{j\ell}]-[\xi^{\pm}_{ik},\xi^{\pm}_{j,\ell+1}]=\pm hB_{ij}(\xi^{\pm}_{ik}\xi^{\pm}_{j\ell}+\xi^{\pm}_{j\ell}\xi^{\pm}_{ik}),$$

$$i\neq j,\ n=1-A_{ij}\Rightarrow \operatorname*{sym}_{k_1,\dots k_n}[\xi^{\pm}_{i,k_1}[\xi^{\pm}_{ik_2},\dots[\xi^{\pm}_{i,k_n},\xi^{\pm}_{j\ell}]]\dots]=0,$$

where A_{ij} is a Cartan matrix of $\mathfrak{a}$, $B_{ij}=1/2(H_i,H_j)$. The connection between the two realizations of Q is described in [54].

There is an analogous realization for the quantized affine Kac-Moody algebras [54]. For example, if $\mathfrak{g}$ is a nontwisted algebra, i.e., $[\mathfrak{g},\mathfrak{g}]$ / (center of $\mathfrak{g}$) = $\mathfrak{a}[\lambda,\lambda^{-1}]$, then $U_h\mathfrak{g}$ admits the following realization. Generators: $c, \mathcal{D}, \xi^{\pm}_{ik}, x_{ik}$, where $i\in\Gamma$, $k\in\mathbb{Z}$. Defining relations:

$$[c,x_{ik}]=[c,\xi^{\pm}_{ik}]=[c,\mathcal{D}]=0,\ [\mathcal{D},x_{ik}]=kx_{ik},\ [\mathcal{D},\xi^{\pm}_{ik}]=k\xi^{\pm}_{ik},$$

$$[x_{ik},x_{j\ell}]=4\delta_{k,-\ell}\kappa^{-1}h^{-2}\operatorname{sh}(khB_{ij})\operatorname{sh}(khc/2),$$

$$[x_{ik},\xi^{\pm}_{j\ell}]=\pm 2(kh)^{-1}\operatorname{sh}(khB_{ij})\exp(\mp|k|hc/4)\xi^{\pm}_{j,k+\ell},$$

$$\xi^{\pm}_{i,k+1}\xi^{\pm}_{j\ell}-e^{\pm hB_{ij}}\xi^{\pm}_{j\ell}\xi^{\pm}_{i,k+1}=e^{\pm hB_{ij}}\xi^{\pm}_{ik}\xi^{\pm}_{j,\ell+1}-\xi^{\pm}_{j,\ell+1}\xi^{\pm}_{ik},$$

$$[\xi^{+}_{ik},\xi^{-}_{j\ell}]=\delta_{ij}h^{-1}\{\psi_{i,k+\ell}e^{hc(k-\ell)/4}-\varphi_{i,k+\ell}e^{hc(\ell-k)/4}\},$$

$$i\neq j,\ n=1-A_{ij},\ q=\exp h/2(H_i,H_i)\Rightarrow \operatorname*{sym}_{k_1,\dots,k_n}\sum_{r=0}^{n}(-1)^r\binom{n}{r}_q$$

$$\cdot q^{-r(n-r)/2}\xi^{\pm}_{i,k_1}\cdots\xi^{\pm}_{ik_r}\xi^{\pm}_{j\ell}\xi^{\pm}_{ik_{r+1}}\cdots\xi^{\pm}_{i,k_n}=0.$$

Here φ_{ip}, ψ_{ip} are defined from the relations $\sum_p\varphi_{ip}u^p=\exp h(1/2x_{i0}+\sum_{p<0}x_{ip}u^p)$, $\sum_p\psi_{ip}u^p=\exp h(1/2x_{i0}+\sum_{p>0}x_{ip}u^p)$. The connection between the two realizations of $U_h\mathfrak{g}$ is described in [54]. We note that $U_h\mathfrak{g}'$ (i.e., $U_h\mathfrak{g}$ "without $\mathcal{D}$" and with the additional relation c = 0) degenerates into $Y(\mathfrak{a})$: if A is the subalgebra of $U_h\mathfrak{g}'\otimes_{\mathbb{C}[[h]]}\mathbb{C}((h))$ generated by $U_h\mathfrak{g}'$ and $h^{-1}\ker\varepsilon$, where f is the composition $U_h\mathfrak{g}'\to U\mathfrak{g}'=U(\mathfrak{a}[\lambda,\lambda^{-1}])\xrightarrow{\lambda=1}U\mathfrak{a}$, then $A/hA=Y(\mathfrak{a})$.

The problem of describing Δ in terms of $\xi^{\pm}_{ik}$ is unsolved both for Yangians and for quantized affine algebras.

7. Quantized Universal Enveloping Algebras and h-Formal Groups

A quantized universal enveloping algebra (QUE-algebra) is a Hopf algebra A over $\mathbb{C}[[h]]$ such that A is a topologically free $\mathbb{C}[[h]]$-module, and A/hA is a universal enveloping algebra. In other words, a QUE-algebra is the quantization of some Lie bialgebra. It is easy to show that a cocommutative QUE-algebra is the h-adic completion of $U\mathfrak{g}$ for some Lie algebra $\mathfrak{g}$ over $\mathbb{C}[[h]]$, which is a topologically free $\mathbb{C}[[h]]$-module (actually $\mathfrak{g}=\{a\in A;\ \Delta(a)=a\otimes 1+1\otimes a\}$).

Another important class of Hopf algebras over $\mathbb{C}[[h]]$ is made up of the QFSH-algebras (QFSH is the abbreviation of "quantum formal series Hopf"). A QFSH-algebra is a topological Hopf algebra B over $\mathbb{C}[[h]]$ such that 1) as a topological $\mathbb{C}[[h]]$-module B is isomorphic to $\mathbb{C}[[h]]$ for some set I, 2) B/hB is isomorphic to $\mathbb{C}[[u_1,u_2,\dots]]$ as a topological algebra.

An h-formal group is the spectrum of a QFSH-algebra. In order to understand h-formal groups in simpler terms, we fix a "coordinate system," i.e., elements $x_i\in B$ such that 1) B/hB is the ring of formal power series in $\bar{x}_i=x_i \bmod h$, 2) $\varepsilon(x_i)=0$, where $\varepsilon: B\to\mathbb{C}[[h]]$ is

the counit. $\Delta(x_i)$ can be represented in the form $F_i(x_1, x_2, \ldots; y_1, y_2, \ldots; h)$, where $y_i = 1 \otimes x_i \in B \otimes B$ and x_i is identified with $x_i \otimes 1 \in B \otimes B$. Thus, an h-formal group is defined by a "quantum group law" F(x, y, h) and "commutation relations," i.e., formulas expressing $h^{-1}[x_i, x_j]$ in the form of a formal series in x_i and h.

Now we discuss the connection between QUE-algebras and QFSH-algebras. First of all the algebra dual to a QUE-algebra is a QFSH-algebra and conversely (in both cases it is necessary to understand the word "dual" in a suitable way). For example, the algebra dual to the QUE-algebra of Example 1 of Sec. 6 is generated by the elements ξ_1 and ξ_2, where $\xi_1(f(x_1, x_2)) = \frac{\partial}{\partial x_1} f(x_1, 0)\big|_{x_1=0}$, $\xi_2(\sum_i \varphi_i(x_1) x_2^i) = \varphi_1(\alpha/2)$. The commutation relation between ξ_1 and ξ_2 has the form $[\xi_1, \xi_2] = -h\beta\xi_2$, and the comultiplication is given by the formulas $\Delta(\xi_1) = \xi_1 \otimes 1 + 1 \otimes \xi_1$, $\Delta(\xi_2) = \xi_2 \otimes \exp(-\frac{\alpha}{2}\xi_1) + \exp(\frac{\alpha}{2}\xi_1) \otimes \xi_2$. The algebra dual to $U_h\, s\ell(n)$ also admits explicit description. We consider a representation $\rho: U_h s\ell(n) \to Mat(n, \mathbb{C}[[h]])$ such that

$$\rho(H_i) = e_{ii} - e_{i+1,i+1}, \rho(X_i^+) = e_{i,i+1}, \rho(X_i^-) = 2h^{-1}\operatorname{sh}(h/2)\, e_{i+1,i},$$

where e_{ij} are matrix units (it is understood that the scalar product on $s\ell(n)$ is equal to TrXY). The matrix elements ρ_{ij}, $1 \le i, j \le n$, belong to $(U_h s\ell(n))^*$. It is easy to show that

$$\rho_{ij}\rho_{il} = e^{-h/2}\rho_{il}\rho_{ij} \quad \text{for } j < l, \tag{16}$$

$$\rho_{ij}\rho_{kj} = e^{-h/2}\rho_{kj}\rho_{ij} \quad \text{for } i < k, \tag{17}$$

$$\rho_{il}\rho_{kj} = \rho_{kj}\rho_{il} \quad \text{for } i<k, l>j, \tag{18}$$

$$[\rho_{il}, \rho_{ki}] = 2\operatorname{sh}(h/2)\,\rho_{ij}\rho_{kl}, \quad \text{for } i>k, l>j \tag{19}$$

$$\sum_{i_1, \ldots i_n} \rho_{1i_1}\rho_{2i_2}\cdots\rho_{ni_n}(-e^{-h/2})^{l(i_1, \ldots i_k)} = 1,$$

where $l(i_1, \ldots, i_n)$ is the number of inversions in the permutation $(i_1, \ldots, i_n)$. Actually $(U_h s\ell(n))^*$ is the quotient of the ring of noncommutative formal power series in $\tilde{\rho}_{ij} = \rho_{ij} - \delta_{ij}$ by the ideal generated by the relations given above. The comultiplication is the usual one: $\Delta(\rho_{ij}) = \sum_k \rho_{ik} \otimes \rho_{kj}$. It is natural to call $\operatorname{spec}(U_h s\ell(n))^*$ the quantized formal SL(n). One can also consider the subalgebra of $(U_h s\ell(n))^*$ consisting of matrix elements of finite-dimensional representations of $U_h s\ell(n)$, i.e., of polynomials in ρ_{ij}. One can consider the spectrum of this subalgebra as the quantized algebraic group SL(n).

It is somewhat unexpected that the category of QFSH-algebras, finitely generated in the sense of topological algebras, is equivalent (and not just antiequivalent) to the category of QUE-algebras, whose classical limits are finite-dimensional Lie bialgebras (the finiteness condition is not too essential here). The following example possibly promotes understanding of this equivalence. We consider the QUE-algebra of Example 1 of Sec. 6, i.e., $[x_1, x_2] = \alpha x_2$, $\Delta(x_1) = x_1 \otimes 1 + 1 \otimes x_1$, $\Delta(x_2) = x_2 \otimes \exp(h/2\beta x_1) + \exp(-h/2\beta x_1) \otimes x_2$. We set $y_i = hx_i$. Then $[y_1, y_2] = \alpha h y_2$, $\Delta(y_1) = y_1 \otimes 1 + 1 \otimes y_1$, $\Delta(y_2) = y_2 \otimes \exp(1/2\beta y_1) + \exp(-1/2\beta y_1) \otimes y_2$, i.e., we have obtained a QFSH-algebra. Here is the general construction. If A is a QUE-algebra, then we set $\delta_1(a) = a - \varepsilon(a)1$, $\delta_2(a) = \Delta(a) - a \otimes 1 - 1 \otimes a + \varepsilon(a) \cdot 1 \otimes 1$ etc. (ϵ is the counit). Then $\{a \in A;\ \delta_n(a) \in h^n A^{\otimes n}\ \forall\, n\}$ is a QFSH-algebra. Conversely, if B is a QFSH-algebra with maximal ideal m, then the h-adic completion of the algebra $\sum h^{-n} m^n$ is a QUE-algebra.

Definition. By the QUE-dual algebra of a QUE-algebra A we mean the QUE-algebra corresponding to A^* in the sense indicated above (or, what is the same, the algebra dual to the QFSH-algebra corresponding to A).

One can show that the classical limit of the QUE-dual algebra with respect to A is the Lie bialgebra dual to the classical limit of A.

8. Square of the Antipode

Let A be a quantized Lie bialgebra $\mathfrak{g}$, S be the antipode of A. In general, $S^2 \neq id$, but $S^2 \equiv id \bmod h$. Here is a description of $h^{-1}(S^2 - id) \bmod h$ in terms of $\mathfrak{g}$. We denote by $\mathcal{D}$ the composition of the cocommutator $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ and the commutator $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. It is easy to show that $\mathcal{D}: \mathfrak{g} \to \mathfrak{g}$ is a differentiation of $\mathfrak{g}$ as a Lie bialgebra (this means that $e^{t\mathcal{D}}$ is an automorphism of $\mathfrak{g}$ as Lie bialgebra) and that $h^{-1}(S^2 - id) \bmod h$ is the unique differentiation of $U\mathfrak{g}$, whose restriction to $\mathfrak{g}$ is equal to $\mathcal{D}/2$.

If $\mathfrak{g} = \mathfrak{a}[u]$ is the bialgebra of Example 3 of Sec. 3, and the scalar product on $\mathfrak{a}$ is equal to the Killing form, then $\mathcal{D} = d/du$. For bialgebras corresponding to nondegenerate solutions of (10), (11), $\mathcal{D}$ becomes equal to d/du after suitable normalization.

9. Theory of Obstructions

Let $\mathfrak{g}$ be a Lie bialgebra. We denote by $\mathcal{D}er^i \mathfrak{g}$ the (i + 2)nd cohomology group of the complex $\operatorname{Ker}(C^*(\mathfrak{g} \oplus \mathfrak{g}^*) \to C^*(\mathfrak{g}) \oplus C^*(\mathfrak{g}^*))$, where $\mathfrak{g} \oplus \mathfrak{g}^*$ is endowed with the Lie algebra structure introduced in Sec. 3, and $C^*(\mathfrak{a})$ is the standard complex $0 \to \mathfrak{a}^* \to \Lambda^2\mathfrak{a}^* \to \Lambda^3\mathfrak{a}^* \to \dots$. Obviously $\mathcal{D}er^i \mathfrak{g} = 0$ for $i < 0$. $\mathcal{D}er^0 \mathfrak{g}$ is the set of differentiations of $\mathfrak{g}$ as Lie bialgebra. One can show that 1) if two quantizations of $\mathfrak{g}$ coincide mod h^n, then their difference $\bmod h^{n+1}$ "belongs" to $\mathcal{D}er^1(\mathfrak{g})$, 2) if $\mathcal{D}er^2 \mathfrak{g} = 0$, then $\mathfrak{g}$ admits quantization. I don't know whether every Lie bialgebra admits quantization.

10. Coboundaries, Triangular and Quasitriangular Hopf Algebras

A pair (A, R), consisting of a Hopf algebra A and an invertible element $R \in A \otimes A$, is called a cobounding Hopf algebra if

$$\Delta'(a) = R\,\Delta(a)\,R^{-1}, \quad a \in A, \tag{20}$$

and $R^{12}R^{21} = 1$, $R^{12}\cdot(\Delta \otimes id)(R) = R^{23}\cdot(id \otimes \Delta)(R)$, $\varepsilon \otimes \varepsilon(R) = 1$. Here $\Delta' = \sigma \circ \Delta$, $\sigma: A \otimes A \to A \otimes A$, $\sigma(x \otimes y) = y \otimes x$, and the symbols R^{12}, R^{21}, R^{23} have the following meaning: if $R = \sum a_i \otimes b_i$, then $R^{21} = \sum b_i \otimes a_i \otimes 1$, $R^{23} = \sum 1 \otimes a_i \otimes b_i$, $R^{12} = \sum a_i \otimes b_i \otimes 1$ (but sometimes $R^{12} = R$). (A, R) is called a quasitriangular Hopf algebra if R satisfies (20) and the equations

$$(\Delta \otimes id)(R) = R^{13}R^{23}, \quad (id \otimes \Delta)(R) = R^{13}R^{12}; \tag{21}$$

a quasitriangular Hopf algebra is called triangular, if $R^{12}R^{21} = 1$. By definition, a cobounding (respectively triangular, quasitriangular) QUE-algebra is a cobounding (respectively triangular, quasitriangular) Hopf algebra (A, R) such that A is a QUE-algebra and $R \equiv 1 \bmod h$.

One can show that if (A, R) is a cobounding (respectively triangular, quasitriangular) QUE-algebra, $r = h^{-1}(R - 1) \bmod h$, and $\mathfrak{g}$ is the classical limit of A, then $r \in \mathfrak{g} \otimes \mathfrak{g}$ (a priori, $r \in U\mathfrak{g} \otimes U\mathfrak{g}$), and $(\mathfrak{g}, r)$ is a cobounding (respectively triangular, quasitriangular) Lie bialgebra. Insufficient space does not permit me to formulate other arguments using the definitions given above.

It is easy to show that 1) a triangular Hopf algebra is cobounding, 2) if (A, R) is a quasitriangular QUE-algebra and $\bar{R} = (R^{12}R^{21})^{-1/2}R$, then $(A, \bar{R})$ is a cobounding QUE-algebra, 3) if (A, R) is quasitriangular, then R satisfies the QYBE, i.e.,

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

4) if (A, R) is a cobounding QUE-algebra and R satisfies the QYBE, then (A, R) is triangular, 5) Eqs. (21) mean that the map $A^* \to A$ defined by the formula $l \mapsto (l \otimes id)(R)$, is a homomorphism of algebras and an antihomomorphism of coalgebras.

In order to understand the different types of Hopf algebras one can use the Tannaka–Krein approach [56], i.e., instead of considering a Hopf algebra A one can consider the category Rep_A of its representations together with the forgetful functor $F:\mathrm{Rep}_A \to \{\text{vector spaces}\}$. We recall that a representation of a Hopf algebra A is a representation of it as an algebra, and the tensor product of representations $\rho_1:A \to \mathrm{End}\, V_1$ and $\rho_2:A \to \mathrm{End}\, V_2$ is equal to the composition $A \xrightarrow{\Delta} A \otimes A \xrightarrow{\rho_1 \otimes \rho_2} \mathrm{End}(V_1 \otimes V_2)$. For any Hopf algebra A the functor $\otimes : \mathrm{Rep}_A \times \mathrm{Rep}_A \to \mathrm{Rep}_A$ is associative. More precisely, for this functor there is an associativity morphism [56], compatible with F. In general the A-modules $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are not isomorphic, but if (20) holds, then R defines an isomorphism between them. Moreover, if $R^{12}R^{21} = 1$, then we get a commutativity morphism [56] for the functor $\otimes : \mathrm{Rep}_A \times \mathrm{Rep}_A \to \mathrm{Rep}_A$. This morphism is not compatible with F, if $R \neq 1$. Condition (21) is nothing but the compatibility of the commutativity and associativity morphisms [56]. Thus, if (A, R) is a triangular Hopf algebra, then Rep_A is a tensor category [56], but F is not a tensorial functor if $R \neq 1$. A result of V. V. Lobashenko means that a tensorial category C together with the functor $F:C \to \{\text{vector spaces}\}$, which has the properties indicated above, is more or less Rep_A for some (uniquely determined) Hopf algebra A.

If (A, Δ_0) is a cocommutative Hopf algebra, and Φ is an invertible element of $A \otimes A$ such that $\Phi^{12}(\Delta_0 \otimes id)(\Phi) = \Phi^{23}(id \otimes \Delta_0)(\Phi)$ then one can get a triangular Hopf algebra (A, Δ, R) by setting $\Delta(a) = \Phi \Delta_0(a) \Phi^{-1}$, $R = \Phi^{21}(\Phi^{12})^{-1}$. One can show that in this way one gets all triangular QUE-algebras (this is natural from the point of view of Tannaka–Krein). In particular, a triangular QUE-algebra is isomorphic (as an algebra) to the universal enveloping algebra of some Lie algebra.

Finally, we mention a version of (20) which plays a key role in the QMIP (cf., e.g., [71]) and which serves as the source of (20) itself. Suppose given a representation $\rho : A \to \mathrm{Mat}(n, \mathbb{C})$. We denote by t_{ij} the matrix elements of ρ. Then $T_{ij} \in A^*$ satisfy the commutation relations $R_\rho \tilde{T} \tilde{\tilde{T}} = \tilde{\tilde{T}} \tilde{T} R_\rho$, where $R_\rho = (\rho \otimes \rho)(R)$, $T = (t_{ij})$, $\tilde{T} = T \otimes E_n$, $\tilde{\tilde{T}}_n = E_n \otimes T$ (E_n being the identity matrix). Moreover, if we are given a family of representations $\rho_\lambda : A \to \mathrm{Mat}(n, \mathbb{C})$, $\lambda \in \mathbb{C}$ then one has

$$R(\lambda, \mu)\, \tilde{T}(\lambda)\, \tilde{\tilde{T}}(\mu) = \tilde{\tilde{T}}(\mu)\, \tilde{T}(\lambda)\, R(\lambda, \mu), \tag{22}$$

where $R(\lambda, \mu) = (\rho_\lambda \otimes \rho_\mu)(R)$, $T(\lambda) = t_{ij}(\lambda)$, $t_{ij}(\lambda)$ are the matrix elements of ρ_λ. A more traditional point of view about (22) is to start with the function $R(\lambda, \mu) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ and consider the algebra B with generators $t_{ij}(\lambda)$, $\lambda \in \mathbb{C}$, $1 \leq i, j \leq n$, and defining relations (22). We note that the formula $\Delta(t_{ij}(\lambda)) = \sum_k t_{ik}(\lambda) \otimes t_{kj}(\lambda)$ defines a Hopf algebra structure on B, where if $R(\lambda, \mu) = (\rho_\lambda \otimes \rho_\mu)(R)$ for some A, $\{\rho_\lambda\}$ and an element $R \in A \otimes A$, satisfying (20), then there is a natural homomorphism $B \to A$. Thus, Hopf algebras already occur implicitly in [71].

11. QMIP from the Point of View of Hopf Algebras

The QMIP is a method of constructing and studying integrable quantum systems, created in 1978-1979 by L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtadzhyan [69-71, 37] as a development of the classical method of the inverse problem and Baxter's work on exactly solvable models of classical statistical mechanics. In addition, the QMIP is closely connected with the theory of factored S-matrices (C. N. Yang and A. B. Zamolodchikov). Here is an account of the origins of the QMIP which is almost standard (the slight difference is that in the standard accounts the Hopf algebras occur implicitly and that instead of (20) one uses (22)).

Let A be a Hopf algebra, $C = \{ l \in A^* : l(ab) = l(ba)\ \forall a, b \}$; C is a subalgebra of A^*. Let us now assume that (20) holds. Then it is easy to show that C is commutative (this is the quantum analog of (9)). We fix a representation $\pi : A^* \to \mathrm{End}\, V$ ("quantum L-operator") and a

number N ("number of nodes of the one-dimensional lattice"). The elements of $V^{\otimes N}$ can be considered as quantum states. The image of C in the representation $\pi^{\otimes N}: A^* \to End\, V^{\otimes N}$ is the commutative algebra of observables. Physicists usually fix an element $H \in C$ and consider $\pi^{\otimes N}(H)$ as the Hamiltonian.

We note that if (A, R) is quasitriangular, then R induces a homomorphism $A^* \to A$ and consequently a representation of A determines a representation of A^*. One usually considers only those representations of A^* which come from representations of A. In this case instead of $\pi^{\otimes N}$ and C one actually works with $\rho^{\otimes N}$ and C', where ρ is the representation of A, and C' is the image of C in A, i.e., $C' = \{(l \otimes id)(R);\ l \in C\}$. In order to use this scheme, it is necessary to construct a quasitriangular Hopf algebra (A, R) and as many representations of A as possible (these representations can be taken as ρ, and their characters belong to C). The Hopf algebras A used in the QMIP are the quantized Lie bialgebras G_s of Example 4 of Sec. 3 (i.e., Lie bialgebras corresponding to solutions of the CYBE). Typical examples are Yangians and quantized affine algebras. On these algebras there is not a quasitriangular, but a "pseudotriangular" structure (cf. Secs. 12 and 13), but this is sufficient for using the scheme cited above.

Deeper aspects of the QMIP such as the description of the spectrum of $\rho^{\otimes N}(C')$ and passage to the limit as $N \to \infty$ (cf. [65, 36, 71]) are not yet understood from the point of view of Hopf algebras.

12. Pseudotriangular Structure on Yangians and Rational Solutions of the QYBE

We recall that one gets $Y(\mathfrak{a})$ if one prequantizes the bialgebra $\mathfrak{g} = \mathfrak{a}[u]$ of Example 3 of Sec. 3, and then sets h = 1. Since $\mathfrak{g}$ is pseudotriangular, it is natural to expect that the algebra $Y(\mathfrak{a})$ is also pseudotriangular. More precisely, we consider operators $T_\lambda: \mathfrak{g} \to \mathfrak{g}$, defined by $T_\lambda f(u) = f(u+\lambda)$. We note that although $r = t/u - v$ does not belong to $\mathfrak{g} \otimes \mathfrak{g}$, the expression $(T_\lambda \otimes id)r = t/u+\lambda-v = \sum_{k=0}^{\infty} t(v-u)^k \lambda^{-k-1}$ is a power series in λ^{-1}, whose coefficients belong to $\mathfrak{g} \otimes \mathfrak{g}$. Now we define automorphisms $T_\lambda: Y(\mathfrak{a}) \to Y(\mathfrak{a})$ by $T_\lambda|_{\mathfrak{a}} = id$, $T_\lambda(J(a)) = J(a) + \lambda a$ for $a \in \mathfrak{a}$. It is natural to expect that there exists a formal series $R(\lambda) = 1 + \sum_{k=1}^{\infty} R_k \lambda^{-k}$, $R_k \in Y(\mathfrak{a}) \otimes Y(\mathfrak{a})$, which behaves as if $R(\lambda)$ were equal to $(T_\lambda \otimes id)R$ for some R, satisfying (20), (21), and the equation $R^{12}R^{21} = 1$. This is actually so [28]. More precisely, there exists a unique series $R(\lambda)$ such that $(\Delta \otimes id)R(\lambda) = R^{13}(\lambda) R^{23}(\lambda)$ and $(T_\lambda \otimes id)\Delta'(a) = R(\lambda)(T_\lambda \otimes id)\Delta(a))R(\lambda)^{-1}$ for $a \in Y(\mathfrak{a})$. Moreover, $R(\lambda_1 - \lambda_2)$ satisfies the QYBE, i.e.

$$R^{12}(\lambda_1-\lambda_2)R^{13}(\lambda_1-\lambda_3)R^{23}(\lambda_2-\lambda_3) = R^{23}(\lambda_2-\lambda_3)R^{13}(\lambda_1-\lambda_3)R^{12}(\lambda_1-\lambda_2).$$

Moreover, $R^{12}(\lambda) R^{21}(-\lambda) = 1$, $(T_\mu \otimes T_\nu)R(\lambda) = R(\lambda+\mu-\nu)$, and the coefficient R_1 is equal to t.

Among other things we have been lucky that $Y(\mathfrak{a})$ is pseudotriangular and not triangular: otherwise $Y(\mathfrak{a})$ would be isomorphic as an algebra to the universal enveloping algebra of some Lie algebra and life would be dull.

If a representation $\rho: Y(\mathfrak{a}) \to Mat(n, \mathbb{C})$ is given, then we get a matrix solution $R(\lambda_1-\lambda_2) = (\rho \otimes \rho)(R(\lambda_1-\lambda_2))$ of the quantum Yang–Baxter equation. It has the form

$$R(\lambda_1-\lambda_2) = 1 + (\rho \otimes \rho)(t)(\lambda_1-\lambda_2)^{-1} + \sum_{n=2}^{\infty} A_k (\lambda_1-\lambda_2)^{-k}. \tag{23}$$

If ρ is irreducible, then $R(\lambda)$ is, up to a scalar factor, a rational function [28].

We now temporarily forget about $Y(\mathfrak{a})$ and take the point of view of a person hoping to construct matrix solutions of the QYBE. According to [45], it is natural to seek solutions of the form $R(\lambda_1-\lambda_2, h) = 1 + h(\rho \otimes \rho)(r(\lambda_1-\lambda_2)) + \sum_{k=2}^{\infty} h^k a_k(\lambda_1 - \lambda_2)$, where ρ is a represen-

tation of $\mathfrak{A}$, and r is a solution of the CYBE with values in $\mathfrak{A}\otimes\mathfrak{A}$. The simplest r has the form $r(\lambda) = t/\lambda$. Since this r is homogeneous, it is natural to require that $\forall\alpha\ R(\alpha\lambda,\alpha h)= R(\lambda,h)$. In this case $R(\lambda, h)$ is uniquely determined by $R(\lambda, 1)$, and $R(\lambda, 1)$ has the form (23). Thus, a natural problem is, for given $\rho: \mathfrak{A}\to \mathrm{Mat}(n,\mathbb{C})$, to describe all solutions of the QYBE of the form (23). Theorem [28]: if ρ is irreducible, then all solutions have the form $(\tilde\rho\otimes\tilde\rho, (R(\lambda_1-\lambda_2))$ for some representation $\tilde\rho: Y(\mathfrak{A})\to \mathrm{Mat}(n,\mathbb{C})$ such that $\tilde\rho\,|\mathfrak{A} = \rho$.

From what was said above there follows the possibility of the problem of describing all irreducible finite-dimensional representations of $Y(\mathfrak{A})$. This problem for $\mathfrak{A}=s\ell(2)$ was solved by V. E. Korepin and V. O. Tarasov [59-61]. For classical $\mathfrak{A}$ specialists on QMIP constructed by more or less elementary means, many R-matrices of the form (23) (cf., e.g., [62, 63] and the appendix to [45]) and consequently many representations of $Y(\mathfrak{A})$. Some results relating to this problem are found in [28, 64]. In [54] with the help of the second realization of $Y(\mathfrak{A})$ (cf. Sec. 6) and the results for $Y(s\ell(2))$ a parametrization of the set of irreducible finite-dimensional representations of $Y(\mathfrak{A})$ is found (in the spirit of Cartan's theorem on the highest weight).

In $Y(\mathfrak{A})$ there are two "large" commutative subalgebras: a "Cartan" subalgebra generated by elements φ_{ik} (cf. Sec. 6) and the subalgebra $\mathfrak{A}$ generated by the coefficients of $(\ell\otimes id)$ $R(\lambda)$, where $\ell\in Y(\mathfrak{A})^*$, $\ell(ab)=\ell(ba)$ $\forall a,b\in Y(\mathfrak{A})$. It is desirable to describe these subalgebras in the irreducible representations of $Y(\mathfrak{A})$. For $\mathfrak{A}$ this problem is especially important (cf. Sec. 11). For the classical $\mathfrak{A}$ and at least some representations of $Y(\mathfrak{A})$ the spectrum of $\mathfrak{A}$ is known [65-68]. The analogy between the Bethe structural equations and the corresponding Cartan matrices (N. Yu. Reshetikhin [68]) lets one hope that the spectrum of $\mathfrak{A}$ can be described for all $\mathfrak{A}$.

In conclusion, I mention a realization of $Y(\mathfrak{A})$, which is often useful and which appeared considerably earlier than the general definition of $Y(\mathfrak{A})$. We fix a nontrivial irreducible representation $\rho: Y(\mathfrak{A})\to \mathrm{Mat}(n,\mathbb{C})$. We set $R(\lambda)=(\rho\otimes\rho)(R(\lambda))$. Then $Y(\mathfrak{A})$ is almost isomorphic to the Hopf algebra A_ρ with generators $t_{ij}^{(k)}$, $1\leqslant i,j\leqslant n$, $k=1,2,\ldots$, defined by the relations $\tilde T(\lambda)\tilde{\tilde T}(\mu)R(\mu-\lambda)=R(\mu-\lambda)\tilde{\tilde T}(\mu)\tilde T(\lambda)$ and the comultiplication $\Delta(t_{ij}(\lambda))=\sum_k t_{ik}(\lambda)\otimes t_{kj}(\lambda)$, where $t_{ij}(\lambda)=\delta_{ij}+\sum_{k=1}^{\infty} t_{ij}^{(k)}\lambda^{-k}$, $T(\lambda)=(t_{ij}(\lambda))$, $\tilde T(\lambda)=T(\lambda)\otimes E_n$, $\tilde{\tilde T}(\lambda)=E_n\otimes T(\lambda)$. More precisely, there is an epimorphism $A_\rho\to Y(\mathfrak{A})$, defined by $T(\lambda)\mapsto(id\otimes\rho)(R(\lambda))$, and to get $Y(\mathfrak{A})$ from A_ρ, it is necessary to add an additional relation of the form $C(\lambda) = 1$, where $c(\lambda)\in A_\rho[[\lambda^{-1}]]$, $\Delta(c(\lambda))=c(\lambda)\otimes c(\lambda)$ and $[a,c(\lambda)]=0$ $\forall a\in A_\rho$. For example, if $\mathfrak{A}=s\ell(n)$, and ρ is a vector representation, then $C(\lambda)$ is the "quantum determinant" $T(\lambda)$ in the sense of [73], and $R(\lambda)$ is proportional to the Yang R-matrix $1+\lambda^{-1}\sigma$, where $\sigma: \mathbb{C}^n\otimes\mathbb{C}^n\to\mathbb{C}^n\otimes\mathbb{C}^n$, $\sigma(x\otimes y)=y\otimes x$.

13. Double. Trigonometric Solutions of the QYBE

If $\mathfrak{g}$ is a Lie bialgebra, then we denote by $\mathfrak{D}(\mathfrak{g})$ the space $\mathfrak{g}\otimes\mathfrak{g}^*$ with the Lie algebra structure introduced in Sec. 3. One can show that 1) the image r of the canonical element of $\mathfrak{g}\otimes\mathfrak{g}^*$ under the inclusion $\mathfrak{g}\otimes\mathfrak{g}^*\subset\mathfrak{D}(\mathfrak{g})\otimes\mathfrak{D}(\mathfrak{g})$ satisfies (7) and consequently, $(\mathfrak{D}(\mathfrak{g}),r)$ is a quasitriangular Lie bialgebra, 2) the inclusion $\mathfrak{g}\subset\mathfrak{D}(\mathfrak{g})$ is a homomorphism of Lie bialgebras, and the inclusion $\mathfrak{g}^*\subset\mathfrak{D}(\mathfrak{g})$ is a homomorphism of algebras and an antihomomorphism of coalgebras. $\mathfrak{D}(\mathfrak{g})$ is called the double of $\mathfrak{g}$. There are important applications of this construction [43].

Now we introduce the concept of quantum double. Let A be a Hopf algebra. We denote by A° the algebra A^* with the opposite comultiplication. One can show that there exists a unique quasitriangular Hopf algebra $(\mathfrak{D}(A),R)$ such that 1) $\mathfrak{D}(A)$ contains A and A° as Hopf subalgebras, 2) R is the image of the canonical element of $A\otimes A^\circ$ under the inclusion $A\otimes A^\circ\subset$

$\mathcal{D}(A) \otimes \mathcal{D}(A)$ 3) the linear map $A \otimes A^{\circ} \to \mathcal{D}(A)$, defined by $a \otimes b \to ab$, is bijective. As a vector space, $\mathcal{D}(A)$ can be identified with $A \otimes A^{\circ}$, and one can find the Hopf algebra structure on $A \otimes A^{\circ}$ by starting from the commutation relations $e_s e^t = \mu_s^{ujn} m^t_{p\ell\kappa} \delta^p_n e^\ell e_j$ and $e^t e_s = \mu_s^{njk} m^t_{k\ell p} \delta^p_n e_j e^\ell$, which follow from 1)-3). Here $\{e_s\}$ and $\{e^t\}$ are dual bases in A and A°, σ, m, and μ are the matrices of the skew antipode $A \to A$, the multiplication map $A \otimes A \otimes A \to A$ and the comultiplication $A \to A \otimes A \otimes A$.

If A is a QUE-algebra, then we shall understand A° in the QUE sense. Then $\mathcal{D}(A)$ is also a QUE-algebra. If $\mathfrak{g}$ is the classical limit of A, then $(\mathcal{D}(\mathfrak{g}), r)$ is the classical limit of $(\mathcal{D}(A), R)$.

Let $\mathfrak{g}, \mathfrak{b}_{\pm}, \mathfrak{f}$ denote the same things as in Example 2 of Sec. 3. Then $\mathcal{D}(\mathfrak{b}_+) = \mathfrak{g} \times \mathfrak{f}$, where the Lie bialgebra structure on $\mathfrak{f}$ is trivial. A quasitriangular (or pseudoquasitriangular) structure on $\mathfrak{g}$ (cf. Sec. 4) is defined by the image of the element $r \in \mathcal{D}(\mathfrak{b}_+) \otimes \mathcal{D}(\mathfrak{b}_+)$ under the natural map $\mathcal{D}(\mathfrak{b}_+) \otimes \mathcal{D}(\mathfrak{b}_+) \to \mathfrak{g} \otimes \mathfrak{g}$. One can show that in the quantum case the situation is analogous. We denote by $U_h \mathfrak{b}_{\pm}$ the subalgebra of $U_h \mathfrak{g}$ generated by f and $X_i^{\pm}$. Then $U_h \mathfrak{b}_- = (U_h \mathfrak{b}_+)^{\circ}$ and $\mathcal{D}(U_h \mathfrak{b}_+) = U_h \mathfrak{g} \otimes U_h \mathfrak{g}$. We denote by $\bar{R}$ the image of $R \in \mathcal{D}(U_h \mathfrak{b}_+) \otimes \mathcal{D}(U_h \mathfrak{b}_+)$ under the natural map $\mathcal{D}(U_h \mathfrak{b}_+) \otimes \mathcal{D}(U_h \mathfrak{b}_+) \to U_h \mathfrak{g} \otimes U_h \mathfrak{g}$. Then $\bar{R}$ defines on $U_h \mathfrak{g}$ a quasitriangular structure, if $\dim \mathfrak{g} < \infty$, and a "pseudoquasitriangular" one if $\dim \mathfrak{g} = \infty$.

If $\mathfrak{g} = s\ell(2)$, and the scalar product in $\mathfrak{g}$ is equal to TrXY, then $R = \sum_{k=0}^{\infty} h^k Q_k(h) \{\exp h/4 [H \otimes H + k(H \otimes 1 - 1 \otimes H)]\} (X^+)^k \otimes (X^-)^k$, where $Q_k(h) = e^{-kh/2} \prod_{r=1}^{k} (e^h - 1) / (e^{rh} - 1)$. In general $\bar{R} = \sum_{\beta \in \mathbb{Z}_+^r} \{\exp h [1/2 t_0 + 1/4 (H_\beta \otimes 1 - 1 \otimes H_\beta)]\} P_\beta$, where t_0 is the element of $\mathfrak{f} \otimes \mathfrak{f}$ corresponding to the canonical scalar product in $\mathfrak{f}$, $\mathbb{Z}_+ = \{n \in \mathbb{Z} ; n \geqslant 0\}$, r is the rank of $\mathfrak{g}$, for each $\beta = (\beta_1, \beta_2, \ldots) \in \mathbb{Z}^r$, $H_\beta = \sum \beta_i H_i$, and $P_\beta \in U_h \mathfrak{b}_+ \otimes U_K \mathfrak{b}_-$ is a polynomial in $u_i = X_i^+ \otimes 1$ and $v_i = 1 \otimes X_i^-$, which is homogeneous in each variable and such that $\deg_{u_i} P_\beta = \deg_{v_i} P_\beta = \beta_i$. Moreover: $P_0 = 1$, $P_\beta \equiv 0 \pmod{h^{k_\beta}}$ $P_\beta \not\equiv 0 \mod h^{k_\beta - 1}$, where $k_\beta = \min\{n;$ H_β we represent in the form of a sum of n positive roots of $\mathfrak{g}\}$, u) are the coefficients of $h^{-|\beta|} P_\beta$ which are rational functions of e^{ch} for some $c \in \mathbb{C}$, where $|\beta| = \sum_i \beta_i$.

If $\dim \mathfrak{g} < \infty$, then $R_\varrho \overset{def}{=} \varrho \otimes \varrho(\bar{R})$ makes sense and satisfies the QYBE for any representation $\varrho : U_h \mathfrak{g} \to \mathrm{Mat}(h, \mathbb{C}[[h]])$. For example, if $\mathfrak{g} = s\ell(n)$, and ρ is the representation mentioned in Sec. 7, then $R_\varrho = \sum_{i \neq j} e_{ii} \otimes e_{jj} + e^{h/2} \sum e_{ii} \otimes e_{ii} + 2\mathrm{sh}(h/2) \sum_{i<j} e_{ij} \otimes e_{ji}$, where e_{ij} are matrix units (one can extract this formula from [52]). We note that (16)-(19) simply mean that $R_\varrho \tilde{T} \tilde{\tilde{T}} = \tilde{\tilde{T}} \tilde{T} R_\varrho$, where $T = (\rho_{ij})$.

Now let $\mathfrak{g}$ be an affine algebra. We set $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] /$ (center of $\mathfrak{g}$). We denote by R' the analog of $\bar{R}$ for $U_h \mathfrak{g}$. Let us assume that there is given a representation $\varrho : U_h \mathfrak{g}' \to \mathrm{Mat}(n, \mathbb{C}[[h]])$. Strictly speaking, the expression $(\varrho \otimes \varrho)(R')$ is meaningless. But if one defines $T_\lambda \in \mathrm{Aut}\, U_h \mathfrak{g}$ by $T_\lambda(X_i^{\pm}) = \lambda^{\pm 1} X_i^{\pm}$ and sets $R'(\lambda) = (T_\lambda \otimes \mathrm{id}) R'$, then $R_\varrho(\lambda) \overset{def}{=} \varrho \otimes \varrho(R'(\lambda))$ is a well-defined formal series in λ such that

$$R_\varrho^{12}(\lambda_1/\lambda_2) R_\varrho^{13}(\lambda_1/\lambda_3) R_\varrho^{23}(\lambda_2/\lambda_3) = R_\varrho^{23}(\lambda_2/\lambda_3) R_\varrho^{13}(\lambda_1/\lambda_3) R_\varrho^{12}(\lambda_1/\lambda_2)$$

With the help of an idea of M. Jimbo [50] one can show that if ρ is irreducible, then $R_\rho(\lambda)$ is a rational function of λ up to a scalar factor. Moreover, one can find this rational function by solving some linear equations [50]. After substituting $\lambda = e^u$ we get a trigono-

metric solution of the QYBE. For solutions corresponding to classical $\mathfrak{g}$ and some ρ, explicit formulas were found by elementary methods in [48, 74-77].

14. Concluding Remarks

Insufficient space does not permit me to discuss the connection between $Y(s\ell(n)), U_h s\ell(n)$ and Hecke algebras (cf. [52, 64]), between Yangians and Hermitian symmetric spaces (cf. [62] and Theorem 7 of [28]), between $Y(s\ell(n))$ and representations of the infinite-dimensional unitary group (G. I. Ol'shanskii). For the same reason I cannot discuss attempts to generalize the concept of superalgebra based on QYBE (D. I. Gurevich [78]) or on triangular Hopf algebras (V. V. Lyubashenko).

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