A.8

Doctrines in Categorical Logic

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Contents

Preface 284
1. Algebraic theories 284
2. Cartesian theories 289
3. Elementary doctrines: quantifiers as adjoints 293
4. Logical categories: coherent logic 295
5. Grothendieck topoi: infinitary coherent logic 298
6. Elementary topoi 302
7. Topoi and axiomatic set theory 308
8. Other fields of research in categorical logic 310
References 311

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Preface

The aim of this paper is to give a survey of category-theoretic methods in logic, in particular in model theory and set theory. We organize the material by increasing richness of the "doctrines" involved. These doctrines are categorical analogues of fragments of logical theories which have sufficient category-theoretic structure for their models to be described as functors. In particular, we deal with the doctrines of equational, cartesian, finitary coherent and infinitary coherent logic. Higher order logic and set theory are touched upon in Sections 6 and 7.

Certain themes run through the various doctrines: invariance of presentation, relative interpretation, completeness, conceptual completeness and generic structures (as well as operations performed on these).

Fundamental to the functorial approach to model theory is the introduction of universes (categories) other than sets where models live. This makes possible the notion of generic structure considered here.

1. Algebraic theories

The introduction of the categorical notion of algebraic theory led to a systematic theory of relative interpretations of one equational theory into another, as well as a theory about the categories (or varieties) of algebras for these, and their relationship. This progress springs from having a presentation-invariant notion of equational (or algebraic) theory.

1.1. HISTORICAL REMARK. It is clear that the two standard ways of presenting the notion of *abelian group*: (i) a set equipped with a binary operation 'plus', a unary operation 'minus' and a nullary operation (constant) 0 (with certain equations satisfied) — or, (ii) a set equipped with a binary 'minus' and a nullary '0' (with certain equations satisfied) — only differ in what operations are considered primitive and which derived. The "equational theories" are the same, or the "totality" of operations (primitive and derived) can be identified in the two cases, and in such a way that the substitution process of operations into other operations are preserved under the identification. To understand this phenomenon in precise terms as an isomorphism between two mathematical structures "consisting of" operations, and equipped with some kind of substitution structure, led to invention of concepts like "Clone of operations" (P. Hall, see COHN [1965]), or "compositeur" (LAZARD [1955]), but these were somewhat

unmanageable notions. The right way of conceiving the totality of operations for an equational theory was found by LAWVERE [1963] who realized that *substitution* should be viewed as the composition of arrows in a certain kind of category:

1.2. DEFINITION. An *algebraic theory* T is a category whose objects are the natural numbers $0, 1, 2, \ldots$ and which for each n is equipped with an n-tuple of maps

$$\text{proj} : n \to 1, \qquad i = 1, \ldots, n$$

making n into the n-fold categorical product of 1, $n = 1^n$.

1.3. EXAMPLE. We define an algebraic theory T with

$$\hom(n, m) = m\text{-tuples of polynomials in } X_1, \ldots, X_n, \text{ with integral coefficients}$$

with substitution of polynomials as composition. (The proj_i is just X_i, considered as a polynomial in the variables $X_1, \ldots, X_n$, $1 \leq i \leq n$.) This T is called the *theory of commutative rings*. For further examples, see 1.14.

1.4. For every algebraic theory, a map $n \to m$ in it can be described by an m-tuple of maps $n \to 1$ because m is an m-fold product of 1; so the maps $n \to 1$ play a special role, they are called the n-ary *operations* of the theory. In the example above, an n-ary operation is a polynomial in n variables, which is also precisely what an n-ary (derived) operation is in any presentation of the equational theory of commutative rings.

Similarly, there exist algebraic theories for all other equational theories like the theory of groups, lattices, Lie-algebras, ...; in these, the n-ary operations are (composite) operations in n variables (that is, are n-ary *terms*), with two such identified if their equality follows from the axioms (basic equations) of the theory. This description indicates how one constructs an algebraic theory (in the sense of Definition 1.2) for each equational theory. This might be called the syntactical way of constructing algebraic theories.

The semantics of algebraic theories, that is, the process which leads from the theory to its category of models, can now be explained in the following way.

1.5. DEFINITION. Let T be an algebraic theory, and $\mathscr{E}$ a category with finite products. The *category of* T-*algebras* or T-*models in* $\mathscr{E}$ is the full subcateg-

ory $\mathrm{Alg}(\mathbb{T}, \mathscr{E})$ of the functor category $[\mathbb{T}, \mathscr{E}]$ whose objects are the finite product preserving functors.

1.6. REMARK. (i) When considering $\mathrm{Alg}(\mathbb{T}, \mathscr{E})$, we call $\mathscr{E}$ the *value* category. If $\mathscr{E} = (\mathrm{Sets})$, we write $\mathrm{Alg}(\mathbb{T})$ for $\mathrm{Alg}(\mathbb{T}, \mathscr{E})$.

(ii) We know the value of an algebra $A : \mathbb{T} \to \mathscr{E}$ at an $n \in \mathbb{T}$ if we know $A(1)$, since $A(n) = A(1)^n$, A being product preserving and n being 1^n. To describe A on the morphisms of $\mathbb{T}$, say, a morphism ("operation") $\omega : 2 \to 1$, amounts to describing a map

$$A(2) = A(1) \times A(1) \xrightarrow{A(\omega)} A(1),$$

that is, a binary operation on the object $A(1)$. So in case $\mathscr{E} = (\mathrm{Sets})$, A interprets abstract binary operations $\omega : 2 \to 1$ as actual binary operations on a single set $A(1)$, called the *underlying set* of the algebra A.

(iii) With these remarks, it is not hard to understand why, in case, say, where $\mathbb{T}$ is the algebraic theory of commutative rings, we have an equivalence of categories $\mathrm{Alg}(\mathbb{T}, (\mathrm{Sets})) \simeq$ Category of commutative rings. One should note that the naturality requirement on transformations $\tau : A \to B$ precisely amounts to requiring that $\tau_1 : A(1) \to B(1)$ is a homomorphism with respect to all the operations of the theory. If, for instance, $\omega : 2 \to 1$ is a binary operation in $\mathbb{T}$, both things amount to commutativity of the diagram

$$\begin{array}{ccccc} A(1) \times A(1) & \simeq & A(2) & \xrightarrow{\omega_A} & A(1) \\ \downarrow{\scriptstyle \tau_1 \times \tau_1} & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_1} \\ B(1) \times B(1) & \simeq & B(2) & \xrightarrow{\omega_B} & B(1) \end{array}$$

(the left-hand square being also commutative because of naturality).

1.7. Relative interpretation

As mentioned, relative interpretations of one equational theory into another can now be dealt with very rationally:

1.8. DEFINITION. A *morphism* of algebraic theories is a functor $f : \mathbb{T} \to \mathbb{T}'$ with $f(n) = n$ and with $f(\mathrm{proj}_i) = \mathrm{proj}'_i$ for all the specified projections.

1.9. EXAMPLE. We give two trivial and one non-trivial example.

(i) Let $\mathbb{T}$ be the theory of abelian groups, $\mathbb{T}'$ the theory of commutative rings. There is an obvious inclusion $\mathbb{T} \to \mathbb{T}'$, in fact $\mathbb{T}(n, 1)$ may be viewed as

the subset of $T'(n, 1)$ (= set of polynomials in n variables) consisting of homogeneous degree 1 polynomials.

(ii) Let T be the theory of groups, T' the theory of abelian groups. There is an obvious 'surjective' $T \to T'$ obtained by identifying two abstract group operations if their equality follows from the commutative law. Studying the quotients of a theory T in fact is equivalent to studying the subvarieties of the variety it defines, to express things in Birkhoff terms (COHN [1965]).

(iii) Let T be the theory of Lie-rings, T' the theory of (associative but not necessarily commutative) rings. There is a morphism $T \to T'$ which sends the binary $[-,-]$ in T into the commutator operation ω in T', $\omega(x, y) = x \cdot y - y \cdot x$.

1.10. If $f : T \to T'$ is a morphism of algebraic theories, one immediately gets a functor "composing with f"

$$f^b : \mathrm{Alg}(T', \mathscr{E}) \to \mathrm{Alg}(T, \mathscr{E}).$$

This feature is common to all kinds of functorial semantics. In the case of algebraic theories (next section), one can say more, namely (LAWVERE [1963]):

1.11. THEOREM. *If the value category $\mathscr{E}$ is* (Sets) *(or any other locally presentable category,* GABRIEL and ULMER [1971]*), the functor f^b has a left adjoint.*

PROOF. For a proof in the (Sets) case we refer to SCHUBERT [1970]. □

1.12. Free algebras

The existence of such can be considered as a corollary of Theorem 1.11, by taking the T in 1.10 to be the algebraic theory whose *only* n-ary operations are the projections, and whose category of algebras is just (Sets). The functor f^b can be identified with $A \mapsto A(1)$, which is a functor $\mathrm{Alg}(T') \to$ (Sets). It has a left adjoint $F :$ (Sets) $\to \mathrm{Alg}(T')$, by 1.11. An algebra of form $F(M)$ (M a set) is called the *free* algebra on the set M of generators. It now turns out that any algebraic theory can be reconstructed from the free algebras:

1.13. THEOREM. *Let T be an algebraic theory. Then T is equivalent to the dual of the full subcategory of* $\mathrm{Alg}(T)$ *defined by the free algebras $F(n)$ ($n = 0, 1, 2, \ldots$).*

PROOF. The functor $\hom_{\mathbb{T}}(n, -) : \mathbb{T} \to (\text{Sets})$ preserves products and is thus a T-algebra. For any other T-algebra $A : \mathbb{T} \to (\text{Sets})$, we have (denoting by Hom the hom-functor of $[\mathbb{T}, (\text{Sets})]$ and thus of $\text{Alg}(\mathbb{T})$):

$$\text{Hom}(\hom_{\mathbb{T}}(n, -), A) \simeq A(n) \simeq \prod_n A(1) \simeq \hom(n, A(1)),$$

the first isomorphism by Yoneda's lemma. Thus $\hom_{\mathbb{T}}(n, -)$ serves as $F(n)$. The second conclusion is now immediate, again by Yoneda. □

1.14. EXAMPLE. The following is an example of an algebraic theory which is neither obtained by a presentation, nor by "semantic" means as in 1.13, but rather by a construction performed in the *category* of algebraic theories. Let $\hat{\mathbb{T}}_0$ be the theory whose n-ary operations are formal power series in n variables and without constant term. They can be substituted into each other without any convergence trouble, since the constant term is missing. Formally, $\hat{\mathbb{T}}_0$ can be obtained as an inverse limit formation in the category of algebraic theories, $\varprojlim \mathbb{T}_{0,n}$, where $\mathbb{T}_{0,n}$ is the algebraic theory for commutative rings without unit and satisfying $\forall x : x^n = 0$. The algebras for $\hat{\mathbb{T}}_0$ are not known, neither is a syntactic presentation.

1.15. Varying the value category

We can consider $\text{Alg}(\mathbb{T}, \mathscr{E})$ for *any* category $\mathscr{E}$ with finite products; in particular when $\mathscr{E} = (\text{Alg}(\mathbb{T}'))^{\text{op}}$. In this case one gets the notion of "T-coalgebra in the category of T'-algebras", see WRAITH [1969] and FREYD [1966], which covers structures like Hopf algebras which cannot be described in terms of classical universal algebra (not even in finitary first-order logic).

1.16. The generic algebra

It is clear that if $p : \mathscr{E} \to \mathscr{E}'$ is a product preserving functor, we get a functor 'transport along p'

$$\text{Alg}(\mathbb{T}, \mathscr{E}) \to \text{Alg}(\mathbb{T}, \mathscr{E}')$$

by composing with p. We give a trivial example of this, which, however, is a case of an important principle which we shall meet in Sections 2, 4 and 5, the principle of *generic structure*. For a given algebraic theory T, we may consider $\text{Alg}(\mathbb{T}, \mathbb{T})$, which makes sense since T has finite products. Among the T-algebras in T, we have the identity functor $\text{id}_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$. It is a *generic* T-algebra in the sense that any other T-algebra $A : \mathbb{T} \to \mathscr{E}$ in any category $\mathscr{E}$ (with finite products) can be obtained by transport of the algebra $\text{id}_{\mathbb{T}}$ along

the product preserving functor A (clearly, since $A \circ \mathrm{id}_T = A$). (The example 1.9(iii) can be seen in this light: we equip (the underlying object 1 of) the generic associative T'-algebra (associative ring) with the structure of a Lie ring.)

However, the category T in which the generic T-algebra lives, is rather poor in categorical structure. In the next section, we shall encounter a richer category $\mathrm{FP}(T)^{\mathrm{op}}$ in which also lives a generic T-algebra (this time for cartesian (= finite limit preserving) functors) — and similarly for each of the stronger doctrines we meet later. The richer contexts make certain constructions on T-algebras possible 'once and for all', namely as constructions on the generic T-algebra.

1.17. Limitations

It is possible to characterize in categorical terms categories which come about as Alg(T) for some algebraic theory T, see LAWVERE [1965a]. Some that do not are the category Cat of small categories, the category of algebraic theories, or the category of partially ordered sets. These can be dealt with in terms of functorial semantics using a somewhat richer notion of "theory": a small cartesian category. This kind of functorial semantics is what GABRIEL and ULMER [1971] is about. We shall motivate cartesian theories by an example of a relative interpretation of one theory into another which cannot be described in the framework of algebraic theories.

2. Cartesian theories

2.1. EXAMPLE (Descartes, roughly). Consider the functor "circle"

$$S^1 : (\text{Commutative rings}) \to (\text{Sets})$$

which to a ring A associates the set $S^1(A)$,

$$S^1(A) = \{(x, y) \in A^2 \mid x^2 + y^2 = 1\}.$$

This set even carries an abelian group structure defined as follows:

$$(x, y) \cdot (x', y') = (x \cdot x' - y \cdot y', x \cdot y' + y \cdot x')$$

(Gauss). Denoting by T and T' the algebraic theories of commutative rings and abelian groups, respectively, we thus have a functor

$$S^1 : \mathrm{Alg}(T) \to \mathrm{Alg}(T'). \tag{2.1}$$

If $\mathscr{E}$ is a cartesian category (= having finite inverse limits), it has

equalizers, and if $A : \mathbf{T} \to \mathscr{E}$ is a commutative ring object, we can use the equalizer of

$$A(X_1^2 + X_2^2) \quad \text{and} \quad A(1) \tag{2.2}$$

(both the arguments of A being viewed as morphisms $2 \to 1$ in $\mathbf{T}$) to define S^1. In this way, the context of (2.1) is extended so as to define a functor

$$S^1 : \mathrm{Alg}(\mathbf{T}, \mathscr{E}) \to \mathrm{Alg}(\mathbf{T}, \mathscr{E})$$

for any *cartesian* category $\mathscr{E}$.

Unlike the functor (Associative algebras) $\to$ (Lie algebras) considered in 1.9(iii), the functor S^1 is not describable by performing the S^1-construction once and for all on the 'generic commutative ring' in $\mathbf{T}$, because $\mathbf{T}$ does not possess the requisite equalizer. This motivates looking for a notion of *cartesian* theory where such constructions can be performed, and investigating the functorial semantics of such:

2.2. DEFINITION. A *cartesian theory* $\mathbf{C}$ is a small category with finite inverse limits. A *model* M for $\mathbf{C}$ with *values* in a category $\mathscr{E}$ with finite inverse limits is a functor $M : \mathbf{C} \to \mathscr{E}$ which preserves these limits; a *morphism* of models, $M \to M$ is any natural transformation from M to M.

2.3. REMARK. Thus we have defined a *category* of models with values in $\mathscr{E}$, frequently denoted $\mathrm{Lex}(\mathbf{C}, \mathscr{E})$ since its objects are the left exact (= finite inverse limit preserving = cartesian) functors, It depends functorially on both $\mathbf{C}$ and $\mathscr{E}$: any left exact functor $\mathscr{E} \to \mathscr{E}$ between the value categories defines a 'transport' functor between the model categories, $\mathrm{Lex}(\mathbf{C}, \mathscr{E}) \to \mathrm{Lex}(\mathbf{C}, \mathscr{E})$. Any left exact functor $\mathbf{C} \to \mathbf{C}$ between cartesian theories ('relative interpretation') induces a functor $\mathrm{Lex}(\mathbf{C}, \mathscr{E}) \to \mathrm{Lex}(\mathbf{C}, \mathscr{E})$ between the model categories. As in the case of algebraic theories, such functors have left adjoints provided $\mathscr{E} = (\mathrm{Sets})$ (or any other locally presentable category). The reader may find a thorough treatment of categorical properties of categories of this form in GABRIEL and ULMER [1971] (in their notation, $\mathrm{Lex}(\mathbf{C}, \mathscr{E}) = \mathrm{St}_\alpha(\mathbf{C}, \mathscr{E})$ with $\alpha = \aleph_0$). Among the theorems in their book is one which we would term a (strong form) of conceptual completeness of cartesian theories:

2.4. THEOREM. *Suppose* $\mathbf{C}$ *and* $\mathbf{C}'$ *are cartesian theories such that* $\mathrm{Lex}(\mathbf{C}, (\mathrm{Sets})) \simeq \mathrm{Lex}(\mathbf{C}', (\mathrm{Sets}))$. *Then* $\mathbf{C} \simeq \mathbf{C}'$.

PROOF. The proof follows from the fact that $\mathbf{C}$ can be reconstructed from

Lex(C, (Sets)) as the dual of the subcategory of (abstractly) finitely presentable objects (GABRIEL and ULMER [1971], §7, in particular 7.10). □

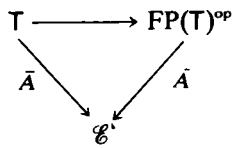
2.5. Comparison with algebraic theories

The clue to the proof that the doctrine of cartesian theories comprises that of algebraic theories also lies in the notion of finitely presented object. In a category of form Alg(T), the finitely presented objects form precisely the smallest subcategory containing all objects of form $F(n)$, and stable under finite colimits. Every finitely presented algebra A can be described as a coequalizer in Alg(T),

$$F(n') \rightrightarrows F(n) \to A,$$

(and a finite colimit of finitely presented algebras is finitely presented). We denote this subcategory FP(T). If α is a functor from the full subcategory of Alg(T) consisting of the objects of form $F(n)$ ($n = 0, 1, \ldots$) (and which we may identify with T^{op}, by 1.13) into a category with finite colimits, then it extends, because of the above coequalizer, in at most one way to a finite colimit preserving functor on FP(T), and a necessary condition that it does, is, of course, that α preserves finite coproducts (since $F(n) + F(m) = F(n + m)$). This necessary condition is also sufficient: we formulate this in the following theorem, where we, however, first change the variance.

2.6. THEOREM. *Let $A : \mathsf{T} \to \mathscr{E}$ be a finite product preserving functor from an algebraic theory into a cartesian category (so A is an algebra for T). Then the following diagram can be filled in with a left exact functor $\bar{A}$ (which is uniquely determined up to isomorphism):*



(The top functor being $n \mapsto F(n)$ as studied also in 1.13.)

PROOF. For the case $\mathscr{E}$ = (Sets), it follows because Alg(T) is locally finitely presentable, thus can be reconstructed as the category of set-valued left exact funtors on the dual of the category of finitely presented objects in it (GABRIEL and ULMER [1971], 7.9). It now easily follows also for $\mathscr{E}$ a functor category $[C^{\mathrm{op}}, (\mathrm{Sets})]$. Further, if the values of A are representable functors the values of $\bar{A}$ will be finite inverse limits of representables, and

will themselves be representable, provided C has finite inverse limits. This proves the theorem for $\mathscr{E} = C$. □

2.7. REMARK. The way to read the theorem (provided the value category has finite inverse limits) is: Every category of algebras for an algebraic theory T can be considered the category of models for a cartesian theory $P = FP(T)^{op}$. The identification identifies A with $\bar{A}$ in the above diagram.

2.8. Scope of the doctrine

As mentioned in 1.17, there are notions, like 'partially ordered set', which can be expressed by functorial semantics of a cartesian theory but not of an algebraic theory. (The appropriate cartesian theory C is in this case the dual of the category of finite partially ordered sets). How this works in detail can be seen in GABRIEL and ULMER [1971], 8.2b, or in the work of the Ehresmann school, where one may find many examples of cartesian theories or the closely related 'esquisses', see BASTIANI and EHRESMANN [1972].

2.9. Syntactic characterization of the doctrine

A sufficient condition for a first order theory to have its category of models (where maps are those that preserve the primitive predicates and operations) of the form Lex(C, (Sets)) for a suitable cartesian theory C, is that the axioms of the theory are simple Horn sentences, that is, of form

$$\forall x\,(\varphi(x) \Rightarrow \exists! y\,\psi(x, y))$$

(x and y are tuples of variables) with φ and ψ conjunctions of atomic formulae — see KEANE [1975]. The theory may be allowed to be many sorted; this corresponds to the fact that a cartesian theory C unlike an algebraic theory T does not have a preferred ground object. — There is a sense in which Keane's condition also goes the other way, but it is not clear to us.

2.10. EXAMPLE (continuation of 2.1). The functor (2.1) is not induced by a morphism of algebraic theories. However, if we take the corresponding cartesian theories, it is. Equivalently, out of the generic ring object in the cartesian theory C of commutative rings, we can construct the circle object (and its group structure), once and for all, that is, we have a *generic circle*. If A^1 denotes the generic ring object in C, the generic S^1 is constructed as the equalizer in C

$$S^1 \to A^2 \rightrightarrows A^1$$

where the two displayed maps from A^2 ($= A^1 \times A^1$) to A^1 are the ring operations $X_1^2 + X_2^2$ and 1, applied on the ring object A^1. Now, recalling that $\mathbf{C}$ = (finitely presented commutative rings)^{op}, and $A^1 = F(1)$, $A^2 = F(2)$, we see that S^1 is the finitely presented commutative ring which is the coequalizer

$$S^1 \leftarrow \mathbf{Z}[X_1, X_2] = F(2) \leftleftarrows \mathbf{Z}[X] = F(1)$$

(the two parallel maps again being given by $X \mapsto X_1^2 + X_2^2$ and $X \mapsto 1$, respectively). So

$$S^1 = \mathbf{Z}[X_1, X_2]/(X_1^2 + X_2^2 - 1).$$

The group structure on S^1, when viewed in $\mathbf{C}$, is a co-group structure (1.15) when viewed in the dual more familiar category, the category of finitely presented commutative rings. As such, it is a map $S^1 \to S^1 \otimes S^1$ (which cannot be described in first-order terms). — This way of thinking on the circle, and other 'algebraic subsets' or 'affine schemes' in affine space A^n, is familiar in modern algebraic geometry, see e.g. DEMAZURE and GABRIEL [1970].

3. Elementary doctrines: quantifiers as adjoints

3.1. Once equational and cartesian logic have been set in a categorical context, it is natural to search for a categorical description of the logic of quantifiers. The basic observation, due to LAWVERE [1965b], [1969] and [1970], is that existential and universal quantification can be seen as left and right adjoints, respectively, of substitution. Lawvere used this description as basis for arriving at a functorial semantics of first-order logic, first in the context of 'elementary theories', see LAWVERE [1965b], further developed by VOLGER [1975], and later in the context of 'hyperdoctrines' and 'elementary existential doctrines', LAWVERE [1970], p. 6. We shall discuss the latter.

3.2. DEFINITION. An *elementary existential doctrine* (*eed*) is given by a base category $\mathbf{T}$ with finite products (whose objects and morphisms one should think of as the *types* and *terms*, respectively), and for each object (type) n, there is given a category $B(n)$, the "category of *attributes*" of that type. (For applications in logic, the attribute category is usually, but not always, just a partially ordered set, the order relation being thought of as entailment.) The basic structure required is now that $B(n)$ depends

contravariantly functorial on n, that is, for $f : n \to m$ a morphism in T, there is given a functor (= order preserving map) $B(f) : B(m) \to B(n)$, thought of as substitution along f, and having a left adjoint $\exists_f$, "existential *quantification* along f".

3.3. REMARK. In Lawvere's original treatment of eed's, $B(n)$ is assumed to be a cartesian closed category, but there are as many variants of the notion as there are fragments of first-order language (roughly).

3.4. EXAMPLE. Any category $\mathscr{E}$ with finite inverse limits and good direct images defines an eed $P(\mathscr{E})$ with $\mathsf{T} = \mathscr{E}$, $B(n)$ = subobject semilattice of $n \in \mathscr{E}$, and $B(f)$ = inverse image formation (pull-back) along f. The $B(n)$'s will have, and the $B(f)$'s preserve, more or less lattice theoretic properties, depending on the exactness properties of $\mathscr{E}$, leading to the variants of the doctrine notion. We shall discuss the applicability of eed's for functorial semantics by considering the boolean case, where we have an (abstract) eed where each $B(n)$ is a boolean algebra, and each $B(f)$ a boolean homomorphism, and by considering only (Sets) as value category; note that $P((\text{Sets}))$ is a boolean eed. Thus:

3.5. DEFINITION. A (set-valued) *model* of the boolean eed (T, B) is a morphism of boolean eed's

$$(\mathsf{T}, B) \to P((\text{Sets})),$$

(where a morphism of boolean eed's $(\mathsf{T}, B) \to (\mathsf{T}', B')$ is a product preserving functor $M : \mathsf{T} \to \mathsf{T}$ between the base categories, as well as, for each $n \in \mathsf{T}$, a boolean homomorphism $B(n) \to B'(M(n))$, satisfying obvious compatibility conditions).

3.6. To relate this notion of model to the notion of model for an ordinary first-order theory $\mathscr{T}$, one has to take (T, B) to be the 'Lindenbaum doctrine' of $\mathscr{T}$, where T is the algebraic theory having no other operations than the projectons (as studied in 1.12) and with $B(n) = \mathscr{T}$-equivalence classes of formulas whose free variables are among $x_1, \ldots, x_n$. Then $B(f)$ (for $f : n \to m$) is defined by means of the syntactic process of substitution. Existential quantification can be defined in terms of the syntactic $\exists$. Sketches of this kind of semantics for eed's are given in LAWVERE [1970] (for the higher order case); see also COSTE [1973].

The reader familiar with polyadic or cylindric algebras may recognize features in the (boolean) eed-approach, and in fact, specific comparison theorems have been proved, JOYAL [1971].

3.7. However, from the viewpoint of categories as theories, eed's have the defect that they are categories T equipped with something extra, the $B(n)$'s, which is not defined in terms of the category structure, except for the case of a subobject eed $P(\mathscr{E})$ (Example 3.4). For which eed's (T, B), or for which first-order theories $\mathscr{T}$ does there exist a sufficiently rich cartesian category R such that (T, B) (or $\mathscr{T}$) has the same models as the subobject eed $P(\mathsf{R})$? Equivalently, when can the $\mathscr{E}$-valued models of (T, B) (or $\mathscr{T}$) be described as functors $\mathsf{R} \to \mathscr{E}$ preserving a certain amount of the categorical structure? This question was first resolved by Joyal–Reyes–Dionne (DIONNE [1973] for some of the fragments); see Example 4.4. Dionne's method was to construct R from $\mathscr{T}$. Better proofs of these results have been given by Benabou–Coste (COSTE [1974]) who construct actual 'good' categories R from eed's satisfying the 'Beck' (or Chevalley) and 'Frobenius' laws (terminology of LAWVERE [1970]), and apply this construction to the Lindenbaum doctrine of $\mathscr{T}$ (for the appropriate fragment of logic). Another technique is due to Freyd via the notion of allegory, a categorical axiomatization of the relation calculus.

4. Logical categories: coherent logic

The search for categorical versions of first-order logic in the spirit of equation theories with categories and models with functors led to the notion of regular category with stable sups, or logical category, for short, of Joyal–Reyes, REYES [1974]. (As pointed out in Section 3, Lawvere and Volger had studied earlier the related notion of "elementary theory" and "elementary doctrine".) In the context of logical categories, existential quantifiers appear as images (in the categorical sense). the connective $\vee$ ("or") as a supremum (of subobjects), whereas $\wedge$ ("and") and substitution (of variables by terms) are cases of inverse limits.

4.1. DEFINITION. A *logical category* T is a cartesian category with

(a) images which are stable under pull-backs,

(b) finite sups of subobjects of a given object which are stable under pull-backs.

4.2. REMARK. (ad (a)) The image of $f: A \to B$ is the smallest subobject $I \to B$ through which f factors. We write $f: A \twoheadrightarrow B$ to mean that the image of f is B itself.

We say that $f: A \twoheadrightarrow B$ is stable, if for every pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow & & \uparrow \\ A' & \xrightarrow[f']{} & B' \end{array}$$

the image of f' is B'.

(ad (b)) We say that $\bigvee_{i\in I} A_i = A$ is stable if for every $B \to A$,

$$\bigvee_{i\in I} B_A \times A_i \simeq B.$$

4.3. DEFINITION. A *logical morphism* between logical categories $f : \mathsf{T} \to \mathsf{T}'$ is a functor which preserves finite inverse limits, images and finite sups.

We let $\mathrm{Mod}_{\mathsf{T}'}(\mathsf{T})$ be the full subcategory of the functor category $[\mathsf{T}, \mathsf{T}']$ consisting of logical morphisms. In the particular case $\mathsf{T}' = (\mathrm{Sets})$, the morphisms are called models of T and we define $\mathrm{Mod}(\mathsf{T}) = \mathrm{Mod}_{(\mathrm{Sets})}(\mathsf{T})$.

4.4. EXAMPLE. Joyal–Reyes–Dionne (DIONNE [1973]) proved that those first-order theories whose categories of models (with primitive-predicate preserving maps as morphisms) are of the form $\mathrm{Mod}_{(\mathrm{Sets})}(\mathsf{T})$ are precisely the *coherent* theories $\mathscr{T}$, as defined by Joyal–Reyes, in REYES [1974]; a first-order theory is *coherent* if it can be presented with axioms of the form

$$\forall x\, (\varphi(x) \Rightarrow \psi(x))$$

where φ and ψ are built from the atomic formulas by means of $\wedge$, $\vee$ and $\exists$, or are $\uparrow$ or $\downarrow$ ('true' or 'false'). The notion of coherent theory will be crucial in Section 5. The hard part of the proof consists in construction of a logical category R starting from $\mathscr{T}$. Basically, there is one object for each concept definable in $\mathscr{T}$, and a map for each provably functional relation.

4.5. EXAMPLE. The theory of commutative local rings is coherent, being axiomatizable by equations, and the two axioms

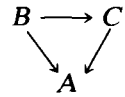
(i) $$\forall x\, (\uparrow \Rightarrow (\exists y : x \cdot y = 1 \vee \exists y : (x-1) \cdot y = 1)),$$

(ii) $$(1 = 0 \Rightarrow \downarrow).$$

(The equations are of the required form again by using $\uparrow$.)

4.6. COMPLETENESS THEOREM (Deligne, GROTHENDIECK [1963–1972], Expose 6; Joyal). *If* T *is a logical category and* $f, g : X \rightrightarrows Y$ *are two different morphism in* T, *then there is a model* M *of* T *such that* $M(f) \neq M(g)$.

PROOF (outline). The key idea is to extend T to a new logical category T' having the property that every $A \twoheadrightarrow 1$ has a section (i.e. T satisfies a weak axiom of choice). The key observation in Joyal's proof is the following: whenever $A \twoheadrightarrow 1$ in T is given, we can add "universally" a formal section by "changing base", i.e., by considering the category T/A whose objects are morphisms $B \to A$ and whose morphisms are commutative triangles



which is again a logical category. The pullback functor $\mathsf{T} \to \mathsf{T}/A$ (which sends B into $\pi_2 : B \times A \to A$) is conservative and, furthermore $A \twoheadrightarrow 1 \in \mathsf{T}$ has the diagonal $\Delta_A : A \to A \times A$ as a section in T/A.

By identifying objects with formulas (as in the example) we recognize the analogue of a key step in Henkin's proof: if A is a formula such that $T \vdash \exists x A$, then $T \cup A(a)$ is a conservative extension of T, where a is a new constant.

The rest of the proof proceeds by taking an appropriate ultrafilter. □

4.7. REMARK. One important question that can be asked in the categorical context is: to what extent does the category of models of T determine T? One way of interpreting this question is to ask whether further logical structure may be introduced in T without changing the category of models of T or whether T is "conceptually complete". We notice at once that logical categories are not "conceptually complete" in general. Indeed, by adding "formally" to T coproducts and quotients by equivalence relations we obtain a logical category $\hat{\mathsf{T}}$ such that the logical inclusion $\mathsf{T} \to \hat{\mathsf{T}}$ induces (via composition) an equivalence $\mathrm{Mod}(\hat{\mathsf{T}}) \xrightarrow{\sim} \mathrm{Mod}(\mathsf{T})$ (see ANTONIUS [1975]). (Clearly, e.g. X/R is uniquely interpreted in (Sets) as the quotient of the interpretation of X by that of R.) We find here a *defect* in the usual formalizations of many-sorted languages where the "implicitly definable" operations of coproducts and quotients are not explicit. Letting a *pretopos* be a logical category with "good" coproducts and quotients by equivalence relations, we have:

4.8. CONCEPTUAL COMPLETENESS (MAKKAI and REYES [1976]). *If a logical functor $f : \mathsf{T} \to \mathsf{T}'$ between pretopoi induces (via composition) an equivalence $\mathrm{Mod}\,\mathsf{T}' \xrightarrow{\sim} \mathrm{Mod}\,\mathsf{T}$, then f is an equivalence.*

PROOF. The proof proceeds by introducing syntactical many-sorted presentations for T, T' whereby f becomes an interpretation of theories.

Compactness and the method of diagrams are then used. A purely categorical proof is still lacking. □

4.9. REMARK. Infinitary logical categories have also been considered and BARR [1974] has proved a completeness theorem for Boolean-valued models. A Gentzen-type axiomatization for this infinitary coherent logic, with applications to completeness theorems (including Barr's) may be found in MAKKAI and REYES [1976].

5. Grothendieck topoi: infinitary coherent logic

Topoi (and pretopoi) appeared in algebraic geometry, where they are ubiquitous via a generalization of the notion of sheaf over a topological space: this is the notion of sheaf over a site, due to Grothendieck.

5.1. DEFINITION. (i) A *site* is a cartesian category $\mathscr{C}$ together with a notion of localization, i.e. for every $A \in |\mathscr{C}|$ we are given a non-empty class $\mathrm{Loc}(A)$ of families of morphisms $(A_i \to A)_{i\in I}$ of $\mathscr{C}$, called the *localizations* of A which are stable under pullbacks in the sense that for every $B \to A$, the family $(A_{i_A} \times B \to B)_{i\in I}$ is a localization of B.

(ii) A *sheaf* over $\mathscr{C}$ is a functor $F : \mathscr{C}^{\mathrm{opp}} \to (\mathrm{sets})$ satisfying the following conditions, for every $(f_i : A_i \to A)_{i\in I} \in \mathrm{Loc}(A)$:

(a) if $\xi, \eta \in F(A)$ are such that $\xi_i = F(f_i)(\xi) = F(f_i)(\eta) = \eta_i$ for all $i \in I$, then $\xi = \eta$.

(b) if $(\xi_i)_{i\in I}$ is a family such that $\xi_i \in F(A_i)$ for all $i \in I$ and is compatible, i.e. in the diagram

$$\xi_i \in F(A_i) \xrightarrow{F(\pi_i)} F(A_{i_A} \times A_j) \xleftarrow{F(\pi_j)} F(A_j) \ni \xi_j$$

obtained via F from

$$A_i \xleftarrow{\pi_i} A_{i_A} \times A_j \xrightarrow{\pi_j} A_j$$

we have $F(\pi_i)(\xi_i) = F(\pi_j)(\xi_j)$ for all $i, j \in I$, then there is $\xi \in F(A)$ such that $\xi_i = F(f_i)(\xi)$, for all $i \in I$.

We let $\mathrm{Sh}(\mathscr{C})$ be the full subcategory of the functor category $[\mathscr{C}^{\mathrm{op}}, (\mathrm{Sets})]$ consisting of sheaves over $\mathscr{C}$.

(iii) A *Grothendieck topos* is a category equivalent to one of the form $\mathrm{Sh}(\mathscr{C})$, for a small site $\mathscr{C}$.

5.2. REMARK. (i) These notions allow us to give a meaning to the intuitive idea that some concepts, e.g. "real-valued continuous functions with open domain" are of local character. In this example, we identify this concept with the functor $C_{\mathsf{R}}: \mathrm{Open}(X)^{\mathrm{op}} \to (\mathrm{Sets})$ which to every open set $U \in \mathrm{Open}(X)$ associates its "extension" $C_{\mathsf{R}}(U) = \{f: U \to \mathsf{R} \mid f \text{ is continuous}\}$. The ordered category $\mathrm{Open}(X)$ may be considered as a site, by defining a localization of U as an open cover of U.

The local character of the concept in question are just the statements (a) and (b).

(ii) A site structure and the corresponding notion of sheaf may be defined on a category $\mathscr{C}$, even if $\mathscr{C}$ does not have finite inverse limits. The definition is technically more involved. Examples of this type appear in model theory (see 5.3(ii)).

(iii) One of the main theorems of topos theory is Giraud's (GROTHENDIECK [1963–1972], Exposé 4) which characterizes a topos as an *∞-pretopos* with a set of generators (where an ∞-pretopos is a category with finite inverse limits, "good" infinite coproducts and quotients by equivalence relations). In particular, any topos has a *canonical* site structure given by $(A_i \to A)_{i \in I}$ is a localization iff the unique map

$$\frac{|\cdot|}{i \in I} A_i \to A$$

is epimorphic. ("Goodness" is precisely the condition that this type of families are localizations.) This is sometimes called the arbitrary cover localization.

5.3. EXAMPLE. (i) Let B be a complete Boolean algebra, considered as a category. We define the canonical localization as follows: $(a_i)_{i \in I}$ covers a iff $a = \bigvee_{i \in I} a_i$.

Then $\mathrm{Sh}(\mathsf{B})$ is equivalent to the category of Boolean-valued sets and Boolean-valued maps of Scott–Solovay's universe $\mathrm{V}^{(\mathsf{B})}$.

This example opens the way for a category-theoretic approach to independence results in set theory: see HIGGS [1976], TIERNEY [1972], BOILEAU [1975], and BUNGE [1974].

(ii) Let C be a small category. There is a so-called $\neg\neg$-localization on it which can be described as follows: a family $\{C_i \to C \mid i \in I\}$ is a $\neg\neg$-localization if for each $B \to C$ there is an $A \to B$ such that for some $i \in I$ we have a commutative diagram

$$\begin{array}{ccc} C_i & \longrightarrow & C \\ \uparrow & & \uparrow \\ A & \longrightarrow & B. \end{array}$$

This $\neg\neg$-localization will be used below.

(iii) Let R be a logical category. We introduce a site structure on R, *the finite cover localization*, by defining for I finite:

$$(A_i \xrightarrow{f_i} A)_{i\in I} \in \mathrm{Loc}(A) \quad \text{iff} \quad A = \bigvee_{i\in I} \mathrm{Image}(f_i).$$

The stability conditions on R guarantee that this is a localization.

5.4. DEFINITION. (i) A functor $F: \mathscr{C} \to \mathscr{D}$ between two sites is a *site morphism* if it preserves the finite inverse limits and takes localizations of $\mathscr{C}$ into localizations of $\mathscr{D}$. We let $\mathrm{Cont}(\mathscr{C}, \mathscr{D})$ be the full subcategory of the functor category $[\mathscr{C}, \mathscr{D}]$ consisting of site morphisms.

(ii) A *geometric morphism* $p: \mathscr{E} \to \mathscr{E}'$ between topoi is a couple (p^*, p_*), where $p^*: \mathscr{E}' \to \mathscr{E}$ is a left exact functor having p_* as a right adjoint (thus p^* is a site morphism for the canonical site structures on $\mathscr{E}$ and $\mathscr{E}'$).

Remark. The significance of coherent logic lies in the fact that its logical operations (i.e., $\exists, \vee, \wedge, \top, \downarrow$) are preserved by the functor p^* for a geometric morphism p. See also 6.13.

5.5. Forcing: the Kripke–Joyal semantics

From the examples it should be clear that topoi have something to do with forcing and generic structures. A. Joyal pointed out that the various notions of forcing (Cohen's, Robinson's, Kripke's) are special cases of forcing with a site of conditions. Even for a site Open X (Example 5.2(i)) the usual definitions break down: in Kripke's semantics, for instance, the validity of a disjunction or existential formula at the "stage" or "time" α are decided "on the spot", i.e. without reference to further "stages" (or "times"). On the other hand, in topology the existence of cross-sections (over the whole space) for sheaves is rare compared to local existence, i.e. existence of sections over coverings. I.e., further "stages" must be taken into account.

5.6. DEFINITION. Let $\mathscr{C}$ be a site, F a sheaf and $a \in F(X)$. $X \Vdash \exists x\, \varphi[a]$ iff there is a localization $(f_i: X_i \to X)_{i\in I}$ of X and a family $(b_i)_{i\in I}$ such that for every $i \in I$, $b_i \in F(X_i)$ and $X_i \Vdash \varphi[b_i, F(f_i)(a)]$.

5.7. REMARK. A similar clause should be used to define $X \Vdash \varphi_1 \vee \varphi_2$. On the other hand, $\wedge$ is decided "on the spot" and $\Rightarrow, \forall$ are dealt with in the usual Kripke fashion, e.g.

$$X \Vdash (\varphi \Rightarrow \psi)(a) \quad \text{iff} \quad \text{for every } f: Y \to X,$$
$$Y \Vdash \varphi[F(f)(a)] \quad \text{implies} \quad Y \Vdash \psi[F(f)(a)];$$

negation is $\Rightarrow \downarrow$, and $\downarrow$ is described by means of objects Y for which $\emptyset \in \mathrm{Cov}(Y)$.

(ii) One can check that this notion of forcing specializes to the usual ones. As an example, assume e.g. $\mathscr{A} \Vdash^* (\varphi \vee \psi)[a]$ in the sense of ROBINSON [1971], i.e.,

$$\forall \mathscr{A} \xrightarrow{f} \mathscr{B} \in \mathrm{Mod}(T)\ \exists \mathscr{B} \xrightarrow{g} \mathscr{L} \in \mathrm{Mod}(T)$$

such that $\mathscr{L} \Vdash^* \varphi[gfa]$ or $\mathscr{L} \Vdash^* \psi[gfa]$. We may assume that $g = \Phi(f)$. Then the family

$$(\mathscr{A} \xrightarrow{\Phi(f) \circ f} \mathscr{L})_f$$

is a "co-localization" for $\neg\neg$. Furthermore, for every $\mathscr{L}$, $\mathscr{L} \Vdash^* \varphi[\Phi(f)fa]$ or $\mathscr{L} \Vdash^* \psi[\Phi(f)fa]$. We now use induction. (This example is taken from unpublished work by Joyal–Reyes.)

(iii) In the particular case that the site is a topos $\mathscr{E}$ (with the canonical localization) we have a way of interpreting languages (even infinitary ones). In particular we may define the notion of a *T-model in $\mathscr{E}$*, for any theory T (in a $L_{\infty\omega}$-language, say), and the *category of T-models in $\mathscr{E}$* as well. In case $\mathscr{E}$ = (Sets), it amounts to this: its objects are set-theoretical models and its morphisms are "algebraic", i.e. functions $f: \mathscr{A} \to \mathscr{B}$ which preserve the basic relations and operation symbols in the sense that $(a_1, \ldots, a_n) \in R^{\mathscr{A}} \Rightarrow (f(a_1), \ldots, f(a_n)) \in R^{\mathscr{B}}$ (with a corresponding clause for operation symbols).

For a semantics in terms of extensions, see 6.8.

5.8. PROPOSITION (Existence of a classifying topos; REYES [1974], in collaboration with Joyal). *For every finitary coherent theory T, there is a topos $\mathscr{E}[T]$ and a T-model M in $\mathscr{E}[T]$ which is generic in the sense that the category of T-models of $\mathscr{D}$ is equivalent (via M) to the category of geometric morphisms of $\mathscr{D}$ into $\mathscr{E}[T]$, for every topos $\mathscr{D}$.*

PROOF (outline). Let R be the logical category associated to T (Example

4.4). Then $\mathscr{E}[T]$ is the category of sheaves for R with the finite cover topology (Example 5.3(iii)). □

5.9. REMARK. (i) The earliest example of the existence of classifying topos was given by HAKIM [1972], who proved that the Zariski topos is the classifying topos for the theory of local rings.

(ii) This theorem remains true for an arbitrary (i.e. infinitary) coherent theory. Furthermore, any topos is the classifying topos of such a theory (see MAKKAI and REYES [1976]).

6. Elementary topoi

The word "elementary" in this context refers to the fact that the notion of elementary topos is an elementary notion: the notion of category with a few properties which are expressible in first order terms in the language of the theory of categories. It is even an "essentially algebraic" notion (FREYD [1972]).

The search for such a concept, as a foundation of mathematics, was initiated in LAWVERE [1964], but a more general notion comprising the notion of Grothendieck topos, and at the same time revealing the logical (even higher-order logical) nature of these, was found in 1969 by Lawvere and Tierney; see LAWVERE [1972]. With the simplifications now possible, KOCK and MIKKELSEN [1974], this very simple and powerful notion can be decribed as follows.

6.1. DEFINITION. An *elementary topos* $\mathscr{E}$ is a category with finite inverse limits and an object Ω which

(i) classifies subobjects,

(ii) is exponentiable.

6.2. REMARK. Here, (i) means that there is a universal subobject *true*: $1 \to \Omega$, such that any subobject $A' \rightarrowtail A$ of any object A comes about as a pull-back

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & \Omega \\ \uparrow & & \uparrow{\scriptstyle true} \\ A' & \longrightarrow & 1 \end{array}$$

for a unique α ("the characteristic function of A'"); and (ii) means that for

each A, there is an object Ω^A ("the power object of A") and a map $\in_A : A \times \Omega^A \to \Omega$, which establishes for each B a $1-1$ correspondence

$$\frac{A \times B \xrightarrow{\hat{\beta}} \Omega}{B \xrightarrow[\check{\beta}]{} \Omega^A} \tag{6.1}$$

in the following way: to $\check{\beta}$, associate

$$A \times B \xrightarrow{1\times\check{\beta}} A \times \Omega^A \xrightarrow{\in_A} \Omega.$$

6.3. EXAMPLE. The model to have in mind is the category of sets, with Ω any two-element set, say the set consisting of the two symbols "true" and "false"; then Ω^A can be identified with the power set of A (since a subset $A' \subseteq A$ can be completely described by its characteristic function $\alpha : A \to \Omega$ given by

$$\alpha(a) = \text{true} \quad \text{if } a \in A;$$
$$= \text{false} \quad \text{if not.}$$

What is important is first that each Grothendieck topos is also a model of these axioms, secondly that essentially all ("intuitionistically valid") higher order naive set theory follows from the simple axioms of Definition 6.1. From 1969 to now, a vast amount of papers etc. demonstrated how various notions and constructions of higher order naive set theory can be based on and carried out in the setting of an elementary topos. Instead of trying to survey all these, we give some detailed examples of constructions.

6.4. EXAMPLE. It is clear that Ω^A depends contravariantly on A, $f : A \to B$ gives rise to $\Omega^f : \Omega^B \to \Omega^A$ (which in (Sets) is the law which to a subset B' of B associates $f^{-1}(B') \subseteq A$). Also, Ω^A carries canonically the structure of an ordered object (in the set case: the inclusion ordering on the set of subsets of A). This structure is a subobject $\leq_A$,

$$\leq_A \rightarrowtail \Omega^A \times \Omega^A.$$

It has a characteristic function

$$\mathrm{ch}(\leq_A) : \Omega^A \times \Omega^A \to \Omega$$

which, according to the exponential adjointness (6.1) corresponds to a map

$$\Omega^A \xrightarrow{\downarrow \mathrm{seg}} \Omega^{\Omega^A}$$

which in the (Sets) case has the effect of associating to the subset $A \subseteq A$

the family of all subsets $A'' \subseteq A$ with $A'' \subseteq A'$ (the "down-segment of A'"). The right adjoint $\forall_f$ of $\Omega^f : \Omega^B \to \Omega^A$ (for $f : A \to B$) can now be described as the composite

$$\Omega^A \xrightarrow{\downarrow \text{seg}} \Omega^{\Omega^A} \xrightarrow{\Omega^{\Omega^f}} \Omega^{\Omega^B} \xrightarrow{\Omega^{\{\,\}}} \Omega^B$$

where $\{\cdot\} : B \to \Omega^B$ corresponds under exponential adjointness to the characteristic map $B \times B \to \Omega$ of the diagonal $B \to B \times B$. In the (Sets)-case, the reader will see that the composite has the following effect on

$$\begin{aligned} &A' \subseteq A \\ &A' \mapsto \{A'' \subseteq A \mid A'' \subseteq A'\} \mapsto \{B' \subseteq B \mid f^{-1}(B') \subseteq A'\} \\ &\quad \mapsto \{b \in B \mid f^{-1}(\{b\}) \subseteq A'\} \\ &\quad = \{b \in B \mid \forall a \in A : f(a) = b \Rightarrow a \in A'\} \end{aligned}$$

which we denote $\forall_f(A')$. Then it is easy set-theoretically to check that we do have a right adjoint for the inverse image formation, i.e. we have

$$f^{-1}(B') \subseteq A' \quad \text{iff} \quad B' \subseteq \forall_f(A').$$

Using this "internal universal quantification", one can get an "intersection"-construction

$$\Omega^{\Omega^A} \xrightarrow{\cap} \Omega^A$$

which in (Sets) has the effect of associating to $\mathcal{F} \subseteq \Omega^A$ the set $\bigcap \mathcal{F} \subseteq A$ (the intersection of all $A' \in \mathcal{F}$). We omit that. Using this "intersection combinator", one can derive the *left* adjoint $\exists_f$ ("direct image formation along f") as the composite

$$\Omega^A \xrightarrow{\uparrow \text{seg}} \Omega^{\Omega^A} \xrightarrow{\Omega^{\Omega^f}} \Omega^{\Omega^B} \xrightarrow{\cap} \Omega^B$$

($\uparrow$ seg defined dually of $\downarrow$ seg). In (Sets) this map takes $A' \subseteq A$ into the intersection of all $B' \subseteq B$ with $f^{-1}(B') \supseteq A'$, which clearly is just $f(A')$. (The descriptions here are due to MIKKELSEN [1976]; it illustrates how one can derive colimit-type notions, like image-formation $\exists_f$, on basis of the axioms. For more about the basic theory, see KOCK and WRAITH [1971], KOCK and MIKKELSEN [1974], MACLANE [1975] or WRAITH [1975].)

We now describe how some more complicated notions of higher naive set theory get very simple formulations in the setting of elementary topoi. Observe that a binary relation on an object A

$$R \rightarrowtail A \times A$$

gives rise to a characteristic function $A \times A \to \Omega$, and then by the exponential adjointness to a map $r : A \to \Omega^A$ ("the down-segment for R"), and conversely, whence we also call a map $r : A \to \Omega^A$ a binary relation on A.

6.5. DEFINITION (OSIUS [1974]). A relation $r : A \to \Omega^A$ is said to be *recursive* (or (*strongly*) *well founded*) if for every B and $g : \Omega^B \to B$ there is a *unique* $f : A \to B$ making the diagram

$$\begin{array}{ccc} \Omega^A & \xrightarrow{\exists_f} & \Omega^B \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array} \tag{6.2}$$

commute; and r is called (*weakly*) *well founded* if for each $g : \Omega^B \to B$, there is *at most* one f making (6.2) commute.

6.6. THEOREM (Mikkelsen; see OSIUS [1975a]). *The notions of strongly and weakly well founded relation agree in any elementary topos.*

PROOF. The quite difficult proof consists of course in constructing an f making (6.2) commute. Theorem 6.6 thus is the assertion that the principle of map-construction by transfinite recursion over (internally) well founded relations is valid inside any elementary topos. □

6.7. REMARK. The fact that the recursion principle is valid in every topos may be interpreted proof-theoretically: it is intuitionistically provable. In fact, Fourman, COSTE [1974], BOILEAU [1975] (in collaboration with Joyal) have all set up formal systems of higher order intuitionistic type theory which are adequate and complete for elementary topoi. Thus elementary topoi may be viewed as a categorical version of intuitionistic higher order logic. Several results may be interpreted and proven in this context. This is discussed in detail in Chapter D.6.

As far as first-order intuitionistic logic in a topos is concerned, formal systems which again are adequate and complete for topoi have been set up by Benabou, Boileau, Coste, Dionne, Joyal, Makkai, Oullet, Reyes, Robitaille-Giguere (see COSTE [1973], OULLET [1974], for Gentzen-type system, and ROBITAILLE-GIGUERE [1975] for a Hilbert-type system).

6.8. Semantics by extensions

The good exactness properties and the cartesian closedness which can be proved for any elementary topos can be used to prove that any first-order

formula (coherent or not) whose primitive relations have been interpreted by "extensions" have themselves "extensions". The notion of *extension* of a formula can be defined as a universal element (in the sense of Kripke–Joyal semantics) satisfying the formula. (We think of $\mathscr{E}$ as a site by taking the finite cover topology.) A systematic description of this technique of forming extensions of formulas and proving inclusion relationships between such has been given by MITCHELL [1972], Benabou (COSTE [1973, 1974]) and OSIUS [1973]. The relationship to Kripke–Joyal semantics is analysed in OSIUS [1975b].

6.9. EXAMPLE. Let R be a ring object in an elementary topos, and form $GL(1, R) \subseteq R$ as the extension of the formula "x is invertible" (or formally "$\exists y : x \cdot y = 1$"). Also, for each natural number n, one may form the extension

$$[[\neg((x_1 = 0) \wedge \cdots \wedge (x_n = 0))]] \subseteq R^n \tag{6.3}$$

as well as the extension

$$[[(x_1 \text{ is invertible}) \vee \cdots \vee (x_n \text{ is invertible})]] \subseteq R^n. \tag{6.4}$$

6.10. DEFINITION. We say that R is a *field* object if for each n, the objects described in (6.3) and (6.4) agree (i.e. if R satisfies: "when n elements are not simultaneously zero, one of them is invertible, and conversely"). (This is a non-coherent notion.)

6.11. EXAMPLE (synthetic geometry, KOCK [1974]). Using the technique of extensions, and other exactness properties of an elementary topos $\mathscr{E}$ with a field object R, one can, following what one does in the (Sets) case, build some of the interesting objects of algebraic geometry, like Grassmannians. We shall illustrate this by constructing the projective $(n-1)$-space $\mathbf{P}^{n-1}(R)$. This is simply (6.3) (or (6.4)) modulo the equivalence relation induced on this object by the obvious action of $GL(1, R)$

$$\mathbf{P}^{n-1}(R) = \left[\left[\bigvee_{i=1}^{n} x_i \text{ is invertible} \right]\right] / GL(1, R).$$

Also, one can carve out of the power object of $\mathbf{P}^{n-1}(R)$ an object of $(n-2)$-planes in $\mathbf{P}^{n-1}(R)$, but it is more accessible to computation to construct this object by linear and multilinear algebra. In the case $\mathbf{P}^2(R)$, for instance, one could define an object $\mathbf{L}$ ("of lines in $\mathbf{P}^2(R)$") purely algebraically. Also using 3×3 determinants, one can next form the extension of the notion "incidence", i.e.,

$$[[\text{line } l \text{ passes through point } Q]] \subseteq \mathsf{L} \times \mathsf{P}^2(R).$$

One has now a model of a certain two-sorted first order structure with sorts "lines" and "points", and one relation, "incidence", and one may ask which sentences in this language hold for the structure thus constructed, in particular, does it satisfy the axioms of the notion of projective plane? It does, provided suitable of the non-equivalent intuitionistic forms of the axioms for "projective plane" are chosen. For details, see KOCK [1974]. With suitable modifications, like saying "invertible" instead of "non-zero" wherever possible, the whole development can be carried through for a local ring object instead of a field object. The interest of this now is when it is carried out for the *generic* local ring object A in the classifying topos $\mathscr{X}$ for local rings (the Zariski topos).

It turns out (KOCK [1974]) that this generic local ring is in fact a *field* object in the sense explained above, so that the construction of $\mathsf{P}^2(A)$ and the other Grassmannians can be carried out using not only the technique of extensions, but also the assumption that A is a field. This $\mathsf{P}^2(A)$, which as an object of $\mathscr{X}$ is a functor from finitely presented commutative rings to sets, agrees with the explicitly described such functor which in DEMAZURE and GABRIEL [1970] is honoured with this name. The theory of extensions allows us to get the geometric objects "by logical means" without having to produce functors and test sheaf-conditions; and it allows questions of *synthetic* geometry to be raised.

6.12. REMARK. Using the object $D \subseteq A$, the extension $[[x^2 = 0]]$, as "the infinitesimal object" as well as the fact that $\mathscr{X}$ is an elementary topos and thus has exponentiation, there seems to be a possibility of having even a synthetic differential geometry in $\mathscr{X}$ and related topoi.

6.13. Arithmetic universe

A complete axiomatization of this notion (due to Joyal) seems not to exist, but the programme for what one wants to be axiomatized is the following: all structure of essentially algebraic nature (FREYD [1972]) which an elementary topos with natural number object has, and which is preserved by inverse image functors f^*. So an arithmetic universe is at least a pretopos with natural number object N (since f^* is a pretopos morphism which preserves N), but since an elementary topos with natural number object has, and f^* preserves, formation of free monoids, free categories on a graph of generators, formation of $\otimes$ product of abelian group objects, ..., all this may be included into the good axiomatics for arithmetic

universe in case it does not already follow from the existence of natural number object in a pretopos (which most of it does not). Joyal suggested: pretopos where free categories on graphs exist as an axiomatization. A *morphism* of arithmetic universe is a functor which preserves the pretopos structure and the added structure (free categories on graphs, say). If one includes formation of $\otimes$ of abelian groups into the structure, one can describe the notion of Hopf-algebra object (ring object A with a map $A \to A \otimes A$ satisfying certain equations) by functorial semantics by arithmetic universe. Classifying topos for the notion of Hopf algebra can be proved to exist.

7. Topoi and axiomatic set theory

Any model m of set theory (ZF, say) defines an elementary topos, $\mathscr{E}(m)$ the category of m-sets, whose objects are the elements ("sets") of m, and topoi arising this way are rather special (having the multitude of, say, Grothendieck topoi in mind). The question arises whether one can characterize topoi of form $\mathscr{E}(m)$ in elementary terms (a non-elementary characterization was given in LAWVERE [1964]).

This can also be viewed as the question of relative interpretation of a suitable extension of the first order theory of elementary topos into ZF, and vice versa.

The question of characterization (or mutual relative interpretation) was solved in 1971 by COLE [1971], MITCHELL [1972], and OSIUS [1974]. It is a matter of building models of set theory, out of a given elementary topos having suitable properties. Which properties are suitable?

First, the following propositions are evident.

7.1. PROPOSITION. *Any topos of form* $\mathscr{E}(m)$ *has the property that the terminal object* 1 *is a generator; and it is non-trivial*: 0 *not isomorphic to* 1.

That 1 is a generator means: if f, g are two different maps $X \rightrightarrows Y$ in $\mathscr{E}(m)$, then there is a map $x : 1 \to X$ such that $f \circ x \neq g \circ x$, and is clear since a map $x : 1 \to X$ corresponds to an element $x \in X$, viewing X as a "set" in m.

Clearly, the more properties of an elementary topos $\mathscr{E}$ one assumes, the stronger model of set theory one can build from it. The main idea of the construction is starting with an elementary topos with very few extra properties, and obtaining a model of a weak set theory. We give Osius'

form (OSIUS [1974]), which in view of 7.1 is "the minimal". The set theory ZO resulting here lies between Zermelo–Thiele and Zermelo–Fraenkel (ZF), see OSIUS [1974].

7.2. THEOREM. *Given a non-trivial elementary topos $\mathscr{E}$ in which* 1 *is a generator. Then one can build a model of the set theory* ZO.

Before we give (a sketch of) the proof, we motivate and give the relevant definition. The main thing is a categorical characterization of transitive sets. If $A \in m$, then A is said to be *transitive* if $a \in A \Rightarrow a \subseteq A$. Denoting the (restriction of the) $\in$-relation on A by R, and reinterpreting R as a map $r : A \to \Omega^A$ in $\mathscr{E}(m)$, we have that r is well-founded (in the sense explained in Section 6) and is a monic map, because $\in$ is well-founded an extensional in Zermelo–Thiele set theory.

We can even get the converse, and more (for m a ZF model).

7.3. THEOREM (Mostowski; see OSIUS [1974]). *In order that a relation $r : A \to \Omega^A$ on $A \in \mathscr{E}(m)$ is isomorphic to the $\in$-relation on a transitive set, it is necessary and sufficient that r is well-founded (Definition* 6.5) *and monic as a map.*

7.4. THEOREM (OSIUS [1974]). *In order that a map $f : A_1 \to A_2$ between two transitive sets in m is the inclusion of $A_1 \subseteq A_2$, it is necessary and sufficient that*

$$\begin{array}{ccc} A_1 & \longrightarrow & A_2 \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle r_2} \\ \Omega^{A_1} & \xrightarrow[\exists_f]{} & \Omega^{A_2} \end{array} \tag{7.1}$$

commutes, where r_1 and r_2 are the $\in$-relations on A_1 and A_2.

So we have a structural characterization of transitive sets and of those maps between them which are inclusions.

7.5. PROOF OF THEOREM 7.2 (outline). We start by taking the conclusions of Theorems 7.3 and 7.4 as definitions. Let $\mathscr{E}$ be a non-trivial elementary topos where 1 is a generator.

7.6. DEFINITION. An object $A \in \mathscr{E}$ equipped with a relation $r : A \to \Omega^A$ is called a *transtive set object* (tr.s.o.) if r is monic and recursive (Definition

6.5). A map $f : A_1 \to A_2$ between two tr.s.o.'s (A_1, r_1) and (A_2, r_2) is called an *inclusion* if (7.1) commutes.

One now goes about proving that the category of tr.s.o.'s in $\mathscr{E}$, with inclusions as mappings, is equivalent (as a category) to a *lattice* (except that a maximal element is missing).

Now a model $m(\mathscr{E})$ for ZO can be constructed as follows: the elements in $m(\mathscr{E})$ are equivalence classes of pairs $((A, r), M)$, where (A, r) is a tr.s.o. and $M : A \to \Omega$ is a map (heuristically, thus M is (characteristic map of) a subset M' of a transitive set A). The equivalence relation is given by means of the lattice structure.

To describe the membership relation, one utilizes the family of maps

$$\{\Omega^A \times A \xrightarrow{\in_A} \Omega \mid A \in |\mathscr{E}|\}$$

given by the topos structure of $\mathscr{E}$.

A more detailed comparison between mutual strengths of certain (stronger) systems of set theory, and some further strengthenings of the notion of elementary topos, is given in the works (OSIUS [1974], COLE [1971] and MITCHELL [1972]).

8. Other fields of research in categorical logic

We have not tried to be comprehensive. We would have liked also to include comments on:

8.1. Model completeness via sheaf categories (MACINTYRE [1973]).

8.2. Logical methods applied to sheaf categories on topological spaces and real number objects in elementary topoi (MULVEY [1974], SCOTT [1968], Stout, Tierney).

8.3. Topos theoretic interpretation of methods of non-standard analysis (KOCK and MIKKELSEN [1974], REYES [1974], TAKAHASHI [1975]).

8.4. Topos theoretic methods in independence proofs in set theory (TIERNEY [1972], BUNGE [1974]) — although this goes against the trend of letting the "topos" notion have the lead over the notion "model of set theory".

8.5. Elementary topoi with a natural number object, and how an internal theory of classifying topos can be formed (Diaconescu, Tierney, Joyal, Johnstone, Lesaffre, De Kinder, WRAITH [1975], where references can be found).

8.6. Combinatorial logic and proof theory in form of cartesian closed categories (LAMBEK [1972], SZABO [1977] and SCOTT [1972]).

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