PLAN FOR TODAY

17/4/2024

PART 1

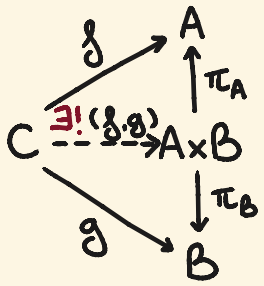
- recap cartesian categories
- (symmetric) monoidal categories
- string diagrams (& do-notation)
- monoids, comonoids & modules
- Yox's theorem

PART 2

- (symmetric) monoidal functors
- coherence & strictification (overview)
- (symmetric) monoidal natural transformations

RECAP: CARTESIAN CATEGORIES

CARTESIAN PRODUCT



$\rightsquigarrow$ A×B is the resource A in parallel with the resource B

CARTESIAN CATEGORY

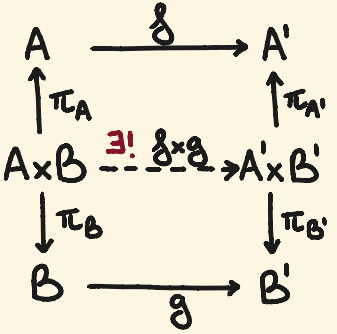
A category with

- a terminal object 1
- for all objects A and B, their cartesian product A×B

RECAP: PROPERTIES OF CARTESIAN PRODUCTS

CARTESIAN PRODUCT FUNCTOR

$$(- \times =) : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$
$$(A, B) \longmapsto A \times B$$
$$(f, g) \longmapsto f \times g$$



$\rightsquigarrow f \times g$ is the process that executes f and g in parallel

ASSOCIATIVITY & UNITALITY

$$\alpha_{A,B,C} : A \times (B \times C) \xrightarrow{\cong} (A \times B) \times C$$
$$\lambda_A : A \xrightarrow{\cong} 1 \times A \quad \text{and} \quad \rho_A : A \xrightarrow{\cong} A \times 1$$

MONOIDAL CATEGORIES : INTRO

Cartesian categories provide an algebra of
sequential $f : A \to B$, $g : B \to C \Rightarrow f ; g : A \to C$
and parallel $f : A \to B$, $f' : A' \to B' \Rightarrow f \times f' : A \times A' \to B \times B'$
composition.

Monoidal categories generalise this aspect of
cartesian categories.
$\rightsquigarrow$ object represent resources
arrows represent processes / computations

MONOIDAL CATEGORIES

STRICT MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, I)$

- $\mathcal{C}$ category
- $(- \otimes =) : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ functor — tensor/ (monoidal product)
- $I \in \mathrm{obj}\,\mathcal{C}$ — (monoidal unit)

such that

- $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ — (associativity)
- $A \otimes I = A$, $I \otimes A = A$ — (unitality)

$\rightsquigarrow$ the objects of $\mathcal{C}$ form a monoid
the arrows of $\mathcal{C}$ form a monoid

MONOIDAL CATEGORIES

MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$

- $\mathcal{C}$ category
- $(- \otimes =) : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ functor — (tensor/monoidal product)
- $I \in \mathrm{obj}\,\mathcal{C}$ — (monoidal unit)

such that

- $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ — (associativity)
- $A \otimes I \cong A$ — (unitality)

  $I \otimes A \cong A$

→ natural isomorphisms + coherence conditions

MONOIDAL CATEGORIES

MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$

- $\mathcal{C}$ category
- $(- \otimes =) : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ functor — tensor/ (monoidal product)
- $I \in \mathrm{Obj}\,\mathcal{C}$ — (monoidal unit)
- natural isomorphisms

$\alpha : - \otimes (= \otimes \equiv) \Longrightarrow (- \otimes =) \otimes \equiv$ — (associator)

$\lambda : id_{\mathcal{C}} \Rightarrow I \otimes id_{\mathcal{C}}$

$\rho : id_{\mathcal{C}} \Rightarrow id_{\mathcal{C}} \otimes I$ — (left and right unitors)

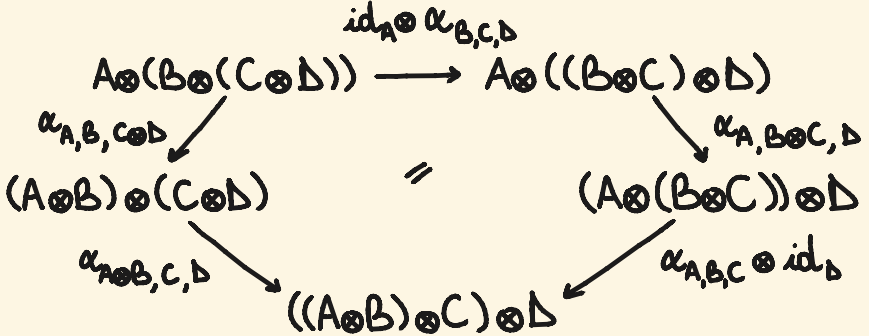
such that

($\pentagon$) ($\triangle$)

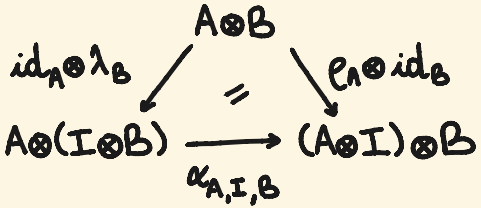
MONOIDAL CATEGORIES

α, λ, ρ such that

(⬠) PENTAGON EQUATION



(△) TRIANGLE EQUATION



MONOIDAL CATEGORIES : EXAMPLES

- Cartesian categories $(\mathcal{C}, \times, 1)$ es. $(\mathbf{Set}, \times, 1)$
  $\rightsquigarrow f \times g$ is f and g
- Cocartesian categories $(\mathcal{C}, +, 0)$ es. $(\mathbf{Set}, +, \varnothing)$
  $\rightsquigarrow f + g$ is f or g
- $(\mathbf{Cat}(\mathcal{C}, \mathcal{C}), \circ, id_{\mathcal{C}})$
- $(\mathbf{Ab}, \otimes, \mathbb{Z})$ of abelian groups with their tensor product

MONOIDAL CATEGORIES : MORE EXAMPLES

- $(\mathbf{Mon}, \otimes, I)$ of monoids (details later)

- $(\mathbf{Mod}, \otimes, I)$ of modules (details later)
  ↓ particular case
- Linear maps between vector or Hilbert spaces
  $(\mathbf{Vect}, \otimes, I)$ and $(\mathbf{Hilb}, \otimes, I)$
  ⇝ PROJECT IDEA: Heunen & Kornell (2022)
  Axioms for the category of Hilbert spaces

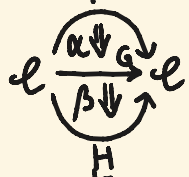
- Many more examples next time
  ↳ $(\mathbf{Par}, \times, 1)$, $(\mathbf{Rel}, \times, 1)$, ...

ENDOFUNCTORS

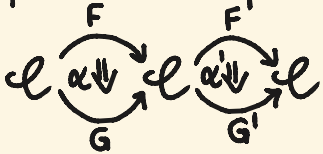
$\mathbf{Cat}(\mathcal{C}, \mathcal{C})$:

objects are functors $F : \mathcal{C} \to \mathcal{C}$

morphisms are natural transformations 

composition is vertical composition 

monoidal product is horizontal composition



monoidal unit is the identity functor

ENDOFUNCTORS

PROPOSITION

$(\mathbf{Lat}(\mathcal{L},\mathcal{L}), \circ, id_{\mathcal{L}})$ is a strict monoidal category.

PROOF

Exercise. □

STRING DIAGRAMS

Convenient syntax for monoidal categories.
Arrows are boxes with possibly many or no inputs/outputs.

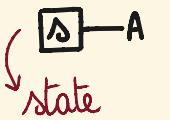
$f : A \to B$



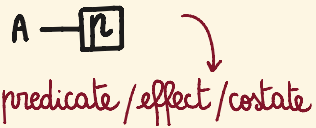
$h : A \otimes B \to C \otimes D \otimes E$



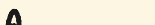
$s : I \to A$



$p : A \to I$



$id_A : A \to A$

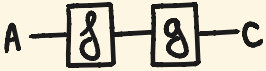


📄 Joyal & Street (1991) The geometry of tensor calculus I

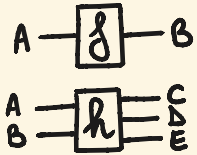
STRING DIAGRAMS

Sequential and parallel composition

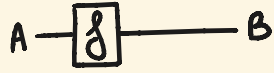
$f ; g : A \to C$

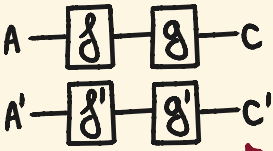


$f \otimes h : A \otimes A \otimes B \to B \otimes C \otimes D \otimes E$



Structural equations become trivial

$f ; id_B =$  $=$  $= f$

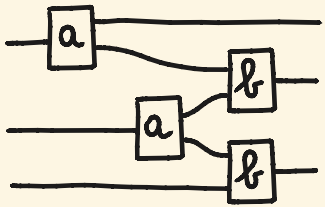
$(f \otimes f') ; (g \otimes g') =$  $= (f ; g) \otimes (f' ; g')$

← interchange law

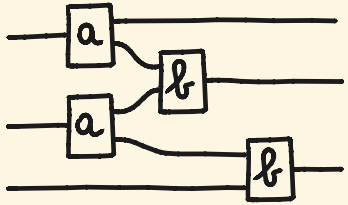
STRING DIAGRAMS ARE NICE

$(a \otimes id \otimes id) ; (id \otimes id \otimes a \otimes id) ; (id \otimes b \otimes b)$

=



=



$= (a \otimes a \otimes id) ; (id \otimes b \otimes id \otimes id) ; (id \otimes id \otimes b)$

STRING DIAGRAMS ARE FORMAL

THEOREM

A well-typed equation between arrows in a monoidal category is a consequence of the axioms iff it holds in string diagrams up to planar isotopy.

⇝ more on this later

SYMMETRIES

In some monoidal categories, resources can be permuted.

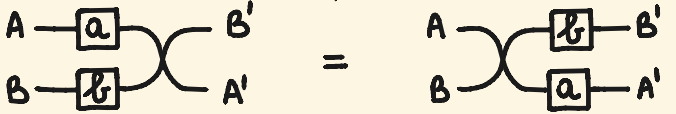
For all objects A and B, there is a symmetry

$$\sigma_{A,B} : A \otimes B \to B \otimes A$$

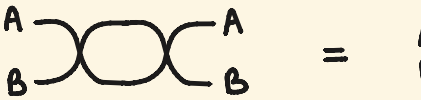


SYMMETRIES

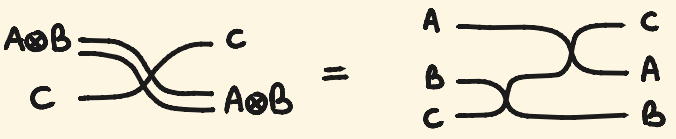
Symmetries satisfy some equations



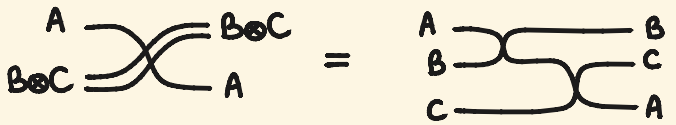
(naturality)



(invertibility)



(⬡ equations)



INTERLUDE : BRAIDINGS

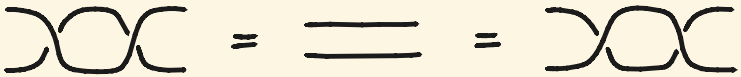
In some monoidal categories, the way of permuting resources matters.

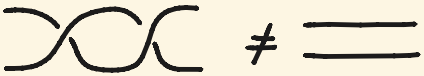
$\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$



$\sigma^{-1}_{A,B} : B \otimes A \longrightarrow A \otimes B$







SYMMETRIC MONOIDAL CATEGORIES

SYMMETRIC MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$

- $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category
- $\sigma : (- \otimes =) \Rightarrow (= \otimes -)$ natural isomorphism

such that

(1) (2)

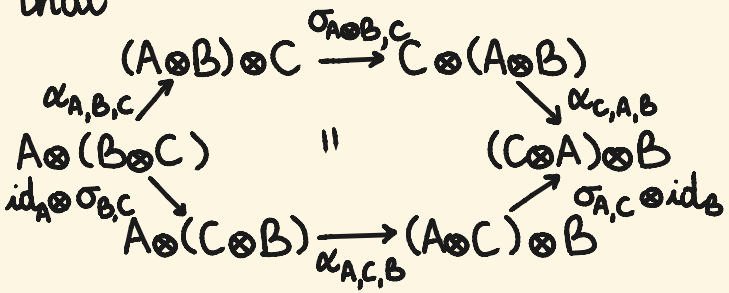
and $\sigma_{B,A} = \sigma^{-1}_{A,B}$.

↝ drop this for braided

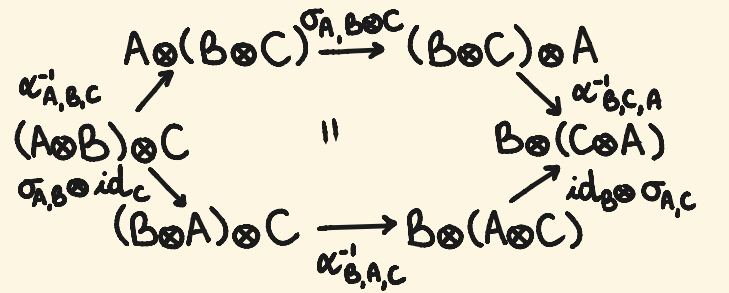
SYMMETRIC MONOIDAL CATEGORIES

σ such that

(1)

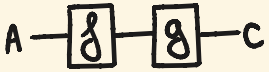


(2)

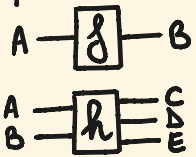


SMCs : SUMMARY

sequential

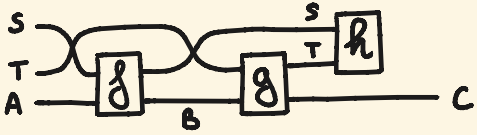


parallel



symmetries





$$(\sigma_{S,T} \otimes id_A);(id_T \otimes g);(\sigma_{T,S} \otimes id_B);(id_S \otimes g);(h \otimes id_C)$$

MONOIDS

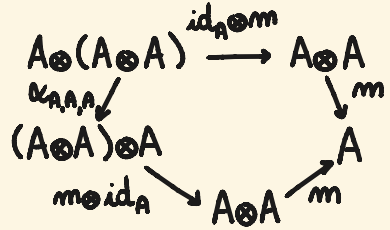
$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.

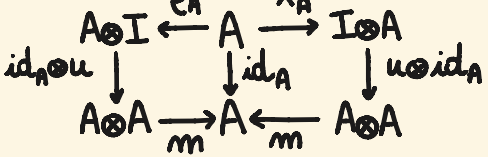
MONOID

(A, m, u)

- $A \in \mathrm{obj}\mathcal{C}$ (carrier)
- $m : A \otimes A \to A$ (multiplication)
- $u : I \to A$ (unit)

that are associative and unital





MONOIDS

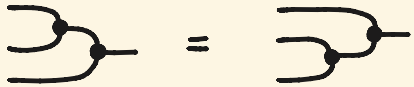
$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.

MONOID

(A, m, u)

- $A \in \mathrm{obj}\mathcal{C}$ (carrier)
-  or  (multiplication)
-  or  (unit)

that are associative and unital





MONOIDS : EXAMPLES

- Monoids in $(\mathcal{S}et, \times, 1)$ are set-theoretic monoids.

ex $(\mathbb{N}, +, 0)$ is a monoid

$(A^*, ::, ())$ is a monoid

- In $(\mathcal{S}et, +, \varnothing)$ there is a canonical monoid on every A

$$[id, id] : A + A \to A \qquad i : \varnothing \to A$$
$$(a, i) \mapsto a$$

- Monoids in $\mathcal{F}at(\mathcal{C}, \mathcal{C})$ are monads (T, μ, η).

- Monoids in the monoidal category $(Ab, \otimes, \mathbb{Z})$ of abelian groups are rings.

COMONOIDS

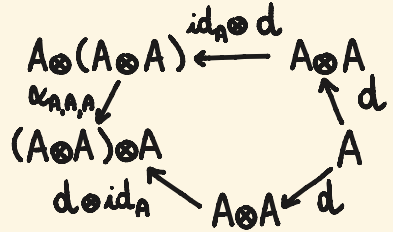
$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.

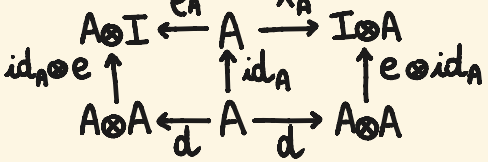
COMONOID $(= \text{monoid in } \mathcal{C}^{op})$

(A, d, e)

- $A \in \text{obj}\mathcal{C}$ (carrier)
- $d : A \to A \otimes A$ (comultiplication)
- $e : A \to I$ (counit)

that are coassociative and counital





COMONOIDS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.

COMONOID (= monoid in $\mathcal{C}^{op}$)

(A, d, e)

- $A \in \mathrm{obj}\,\mathcal{C}$ (carrier)
- 

(comultiplication)
- 

(counit)

that are coassociative and counital





COMONOIDS : EXAMPLES

- In $(\mathcal{S}et, \times, 1)$ there is a canonical comonoid on every A

$$\langle id, id \rangle : A \to A \times A \qquad a \mapsto (a, a)$$

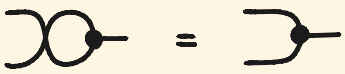
$$! : A \to 1 \qquad a \mapsto *$$

- Comonoids in $\mathcal{C}at(\mathcal{C}, \mathcal{C})$ are comonads (T, δ, ε).

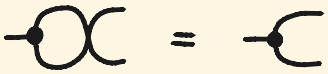
COMMUTATIVITY

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.

a monoid $(A, \curlyvee, \bullet)$ is commutative if



a comonoid $(A, \curlywedge, \bullet)$ is cocommutative if

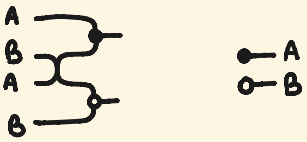


ex $(\mathbb{N}, +, 0)$ $(A, \langle id, id \rangle, !)$

NON-ex monads comonads

COMPOSING (CO)MONOIDS

In a symmetric monoidal category, monoids compose.
For two monoids $(A, \exists, \bullet)$ and $(B, \exists, \circ)$,
there is a monoid on $A \otimes B$



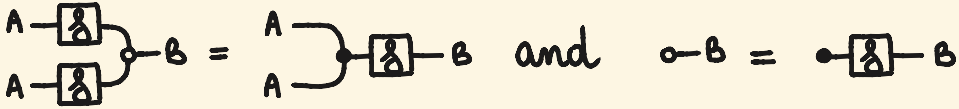
⚠ when the monoidal category is not symmetric,
we need distributive laws.
⇝ PROJECT IDEA: distributive laws of monads & liftings

MORPHISMS OF (CO)MONOIDS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.
$(A, \curlyvee, \bullet)$ and $(B, \curlyvee, \circ)$ monoids.

MONOID MORPHISM

$f : (A, \curlyvee, \bullet) \longrightarrow (B, \curlyvee, \circ)$ is an arrow $f : A \to B$ such that



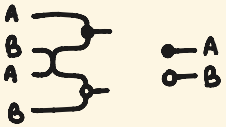
CATEGORY OF MONOIDS

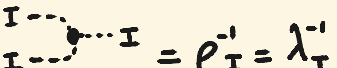
$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.

$\mathrm{Mon}(\mathcal{C})$:

objects are monoids

arrows are morphisms of monoids

monoidal product is composition of monoids 

monoidal unit is the monoid on I 

$= \rho_I^{-1} = \lambda_I^{-1}$

CATEGORY OF MONOIDS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.

PROPOSITION

$\mathrm{Mon}(\mathcal{C})$ is a symmetric monoidal category.

PROOF

Exercise. □

MODULES

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.
$(M, \supset\!\!-, \bullet\!\!-)$ monoid.

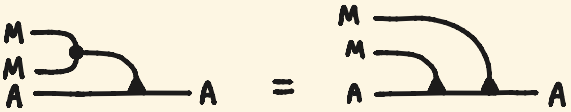
MODULE OVER A MONOID

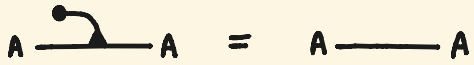
A module (A, a) over a monoid $(M, \supset\!\!-, \bullet\!\!-)$ is

- $A \in \mathrm{obj}\,\mathcal{C}$ (carrier)
- $a : M \otimes A \longrightarrow A$ (action)



such that



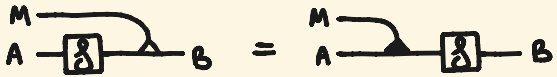


MODULE MORPHISMS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.
(M, μ, η) monoid.
(A, λ) and (B, λ) modules over (M, μ, η).

MODULE MORPHISM

$f : (A, \lambda) \longrightarrow (B, \lambda)$ is
$f : A \to B$ such that



BIALGEBRAS

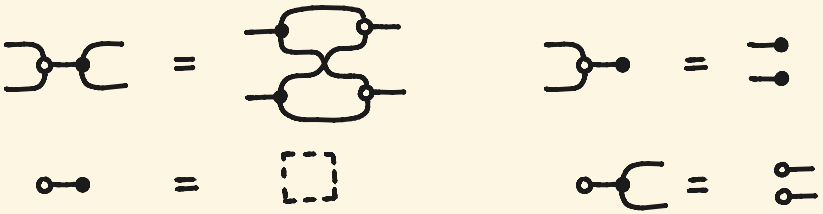
Modules over bialgebras can be tensored.

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.

BIALGEBRA (OR BIMONOID)

$(M, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra if

- (M, μ, η) monoid
- (M, Δ, ε) comonoid
- bialgebra axioms (= the monoid is a comonoid morphism
  = the comonoid is a monoid morphism)



BIALGEBRAS : EXAMPLES

- $(\mathbb{N}, +, 0, \text{copy}, \text{discard})$
- any monoid in Set, with copy and discard
- any monoid in a cartesian category, with copy and discard
- the category $(\mathsf{Ab}, \oplus, 0)$ of abelian groups with their direct product is cartesian (and cocartesian)

PRODUCT OF MODULES

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.
$(M, \triangleright, \circ, \mathcal{C}, \bullet)$ bialgebra.
$(A, \blacktriangleright)$ and $(B, \blacktriangleright)$ modules over $(M, \triangleright, \circ)$.

There is a module on $A \otimes B$ whose action is



NOTE: this is not the tensor product of modules, which requires coequalisers in $\mathcal{C}$.

CATEGORY OF MODULES

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.
$(M, \triangleright, \circ, \multimap, \bullet)$ bialgebra.

PROPOSITION

Modules over $(M, \triangleright, \circ)$ form a symmetric monoidal category $(M\text{-}\mathrm{Mod}, \otimes, I)$.

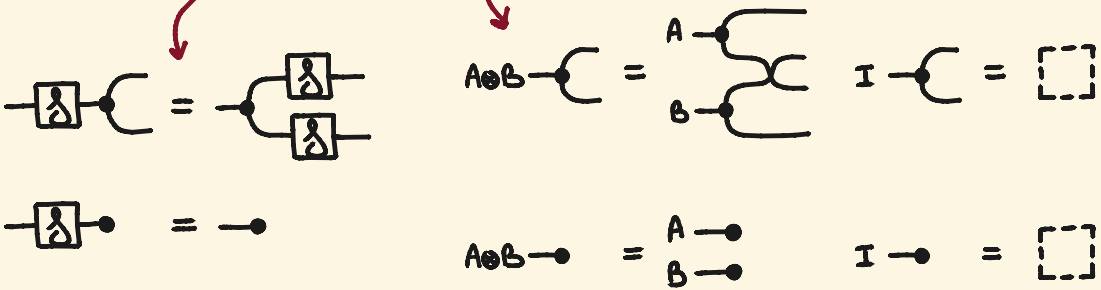
PROOF

Exercise. □

FOX'S THEOREM

THEOREM

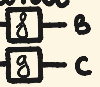
A symmetric monoidal category is cartesian iff every object has a natural and coherent comonoid structure.



Fox (1976) Coalgebras and cartesian categories

FOX'S THEOREM

PROOF SKETCH

(⇐) The maps to the terminal object are $!_A := A \multimap$.
The projection maps are $\pi_A := A, B \multimap A$ and $\pi_B := A, B \multimap B$.
The pairing maps are $\langle f, g \rangle :=$ 

.

(⇒) The copy maps are $A \multimap A, A := \langle 1_A, 1_A \rangle$.
The discard maps are $A \multimap := !_A$.
Naturality follows from the universal properties. □

COPY-DISCARD CATEGORIES

CD-CATEGORY

A copy-discard category is a symmetric monoidal category where every object has a coherent cocommutative comonoid structure.

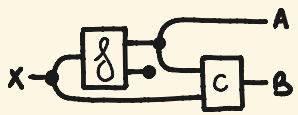
ex $(\mathbf{Par}, \times, 1)$, $(\mathbf{Rel}, \times, 1)$
+ more next time

⇝ PROJECT IDEAS: cartesian restriction categories
(partial) Markov categories
cartesian bicategories of relations

Corradini & Gadducci (1999)

INTERLUDE: DO-NOTATION

Morphisms in copy-discard categories can be specified by do-notation.




```
cond(x) = do
  f(x) → (a, b)
  c(a, x) → b'
  return (a, b')
```



⇝ PROJECT IDEA: premonoidal categories & do-notation
premonoidal vs monoidal categories

PART 1

- recap cartesian categories
- (symmetric) monoidal categories
- string diagrams (& do-notation)
- monoids, comonoids & modules
- Yox's theorem

PART 2

- (symmetric) monoidal functors
- coherence & strictification (overview)
- (symmetric) monoidal natural transformations

MONOIDAL FUNCTORS

$(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \boxtimes, J)$ strict monoidal categories

STRICT MONOIDAL FUNCTOR

$F : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \boxtimes, J)$ is
a functor $F : \mathcal{C} \to \mathcal{D}$ such that

- $J = F(I)$
- $F(A) \boxtimes F(B) = F(A \otimes B)$

$\rightsquigarrow$ strict monoidal functors are functors that strictly preserve the monoidal structure

MONOIDAL FUNCTORS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ monoidal categories

STRONG MONOIDAL FUNCTOR

$(F, e, m) : (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho) \longrightarrow (\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that

- $J \cong F(I)$
- $F(A) \boxtimes F(B) \cong F(A \otimes B)$

↳ natural isomorphisms + coherence conditions

⇝ strong monoidal functors are functors that preserve the monoidal structure up to isomorphism

MONOIDAL FUNCTORS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ monoidal categories

STRONG MONOIDAL FUNCTOR

$(F, e, m) : (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho) \longrightarrow (\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ is

- a functor $F : \mathcal{C} \to \mathcal{D}$
- natural isomorphisms $e : J \longrightarrow F(I)$

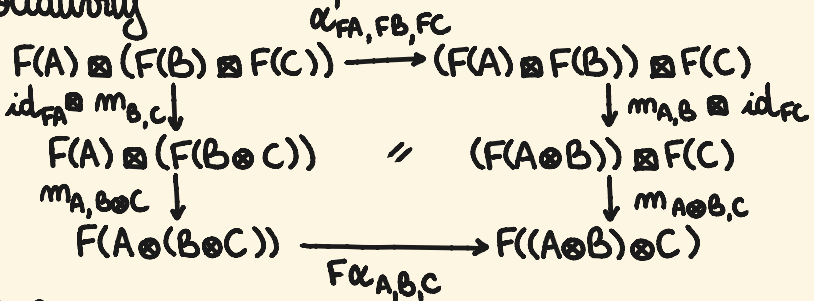
  and $m : F(-) \boxtimes F(=) \longrightarrow F(- \otimes =)$

  that are associative and unital.

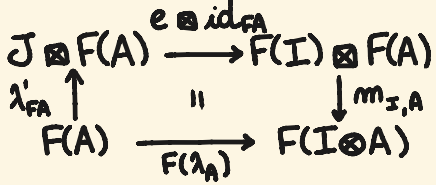
MONOIDAL FUNCTORS

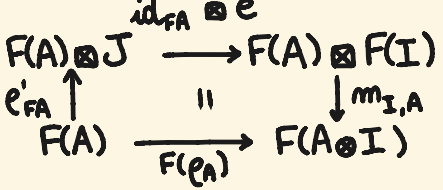
e, m such that

- associativity



- unitality





MONOIDAL FUNCTORS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ monoidal categories

LAX MONOIDAL FUNCTOR

$(F, e, m) : (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho) \longrightarrow (\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ is

- a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$
- natural transformations $e : J \longrightarrow F(I)$

  and $m : F(-) \boxtimes F(=) \longrightarrow F(- \otimes =)$

  that are associative and unital.

$\rightsquigarrow$ lax monoidal functors are functors that laxly preserve the monoidal structure

CATEGORIES OF MONOIDAL CATEGORIES

- StrMonCat of strict monoidal categories and strict monoidal functors

- MonCat of monoidal categories and strong monoidal functors

MONOIDAL CATEGORIES : STRICTIFICATION

MONOIDAL EQUIVALENCE

A monoidal equivalence is a monoidal functor whose underlying functor is an equivalence.

THEOREM (STRICTIFICATION OF MONOIDAL CATEGORIES)

Every monoidal category $\mathcal{C}$ is monoidally equivalent to a strict one $S(\mathcal{C})$.

$\rightsquigarrow$ we can forget about α, λ, ρ

MONOIDAL CATEGORIES : COHERENCE

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category

$\mathcal{C}_s \subseteq \mathcal{C}$ smallest monoidal subcategory of $\mathcal{C}$ containing, for all objects A, B and C, $\alpha_{A,B,C}$, λ_A and ρ_A

THEOREM (COHERENCE FOR MONOIDAL CATEGORIES)

For any two objects X and Y in $\mathcal{C}_s$, the set $\mathcal{C}_s(X,Y)$ either

- is empty, or
- contains exactly one element

$\rightsquigarrow$ any well-typed equation between arrows in $\mathcal{C}_s$ holds

COHERENCE & STRICTIFICATION IN PRACTICE

Take a monoidal category $\mathcal{C}$.

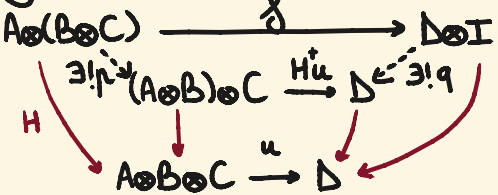
Suppose we have $f, g : X \to Y$ in $\mathcal{C}$
and we want to show $f = g$.

Consider the equivalence $H : \mathcal{C} \to S(\mathcal{C})$.

Consider $u := H(f)$ and $v := H(g)$ in $S(\mathcal{C})$.

If we show $u = v$, we obtain $f = g$ because
$\exists!$ iso $p : X \to H^+H(X)$ and $q : Y \to H^+H(Y)$ by coherence

$\Rightarrow f = p ; H^+u ; q^{-1} = p ; H^+v ; q^{-1} = g$



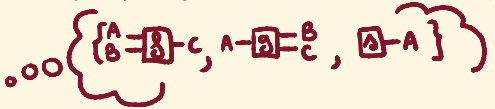
STRING DIAGRAMS ARE FREE



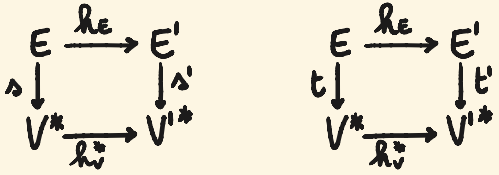
String diagrams are constructed over a monoidal signature

MONOIDAL SIGNATURE

$\Sigma = (E \overset{s,t}{\rightrightarrows} V^*)$



A morphism $h: \Sigma \to \Sigma'$ is $(h_E : E \to E', h_V : V \to V')$ such that



RECAP: STRING DIAGRAMS ARE FORMAL

String diagrams over $U(S(\mathcal{C}))$ are sound and complete for $S(\mathcal{C})$

Equalities in $S(\mathcal{C})$ show equalities in $\mathcal{C}$

$\Rightarrow$ Equalities of string diagrams show equalities in $\mathcal{C}$

SYMMETRIC MONOIDAL FUNCTORS

$(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \boxtimes, J)$ symmetric monoidal categories

SYMMETRIC MONOIDAL FUNCTOR

A monoidal functor $(F, e, m) : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{D}, \boxtimes, J)$ is symmetric if

$$\begin{array}{ccc} F(A) \boxtimes F(B) & \xrightarrow{m_{A,B}} & F(A \otimes B) \\ {\scriptstyle \sigma'_{FA,FB}}\downarrow & = & \downarrow{\scriptstyle F\sigma_{A,B}} \\ F(B) \boxtimes F(A) & \xrightarrow[m_{B,A}]{} & F(B \otimes A) \end{array}$$

COHERENCE & STRICTIFICATION

As for monoidal categories, but we need to take care of permutations: $\sigma_{A,A} \neq id_A \otimes id_A$  .

→ all formal well-typed equations between morphisms constructed by $\alpha, \lambda, \rho, \sigma$ and with the same underlying permutation hold

STRICTIFICATION (SMCs)

Every symmetric monoidal category is symmetric monoidally equivalent to a symmetric strict one.

MONOIDAL NATURAL TRANSFORMATIONS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ monoidal categories

$(F, m, e), (G, m', e') : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \boxtimes, J)$ monoidal functors

MONOIDAL NATURAL TRANSFORMATION

$\beta : (F, m, e) \Rightarrow (G, m', e')$ is

a natural transformation $\beta : F \Rightarrow G$

such that

$$\begin{array}{ccc} F(A) \boxtimes F(B) & \xrightarrow{\beta_A \boxtimes \beta_B} & G(A) \boxtimes G(B) \\ \downarrow{\scriptstyle m_{A,B}} & = & \downarrow{\scriptstyle m'_{A,B}} \\ F(A \otimes B) & \xrightarrow[\beta_{A \otimes B}]{} & G(A \otimes B) \end{array}$$

$$\begin{array}{ccc} & J & \\ {\scriptstyle e}\swarrow & = & \searrow{\scriptstyle e'} \\ F(I) & \xrightarrow[\beta_I]{} & G(I) \end{array}$$

When F, G are symmetric, so is β.