# PLAN FOR TODAY

24/4/2024

## PART 1

- recap : monads
- Kleisli categories
- distributive laws
- Kleisli are free algebras

## PART 2

- monoidal monads
- strength & commutativity

## PART 3

- structure in Par, Rel
- structure in Stoch, SubStoch

# RECAP : MONADS

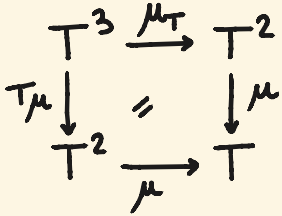
MONAD

$(T, \mu, \eta)$ is

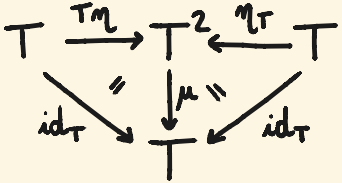
- $T : \mathcal{C} \to \mathcal{C}$ endofunctor
- $\mu : T \circ T \to T$ natural transformation (multiplication)
- $\eta : id_{\mathcal{C}} \to T$ natural transformation (unit)

such that

(associativity)



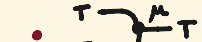

(unitality)



# RECAP : MONADS

MONAD (= monoid in endofunctors)

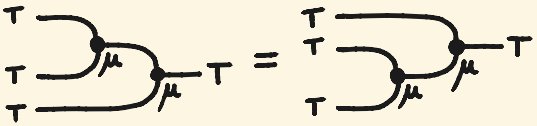
$(T, \mu, \eta)$ is

- $T \in (\mathrm{Cat}(\mathcal{C}, \mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}})$
- 

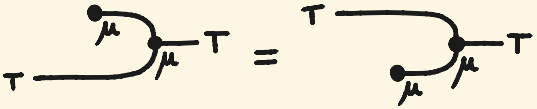
(multiplication)
- 

(unit)

such that



(associativity)



(unitality)

# MONADS: EXAMPLES

## Exception & maybe monads

$\mathcal{C}$ category with coproducts

$E \in \mathrm{Obj}\,\mathcal{C}$

Ex: $\mathcal{C} \longrightarrow \mathcal{C}$, $A \longmapsto A+E$ is a monad with

multiplication and unit given by injections:

$\mu_A : A+E+E \longrightarrow A+E$ and $\eta_A : A \longrightarrow A+E$

$\mu_A := id_A + [i_E, i_E]$ $\qquad$ $\eta_A := i_A$

$\rightsquigarrow E = 1$ (terminal object) $\Rightarrow$ maybe monad

# MONADS: EXAMPLES

## Powerset monad

$$\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}, \quad A \longmapsto \{X \subseteq A\}$$

$$\mu_A : \mathcal{P}^2 A \longrightarrow \mathcal{P} A, \quad X \longmapsto \bigcup_{Y \in X} Y$$

$$\eta_A : A \longrightarrow \mathcal{P}(A), \quad a \longmapsto \{a\}$$

## Distribution monad (finitary)

$$\mathcal{D} : \mathbf{Set} \longrightarrow \mathbf{Set}, \quad A \longmapsto \{\sigma : A \to [0,1] \mid \operatorname{supp}\sigma \text{ finite} \wedge \sum_a \sigma(a) = 1\}$$

$$\mu_A : \mathcal{D}^2 A \longrightarrow \mathcal{D} A, \quad \varphi \longmapsto \sum_\sigma \varphi(\sigma) \cdot \sigma(-)$$

$$\eta_A : A \longrightarrow \mathcal{D} A, \quad a \longmapsto \delta_a$$

↳ with $\leq 1$ we get subdistributions

# MONADS: EXAMPLES

## State monad

$(\mathcal{C}, \times, 1, (-)^{(=)})$ cartesian closed category

$S \in \mathrm{obj}\mathcal{C}$ $\rightsquigarrow$ global state

$$St : \mathcal{C} \longrightarrow \mathcal{C}$$
$$A \longmapsto (S \times A)^S$$

$$\mu_A : (S \times (S \times A)^S)^S \longrightarrow (S \times A)^S$$
$$f : S \to S \times (S \to S \times A) \longmapsto \pi_2(f)(\pi_1(f)(-)) = \lambda s.\, f_1(f_2(s))$$

$s \mapsto g(t)$ for $f(s) = (t, g)$

$$\eta_A : A \longrightarrow (S \times A)^S$$
$$a \longmapsto \langle id_S, a \rangle = \lambda s.\, (s, a)$$

PROJECT IDEA: promonads for state on monoidal non-cartesian non-closed categories.

# KLEISLI CATEGORIES: MOTIVATION

Monads add effects: $TA$ = "T-effects on A"

ex $A+1$ adds failure to the set of resources

$\mathcal{D}(A)$ gives probability distributions over resources

We would like to consider computations $A \to T(B)$

= "computations with T-effects"

ex $f : A \to B+1$ are partial functions

$f : A \to \mathcal{D}(B)$ are stochastic maps

$\rightsquigarrow f(b|a) := f(a)(b)$ = "probability of $b$ given $a$"

# KLEISLI CATEGORIES

$(T, \mu, \eta)$ monad on $\mathcal{C}$

## KLEISLI CATEGORY

$\mathrm{Kl}(T, \mu, \eta)$ category where (usually just $\mathrm{Kl}T$)

- objects are objects of $\mathcal{C}$
- morphisms $A \nrightarrow B$ are $A \to TB$ in $\mathcal{C}$
- composition $A \overset{f}{\nrightarrow} B \overset{g}{\nrightarrow} C := A \overset{f}{\to} TB \overset{Tg}{\to} T^2C \overset{\mu_C}{\to} TC$
- identities $A \overset{1_A}{\nrightarrow} A := A \overset{\eta_A}{\to} TA$

# KLEISLI CATEGORIES

$(T, \mu, \eta)$ monad on $\mathcal{C}$

**PROPOSITION**

$\mathrm{Kl}T$ is a category.

**PROOF**

Exercise. Use associativity & unitality of $(T, \mu, \eta)$. □

**PROJECT IDEA**: comonads & coKleisli categories

↳ ex streams on cartesian categories

# INTERLUDE: MONADS IN KLEISLI FORM

MONAD IN KLEISLI FORM

$(T, (-)^T, \eta)$ is

- a function $T : \mathrm{obj}\mathcal{C} \to \mathrm{obj}\mathcal{C}$
- an operation $(-)^T : \mathcal{C}(A, TB) \to \mathcal{C}(TA, TB)$ — Kleisli extension
- a family $\eta_A : A \to TA$ for $A \in \mathrm{obj}\mathcal{C}$

such that

1. $(\eta_A)^T = id_{TA}$
2. $\eta_A ; f^T = f$
3. $f^T ; g^T = (f ; g^T)^T$

# INTERLUDE: MONADS IN KLEISLI FORM

**PROPOSITION**

Monads are the same thing as monads in Kleisli form.

**PROOF**

Exercise. □

**KLEISLI CATEGORY**

$\mathrm{Kl}(T, (-)^T, \eta)$ category where

- objects are objects of $\mathcal{C}$
- morphisms $A \nrightarrow B$ are $A \to TB$ in $\mathcal{C}$
- composition $A \overset{f}{\nrightarrow} B \overset{g}{\nrightarrow} C := A \overset{f}{\to} TB \overset{g^T}{\to} TC$
- identities $A \overset{1_A}{\nrightarrow} A := A \overset{\eta_A}{\to} TA$

# EXAMPLES

- $\mathrm{Kl}(-+1) \simeq \mathbf{Par}$ category of sets and partial functions
- $\mathrm{Kl}(\mathcal{D}) \simeq \mathbf{Stoch}$ category of sets and stochastic maps

  $\rightsquigarrow$ $\mathcal{G} : \mathbf{Meas} \to \mathbf{Meas}$ Giry monad and $\mathrm{Kl}\mathcal{G}$ for 'continuous' stochastic maps
- $\mathrm{Kl}(\mathcal{P}) \simeq \mathbf{Rel}$ category of sets and relations
- $\mathrm{Kl}(\mathbf{St})$ category of sets and stateful functions

  $\rightsquigarrow$ $A \nrightarrow B \simeq S \times A \to S \times B$

# COMPOSING MONADS

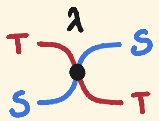
Subdistributions are distributions with failure:

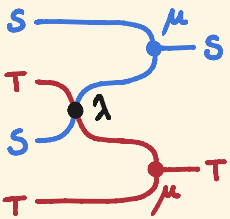
$$\mathcal{D}(A+1) \simeq \mathcal{D}_{\leq 1}(A) := \{\sigma : A \to [0,1] \mid \mathrm{supp}\,\sigma \text{ finite} \wedge \textstyle\sum_a \sigma(a) \leq 1\}$$

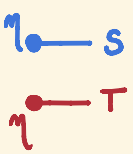
$(\mathcal{L}at(\mathcal{C},\mathcal{C}), \circ, id_{\mathcal{C}})$ is not symmetric

⇒ we cannot compose monads in general

⇒ we need a well-behaved 'swap' in $\mathcal{L}at(\mathcal{C},\mathcal{C})$



# DISTRIBUTIVE LAWS

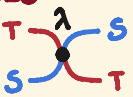
$(\mathcal{C}, \otimes, I)$ monoidal category

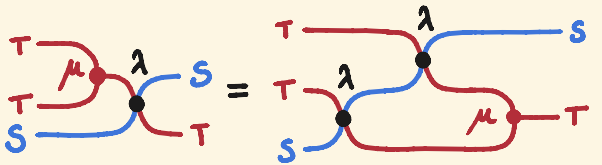
$(T, \supset, \bullet)$ and $(S, \supset, \bullet)$ monoids

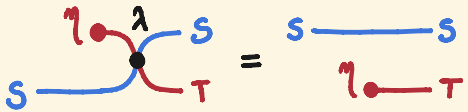
## DISTRIBUTIVE LAW OF MONOIDS

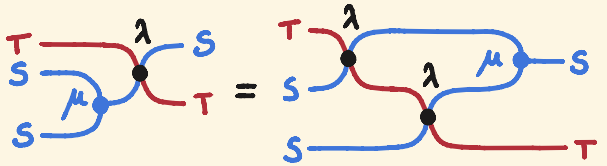
$\lambda : T \otimes S \to S \otimes T$

such that



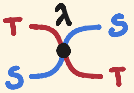



# DISTRIBUTIVE LAWS

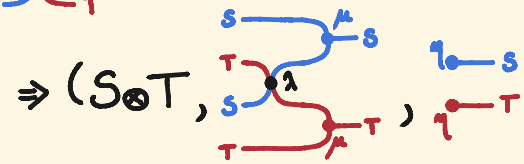
$(\mathcal{C}, \otimes, I)$ monoidal category

$(T, \mu, \eta)$ and $(S, \mu, \eta)$ monoids

**PROPOSITION**



distributive law



is a monoid

**PROOF**

Exercise. □

**PROJECT IDEA**: Weak distributive laws

# DISTRIBUTIVE LAWS: EXAMPLES

- $\lambda_A : \mathcal{D}(A) + 1 \longrightarrow \mathcal{D}(A+1)$ ⇝ subdistributions
  $\lambda_A(\sigma) := \sigma \qquad \lambda_A(\bot) := \delta_\bot$

- $\lambda_A : \mathcal{P}_{NE}(A+1) \longrightarrow \mathcal{P}_{NE}(A) + 1 \simeq \mathcal{P}$ ⇝ relations
  $\lambda_A(X) := X \setminus \{\bot\} \qquad \lambda_A(\{\bot\}) := \bot$

- $\lambda_A : \mathcal{P}_{NE}(A) + 1 \longrightarrow \mathcal{P}_{NE}(A+1)$ ⇝ 'relations' with explicit failure
  $\lambda_A(X) := X \qquad \lambda_A(\bot) := \{\bot\}$

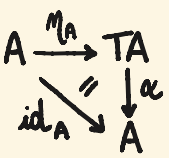
PROJECT IDEA: Zwart & Marsden (2022)
No-go theorems for distributive laws

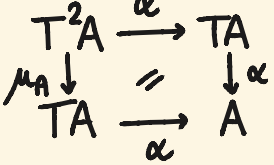
# RECALL: FREE ALGEBRAS

$(T, \mu, \eta)$ monad on $\mathcal{C}$

**ALGEBRA FOR A MONAD**

$(A, \alpha : TA \to A)$ such that




**FREE ALGEBRA**

$(TA, \mu_A : T^2A \to A)$

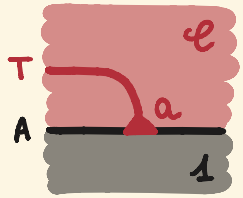
**PROPOSITION**

Free algebras are algebras.

**PROOF**

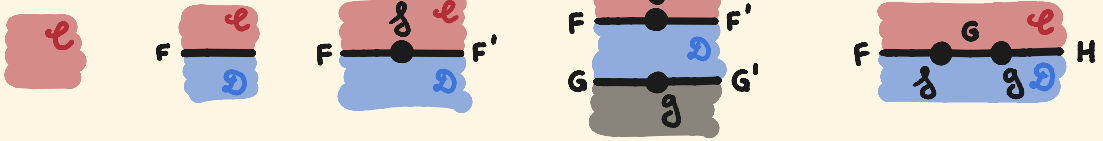
Exercise. □

# INTERLUDE: COLOURING REGIONS




specify morphisms in bicategories
(PROJECT IDEA)

0-cells 1-cells 2-cells parallel sequential




$F : \mathcal{C} \to \mathcal{D}$ $f : F \Rightarrow F'$ $f \otimes g : F \otimes G \to F' \otimes G'$ $f ; g : F \to H$

ex Cat with categories, functors, natural transformations

# ALGEBRAS ARE MODULES

$(T, \eta, -\!\!\bullet)$ monad on $\mathcal{C}$

## ALGEBRA FOR A MONAD

$(A, \alpha : TA \to A)$ such that




$\rightsquigarrow$ objects of $\mathcal{C}$ are functors $A : 1 \to \mathcal{C}$
arrows $A \to B$ are natural transformations $f : A \Rightarrow B$

## FREE ALGEBRA

$(TA, \mu : TTA \to TA)$



# FREE ALGEBRAS ARE KLEISLI

$T : \mathcal{C} \to \mathcal{C}$ monad

## CATEGORIES OF ALGEBRAS

- $\mathrm{Alg}_T$ of algebras and their morphisms
- $\mathrm{FAlg}_T \subseteq \mathrm{Alg}_T$ of free algebras and their morphisms
  (↑ full)

## THEOREM

There's an equivalence of categories

$$\mathrm{FAlg}_T \simeq \mathrm{Kl}(T)$$

# FREE ALGEBRAS ARE KLEISLI

PROOF SKETCH

Define the candidate equivalence $F : Kl(T) \to FAlg_T$:

- on objects $F(A) := (TA, \mu_A)$
- on morphisms $f : A \nrightarrow B \equiv f : A \to TB$
  as $F(f) := Tf ; \mu_B = f^\dagger : TA \to TB$

We need to show:

1. F is well-defined (i.e. $F(f)$ is an algebra morph)
2. F is a functor
3. F is an equivalence (e.g. F is full, faithful and essentially surjective on objects)

Details as exercise. □

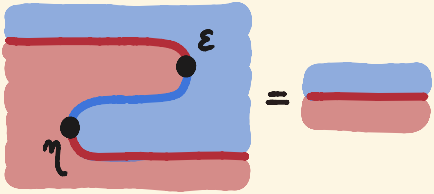
# RECALL: ADJUNCTIONS & MONADS

ADJUNCTION

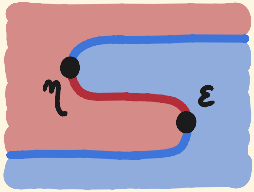
$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ is 

and 

such that



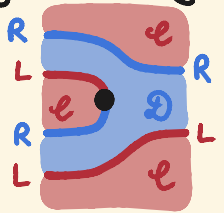
=

and


=


THEOREM (PAIR OF PANTS MONADS)

Every adjunction gives a monad.

$(R \circ L$ , 

, 

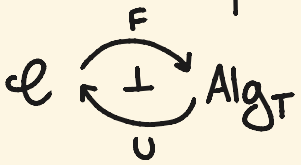
$)$

# RECALL: ADJUNCTIONS & MONADS

$T : \mathcal{C} \to \mathcal{C}$ monad

**THEOREM**

Every monad arises from an adjunction.




$F(A) := (TA, \mu_A)$ $\qquad$ $F(f) := T(f)$

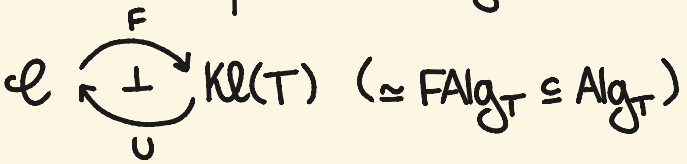
$U(A, \alpha) := A$ $\qquad$ $U(f) := f$

# KLEISLI ADJUNCTION

$T : \mathcal{C} \to \mathcal{C}$ monad

**THEOREM**

Every monad arises from an adjunction.




**PROOF SKETCH**

$F(A) := A$ $\qquad F(f) := f ; \eta_B$

$U(A) := TA$ $\qquad U(f) := Tf ; \mu_B = f^\dagger$

Details as exercise. □

# ADJUNCTIONS FOR A MONAD




PROJECT IDEAS: Beck's monadicity theorem
category of adjunction-resolutions of a monad

PART 1

- recap : monads
- Kleisli categories
- distributive laws
- Kleisli are free algebras

PART 2

- monoidal monads
- strength & commutativity

PART 3

- structure in Par, Rel
- structure in Stoch, SubStoch

# MONOIDAL KLEISLI CATEGORIES

$T : \mathcal{C} \to \mathcal{C}$ monad

$\mathrm{Kl}T$ $\rightsquigarrow$ category of computations with $T$-effects

Q: can we compose these computations in parallel ?
= is $\mathrm{Kl}T$ monoidal ?

$$f \boxtimes f' := A \otimes A' \xrightarrow{f \otimes f'} TB \otimes TB' \xrightarrow{?} T(B \otimes B')$$

A: YES, if
$\begin{cases} (\mathcal{C}, \otimes, I) \text{ monoidal category} \\ (T, \mu, \eta) \text{ monoidal monad} \end{cases}$ (a.k.a. commutative monad)

# MONOIDAL MONADS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category

## MONOIDAL MONAD

Monoid in $\mathrm{MonCat}(\mathcal{C}, \mathcal{C})$ (with lax monoidal functors).

# MONOIDAL MONADS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category

## MONOIDAL MONAD

Monoid in $\mathbf{MonCat}(\mathcal{C}, \mathcal{C})$:

- $(T, m, e)$ lax monoidal endofunctor on $(\mathcal{C}, \otimes, I)$
- $\begin{cases} \mu : T^2 \Rightarrow T \\ \eta : id_{\mathcal{C}} \Rightarrow T \end{cases}$ monoidal natural transformations satisfying associativity and unitality.

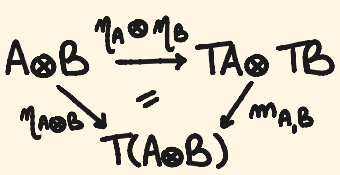
# MONOIDAL MONADS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category

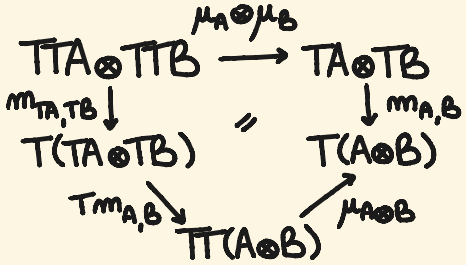
**MONOIDAL MONAD (USEFUL DEFINITION)**

$(T, \mu, \eta, m)$ is

- $(T, \mu, \eta)$ monad on $\mathcal{C}$
- $m : T(-) \otimes T(=) \Rightarrow T(- \otimes =)$ natural transformation satisfying




and



# MONOIDAL KLEISLI CATEGORIES

**THEOREM**

The Kleisli category of a monoidal monad is monoidal.

**PROOF SKETCH**

$T$ monoidal monad on $(\mathcal{C}, \otimes, I)$.

Define a candidate monoidal structure on $\mathcal{K}\ell T$:

- monoidal unit is $I$
- monoidal product on objects is $A \boxtimes B := A \otimes B$
- monoidal product on arrows is

$$f \boxtimes g' := A \otimes A' \xrightarrow{f \otimes g'} TB \otimes TB' \xrightarrow{m_{B,B'}} T(B \otimes B')$$

Details as exercise. □

# EXAMPLES

- Exceptions & maybe

  $(E, \cdot, u)$ monoid $\rightsquigarrow (1, !, !)$ is always a monoid

  $$m_{A,B} : (A+E) \times (B+E) \longrightarrow A \times B + A \times E + E \times B + E \times E \longrightarrow A \times B + E$$

- Powerset

  $$m_{A,B} : \mathcal{P}A \times \mathcal{P}B \longrightarrow \mathcal{P}(A \times B), \quad (X, Y) \longmapsto X \times Y$$

- Distributions

  $$m_{A,B} : \mathcal{D}A \times \mathcal{D}B \longrightarrow \mathcal{D}(A \times B), \quad (\sigma, \tau) \longmapsto \sigma \cdot \tau$$

# EXAMPLES

- NON-EXAMPLE: state monads

$$(S \times A)^S \times (S \times B)^S \xrightarrow{?} (S \times A \times B)^S$$ (details later)

- every monad is monoidal wrt coproducts

**PROPOSITION**

$$\begin{cases} (T, \mu, \eta) \text{ monad on } \mathcal{C} \\ \mathcal{C} \text{ has coproducts} \end{cases}$$

$\Rightarrow T$ is monoidal wrt $(\mathcal{C}, +, 0)$

**PROOF SKETCH**

$m_{A,B} : TA + TB \xrightarrow{[Ti_A, Ti_B]} T(A+B)$ . Details as exercise. □

# STRONG & COMMUTATIVE MONADS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category

## STRONG MONAD

$(T, \mu, \eta, s^l, s^r)$ is

- $(T, \mu, \eta)$ monad on $\mathcal{C}$
- $\begin{cases} s^l : (-) \otimes T(=) \Rightarrow T(- \otimes =) \\ s^r : T(-) \otimes (=) \Rightarrow T(- \otimes =) \end{cases}$ natural transformations

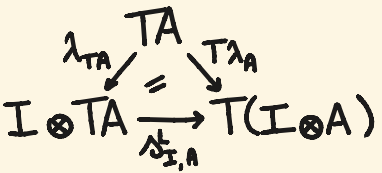
that play well with unitors, associator, monad multiplication and unit.

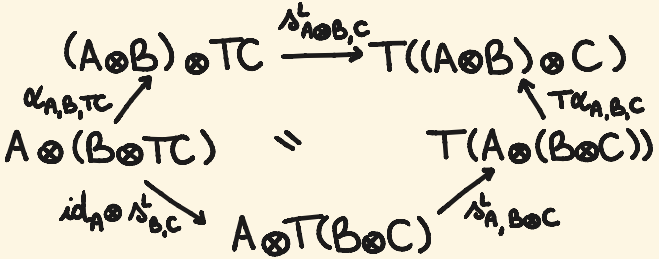
When $\mathcal{C}$ is symmetric, $s^l$ is enough:

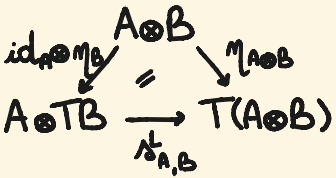
$$s^r_{A,B} := TA \otimes B \xrightarrow{\sigma} B \otimes TA \xrightarrow{s^l} T(B \otimes A) \xrightarrow{T\sigma} T(A \otimes B)$$

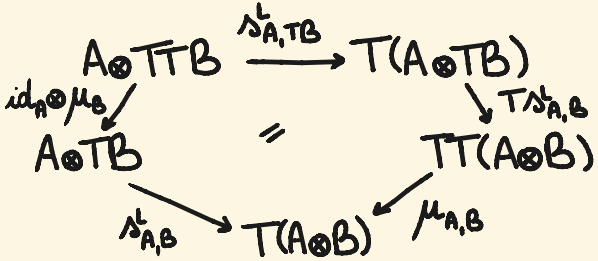
# STRONG & COMMUTATIVE MONADS

$st^L$, $st^R$ such that




+ similar ones for $st^R$

# STRONG & COMMUTATIVE MONADS

**PROPOSITION**

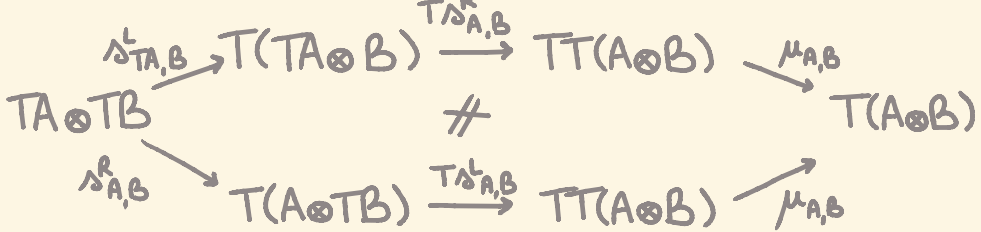
Every monad on $(\mathbf{Set}, \times, 1)$ is strong.

$$st^{L}_{A,B} := A \times TB \xrightarrow{u \times id} (B \to A \times B) \times TB \xrightarrow{ev} T(A \times B)$$

$$(a, b) \longmapsto (\langle a\ id_B \rangle\ b) \longmapsto T(\langle a, id_B \rangle)(b)$$

**THEOREM**

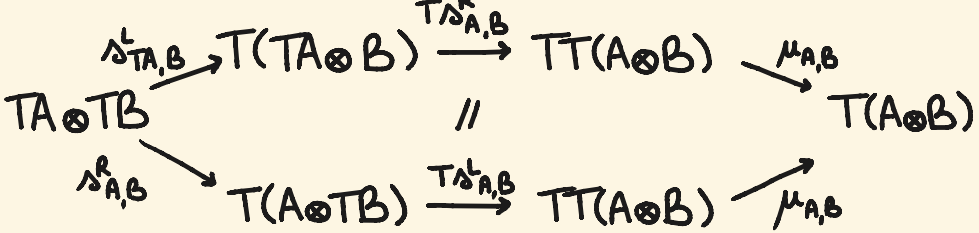
For a strong monad $T$, $Kl(T)$ is premonoidal.



# STRONG & COMMUTATIVE MONADS

## COMMUTATIVE MONAD

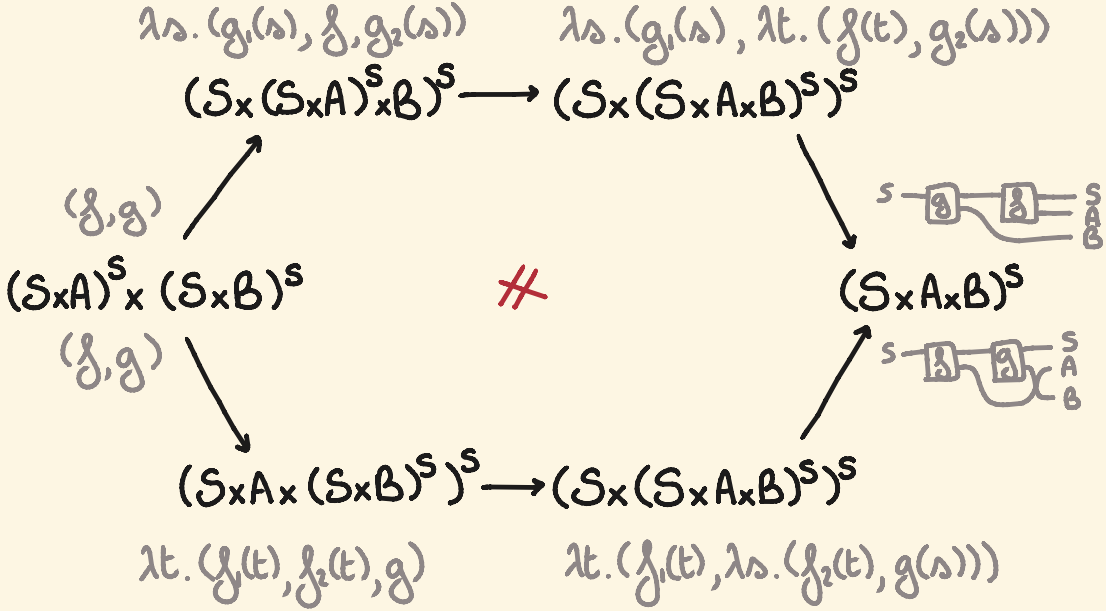
$(T, \mu, \eta, s^L, s^R)$ strong monad such that




## THEOREM

Commutative monads are the same thing as monoidal monads.

# STATE MONADS : A NON-EXAMPLE



PART 1

- recap : monads
- Kleisli categories
- distributive laws
- Kleisli are free algebras

PART 2

- monoidal monads
- strength & commutativity

PART 3

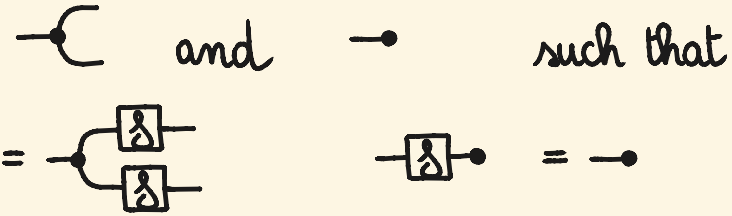
- structure in Par, Rel
- structure in Stoch, SubStoch

# RECALL: FOX & STRUCTURE IN SET

## FOX'S THEOREM

$(\mathcal{C}, \otimes, I)$ cartesian $\Leftrightarrow$ every object has a coherent natural comonoid structure.

- $(\mathbf{Set}, \times, 1)$ is cartesian
  $\Rightarrow$ we have copy and discard

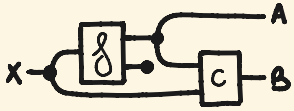


# STRUCTURE IN KLEISLI CATEGORIES

$(T, \mu, \eta)$ monad on $\mathcal{S}et$

$i : \mathcal{S}et \hookrightarrow Kl_T$

$\Rightarrow Kl_T$ is a copy-discard category

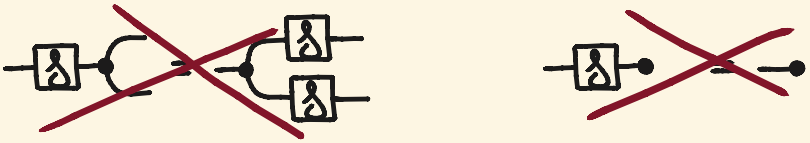



```
cond(x) = do
  g(x) → (a, b)
  c(a, x) → b'
  return (a, b')
```

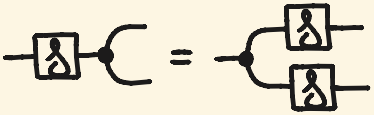
$\rightsquigarrow$ variables can be copied and discarded

# DETERMINISTIC & TOTAL ARROWS

In general, in KlT

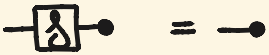


Copiable arrows are deterministic



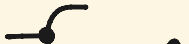
Discardable arrows are total

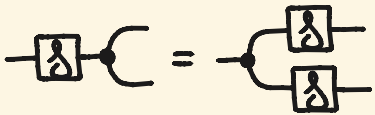
(causal in quantum)



# PARTIAL FUNCTIONS

$\mathbf{Par} \simeq \mathbf{Kl}(-+1)$

 and  such that



There's also the equality check

$$\supset : A \times A \longrightarrow A+1$$
$$(a,a') \longmapsto \begin{cases} a & \text{if } a=a' \\ \bot & \text{if } a \neq a' \end{cases}$$

# AXIOMS FOR PARTIAL FUNCTIONS

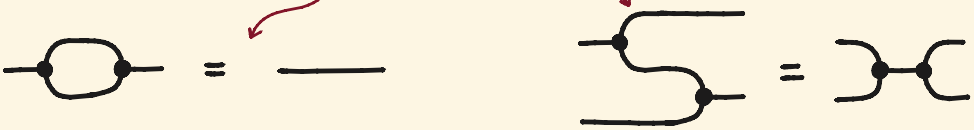


cocommutative comonoid



associative

satisfying the special Frobenius equations

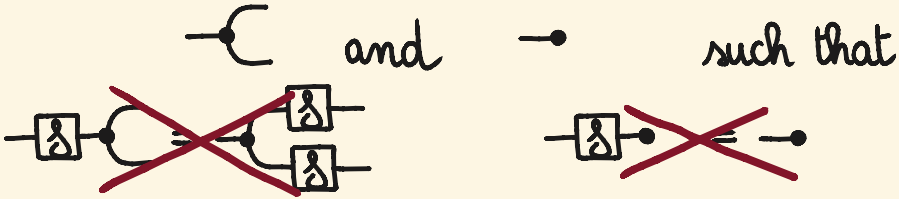


PROJECT IDEA: discrete cartesian restriction categories

Cockett & Lack (2002-2007)
Restriction categories I-III

# RELATIONS

$\mathbf{Rel} \simeq \mathbf{Kl}\,\mathcal{P}$

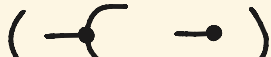



There's also a monoid $(\supset, \bullet)$

$$\supset : A \times A \longrightarrow \mathcal{P}(A), \quad (a,a') \longmapsto \begin{cases} \{a\} & \text{if } a = a' \\ \varnothing & \text{if } a \neq a' \end{cases}$$

$$\bullet : 1 \longrightarrow \mathcal{P}(A), \quad * \longmapsto A$$

# AXIOMS FOR RELATIONS

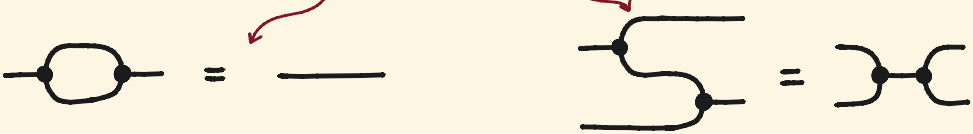


cocommutative comonoid



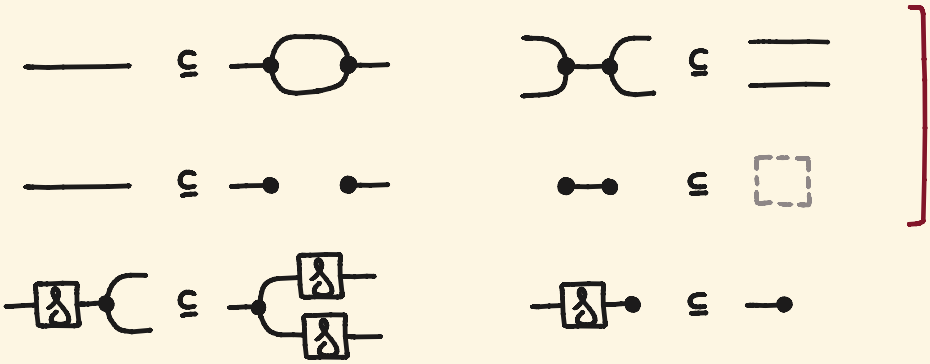
commutative monoid

satisfying the special Frobenius equations



# AXIOMS FOR RELATIONS

comonoid ⊣ monoid

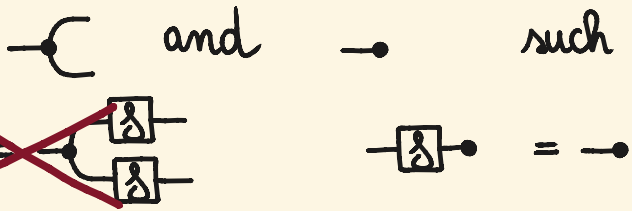


PROJECT IDEAS: Cartesian bicategories of relations

- Carboni & Walters (1987)
- a lot of work by Bonchi & coauthors

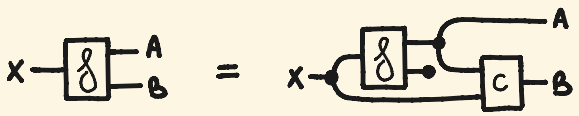
# STOCHASTIC FUNCTIONS

Stoch ≃ KlD



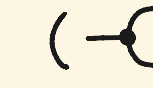

There are conditionals :

$\forall\, f : X \to A \otimes B \quad \exists\, c : A \otimes X \to B \quad$ st




$$f(a,b|x) = c(b|a,x) \cdot \sum_{b' \in B} f(a,b'|x)$$

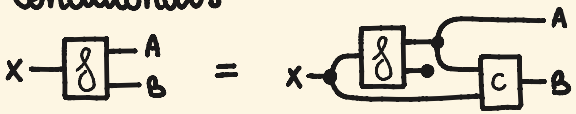
# AXIOMS FOR STOCHASTIC FUNCTIONS


cocommutative comonoid

arrows are total



conditionals




PROJECT IDEAS: Markov categories — Fritz (2020)

Quasi-Borel spaces & lazyPPL

Heunen, Kammar, Staton, Yang (2017)

# PARTIAL STOCHASTIC FUNCTIONS

$\mathbf{SubStoch} \simeq \mathbf{Kl}\,\mathcal{D}(-+1)$

$f : X \nrightarrow A$ gives

$\begin{cases} f(a|x) = \text{"probability of } a \text{ given } x\text{"} \\ f(\perp|x) = \text{"probability of failure given } x\text{"} \end{cases}$

$\rightsquigarrow$ inherit structure from Par and Stoch

# AXIOMS FOR PARTIAL STOCHASTIC FUNCTIONS

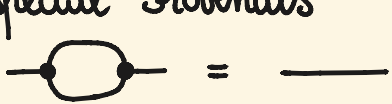


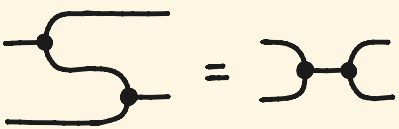
cocommutative comonoid



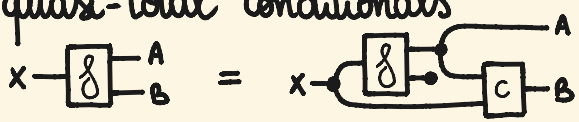
associative

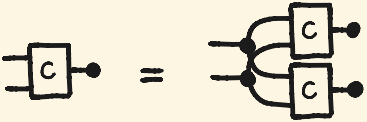
special Frobenius




quasi-total conditionals



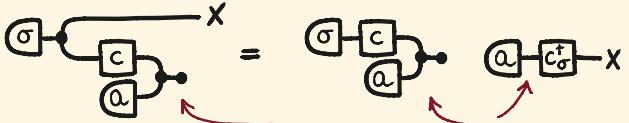




PROJECT IDEA: Partial Markov categories

EDL & Román (2023)

# SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^\dagger_\sigma$ evaluated on $a$.

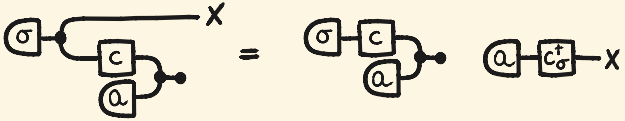



$$P(X=x \mid A=a) = \frac{P(A=a \mid X=x) \cdot P(X=x)}{\sum_{y \in X} P(A=a \mid X=y) \cdot P(X=y)}$$

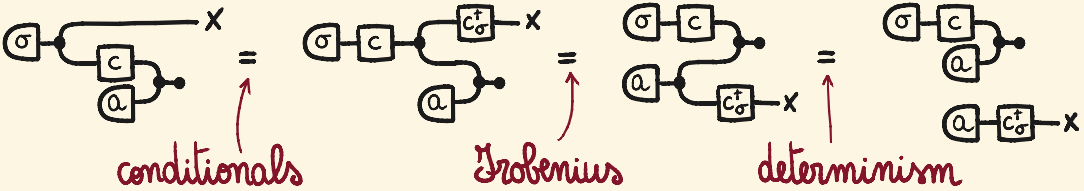
↘ classical formula for Bayes theorem

# SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^\dagger_\sigma$ evaluated on $a$.




PROOF

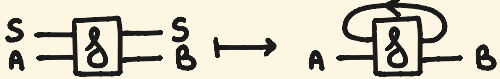



□

# BONUS

# FEEDBACK LOOPS

A feedback monoidal category is
a monoidal category $(\mathcal{C}, \otimes, I)$
with an operation $\mathrm{Fbk} : \mathcal{C}(S \otimes A, S \otimes B) \to \mathcal{C}(A, B)$




+ axioms

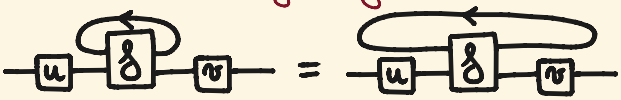
PROJECT IDEAS: feedback, Mealy machines & trace semantics

- Katis, Sabadini, Walters (1997, 2000, 2002)
- Bonchi, Piedeleu, Sobociński, Zanasi (2014, 2015, 2017)
- ESL, de Felice, Román (2022)

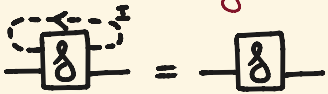
# FEEDBACK LOOPS

FBK such that

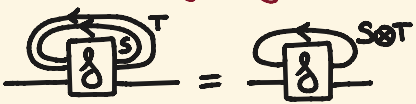
(tightening)



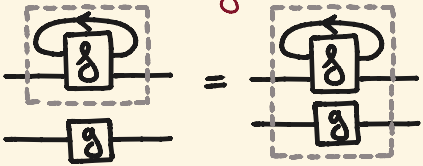

(vanishing)




(joining)




(strength)




(sliding)

