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EMMY NOETHER AND ALGEBRAIC TOPOLOGY

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In 1925, Heinz Hopf (1894–1971), who was then starting the brilliant career which made him one of the key figures in the development of Algebraic Topology in our century, decided to spend a year at Göttingen before taking his 'habilitation' to become a 'Privadozent' in Berlin. It was there that he met the Russian topologist P. Alexandroff, with whom he struck a lifelong friendship, and that both fell under the influence of Emmy Noether [2].

She was then in the process of entirely reshaping algebra in what became known a few years later, through van der Waerden's book, as "Modern Algebra', systematically giving pre-eminence to concepts over computations. Concerning in particular linear algebra, she was liberating it from the plague of matrices and determinants from which it had been suffering for a century, replacing these cumbersome and geometrically meaningless tools by the intrinsic ideas of modules and homomorphisms.

Since the pioneering work of Poincaré in 1900, one could attach to a compact space provided with a 'triangulation' a system of integers which had been shown to depend only on the topology of the space, and not on its triangulation [1].

A (simplicial) triangulation of a compact space X is a finite partition of X into *p-simplices* e_p^i with $0 \leq p \leq n$; the closure in X of a p-simplex e_p^i is homeomorphic with a closed euclidian simplex of dimension p; e_p^i is the image of the interior of that euclidian simplex, and the images of its facets (of each dimension $\leq p-1$) constitute a partition of the boundary of e_p^i in X, and are themselves simplices of the triangulation; finally each simplex is oriented by a choice of the order of its vertices. To each pair (e_p^i, e_{p-1}^j) of a p-simplex and a $(p-1)$-simplex is attached a number (ε_p^{ij}) equal to 0 if e_{p-1}^j is not a face of e_p^i, to $+1$ (resp. -1) if it is such a face and its orientation coincides with (resp. is opposite to) the orientation induced on that face from the one of e_p^i. Let α_p be the number of p-simplices in the triangulation; Poincaré considered the $(\alpha_p \times \alpha_{p-1})$-matrix $\mathbf{E}_p = (\varepsilon_p^{ij})$; classical theorems of matrix algebra prove the existence of unimodular square matrices $\mathbf{P}$, $\mathbf{Q}$, of order α_p and α_{p-1} respectively and with integer entries, such that in the matrix $\mathbf{P}\mathbf{E}_p\mathbf{Q} = (\varrho_{ij})$, all the ϱ_{ij} with $i \neq j$ are 0, and each integer ϱ_{ii} divides $\varrho_{i+1,i+1}$.

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The number b_p of the ϱ_{ii} which are not 0 (the 'rank' of $\mathbf{E}_p$) is the p-th *Betti number* of X, and the integers ϱ_{ii} which are different from 0 and 1 are the *torsion coefficients* of X; they are the topological invariants of X introduced by Poincaré.

At that time Emmy Noether was quite familiar with the translation of the theory of matrices into intrinsic terms: she therefore had no difficulty in discerning in the Poincaré method a formulation in terms of $\mathbb{Z}$-modules; the p-chains: $\sum_i t_i e_p^i$, formal combinations of p-simplices with integer coefficients t_i, are the elements of a free $\mathbb{Z}$-module C_p, of which $(e_p^i)_{1\leq i\leq \alpha_p}$ constitute a basis; the 'boundary' mapping

$$\delta_p : \sum_i t_i e_p^i \to \sum_i \sum_j t_i \varepsilon_p^{ij} e_{p-1}^j$$

is a homomorphism of C_p into C_{p-1}, with the fundamental property

$$\delta_{p-1} \circ \delta_p = 0. \tag{1}$$

Therefore the image $\operatorname{Im} \delta_{p+1}$ and the kernel $\operatorname{Ker} \delta_p$ are submodules of C_p, such that $\operatorname{Im} \delta_{p+1} \subset \operatorname{Ker} \delta_p$, and the consideration of Betti numbers and torsion coefficients is equivalent to that of the $\mathbb{Z}$-modules

$$H_p = \operatorname{Ker} \delta_p / \operatorname{Im} \delta_{p+1}.$$

H. Hopf accordingly used these '*homology modules*' to express his results on the Lefschetz trace formula in a paper written in 1928 [3].

This seemingly innocuous modification was to have far reaching consequences both for the ulterior evolution of Algebraic Topology, and later for the birth of Homological Algebra, since it was clear that the definition of homology modules could at once be extended to what are now known as 'chain complexes', namely arbitrary sequences of modules (C_p) over any ring, and module homomorphisms δ_p satisfying (1).

As a matter of fact, this was already done by W. Mayer in a paper of 1929 [4] for the special case in which the C_p are free $\mathbb{Z}$-modules with finite basis. He was at that time in Vienna and did not mention Emmy Noether at all in this paper. However, by that time the spirit of 'modern algebra' had spread to many German universities under the efforts of E. Artin, R. Baer, R. Brauer, H. Hasse, W. Krull and J. Von Neumann; it is not unlikely that it could also have reached Vienna.

References

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