Algebraic Geometry*

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Notice

This article comprises lecture notes containing most of the material presented by Professor Jean Dieudonné in a course given at the University of Maryland during the months of March–May 1962 on the first three chapters of A. Grothendieck's "Éléments de Géométrie Algébrique" [Publications Mathématiques, Institut des Hautes Études Scientifiques, Nos. 4, 8, 11 (1960-1961)]. The reader of the present article should be familiar with the main definitions and results of the first four paragraphs (pp. 80–127) and of Sections 3-4 of § 5 (pp. 132–136) of Chapter I of Grothendieck's "Éléments," as well as with the rudiments of commutative algebra, homological algebra, and sheaf theory. The complete reference list for this article will be found at the end of the following paper in this issue.

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Introduction

We shall first review the topic of algebraic geometry from its naive beginnings to the curious and outlandish fashion in which it is treated nowadays.

In the elements of calculus a plane curve is defined as the set of points (x, y) which satisfy the equation $P(x, y) = 0$, where P is a polynomial with coefficients in the real field $\mathbf{R}$. Similarly a surface in 3-space is the set of points (x, y, z) which satisfy the equation $P(x, y, z) = 0$, where $P \in \mathbf{R}[X, Y, Z]$. There exist, however, curves which according to this definition have no points at all, like for instance $x^2 + y^2 + 1 = 0$. The mathematicians of the 18th century thought to eliminate this inconvenience by introducing imaginary points of the curve, i.e., points $(x, y) \in \mathbf{C}^2$ which satisfy $P(x, y) = 0$. They created hereby a great confusion since considering complex points of curves defined by equations with real coefficients they mixed up two completely different problems. We now know that the right thing to do is to consider a field K, an equation $P(x, y) = 0$ where P is a polynomial with coefficients in K and to look for solutions $(x, y) \in K^2$. The curves defined over different fields are to be considered as different curves, even if their equations have the same form.

What is the reason for considering an arbitrary field K instead of just the fields $\mathbf{R}$ and $\mathbf{C}$? A great number of problems have conduced to arbitrary fields, especially since the year 1930. Thus Fermat's last theorem states that the surface $x^n + y^n + z^n = 0$ $(n \geqslant 3)$ has no non trivial points over the field $\mathbf{Q}$ of rational numbers. The situation is similar for all diophantine equations. An important method in the theory of these equations is the reduction modulo p, where p is a prime number. Thus in the case of Fermat's last theorem we ask whether the congruence

$$x^n + y^n + z^n \equiv 0 \qquad (\mathrm{mod}\ p)$$

has a solution in integers, which is the same thing as to ask for solutions of the equation $x^n + y^n + z^n = 0$ over the finite field $\mathbf{F}_p = \mathbf{Z}/(p)$. We can also consider congruences modulo p^k, i.e. solutions in $\mathbf{Z}/(p^k)$ which is not a field any more but a ring. Finally we can ask for solutions which satisfy the congruences mod. p^k for every strictly positive integer k, which is the same as asking for solutions of $x^n + y^n + z^n = 0$ in the ring of p-adic integers $\mathbf{Z}_p = \varprojlim \mathbf{Z}(p^k)$ = the projective limit of the rings $\mathbf{Z}/(p^k)$. Hence we are led to consider the field $\mathbf{Q}_p$ of p-adic numbers.

The method of algebraic geometry over finite fields with characteristic

p is very powerful. Thus after André Weil proved the Riemann conjecture for algebraic curves, he has obtained a number of immediate corollaries concerning diophantine equations which have never been proved otherwise. For instance he obtained the inequality

$$\left| \sum_{x=1}^{p-1} \exp\left[\frac{2\pi i}{p}\left(x + \frac{1}{x}\right)\right] \right| \leqslant c p^{1/2}.$$

The best estimate known before was $O(p^{\alpha})$ with $\frac{1}{2} < \alpha < 1$.

Thus we are led to consider an arbitrary field K and points of K^n which satisfy a certain family of polynomial equations $P_\alpha(x_1, \ldots, x_n) = 0$ $(\alpha \in I)$ with coefficients in K. The set V of all such points form a so-called algebraic affine variety. The first remark to be made is that while the polynomials P_α determine the variety V, conversely the variety in no way determines the system P_α since the points of V satisfy all equations of the form $\sum_\alpha R_\alpha P_\alpha = 0$. This was noticed at a very early stage since already in three-dimensional space a curve can be defined be infinitely many different systems of equations. Thus we are led to consider, instead of systems of polynomial equations, ideals in the polynomial ring. Given an ideal $\mathfrak{a}$ in $K[X_1, \ldots, X_n]$, let $V(\mathfrak{a})$ be the set of all points $(x_1, \ldots, x_n) \in K^n$ which satisfy the equation $P(x_1, \ldots, x_n) = 0$ for every $P \in \mathfrak{a}$. Conversely let A be any set in K^n and consider the set $I(A)$ of all polynomials P such that $P(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in A$. The set $I(A)$ is clearly an ideal and thus we have associated ideals and varieties to each other. This association is, however, not very satisfactory. Indeed, if we start out with an ideal $\mathfrak{a}$, then $\mathfrak{a} \subset I(V(\mathfrak{a}))$, but in general $\mathfrak{a} \neq I(V(\mathfrak{a}))$.

The situation is slightly better if the field K is algebraically closed, for in this case we have Hilbert's nullstellensatz: if the ideal $\mathfrak{a}$ is not the whole ring $K[X_1, \ldots, X_n]$, then there exists a point $x = (x_1, \ldots, x_n) \in K^n$ such that $x \in V(\mathfrak{a})$, i.e., $V(\mathfrak{a}) \neq \varnothing$, p. 260. From here we can deduce, using the classical trick of Rabinowitsch, that

$$I(V(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a}),$$

where $\mathfrak{r}(\mathfrak{a})$ *is the root of* $\mathfrak{a}$, i.e. *the set of all polynomials such that some power belongs to* $\mathfrak{a}$.

Proof. Clearly $\mathfrak{r}(\mathfrak{a}) \subset I(V(\mathfrak{a}))$. Conversely, let $f \in I(V(\mathfrak{a}))$ and let T be a new indeterminate. Let us consider the ideal $\mathfrak{A}$ in the ring $K[X_1, \ldots, X_n, T]$ generated by $1 - Tf(X_1, \ldots, X_n)$ and all the polynomials $Q(X_1, \ldots, X_n) \in \mathfrak{a}$. These polynomials have no common root,

since if $x = (x_1, ..., x_n)$ is a root of all polynomials $Q \in \mathfrak{a}$, then by our assumption $f(x_1, ..., x_n) = 0$ and $1 - Tf(x_1, ..., x_n) = 1$. Hence by the nullstellensatz the ideal $\mathfrak{A}$ is the whole ring $K[X_1, ..., X_n, T]$ and thus 1 is a linear combination of the form

$$1 = g(T, X_1, ..., X_n)(1 - Tf(X_1, ..., X_n)) + \sum_i g_i(T, X_1, ..., X_n) Q_i(X_1, ..., X_n).$$

Setting $T = 1/f(X_1, ..., X_n)$ we obtain an identity in the field $K(X_1, ..., X_n)$:

$$1 = \sum_i g_i\left(\frac{1}{f(X_1, ..., X_n)}, X_1, ..., X_n\right) Q_i(X_1, ..., X_n).$$

Multiplying by the highest power of $f(X_1, ..., X_n)$ which occurs in the denominators on the right-hand side, we obtain an identity

$$f^\mu = \sum_i h_i(X) Q_i(X)$$

which proves that $f^\mu \in \mathfrak{a}$, i.e., $f \in \mathfrak{r}(\mathfrak{a})$.

It can be seen in particular that $\mathfrak{r}(\mathfrak{a})$ is a prime ideal if and only if $V(\mathfrak{a})$ is irreducible. For ideals such that $\mathfrak{a} = \mathfrak{r}(\mathfrak{a})$, the correspondence between $\mathfrak{a}$ and V is bijective. Thus in the hands of Hilbert and his followers (E. Noether, W. Krull, B. L. van der Waerden) algebraic geometry has become the study of polynomial ideals. This school has flourished from 1920 to the publication of Weil's "Foundations" in 1947 and its results can be found in the books of van der Waerden ("Einführung in die algebraische Geometrie," 1939) and Gröbner ("Moderne algebraische Geometrie," 1949).

However, the situation is still not satisfactory. The main objection is that what the theory really studies is not the algebraic variety itself but the algebraic variety immersed in a certain space. Thus a circle if considered in the plane is defined by the ideal $\mathfrak{a}$ generated by $X^2 + Y^2 - 1$ in $K[X, Y]$. However, the same circle, if considered in 3-space, is defined by the ideal generated by the polynomials $X^2 + Y^2 - 1$ and Z in $K[X, Y, Z]$. The solution to this dilemma has been seen already by Riemann and it consists in considering the quotient ring $A = K[X_1, ..., X_n]/\mathfrak{a}$; in the above two examples the two rings obtained in this way are isomorphic. The elements of the ring A are polynomials computed mod $\mathfrak{a}$, i.e., two polynomials P_1 and P_2 define the same elements of A if they define the same function on V. Thus the ring A can be thought of as the ring of functions on the variety V.

Riemann expressed himself, of course, in a different language. He considered a rational function $f(x, y)$ and for every fixed x the roots $y_1(x),..., y_n(x)$ of the equation $f(x, y) = 0$. The value of a rational function at a point (x, y_i) is then given by $P(x, y_i(x))/Q(x, y_i(x))$. The trouble with this language is that the $y_i(x)$ are multiple valued functions, i.e., no functions at all. One really has to work with the integral ring $K[X_1 ,..., X_n]/\mathfrak{p}$, $\mathfrak{p}$ prime, and its quotient field. Later Dedekind and Weber translated Riemann's theory into algebraic language (*J. Reine Angew. Math.* **92** (1882), 181–290).

Thus we are led to associate with every variety V a ring of finite type $A = K[X_1 ,..., X_n]/\mathfrak{a}$, where, for the sake of simplicity, we assume that the field K is algebraically closed. Our problem is to establish a one-to-one correspondence between algebraic and geometric objects. Given a point $z \in V$, we associate with it the set of all functions on V which vanish at z; this set is a maximal ideal in A. A subvariety W of V is defined by an ideal $\mathfrak{b}$ which contains $\mathfrak{a}$: the set of all functions on V which vanish on W forms an ideal in A, and we associate this ideal with the subvariety W.

In the opposite direction, given an algebraically closed field K and a ring of finite type $A = K[X_1 ,..., X_n]/\mathfrak{a}$, we want to associate with it a variety V. By what precedes, it is natural to consider as the points of V the elements of the maximal spectrum $\mathrm{Specm}(A)$ of A, i.e., the maximal ideals of A. This, however, will certainly not give a one-to-one correspondence between algebraic and geometric objects, since to any field K there would correspond the variety consisting of one point. A way to correct this situation is again suggested by Riemann's approach who considered the rings A_z formed by the functions on V which have no poles at z. Thus we should take as elements of the geometric object we want to associate with the ring A, the pairs $(\mathfrak{m}, A_{\mathfrak{m}})$, where $\mathfrak{m}$ is a maximal ideal of A and $A_{\mathfrak{m}}$ is the local ring at $\mathfrak{m}$. In this way, for different fields we obtain different pairs (z, K).

The situation is still not satisfactory, since there exist non-isomorphic rings such that the corresponding sets of pairs $(\mathfrak{m}, A_{\mathfrak{m}})$ are the same. For example if we take for varieties over $\mathbf{C}$ the hyperbola V_1 defined by the equation $xy = 1$ and the parabola V_2 defined by $y = x^2$, then the corresponding rings are $A_1 = \mathbf{C}[X, Y]/(XY - 1)$ and $A_2 = \mathbf{C}[X, Y]/(X^2 - Y)$. Now A_1 is isomorphic to $\mathbf{C}[X, X^{-1}]$, A_2 is isomorphic to $\mathbf{C}[X]$ and the last two rings are not isomorphic since in the second one the sum of invertible elements is an invertible element or zero and in the first one this is not true. However, it is easy to see that for either ring the local

ring at a maximal ideal is isomorphic to $(\mathbf{C}[X])_{(X)}$ = the local ring of $\mathbf{C}[X]$ at the maximal ideal (X).

The way out of this dilemma is to glue together the pairs $(\mathfrak{m}, A_{\mathfrak{m}})$ in a sensible way instead of considering them as a discrete collection of objects. This can be done using Leray's theory of sheaves. First we define a topology on the set Specm(A), the so-called Zariski topology. (Observe that in the case of the real or complex field, we have on the variety the topology induced by $\mathbf{R}^n$ or $\mathbf{C}^n$, but this cannot be generalized to an arbitrary field K.) Given an ideal $\mathfrak{a}$ in A we set

$$V(\mathfrak{a}) = \text{the set of all maximal ideals which contain } \mathfrak{a}$$

and we take for the closed sets of the Zariski topology the sets $V(\mathfrak{a})$. In other words, the closed sets of $V = \text{Specm}(A)$ are the subvarieties of V. The Zariski topology is very coarse, it is T_1 (Fréchet) but only exceptionally T_2 (Hausdorff). Now we want to construct on Specm(A) a sheaf whose stalk at the point $\mathfrak{m}$ is $A_{\mathfrak{m}}$. We take for a basis of the open sets in V the sets of the form $D(f) = \{\mathfrak{m} \mid f \notin \mathfrak{m}\}$, where $f \in A$. The sets $D(f)$ can also be interpreted in another way. Write now x for an arbitrary element of $V = \text{Specm}(A)$ and denote by $f(x)$ the class of $f \in A$ modulo the maximal ideal $\mathfrak{m}_x$ corresponding to x. In particular $f(x) = 0$ if and only if $f \in \mathfrak{m}_x$. (This corresponds to the fact that if A is the ring of functions on the variety V, then with each point $x \in V$ we associate the maximal ideal of A formed by the functions which vanish at x.) Then we can also write $D(f) = \{x \mid f(x) \neq 0\}$. We obtain a presheaf of rings $\tilde{A}$ on Specm(A) if we attach to every open set $D(f)$ the ring $\Gamma(D(f), \tilde{A}) = A_f = \{a/f^n \mid a \in A,\ n \geqslant 0\}$ (Grothendieck, "Eléments," 0_1, 1.2.3). Intuitively A_f is the set of functions which have poles at most on the set $V(f) = \{x \mid f(x) = 0\}$ where f vanishes, i.e. which are regular on $D(f)$. The restriction maps for $\tilde{A}$ are readily defined and the axioms of sheaves are easily checked. Thus we have attached to the ring A a geometric object, namely the ringed space $(V, \tilde{A})$. It is easy to see that $A_{\mathfrak{m}}$ is the inductive limit of the rings A_f, where $f \notin \mathfrak{m}$. Finally A is isomorphic to $\Gamma(V, \tilde{A})$ and thus the correspondence between rings A and ringed spaces $(V, \tilde{A})$ is one-to-one.

We have now obtained a satisfactory solution to our problem in the case of affine geometry. However, already the geometers of the 18th century have realized that affine geometry does not reflect correctly the intuitive geometric picture. Thus similar objects, like conic sections, look completely different, and Bezout's theorem, according to which a curve of degree m and a curve of degree n have mn points of intersection

if counted with their multiplicities, does not hold (e.g., a line and a conic). Early in the 19th century projective space was introduced and there general theorems (e.g., Bezout's theorem, at least over an algebraically closed field) were found to be true. Consequently in the ensuing period mostly projective algebraic geometry was developed. Consider the n-dimensional projective space $\mathbf{P}_n(K)$ over the field K which is defined as the quotient set K^{n+1}/Δ where Δ is the equivalence relation in K^{n+1} whose equivalence classes are lines going through the origin. The varieties in $\mathbf{P}_n(K)$ correspond to cones in K^{n+1}, i.e., are defined by a system of equations $P_\alpha(x_0, x_1, \ldots, x_n) = 0$ $(\alpha \in I)$, where the polynomials P_α are homogeneous. The first trouble arises if, following the pattern set by the affine case, we want to define the functions on the variety V as equivalence classes of polynomials, since all such functions turn out to be constants. Once more it was Riemann who circumvented this difficulty. Instead of considering functions which are regular on the whole variety, he allowed poles, i.e., considered rational functions. Two such rational functions

$$\frac{Q(x_0, x_1, \ldots, x_n)}{R(x_0, x_1, \ldots, x_n)} \quad \text{and} \quad \frac{Q'(x_0, x_1, \ldots, x_n)}{R'(x_0, x_1, \ldots, x_n)}$$

were considered to be equal if $QR' - RQ'$ belonged to the homogeneous ideal generated by the polynomials P_α. It turns out that these rational functions form a field with a reasonable degree of transcendence. Something is lost however: isomorphic fields can correspond to different varieties, as in the case of rational (unicursal) curves.

This method of attack has given rise to birational geometry, which disregards the finer geometric properties of the objects under study but has given a great number of interesting results in the hands of German, French, and Italian geometers. Birational invariants were discovered, the first of them by Riemann himself, namely the genus of a curve.

To go beyond this stage it is necessary to consider projective space in a different way. In fact $\mathbf{P}_n(K)$ can be obtained by gluing together affine spaces. Let H_i be the hyperplane defined by $x_i = 0$ $(0 \leqslant i \leqslant n)$; then $E_i = \complement H_i$ is isomorphic to K^n, the isomorphism being defined by

$$(x_0, x_1, \ldots, x_i, \ldots, x_n) \to \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right).$$

The set $E_i \cap E_j$ is an open variety in both E_i and E_j. Thus to obtain $\mathbf{P}_n(K)$ we have to take $n + 1$ affine spaces E_i, in each of these certain

subvarieties H_{ij} $(j \neq i)$ and identify $H_{ij} \subset E_i$ with $H_{ji} \subset E_j$. This procedure has been used before in algebraic topology and to define differentiable or analytic manifolds, but the idea to use it in algebraic geometry is due to Weil. Once in possession of this idea Weil introduced abelian varieties (which turned out much later to be projective varieties) and could prove the Riemann hypothesis for algebraic curves over finite fields. To define his abstract varieties Weil considers a system V_α of algebraic varieties and for each pair V_α, V_β subvarieties $W_{\alpha\beta} \subset V_\alpha$ and $W_{\beta\alpha} \subset V_\beta$, such that $W_{\alpha\beta}$ and $W_{\beta\alpha}$ are isomorphic. The abstract variety is then obtained by identifying $W_{\alpha\beta}$ and $W_{\beta\alpha}$. Weil could extend to these varieties all the results known for affine and projective varieties. However, the detailed construction of the abstract varieties is an extremely long, tedious, and cumbersome process.

Just as in the case of differentiable and analytic manifolds, the simplification was achieved by the use of sheaves. A differentiable manifold can be defined as a ringed space $(V, \mathscr{F})$ in which every point $x \in V$ has a neighborhood U such that $\mathscr{F} \mid U$ is isomorphic to the sheaf of all differentiable functions on an open set of $\mathbf{R}^n$. This definition has the advantage to be intrinsic and can easily be seen to be equivalent to the old one with atlases. In 1955, J. P. Serre has turned his attention from complex analytic manifolds and analytic spaces to algebraic geometry and found that the same definition can be given here (Faisceaux algébriques cohérents, *Ann. Math.* **61** (1955), 197–278). Thus an algebraic variety is a ringed space which is locally isomorphic to the ringed space of an affine variety. This definition was finally modified by Grothendieck in two points.

In the first place Serre still only considered rings of finite type $A = K[X_1, \ldots, X_n]/\mathfrak{a}$ (Weil even took $\mathfrak{a}$ prime, i.e., his varieties were irreducible). Grothendieck (and independently Cartier) had the idea to consider a completely arbitrary commutative ring A with unit element.

Another point still caused some difficulty. With a ring A we associated a ringed space $(V, \tilde{A})$ and it is desirable that this correspondence be a functor from the category of rings into the category of ringed spaces. A morphism $\Psi : (V, \mathscr{O}) \to (V', \mathscr{O}')$ of ringed spaces is a pair $\Psi = (\psi, \theta)$, where $\psi : V \to V'$ is a continuous map and $\theta : \mathscr{O}' \to \mathscr{O}$ is a ψ-morphism (Grothendieck, "Eléments," 0_I, 4.1.1). If $\varphi : A \to B$ is a homomorphism of rings (transforming, as always, the unit element of A into the unit element of B), then the only sensible way to define

$$\psi : W = \mathrm{Specm}(B) \to V = \mathrm{Specm}(A)$$

is as follows: take $\mathfrak{n} \in \mathrm{Specm}(B)$, i.e., let $\mathfrak{n}$ be a maximal ideal in B and consider $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$. It turns out, however, that $\varphi^{-1}(\mathfrak{n})$ is not necessarily a maximal ideal of A at all. Thus if $i : \mathbf{Z} \to \mathbf{Q}$ is the canonical imbedding, (0) is a maximal ideal in $\mathbf{Q}$, but $i^{-1}(0) = (0)$ is not a maximal ideal in $\mathbf{Z}$.

In this way the maximal spectrum $\mathrm{Specm}(A)$ had to be replaced by the prime spectrum $\mathrm{Spec}(A)$ and now $\varphi : A \to B$ defines a map $\psi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ since if $\mathfrak{p}$ is a prime ideal in B, then $\varphi^{-1}(\mathfrak{p})$ is a prime ideal in A. The intuitive picture is somewhat more confusing since maximal ideals correspond to points, whereas we include also all irreducible subvarieties, which correspond to prime ideals. It is clear that for each $\mathfrak{p} \in \mathrm{Spec}(A)$ we have a homomorphism $\varphi_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ which together define a ψ-morphism $\tilde{A} \to \tilde{B}$ of sheaves of rings. Thus $A \rightsquigarrow (\mathrm{Spec}(A), \tilde{A})$ is indeed a functor as desired (Grothendieck, "Eléments," I, 1.6.1).

Interpretation of Some Notions of Classical Algebraic Geometry in the Language of Schemata

As we have mentioned before, the *points* of a preschema $(X, \mathcal{O}_X)$ are not only the points of the variety V corresponding to it but also all irreducible subvarieties of V. The points of X which correspond to points of V are the closed points x of X, i.e., such that $\{x\} = \overline{\{x\}}$. In the case of an affine schema $\mathrm{Spec}(A)$, the closed points x correspond to maximal ideals $\mathfrak{j}_x$ of A (Grothendieck, "Eléments," I, 1.1.7). The variety in the sense of Serre is the subspace formed by all closed points with the topology induced on it by X.

To an *irreducible subvariety* of the affine variety V there corresponds a prime ideal $\mathfrak{p}$ of A and thus an irreducible closed set $V(\mathfrak{p})$ of $X = \mathrm{Spec}(A)$ (Grothendieck, "Eléments," I, 1.1.14). Each irreducible closed set F in X has a unique generic point x such that $F = \overline{\{x\}}$, namely the point $x = \mathfrak{j}_x = \mathfrak{j}(F)$ (*ibid.*). All points $y \in \overline{\{x\}}$ are specializations of x, i.e., $\mathfrak{j}_x \subset \mathfrak{j}_y$ or, in other words, the irreducible subvariety W_y corresponding to $\mathfrak{j}_y$ is a subvariety of W_x. This notion of generic point is completely different from that of Weil.

Next we consider the notion of *dimension*. Let E be a topological space. We consider increasing chains $F_0 \subset F_1 \subset \cdots \subset F_k$ of length k of irreducible, non-empty, closed sets in E. The dimension of E is by definition the maximal length of all such chains (Godement, "Théorie des faisceaux," II, 4.15, p. 198). This definition corresponds to the intuitive

idea of dimension in an algebraic variety. Thus for instance in a three-dimensional variety we have maximal chains consisting of a point, a curve through the point, a surface through the curve, and finally the space itself. If the space is an affine schema, then each irreducible closed set F_i has a unique generic point x_i, hence the chain can be written $\overline{\{x_0\}} \subset \overline{\{x_1\}} \subset \cdots \subset \overline{\{x_k\}}$ and to this there corresponds a chain of prime ideals $\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_k$ in A. Irreducible components of the variety correspond to minimal prime ideals of A (cf. Grothendieck, "Eléments," I, 1.1.14). Given an irreducible variety W corresponding to a prime ideal $\mathfrak{p}$ of A, the dimension of W is the maximal length of a chain $\mathfrak{p} = \mathfrak{p}_k \subset \mathfrak{p}_{k-1} \subset \cdots \subset \mathfrak{p}_0$ of strictly increasing prime ideals of A.

This leads to the definition of the (Krull) dimension of a ring A as the largest number k such that there exists a strictly increasing chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_k$ of A (Samuel, "Progrès récents d'algèbre locale," III. 2, p. 66). We can now speak of the dimension of the local ring $\mathcal{O}_x = A_{\mathfrak{p}}$ at $\mathfrak{p}$. At a generic point x the dimension is zero, since $A_{\mathfrak{p}}$ has only one prime ideal, namely the nilradical. If $X = \mathrm{Spec}(A)$ is irreducible and A is an integral ring, than at a generic point x the local ring $\mathcal{O}_x$ is a field, namely the field of fractions of A (Grothendieck, "Eléments," I, 7.1.5). Corresponding to a maximal chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_k$ of prime ideals of A we have a sequence of local rings $A_{\mathfrak{p}_0}$, $A_{\mathfrak{p}_1}$,..., $A_{\mathfrak{p}_k}$, where $\dim A_{\mathfrak{p}_0} = 0$, $\dim A_{\mathfrak{p}_1} = 1$,..., $\dim A_{\mathfrak{p}_k} = k$.

Analyzing the classical definition of a *simple point* of a variety (Zariski, The concept of a simple point of an abstract algebraic variety, *Trans. Am. Math. Soc.* **62** (1947), pp. 1–52; or Lang, "Introduction to Algebraic Geometry," Chap. VIII) one is led to say that $x \in X$ is simple if $\mathcal{O}_x$ is a regular local ring. Let us recall the definition of these rings. Let A be a noetherian local ring and $\mathfrak{m}$ its maximal ideal. Let n be the minimal number of generators of $\mathfrak{m}$ and d the dimension of A. In general $d \leqslant n$, if $d = n$ we say that A is a regular local ring. Observe that n is also the dimension of the vector space $\mathfrak{m}/\mathfrak{m}^2$ over the field $A/\mathfrak{m}$. This follows from Nakayama's lemma, which states the following: let A be a not necessarily commutative ring, $\mathfrak{R}$ the radical of A, M a module of finite type over A. If N is a submodule of M such that $M = N + \mathfrak{R}M$, then $M = N$ (Bourbaki, "Algèbre," Chap. VIII, § 6, No. 3, Cor. 2 of Prop. 6). To prove the assertion, let x_1,..., $x_n \in \mathfrak{m}$ be such that their classes $\bar{x}_1$,..., $\bar{x}_n$ modulo $\mathfrak{m}^2$ generate $\mathfrak{m}/\mathfrak{m}^2$, then we have to show that x_1,..., x_n generates $\mathfrak{m}$. Set $\mathfrak{n} = Ax_1 + \cdots + Ax_n$, then $\mathfrak{n} \subset \mathfrak{m}$ and $\mathfrak{m} = \mathfrak{n} + \mathfrak{m}^2$. Since $\mathfrak{m}$ is the radical of A, we obtain by Nakayama's lemma that $\mathfrak{m} = \mathfrak{n}$.

Intuitively, if A is the ring of functions regular at a point x of a variety and $\mathfrak{m}$ is its maximal ideal consisting of the functions which vanish at x, then $\mathfrak{m}^2$ is the ideal consisting of those functions whose development starts with quadratic terms and the elements of $\mathfrak{m}/\mathfrak{m}^2$ are the equivalence classes of functions with the same linear terms and no constant terms. The fact that A is a regular local ring means that the dimension of the vector space spanned by the gradients at x of the functions vanishing at x equals the dimension of the variety.

To see an example of a non-simple point, consider the curve $x = t^2$, $y = t^3$, which has a cusp at the origin $(0, 0)$. The polynomial ring $K[X, Y]$ has dimension two; however, this fact is not so trivial as one might think (Samuel, "Progrès récents d'algèbre locale," III.3, Th. 8, p. 83): indeed, Nagata has given an example of a non-noetherian ring R of dimension 1 such that $R[X]$ has dimension three. The ring $A = K[X, Y]/(Y^2 - X^3)$ has dimension 1, and the maximal ideal $((X) + (Y))/(Y^2 - X^3)$ of the functions vanishing at the origin can be generated by two elements but not by one. It follows that the local ring at $(0, 0)$ has dimension 1 but its maximal ideal cannot be generated by fewer than two elements.

A *divisor* on a variety V of dimension n can be thought of roughly as an element of the free abelian group generated by the irreducible $(n - 1)$-dimensional subvarieties W of V, i.e., as an expression of the form $\sum_i n_i W_i$, where the n_i are integers. In the affine case if the variety is a non-singular curve C, hence A a Dedekind ring (i.e. every prime ideal maximal), a divisor is an expression of the form $\sum_{i=1}^{q} n_i a_i$, where the a_i $(1 \leqslant i \leqslant q)$ are points of C. To this divisor there corresponds a fractional ideal $\mathfrak{p}_1^{n_1}\mathfrak{p}_2^{n_2} \cdots \mathfrak{p}_q^{n_q}$ of A. Conversely, since every fractional ideal of A can be written in this form, there corresponds to it a divisor on C and thus we have a one-to-one correspondence between divisors on C and fractional ideals of A. (This correspondence exists only in the affine case, not in the projective one.) It follows that the notion of a divisor can be considered as a generalization of that of an ideal.

In the classical case an irreducible subvariety of dimension $n - 1$ is given locally by the vanishing of a nonzero polynomial f, i.e., as a set $W = \{x \mid f(x) = 0\}$. If we have a *positive* divisor $\sum_i n_i W_i$ $(n_i \geqslant 0)$ and f_i is the irreducible polynomial which locally defines W_i, then $\sum_i n_i W_i$ will represent a not necessarily irreducible variety locally defined by the equation

$$f_1^{n_1} f_2^{n_2} \cdots f_q^{n_q} = 0.$$

Thus locally a positive divisor is defined by an equation $f = 0$, where f is a not necessarily irreducible polynomial. To define a positive divisor globally on the variety X we must assign to each point $x \in X$ an open neighborhood U and an element $f_U \in \Gamma(U, \mathcal{O}_X - \{0\})$. Furthermore in the intersection $U \cap W$ of two such neighborhoods f_U and f_W must define the same subvariety, i.e., we must have $f_U/f_W \in \Gamma(U \cap W, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the subsheaf of invertible elements of $\mathcal{O}_X$. Thus the collection (f_U) defines a section of the quotient sheaf $(\mathcal{O}_X - \{0\})/\mathcal{O}_X^*$ of multiplicative groups, i.e., a positive divisor D can be defined as an element $D \in \Gamma(X, (\mathcal{O}_X - \{0\})/\mathcal{O}_X^*)$.

To define an arbitrary divisor we shall suppose for the sake of simplicity that $(X, \mathcal{O}_X)$ is an *integral* preschema (Grothendieck, "Eléments," I, 2.1.8 and below, p. 255). This means that for every affine open set U the ring $\Gamma(U, \mathcal{O}_X)$ is integral and this can be considered as a definition of an integral preschema (Grothendieck, "Eléments," I, 5.1.4). Let $R(U)$ be the field of fractions of $\Gamma(U, \mathcal{O}_X)$. The fields $R(U)$ determine a quasi-coherent $\mathcal{O}_X$-Module $\mathcal{R}_X = \mathcal{R}(X)$, called the sheaf of rational functions on the preschema X (Grothendieck, "Eléments," I, 7.3.2). To define the restriction homomorphisms $\rho_{V,U} : R(U) \to R(V)$ for $U \supset V$ we can suppose that $U = \mathrm{Spec}(A)$, where A is an integral ring, and $V = D(f)$, where $f \in A$, $f \neq 0$. Then $\Gamma(U, \mathcal{O}_X) = A$, $\Gamma(V, \mathcal{O}_X) = A_f$ and since A is integral, we have $A \subset A_f \subset R(U)$. Thus the fields of fractions of A and A_f are the same, and $\rho_{V,U}$ is the identical isomorphism. This proves that $\mathcal{R}_X$ is a simple sheaf on X.

A divisor D is then defined by assigning to each point $x \in X$ an open neighborhood U and an element $f_U \in \Gamma(U, \mathcal{R}_X^*)$, where $\mathcal{R}_X^* = \mathcal{R}_X - \{0\}$, with the further condition that in the intersection $U \cap W$ of two such neighborhoods $f_U/f_W \in \Gamma(U \cap W, \mathcal{O}_X^*)$. In other words, a divisor D is an element of the group $\Gamma(X, \mathcal{R}_X^*/\mathcal{O}_X^*)$.

A principal divisor is classically a divisor associated to a nonzero rational function f defined over the whole variety V. In our case let $f \in \Gamma(X, \mathcal{R}_X^*)$. The canonical epimorphism $\mathcal{R}_X^* \to \mathcal{R}_X^*/\mathcal{O}_X^*$ defines a homomorphism of groups $\Gamma(X, \mathcal{R}_X^*) \to \Gamma(X, \mathcal{R}_X^*/\mathcal{O}_X^*)$. The image (f) of f under this homomorphism is by definition the principal divisor defined by f. From the exact sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{R}_X^* \to \mathcal{R}_X^*/\mathcal{O}_X^* \to 0,$$

we obtain the exact cohomology sequence

$$0 \to \Gamma(X, \mathcal{O}_X^*) \to \Gamma(X, \mathcal{R}_X^*) \to \Gamma(X, \mathcal{R}_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*).$$

A divisor D is therefore a principal divisor if and only if its image in $H^1(X, \mathcal{O}_X^*)$ is zero. The image of a divisor in $\Gamma(X, \mathcal{R}_X^*/\mathcal{O}_X^*)/\Gamma(X, \mathcal{R}_X^*)$ is called its divisor class. By exactness the group of divisor classes is a subgroup of $H^1(X, \mathcal{O}_X^*)$. If X is an *integral preschema*, then $\mathcal{R}_X$ is a simple sheaf, $H^1(X, \mathcal{R}_X^*) = 0$ and therefore $H^1(X, \mathcal{O}_X^*)$ is isomorphic to the group of divisor classes.

We shall now show that we can associate with every divisor D an invertible sheaf $\mathcal{L}(D)$, but first let us recall a few definitions. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-Module $\mathcal{L}$ is *free* if it is isomorphic to the direct sum $\mathcal{O}_X^n$. It is *locally free* if each point $x \in X$ has an open neighborhood U such that $\mathcal{L} \mid U$ is isomorphic to an $(\mathcal{O}_X \mid U)$-Module $\mathcal{O}_X^n \mid U$ for some positive integer n (Grothendieck, "Eléments," 0_I, 5.4.1). The number n is not necessarily the same for every U but it is constant on each connected component of X. Indeed, the dimension of a module over a commutative ring is well defined and therefore in the intersection of two neighborhoods the restrictions must have the same dimension. If n is constant on X, we say that $\mathcal{L}$ is of *rank n*. A locally free sheaf is quasi-coherent. On a preschema $\mathcal{L}$ is perfectly determined by the modules $\Gamma(U, \mathcal{L})$ for affine open sets U and therefore $\mathcal{L}$ will be locally free of rank n if $\Gamma(U, \mathcal{L})$ is isomorphic to $\Gamma(U, \mathcal{O}_X)^n$ for a basis formed by affine open sets U. If $\mathcal{L}$ is locally free, then for every sufficiently small open set U the $\Gamma(U, \mathcal{O}_X)$-module $\Gamma(U, \mathcal{L})$ is free. Locally free Modules need not be free, not even for affine schemata. Already for a Dedekind ring A there can exist an A-module M such that $\tilde{M}$ is locally free but not free (Bourbaki, "Algèbre commutative," Chap. II, § 5). It can be shown that the locally free Modules are precisely the projective objects in the category of Modules.

The dual $\check{\mathcal{F}}$ of an $\mathcal{O}_X$-Module $\mathcal{F}$ is by definition $\check{\mathcal{F}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. Given two $\mathcal{O}_X$-Modules $\mathcal{F}$ and $\mathcal{G}$ there exists a canonical homomorphism $\varphi : \check{\mathcal{F}} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ (Grothendieck, "Eléments," 0_I, 5.4.2). Indeed, if A is a ring and M, N are two A-modules, there exists a homomorphism $\check{M} \otimes_A N \to \mathrm{Hom}_A(M, N)$ which to $x' \otimes y \in \check{M} \otimes N$ associates the homomorphism $u_{x',y} : M \to N$ such that $u_{x',y}(x) = \langle x, x' \rangle y$ for $x \in M$. Thus for each open set $U \subset X$ we have a homomorphism

$$\Gamma(U, \mathcal{F})^\vee \underset{\mathcal{O}_X(U)}{\otimes} \Gamma(U, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_X(U)}(\Gamma(U, \mathcal{F}), \Gamma(U, \mathcal{G})).$$

These homomorphisms being compatible with the restrictions define a homomorphism of presheaves and thus of sheaves. Now, if $\mathcal{F}$ is locally free, then φ is bijective. Indeed, locally we have $\mathcal{F} \simeq \mathcal{O}_X^n$, but

then $\check{\mathscr{F}} \otimes \mathscr{G} \simeq \mathscr{O}_X^n \otimes \mathscr{G} \simeq \mathscr{G}^n$, $\mathscr{H}om(\mathscr{G}, \mathscr{F}) \simeq \mathscr{H}om(\mathscr{O}_X^n, \mathscr{G}) \simeq \mathscr{G}^n$, and under these isomorphisms, φ is transformed into the identity map $\mathscr{G}^n \to \mathscr{G}^n$.

We insert here a word of caution. In general if we want to prove that two sheaves $\mathscr{F}$ and $\mathscr{G}$ are isomorphic, we first define a homomorphism $\theta : \mathscr{F} \to \mathscr{G}$ and then prove that locally (or for each stalk) θ is an isomorphism. It is necessary, however, to ascertain that θ is globally defined. For instance for a locally free sheaf $\mathscr{F}$ we have local isomorphisms $\mathscr{F}(U) \simeq \mathscr{O}_X(U)^n$, but this does not mean that $\mathscr{F}$ is isomorphic to $\mathscr{O}_X^n$. In the above proof that φ is an isomorphism, φ was of course given globally *a priori*.

A locally free Module $\mathscr{L}$ of rank 1 is called an *invertible* Module. The explanation of this terminology lies in the fact that $\check{\mathscr{L}} \otimes_{\mathscr{O}_X} \mathscr{L}$ is isomorphic to $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L})$ by the preceding, and, as we shall show in a moment, $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L})$ is isomorphic to $\mathscr{O}_X$, i.e., we have $\check{\mathscr{L}} \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{O}_X$. Now $\mathscr{O}_X$ is the unit element for tensor multiplication, i.e., we have $\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{F} \simeq \mathscr{F}$ for any $\mathscr{O}_X$-Module $\mathscr{F}$ (Godement, "Théorie des faisceaux," II. 2.8, p. 137). To prove the isomorphism $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}) \simeq \mathscr{O}_X$, define first a homomorphism $\omega : \mathscr{O}_X \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})$ for any $\mathscr{O}_X$-Module $\mathscr{F}$. Let $s \in \Gamma(U, \mathscr{O}_X)$, then

$$u_s \in \mathrm{Hom}_{\mathscr{O}_X(U)}(\Gamma(U, \mathscr{F}), \Gamma(U, \mathscr{F}))$$

is defined by $u_s(t) = st$ for $t \in \Gamma(U, \mathscr{F})$. The maps $s \rightsquigarrow u_s$ then clearly define the desired homomorphism ω. If now $\mathscr{L}$ is invertible, then ω is an isomorphism, since for sufficiently small open sets U we have $\Gamma(U, \mathscr{L}) \simeq \Gamma(U, \mathscr{O}_X)$ and

$$\omega(U) : \Gamma(U, \mathscr{O}_X) \to \mathrm{Hom}_{\Gamma(U, \mathscr{O}_X)}(\Gamma(U, \mathscr{O}_X), \Gamma(U, \mathscr{O}_X))$$

is an isomorphism. The required isomorphism is then

$$\omega^{-1} : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}) \to \mathscr{O}_X .$$

In virtue of the preceding property we shall also write $\mathscr{L}^{\otimes(-1)}$ for $\check{\mathscr{L}}$. In general, if $\mathscr{F}$ is an $\mathscr{O}_X$-Module, we write $\mathscr{F}^{\otimes n}$ for $\mathscr{F} \otimes \cdots \otimes \mathscr{F}$ n times. if n is a strictly positive integer, and $\mathscr{F}^{\otimes 0} = \mathscr{O}_X$. If $\mathscr{L}$ is an invertible $\mathscr{O}_X$-Module, we define also $\mathscr{L}^{\otimes(-n)} = (\mathscr{L}^{\otimes(-1)})^{\otimes n} = \check{\mathscr{L}}^{\otimes n}$. We then have the rule $\mathscr{L}^{\otimes m} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n} = \mathscr{L}^{\otimes(m+n)}$ for $m, n \in \mathbf{Z}$. All the Modules $\mathscr{L}^{\otimes n}$ are invertible.

Tate has proved that if $\mathscr{O}_X$ and $\mathscr{L}$ are coherent $\mathscr{O}_X$-Modules, and the $\mathscr{O}_x$ are local rings, then if there exists an $\mathscr{O}_X$-Module $\mathscr{G}$ such that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{G} \simeq \mathscr{O}_X$, $\mathscr{L}$ is invertible and $\mathscr{G} = \check{\mathscr{L}}$ (Grothendieck, "Eléments," 0_I, 5.4.3). If $\mathscr{L} \simeq \mathscr{L}'$ and these Modules are invertible, then $\mathscr{L} \otimes \mathscr{L}'^{\otimes(-1)} \simeq \mathscr{O}_X$.

Now we come to the relation between divisors and invertible $\mathcal{O}_X$-Modules. Let D be a divisor represented by a collection $f_U \in \Gamma(U, \mathcal{R}_X^*)$. We shall construct an invertible $\mathcal{O}_X$-Module $\mathcal{L}(D)$ by defining the module $\Gamma(U, \mathcal{L}(D))$ as the module over $A_U = \Gamma(U, \mathcal{O}_X)$ generated by f_U, i.e., $\Gamma(U, \mathcal{L}(D)) = A_U f_U$. Let $U \supset V$ and let $\sigma_{V,U}$ be the restriction map $A_U \to A_V$ in the sheaf $\mathcal{O}_X$. Then the restriction map $\rho_{V,U} : A_U f_U \to A_V f_V$ is defined by

$$\rho_{V,U}(\alpha f_U) = \sigma_{V,U}(\alpha) \frac{f_U \mid V}{f_V} f_V$$

and this is a well defined homomorphism, since $(f_U \mid V)/f_V \in \Gamma(V, \mathcal{O}_X^*)$. Clearly the A_U-modules $\Gamma(U, \mathcal{L}(D))$ and the restriction maps $\rho_{V,U}$ define a sheaf $\mathcal{L}(D)$.

The divisor D is principal if and only if $\mathcal{L}(D)$ is isomorphic to $\mathcal{O}_X$. Indeed, if $D = (f)$ and for an open set U we write $f_U = f \mid U$, then the homomorphism $\alpha \rightsquigarrow \alpha f_U$ is locally an isomorphism, but since it is defined globally, it is an isomorphism also globally. Conversely a homomorphism $u: \mathcal{O}_X \to \mathcal{L}(D)$ is defined by a section $f \in \Gamma(X, \mathcal{L}(D)) \subset \Gamma(X, \mathcal{R}_X^*)$ and $u(\alpha) = \alpha f_U$ for $\alpha \in A_U$, where $f_U = f \mid U$ (Grothendieck, "Eléments," 0_I, 5.1.1). If u is an isomorphism, then we have clearly $D = (f)$.

More generally two divisors D and D' are equivalent, i.e., belong to the same divisor class, if and only if $\mathcal{L}(D)$ and $\mathcal{L}(D')$ are isomorphic.

We use the additive notation in the group $\Gamma(X, \mathcal{R}_X^*/\mathcal{O}_X^*)$ of divisors, i.e., if the divisor D is defined by the elements $f_U \in \Gamma(X, \mathcal{R}_X^*)$ and D' is defined by f'_U then $D + D'$ is defined by $f_U f'_U$. It follows that $\mathcal{L}(D + D')$ is isomorphic to $\mathcal{L}(D) \otimes_{\mathcal{O}_X} \mathcal{L}(D')$. Indeed, we have $\Gamma(U, \mathcal{L}(D)) = A_U f_U$, $\Gamma(U, \mathcal{L}(D')) = A_U f'_U$, $\Gamma(U, \mathcal{L}(D + D')) = A_U f_U f'_U$. Since f_U and f'_U are free elements, we have an isomorphism $A_U f_U \otimes A_U f'_U \xrightarrow{\sim} A_U f_U f'_U$ which is essentially the canonical isomorphism $A \otimes_A A \xrightarrow{\sim} A$ defined by $\alpha \otimes \beta \rightsquigarrow \alpha\beta$ (Bourbaki, "Algèbre," Chap. III, § 1. No. 3, Corr. of Prop. 5). To prove that the collection of these isomorphisms defines globally an isomorphism of sheaves, we have to check that for $U \supset V$ the diagram

$$\begin{array}{ccc} A_U f_U \otimes A_U f'_U & \longrightarrow & A_U f_U f'_U \\ \downarrow & & \downarrow \\ A_V f_V \otimes A_V f'_V & \longrightarrow & A_V f_V f'_V \end{array}$$

is commutative. But this follows immediately from

$$(f_U \mid V)(f'_U \mid V) = f_U f'_U \mid V.$$

Next we have $\mathscr{L}(-D) = \mathscr{L}(D)^{\otimes(-1)}$. Indeed, $\Gamma(U, \mathscr{L}(D)) = A_U f_U$ and by definition $\Gamma(U, \mathscr{L}(D)^\vee) = \Gamma(U, \mathscr{L}(D))^\vee$. If D is defined by f_U, then $-D$ is defined by f_U^{-1} and we have to define a natural homomorphism $(A_U f_U)^\vee \to A_U f_U^{-1}$. Consider the diagram

$$\begin{array}{ccccc} A_U f_U & & (A_U f_U)^\vee & \xrightarrow{\xi \circ i \circ {}^t\chi} & A_U f_U^{-1} \\ {\scriptstyle\chi}\uparrow & & \downarrow{\scriptstyle {}^t\chi} & & \uparrow{\scriptstyle\xi} \\ A_U & {\scriptstyle i} & A_U & \xrightarrow{\ i\ } & A_U \end{array}$$

where χ is the isomorphism defined by $\chi(1) = f_U$, ${}^t\chi$ is its transpose, i is the identity map and ξ is the isomorphism defined by $\xi(1) = f_U^{-1}$. Then $\xi \circ i \circ {}^t\chi$ is the required homomorphism and it is clearly an isomorphism. It is easy to check that for $V \subset U$ these isomorphisms are compatible with the restriction maps and therefore we have an isomorphism $\mathscr{L}(-D) = \mathscr{L}(D)^{\otimes(-1)}$.

We have furthermore $\mathscr{L}(mD) \simeq \mathscr{L}(D)^{\otimes m}$. Thus the properties of the group of divisor classes are reflected in the properties of the group of equivalence classes of invertible $\mathscr{O}_X$-Modules $\mathscr{L}(D)$.

We can also define in the present setting the analogues of the classical vector spaces $L(D)$ which also play an important role in the investigations of Serre and Grothendieck. We say that $D > 0$ if the functions f_U which define it locally have no poles, i.e., if $f_U \in \Gamma(U, \mathscr{O}_X - \{0\})$. More generally $D > D'$ if $D - D' > 0$. Now given a divisor D the elements of $L(D)$ are the principal divisors (g) which satisfy $(g) > D$. Locally, if D is defined by a collection $f_U \in \Gamma(U, \mathscr{R}_X^*)$, then the condition $(g) > D$ means that $g \mid U = g_U f_U$, where g_U has no poles, i.e., $g_U \in A_U = \Gamma(U, \mathscr{O}_X)$. It follows that $g \mid U \in A_U f_U = \Gamma(U, \mathscr{L}(D))$ and thus $g \in \Gamma(X, \mathscr{L}(D))$. Therefore we have $L(D) = \Gamma(X, \mathscr{L}(D))$.

In classical algebraic geometry, if X is a projective variety, then $L(D)$ is finite dimensional. We shall prove that if X is "projective" over a field k and $\mathscr{F}$ is a coherent $\mathscr{O}_X$-Module, then $\Gamma(X, \mathscr{F})$ has finite dimension over k (see p. 308).

The modules $L(D)$ are the substitute for the notion of "complete linear system" in classical algebraic geometry. Finally let us observe that the idea to associate a sheaf $\mathscr{L}(D)$ with every divisor D is just a variant of Weil's idea to associate a line bundle with a divisor (Weil, "Fibre Spaces in Algebraic Geometry," Chicago 1952; Hirzebruch, "Neue topologische Methoden in der algebraischen Geometrie," § 15.2, p. 111).

Special Features of Grothendieck Theory

The first great advantage of considering an algebraic variety as a ringed space is the possibility to use cohomology just as for differentiable or analytic manifolds. This, however, was already done by Serre, so that in this direction the progress is not very significant. Let us observe that Serre had to work with Čech theory, which has the defect that in general the cohomology sequence is not exact. He circumvented this difficulty by proving that for coherent sheaves the cohomology sequence is exact. However, one often needs to consider quasi-coherent sheaves. In 1955, Grothendieck introduced his cohomology theory, in which the cohomology sequence is always exact and this solved the difficulties once and for all.

The more characteristic features of Grothendieck's theory are the following.

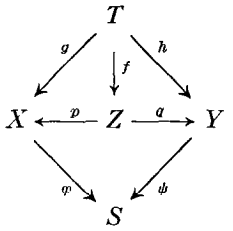
A. A Complete Dictionary of Commutative Algebra in Terms of Affine Schemata

To every ring A there corresponds functorially an affine schema $(\mathrm{Spec}(A), \tilde{A})$ and to every A-module M an $\tilde{A}$-Module $\tilde{M}$. Thus all operations of commutative algebra can be translated into geometric terms and algebraic geometry becomes a generalization of commutative algebra. In Serre's theory the rings were finitely generated over a field, here they are completely arbitrary (commutative with unit element).

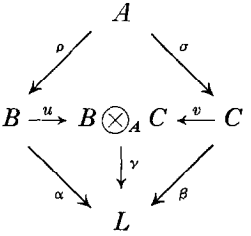
Thus let A and B be two rings and $\varphi : A \to B$ a homomorphism. Let M be an A-module, then $M \otimes_A B$ can be considered as the B-module obtained from M by extending the ring of scalars (cf. Bourbaki, "Algèbre," Chap. III, § 2, No. 1; in this reference A is taken to be a subring of B, but this is unnecessarily restrictive). We have a continuous map $f = {}^a\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and $f^*(\tilde{M})$ is isomorphic to the $\tilde{B}$-Module $(M \otimes_A B)^\sim$ (Grothendieck, "Eléments," I, 1.6.5). Conversely, if N is a B-module, the homomorphism φ defines on N a structure of A-module, setting $a.n = \varphi(a).n$ for $a \in A$, $n \in N$. Considered as an A-module, N will be denoted by $N_{[\varphi]}$ (Grothendieck, "Eléments," 0_I, 1.0.2). Then $f_*(\tilde{N})$ is isomorphic to $(N_{[\varphi]})^\sim$ (Grothendieck, "Eléments," I, 1.6.3). Hence the operations of extending and restricting the ring of scalars are translated into taking the inverse and direct image of sheaves.

B. *Formation of Products*

In the category of preschemata over a preschema S the product of two S-preschemata always exists in the sense in which it is defined for an arbitrary category (Grothendieck, "Eléments," I, 3.2.6). Given two S-preschemata X and Y with structural morphisms $\varphi : X \to S$, $\psi : Y \to S$, there exists a triple (Z, p, q), $p : Z \to X$, $q : Z \to Y$, such that for each S-preschema T and S-morphisms $g : T \to X$, $h : T \to Y$, such that $\varphi \circ g = \psi \circ h$, there exists a unique $f : T \to Z$ such that the diagram



is commutative. The preschema Z is denoted by $X \times_S Y$. In the case of affine schemata $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(C)$, $T = \mathrm{Spec}(L)$, the existence of Z follows by reversing the arrows from the commutative diagram



where $u(b) = b \otimes 1$, $v(c) = 1 \otimes c$, $\gamma(b \otimes c) = \alpha(b)\beta(c)$ (Grothendieck, "Eléments," I, 3.2.2). We say that (Z, p, q) is the pull-back of (φ, ψ, S) and that $(u, v, B \otimes_A C)$ is the push-up of (A, ρ, σ).

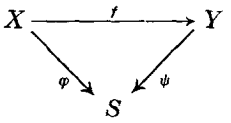
The direct product of two S-preschemata yields an example of a representable functor (Grothendieck, "Eléments," 0_{III}, 8.1.8). Indeed, let X and Y be two S-preschemata and let P be the functor from the category of S-preschemata into the category ***Ens*** of sets which to each S-preschema T assigns the set $P(T) = \{(g, h)\}$ of pairs of S-morphisms $g : T \to X$, $h : T \to Y$ such that $\varphi \circ g = \psi \circ h$, where φ and ψ are the structural morphisms of X and Y. Then there exists a unique S-preschema

Z such that P is isomorphic to the natural functor $\mathbf{h}_Z$, i.e., $P(T) = \mathbf{h}_Z(T) = \mathrm{Hom}(T, Z)$ for all T. We say that (Z, p, q) represents the functor P.

The underlying space of $X \times_S Y$ is in general not the set theoretic product of the spaces X and Y, even in the case when S, X, Y are prime spectra of fields. If $x \in X$, $y \in Y$, then there exists a point $z \in Z$ for which $p(z) = x$, $q(z) = y$ if and only if $\varphi(x) = \psi(y)$ (Grothendieck, "Eléments," I, 3.4.7). There might, however, exist more than one point in Z which projects into (x, y). For instance if $S = \mathrm{Spec}(k)$, $X = \mathrm{Spec}(K)$ and $Y = \mathrm{Spec}(K')$, where k, K, K' are fields, then these spectra all have one point, but $X \times_S Y = \mathrm{Spec}(K \otimes_k K')$ has in general more than one point. More precisely, if $x \in X$, $y \in Y$ are such that $s = \varphi(x) = \psi(y)$ and $\kappa(s)$, $\kappa(x)$, $\kappa(y)$ are the residual fields $(\mathcal{O}_S)_s/\mathfrak{m}_s$, $(\mathcal{O}_X)_x/\mathfrak{m}_x$, $(\mathcal{O}_Y)_y/\mathfrak{m}_y$, then there is a one-to-one correspondence between the points $z \in Z$ such that $p(z) = x$, $q(z) = y$ and the set of types of composite extensions (Bourbaki, "Algèbre," Chap. VIII, § 8, No. 1, def. 1) of $\kappa(x)$ and $\kappa(y)$ considered as extensions of $\kappa(s)$ (Grothendieck, "Eléments," I, 3.4.9). On the other hand, the types of composite extensions of $\kappa(x)$ and $\kappa(y)$ are in one-to-one correspondence with the prime ideals of $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$ (Bourbaki, "Algèbre," Chap. VIII, § 8, No. 1, Prop. 1), hence the points which project into (x, y) are essentially the elements of $\mathrm{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y))$. In the case considered by Serre when X and Y are spectra of integral rings finitely generated over an algebraically closed field $k = \kappa(s)$, x and y are closed points, hence $\kappa(x) = \kappa(y) = k$ and therefore over each (x, y) there lies exactly one point $z \in Z$. In this case the underlying set of Z can be identified with the set theoretic product of the underlying sets of X and Y, but the Zariski topology on Z is much finer than the product topology.

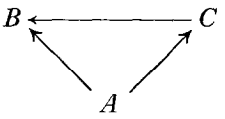
C. *Relativization and Change of Basis*

In Grothendieck's theory a preschema X is usually given over a second preschema S, i.e., with a structural morphism $\varphi : X \to S$. A morphism $f : X \to Y$ of two S-preschemata with structural morphisms $\varphi : X \to S$, $\psi : Y \to S$ is an S-morphism if the diagram



is commutative (Grothendieck, "Eléments," I, 2.5.2). This corresponds

to the notion of A-homomorphisms of two A-algebras B and C where we want that the diagram



should be commutative. Every preschema X is in a unique way a preschema over Spec($\mathbf{Z}$), since for every ring A there exists a homomorphism $\mathbf{Z} \to A$.

Each notion has now an absolute and a relative aspect and in particular a morphism $f : X \to Y$ defines a Y-preschema X. Thus $f : X \to Y$ is a separated morphism if the diagonal morphism $\Delta_{X|Y} : X \to X \times_Y X$ is a closed immersion (Grothendieck, "Eléments," I, 5.4.1). We say that X is a separated preschema or a schema, if the morphism $X \to \mathrm{Spec}(\mathbf{Z})$ is separated. We shall have similar definitions for affine morphisms, etc. If $f : X \to Y$ and $g : Y \to Z$ are morphisms having certain properties, one of the important problems of the theory is to determine whether $g \circ f$ has the same properties.

If B is an A-algebra (i.e. we have a homomorphism $A \to B$) and $\varphi : A \to A'$ is a homomorphism, then $B' = B \otimes_A A'$ is the A'-algebra obtained from B by extending the ring of scalars. We have the commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & B' = B \otimes_A A' \\ \uparrow & & \uparrow \\ A & \xrightarrow{\varphi} & A' \end{array}$$

(Bourbaki, "Algèbre," Chap. III, § 3, No. 4). Similarly if X is an S-preschema and $\varphi : S' \to S$ a morphism, then $X_{(S')} = X \times_S S'$ can be considered as the S'-preschema obtained from X by extending the basic preschema from S to S'. We have the commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{p} & X_{(S')} = X \times_S S' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & \xleftarrow{\varphi} & S' \end{array}$$

where $f : X \to S$ is the structural morphism, p and f' the two projection morphisms (Grothendieck, "Eléments," I, 3.3.6). Here again the question is whether f' will inherit the properties of f. For instance if $f : X \to S$ is a separated morphism, then $f' : X_{(S')} \to S'$ is also separated (Grothendieck, "Eléments," I, 5.5.1). Conversely, if $\varphi : S' \to S$

is faithfully flat (Grothendieck, "Eléments," 0_1, 6.7.8) and $f' : X_{(S')} \to S'$ is separated, then $f : X \to S$ is separated (Grothendieck, "Eléments," IV, 2.7.1).

The reduction of a preschema modulo a prime number p, a thing which the classical theory never managed to define properly, is just a special case of the extension of the basic preschema. Indeed, we have a canonical epimorphism $\mathbf{Z} \to \mathbf{Z}/(p)$ and hence the commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{Z}) & \longleftarrow & \mathrm{Spec}(\mathbf{Z}/(p)). \end{array}$$

Many other things can also interpreted as a product. For instance let $f : X \to Y$ be a morphism, Z a subpreschema of Y. Then $X \times_Y Z$ can be considered as the subpreschema $f^{-1}(Z)$ of X (Grothendieck, "Eléments," I, 4.4.1).

Let $(X, \mathcal{O}_X)$ be the affine schema $X = \mathrm{Spec}(A)$. For each $x \in X$ the local ring $\mathcal{O}_x$ is a ring of fractions of A and therefore we have a homomorphism $A \to \mathcal{O}_x$. To this homomorphism there corresponds a morphism $\mathrm{Spec}(\mathcal{O}_x) \to X$. We say that $\mathrm{Spec}(\mathcal{O}_x)$ is the local schema of X at the point x (Grothendieck, "Eléments," I, 2.4.1). The map $\mathrm{Spec}(\mathcal{O}_x) \to X$ is a homeomorphism from the underlying topological space of $\mathrm{Spec}(\mathcal{O}_x)$ onto the subspace S_x of X consisting of the points y such that $x \in \overline{\{y\}}$, i.e., the generizations of x (Grothendieck, "Eléments," I, 2.4.2). $\mathrm{Spec}(\mathcal{O}_x)$ is not a subpreschema of X since S_x is exactly the opposite of a closed set: S_x contains all generizations of x while a closed set contains all specializations of its points. In fact S_x is the intersection of all neighborhoods of x. Since $\mathcal{O}_x$ is a local ring, $\mathrm{Spec}(\mathcal{O}_x)$ has only one closed point x' (identified to x) and all other points y' of $\mathrm{Spec}(\mathcal{O}_x)$ are generizations of x', i.e., $x' \in \overline{\{y'\}}$. There is an isomorphism between the sheaf of $\mathrm{Spec}(\mathcal{O}_x)$ and the sheaf induced by $\mathcal{O}_X$ on S_x.

Let $f : X \to Y$ be a morphism, y a point of Y. If $\kappa(y)$ is the residual field of the local ring $\mathcal{O}_y$ we have an epimorphism $\mathcal{O}_y \to \kappa(y)$ and thus a morphism of affine schemata $\mathrm{Spec}(\kappa(y)) \to \mathrm{Spec}(\mathcal{O}_y)$. We obtain the commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \times_Y \mathrm{Spec}(\mathcal{O}_y) & \longleftarrow & X \times_Y \mathrm{Spec}(\kappa(y)) \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & \mathrm{Spec}(\mathcal{O}_y) & \longleftarrow & \mathrm{Spec}(\kappa(y)). \end{array}$$

In the bottom row all arrows are homeomorphisms on subspaces. The inverse image $f^{-1}(\mathrm{Spec}\ \mathscr{O}_y)$ as a $\mathrm{Spec}(\mathscr{O}_y)$-preschema will be considered to be $X \times_Y \mathrm{Spec}(\mathscr{O}_y)$. The underlying topological space of $X \times_Y \mathrm{Spec}(\mathscr{O}_y)$ is homeomorphic to the topological space $f^{-1}(\mathrm{Spec}(\mathscr{O}_y))$ and the local rings of the two spaces are the same (Grothendieck, "Eléments," I, 3.6.5). Similarly the fiber $f^{-1}(y)$ as a $\mathrm{Spec}(\kappa(y))$-preschema is defined as $X \times_Y \mathrm{Spec}(\kappa(y))$. As topological spaces $X \times_Y \mathrm{Spec}(\kappa(y))$ and $f^{-1}(y)$ are homeomorphic (Grothendieck, "Eléments," I, 3.6.2). The local ring at a point x of the fiber is $(\mathscr{O}_{f^{-1}(y)})_x = \mathscr{O}_x \otimes_{\mathscr{O}_y} \kappa(y) \simeq \mathscr{O}_x/\mathfrak{m}_y\mathscr{O}_x$.

D. The Introduction of Nilpotent Elements

In classical algebraic geometry all rings were integral. In Serre's approach zero divisors were permitted, but the rings were reduced, i.e., there were no nilpotent elements. Grothendieck considers completely arbitrary commutative rings with unit element. It is important to consider nilpotent elements since they introduce themselves naturally. For instance $\mathbf{Z}/(p^n)$ has nilpotent elements for $n > 1$ and $B \otimes_k C$ may have nilpotent elements even if k, B, C are fields. But the consideration of nilpotent elements is also useful. Take for instance the local ring A of all infinitely differentiable functions (or of all analytic functions which have no poles at the origin). If $\mathfrak{m}$ is the maximal ideal of A consisting of all functions which vanish at the origin, then $\mathfrak{m}^2$, $\mathfrak{m}^3$,... are the functions whose 1st, 2nd,... derivatives vanish at the origin. Thus the study of $A/\mathfrak{m}^n$ amounts to the study of the Taylor development of the functions, i.e., of their local differential properties. The study of differential properties of functions is of paramount importance in algebraic geometry. Similarly one can also study the infinitesimal neighborhoods of a fiber by considering the local rings $\mathscr{O}_x/\mathfrak{m}_y^n\mathscr{O}_x$.

Reduced Preschemata

(See Grothendieck, "Eléments," I, 5.1.) The nilradical $\mathfrak{N} = \mathfrak{r}(0)$ of a ring A is the set of its nilpotent elements. A is reduced if $\mathfrak{N} = (0)$. For any ring A, the quotient $A/\mathfrak{N}$ of A modulo its nilradical is reduced (Grothendieck, "Eléments," 0_{I}, 1.1.1). A preschema $(X, \mathscr{O}_X)$ is *reduced* if each $\mathscr{O}_x$ is reduced. We associate now with each preschema a reduced preschema.

Proposition (Grothendieck, "Eléments," I, 5.1.1). *Let $(X, \mathscr{O}_X)$ be a preschema, $\mathscr{B}$ a quasi-coherent $\mathscr{O}_X$-Algebra. There exists a unique quasi-*

coherent Ideal $\mathscr{N}$ in $\mathscr{B}$ such that for each $x \in X$ the stalk $\mathscr{N}_x$ is the nilradical of $\mathscr{B}_x$. If $X = \mathrm{Spec}(A)$, $\mathscr{O}_X = \tilde{A}$, $\mathscr{B} = \tilde{B}$, then $\mathscr{N} = \tilde{\mathfrak{N}}$, where $\mathfrak{N}$ is the nilradical of B.

The ideal $\mathscr{N}$ is the Nilradical of $\mathscr{B}$. The Nilradical of $\mathscr{O}_X$ will be denoted by $\mathscr{N}_X$. It determines a closed subpreschema of X (Grothendieck, "Eléments," I, 4.1.2) which as a topological space is identical with X and whose sheaf is $\mathscr{O}_X/\mathscr{N}_X$. This closed preschema is therefore reduced, it is called the reduced preschema associated with X and denoted by X_{red} (Grothendieck, "Eléments," I, 5.1.3). There exists a canonical injection morphism $f = (\psi, \theta) : X_{\mathrm{red}} \to X$, where $\psi = 1_X$ is the identity map of the underlying topological spaces and $\theta : \mathscr{O}_X \to \mathscr{O}_X/\mathscr{N}_X$ is the canonical epimorphism. X_{red} is the smallest closed subpreschema of X having the same underlying space as X and also the only reduced subpreschema of X having the same underlying space (Grothendieck, "Eléments," I, 5.1.2).

A preschema $(X, \mathscr{O}_X)$ is *integral* if it is irreducible and reduced (Grothendieck, "Eléments," I, 2.1.8). The first condition means that the underlying topological space X is irreducible, i.e., the intersection of two non-empty open sets is not empty (Grothendieck, "Eléments," 0_{I}, 2.1.1) and is purely topological. If X is irreducible, then obviously X_{red} is integral.

A preschema $(X, \mathscr{O}_X)$ is integral if and only if for every affine open set U the ring $\Gamma(U, \mathscr{O}_X)$ is integral. Indeed, if every $\Gamma(U, \mathscr{O}_X)$ is integral, then it is in particular reduced and thus X is reduced. Furthermore U is irreducible since the affine schema corresponding to the union of two disjoint open sets is the sum (Grothendieck, "Eléments," I, 3.1) of two affine schemata and a direct sum of two rings always has zero divisors. Thus X has a covering by irreducible open sets two of which always intersect and hence is itself irreducible (Grothendieck, "Eléments," 0_{I}, 2.1.4). Conversely, if X is irreducible then U is irreducible and therefore the nilradical of $\Gamma(U, \mathscr{O}_X)$ is prime (Grothendieck, "Eléments," I, 1.1.13). Since X is reduced, $\Gamma(U, \mathscr{O}_X)$ is reduced, i.e., its nilradical is (0). Thus (0) is a prime ideal in $\Gamma(U, \mathscr{O}_X)$ which means that there are no zero divisors, i.e., the ring is integral.

If X is an integral preschema, then for every $x \in X$ the local ring $\mathscr{O}_x$ is integral. A preschema $(X, \mathscr{O}_X)$ is *locally integral* if each point $x \in X$ has a neighborhood U such that the preschema induced on U by $(X, \mathscr{O}_X)$ is integral (Grothendieck, "Eléments," I, 2.1.8). If X is locally noetherian (see p. 261 or Grothendieck, "Eléments," I, 6.1.1), and each $\mathscr{O}_x$ is integral, then X is locally integral (Grothendieck, "Eléments," I, 6.1.13).

Given a morphism of preschemata $f = (\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ there exists a unique morphism $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ such that the diagram

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative (Grothendieck, "Eléments," I, 5.1.5). If $f_{\text{red}} = (\psi', \theta')$, then $\psi' = \psi$ and θ' is the natural homomorphism $\mathcal{O}_Y/\mathcal{N}_Y \to \psi_*(\mathcal{O}_X/\mathcal{N}_X)$.

If X and Y are two S-preschemata, then $X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}} = X_{\text{red}} \times_S Y_{\text{red}}$ (Grothendieck, "Eléments," I, 5.1.7) and $(X \times_S Y)_{\text{red}} = (X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}})_{\text{red}}$. However, if X and Y are reduced, $X \times_S Y$ is not necessarily reduced, since if A and B are reduced rings $A \otimes B$ might have nilpotent elements (Grothendieck, "Eléments," I, 5.1.8).

Proposition (Grothendieck, "Eléments," I, 5.2.1). *Let $(X, \mathcal{O}_X)$ be a preschema and Y a locally closed subspace of the underlying topological space of X. Then there exists a unique reduced subpreschema of X whose underlying space is Y.*

In general there can exist several subpreschemata having Y as the underlying topological space.

Proposition (Grothendieck, "Eléments," I, 5.2.2). *Let X be a reduced preschema $f : X \to Y$ a morphism, Z a closed subpreschema of Y such that $f(X) \subset Z$ and $j : Z \to Y$ the injection morphism. Then f can be factored in $X \xrightarrow{g} Z \xrightarrow{j} Y$.*

Finiteness Conditions

Most really deep theorems in algebraic geometry require some finiteness condition. We shall start with finiteness conditions in algebra. There are three important and essentially different cases to consider:

(1) a finitely generated module over a ring A;

(2) a finitely generated algebra over a ring A;

(3) a finitely generated field over a subfield k.

The meaning of this terminology in each case is the following:

(1) A module M is finitely generated or of finite type over a ring A if there exist elements $c_1, \ldots, c_n \in M$ such that every $x \in M$ can be written as $x = \sum_{i=1}^{n} \alpha_i c_i$, $\alpha_i \in A$. This means that we have an exact sequence $A^n \to M \to 0$.

(2) An algebra B is finitely generated or of finite type over a ring A if there exist elements $c_1, \ldots, c_n \in B$ such that every $x \in B$ can be written as a polynomial $x = P(c_1, \ldots, c_n)$ in the c_i with coefficients in A. This means that we have a surjective homomorphism $A[X_1, \ldots, X_n] \to B$.

(3) A field K is finitely generated over the subfield k if there exist elements $c_1, \ldots, c_n \in K$ such that every element $x \in K$ can be written as a rational function $x = P(c_1, \ldots, c_n)/Q(c_1, \ldots, c_n)$ of the c_i with coefficients in k.

Now an algebra B over A is in particular a module over A. If B is finitely generated over A *as a module*, we say that B is *finite* over A. Clearly if B is finite over A then B is of finite type over A. The converse, however, is not true, as shown by the example $A[X]$ which as an algebra over A is generated by X and as a module over A by the elements $1, X, X^2, \ldots, X^n, \ldots$.

Definition. *An algebra B over A is integral over A if every $y \in B$ satisfies an equation of integral dependence*

$$y^n + a_1 y^{n-1} + \cdots + a_n = 0$$

with $a_i \in A$ $(1 \leqslant i \leqslant n)$ and leading coefficient 1.

Theorem. *If the algebra B is integral and of finite type over A then it is finite.*

Indeed, if $c_1, \ldots, c_p$ is a system of generators of the algebra B and n_i $(1 \leqslant i \leqslant p)$ is the degree of the equation of integral dependence satisfied by c_i, then every $x \in B$ is a linear combination of the monomials $c_1^{k_1} \cdots c_p^{k_p}$, where $0 \leqslant k_i < n_i$, since the higher powers of the elements c_i can be eliminated with the help of the equation of integral dependence.

A field K can be finitely generated over a subfield k without being a finitely generated algebra over k. For instance the field $k(X)$ of rational fractions with coefficients in k is generated as a field by X but is not finitely generated as an algebra. However, if the field k is an algebra of finite type over k, then it is finite over k and indeed an algebraic extension

of finite degree of k (p. 260). Obviously if K is an algebra of finite type over k, then K is finitely generated over k.

A (commutative) ring A is *noetherian* if every ideal is a finitely generated module over A, or equivalently if every increasing chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_n \subset \cdots$ of ideals becomes stationary, i.e., $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \ldots$ for some n (ascending chain condition). Hilbert's basissatz states that an algebra of finite type over a noetherian ring is itself noetherian. If A is a noetherian ring, M a finitely generated module over A, then every submodule of M is also finitely generated.

A subring of a noetherian ring is not necessarily noetherian. For instance the ring $K[X_1, \ldots, X_n, \ldots]$ of polynomials in infinitely many indeterminates is not noetherian but its field of fractions certainly is noetherian. The quotient $A/\mathfrak{a}$ of a noetherian ring A modulo an ideal is noetherian.

The tensor product of two noetherian rings is not necessarily noetherian. For example, if k is an imperfect field, then $k^{p^{-\infty}} \otimes_k k^{p^{-\infty}}$ is not noetherian.

A (commutative) ring A is *artinian* if given a decreasing chain $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_n \supset \cdots$ of ideals there exists an index n such that $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \cdots$ (descending chain condition). Every artinian ring is noetherian (Bourbaki, "Algèbre," Chap. VIII, § 6, No. 5, Corr. 3 of Prop. 12).

An ideal in which every element is nilpotent is called a nilideal (Bourbaki, "Algèbre," Chap. VIII, § 6, No. 1, def. 1). In a noetherian ring every nilideal, and in particular the nilradical, is nilpotent. In an artinian ring even the radical $\mathfrak{R}$ is nilpotent (C. Hopkins), this means much more, since the nilradical $\mathfrak{N}$ is the intersection of all prime ideals, while $\mathfrak{R}$ is the intersection of all maximal ideals (Bourbaki, "Algèbre," Chap. VIII, § 6, No. 4, Th. 3).

The quotient $A/\mathfrak{a}$ of an artinian ring A modulo an ideal $\mathfrak{a}$ is artinian. A subring of an artinian ring is, however, not necessarily artinian, since it may even be non noetherian.

If A is an artinian ring and $\mathfrak{R} = \mathfrak{R}(A)$ its radical, then $A/\mathfrak{R}$ is semisimple (Bourbaki, "Algèbre," Chap. VIII, § 6, No. 4, Corr. 3 of Th. 4) and hence the direct sum of a finite number of fields (Bourbaki, "Algèbre," Chap. VIII, § 6, No. 4, Prop. 9).

A ring A is artinian if and only if it is of finite length as an A-module, i.e., it has a Jordan–Hölder series (Bourbaki, "Algèbre," Chap. VIII, § 2, No. 1, Prop. 3 and § 6, No. 5, Corr. 2 of Prop. 12). A ring is an artinian local ring if and only if it is a local ring, its maximal ideal $\mathfrak{m}$ is nilpotent

and it is noetherian. Indeed, if A is local, then $\mathfrak{m}$ is the radical of A, hence if A is artinian, $\mathfrak{m}$ is nilpotent. Conversely, if $\mathfrak{m}$ is nilpotent, then A has a natural filtration $A \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^n = (0)$. Now, since A is noetherian, every $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is a finitely generated vector space over the field $A/\mathfrak{m}$ and has therefore finite length. Thus the length of A is also finite since it is the sum of the lengths of the $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ $(j = 0,..., n-1)$.

A (commutative) ring which has only a finite number of maximal ideals is said to be *semi-local.* An artinian semi-local ring is the direct product of a finite number of artinian local rings. This follows by lifting idempotents from the fact that $A/\mathfrak{R}(A)$ is the direct sum of a finite number of fields and that $\mathfrak{R}(A)$ is nilpotent. It is not true, however, that an arbitrary semi-local ring is the product of local rings.

In an artinian ring every prime ideal is maximal (Samuel–Zariski, "Commutative Algebra," Vol. I, Chap. IV, § 2, p. 203). Conversely, if A is a noetherian ring in which *every* prime ideal is maximal, then A is artinian. In other words, a noetherian ring of dimension zero (p. 242) is artinian. (The ring $\mathbf{Z}$, which is noetherian but not artinian, seems to contradict this theorem. However, in $\mathbf{Z}$ the ideal (0) is prime but not maximal). Let us sketch the proof. By Zorn's lemma every prime ideal contains a minimal prime ideal and in a noetherian ring the number of minimal prime ideals is finite (p. 262). Let $\mathfrak{m}_1,..., \mathfrak{m}_n$ be the minimal prime ideals of A. Since every prime ideal is maximal, these are the only prime ideals in A and the radical and the nilradical of A are both equal to $\mathfrak{N} = \bigcap_{i=1}^n \mathfrak{m}_i$. We also have $\mathfrak{m}_i + \mathfrak{m}_j = A$ for $i \neq j$. By the *Chinese remainder theorem* $A/\mathfrak{N}$ is isomorphic to the direct product of fields $\prod_{i=1}^n A/\mathfrak{m}_i$. The nilradical $\mathfrak{N}$ is nilpotent, hence we have the filtration $A \supset \mathfrak{N} \supset \mathfrak{N}^2 \supset \cdots \supset \mathfrak{N}^k = (0)$. Now $\mathfrak{N}^j/\mathfrak{N}^{j+1}$ is a module over the ring $A/\mathfrak{N}$ and $A/\mathfrak{N}$ is artinian since it is the product of a finite number of fields (Bourbaki, "Algèbre," Chap. VIII, § 2, No. 3, Prop. 5). On the other hand $\mathfrak{N}^j/\mathfrak{N}^{j+1}$ is finitely generated, because A is noetherian, hence $\mathfrak{N}^j/\mathfrak{N}^{j+1}$ is of finite length (Bourbaki, "Algèbre," Chap. VIII, § 6, No. 5, Corr. 1 of Prop. 12). Thus the length of A, which is the sum of the lengths of the $\mathfrak{N}^j/\mathfrak{N}^{j+1}$, is also finite, hence A is artinian.

The going-up theorem of Cohen–Seidenberg (proved by Krull in a special case). Let A be a commutative ring, ρ a homomorphism from A into another ring B, so that B can be considered as an algebra over A and suppose that ρ is *injective.* Assume furthermore that B is integral over A (p. 257). Then for every prime ideal $\mathfrak{p}$ in A, there exists a prime

ideal $\mathfrak{p}'$ in B such that $\rho^{-1}(\mathfrak{p}') = \mathfrak{p}$ (Samuel–Zariski, "Commutative Algebra," Vol. I, Chap. V, § 2, Th. 3, p. 257). Geometrically this means that the associated map ${}^a\rho : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

Noether's Normalization Lemma. *Let k be a field, A a k-algebra of finite type, $A \neq \{0\}$. Then there exists a system $x_1, \ldots, x_r$ of elements in A, algebraically independent over k, such that A is integral over $A' = k[x_1, \ldots, x_r]$* (Samuel–Zariski, "Commutative Algebra," Vol. I, Chap. V, § 4, Th. 8, p. 266). *Since A is of finite type and integral over A', it is finite over A'* (p. 257).

Let us now see some consequences of these two results. A is a k-algebra of finite type, $A \neq \{0\}$.

(1) *Let $\bar{k}$ be the algebraic closure of k. There exist k-homomorphisms $A \to \bar{k}$.*

Proof. Since the $x_1, \ldots, x_r$ are free, there exists a k-homomorphism $\varphi : A' \to \bar{k}$. Let $\mathfrak{p}' = \mathrm{Ker}(\varphi)$, then $A'/\mathfrak{p}'$ is isomorphic to a subring of k and hence is an integral ring, i.e., $\mathfrak{p}'$ is prime. By the going-up theorem there exists a prime ideal $\mathfrak{p}$ in A such that $\mathfrak{p} \cap A' = \mathfrak{p}'$, since A is integral over A'. Now $A'/\mathfrak{p}'$ is a subring of $A/\mathfrak{p}$ and $A/\mathfrak{p}$ is finite over $A'/\mathfrak{p}'$. It follows that the field of fractions K of $A/\mathfrak{p}$ is finite over the field of fractions of $A'/\mathfrak{p}'$ and thus, since $\bar{k}$ is algebraically closed, there exists a homomorphism $K \to \bar{k}$ extending the map $A'/\mathfrak{p}' \to \bar{k}$ (Bourbaki, "Algèbre," Chap. V, § 4, No. 2, Th. 1).

(2) *Let $\mathfrak{m}$ be a maximal ideal of A, then the field $A/\mathfrak{m}$ is finite over k. In particular, if K is a field of finite type as a ring over k, then taking $\mathfrak{m} = (0)$ we see that K is finite over k* (p. 258).

Indeed, there exists a k-homomorphism $A/\mathfrak{m} \to \bar{k}$, which is necessarily a k-monomorphism. Thus $A/\mathfrak{m}$ is algebraic over k. Since it is finitely generated, it is finite over k (Bourbaki, "Algèbre," Chap. V, § 3, No. 2, Prop. 5).

(3) **Hilbert's nullstellensatz** (p. 235). *Let $\mathfrak{a}$ be an ideal in $A' = k[X_1, \ldots, X_n]$, $\mathfrak{a} \neq A'$. There exist elements $\xi_1, \ldots, \xi_n \in \bar{k}$ such that $P(\xi_1, \ldots, \xi_n) = 0$ for all $P \in \mathfrak{a}$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal in A' such that $\mathfrak{a} \subset \mathfrak{m}$. By 1) there exists a k-homomorphism $\varphi : \mathfrak{m} \to \bar{k}$. Set $\xi_i = \varphi(X_i)$ for $1 \leqslant i \leqslant n$. Then $P(\xi_1, \ldots, \xi_n) = 0$ for all $P \in \mathfrak{m}$ and *a fortiori* for all $P \in \mathfrak{a}$.

(4) *Let A be a finitely generated algebra over the field k. An element $x \in A$ is nilpotent if and only if $f(x) = 0$ for every homomorphism $f : A \to \bar{k}$.*

Proof. If x is nilpotent in A, then $f(x)$ is nilpotent in $\bar{k}$ and since $\bar{k}$ is a field, $f(x) = 0$. Conversely suppose that x is not nilpotent in A. Then the ring of fractions $A_x = A[1/x]$ is not $\{0\}$ (Grothendieck, "Eléments," 0_I, 1.2.3). On the other hand A_x is a finitely generated algebra over k and so satisfies the hypotheses of (1), i.e., there exists a k-homomorphism $g : A[1/x] \to \bar{k}$. Let $h : A \to A[1/x]$ be the canonical map $a \rightsquigarrow a/1$, and $f = g \circ h$, then $f(x)g(1/x) = g(x/1)g(1/x) = g(1) = 1$ and thus $f(x) \neq 0$.

(5) *Let $\mathfrak{a}$ be an ideal of $A' = k[X_1, \ldots, X_n]$, $\mathfrak{a} \neq A'$. If $Q \in A'$ vanishes for all zeros of $\mathfrak{a}$, then $Q^\mu \in \mathfrak{a}$ for some μ* (p. 235).

Proof. Consider a homomorphism $\varphi : A'/\mathfrak{a} \to k$. If $\xi_i = \varphi(X_i)$, then $P(\xi_1, \ldots, \xi_n) = 0$ for all $P \in \mathfrak{a}$ and thus by assumption $Q(\xi_1, \ldots, \xi_n) = 0$, i.e. $\varphi(Q) = 0$. By (4) Q is nilpotent in $A'/\mathfrak{a}$, i.e., some power of Q belongs to $\mathfrak{a}$.

We now turn to finiteness conditions in the geometric language.

A preschema $(X, \mathcal{O}_X)$ is *noetherian* if there exists a finite covering of X by affine open sets U_i such that each ring $\Gamma(U_i, \mathcal{O}_X)$ is noetherian. A preschema $(X, \mathcal{O}_X)$ is *locally noetherian* if every point $x \in X$ has an affine open neighborhood U such that $\Gamma(U, \mathcal{O}_X)$ is a noetherian ring (Grothendieck, "Eléments," I, 6.1.1).

If $(X, \mathcal{O}_X)$ is a noetherian preschema then the topological space X is noetherian, i.e., if $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$ is a descending chain of closed sets, then $F_n = F_{n+1} = \cdots$ for some n (Grothendieck, "Eléments," I, 6.1.2). The converse is false, indeed let V be a valuation ring with non-discrete valuation v, having for instance $\mathbf{R}$ as the group of values. Then taking the ideals $\{x \mid v(x) > \alpha\}$ we obtain an infinite ascending chain of ideals, i.e. V is not noetherian. However, V has exactly two prime ideals, (0) and the maximal ideal $\mathfrak{m} = \{x \mid v(x) > 0\}$. Thus $X = \mathrm{Spec}(V)$ has two points and is a noetherian space. The point a corresponding to (0) is open and a generic point, i.e., $X = \overline{\{a\}}$, the point b corresponding to $\mathfrak{m}$ is closed, i.e., $\overline{\{b\}} = b$.

An affine schema $X = \mathrm{Spec}(A)$ is noetherian if and only if A is noetherian (Grothendieck, "Eléments," I, 6.1.3). A subpreschema of a noetherian (resp. locally noetherian) preschema is noetherian (resp. locally noetherian) (Grothendieck, "Eléments," I, 6.1.4).

A noetherian space has only finitely many irreducible components (Grothendieck, "Eléments," 0_I, 2.2.5). This gives a topological proof of the fact that in a noetherian ring there are only finitely many minimal prime ideals (p. 259).

The Nilradical $\mathcal{N}_X$ of a noetherian preschema $(X, \mathcal{O}_X)$ is nilpotent (Grothendieck, "Eléments," I, 6.1.6).

Proposition (Grothendieck, "Eléments," I, 6.2.2). *Let $(X, \mathcal{O}_X)$ be a preschema. The following conditions are equivalent*:

(a) $X = \mathrm{Spec}(A)$, *where A is an artinian ring.*

(b) *X is noetherian with a discrete underlying space.*

(c) *X is noetherian and the underlying space is T_1* (Fréchet).

These conditions imply that X is finite and A is the direct sum of a finite number of artinian local rings.

If $(X, \mathcal{O}_X)$ satisfies the previous conditions it is said to be an *artinian* preschema. Let us sketch the proof of the implication c) $\Rightarrow$ a). Let $X = \bigcup_{i=1}^{n} U_i$, where U_i are affine open sets and the $\Gamma(U_i, \mathcal{O}_X)$ are noetherian rings. Clearly U_i is a T_1 space, hence every prime ideal in $\Gamma(U_i, \mathcal{O}_X)$ is maximal, from where it follows that $\Gamma(U_i, \mathcal{O}_X)$ is an artinian ring (p. 259). Since U_i is quasi-compact it is necessarily finite and thus X is a discrete finite set. $X = \mathrm{Spec}(A)$ where A is the finite sum of artinian local rings.

Definition (Grothendieck, "Eléments," I, 6.3.1). *A morphism $f: X \to Y$ is of finite type if there exists a covering (V_α) of Y by affine open sets such that $f^{-1}(V_\alpha)$ is the finite union of affine open sets $U_{\alpha i}$ such that every ring $A(U_{\alpha i})$ is a finitely generated algebra over $A(V_\alpha)$. In this case we also say that X is a preschema of finite type over Y.*

Theorem (Grothendieck, "Eléments," I, 6.3.3). *$f: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is of finite type if and only if B is a finitely generated algebra over A.*

We say that a property $\boldsymbol{P}$ of morphisms $f: X \to Y$ is *local* over Y when $\boldsymbol{P}$ holds for Y if and only if it holds for every open subpreschema of Y. Thus the property "$f^{-1}(V_\alpha)$ is the finite union of affine open sets $U_{\alpha i}$ such that every $A(U_{\alpha i})$ is of finite type over $A(V_\alpha)$" is local over Y, i.e., the notion of preschema of finite type over Y is local over Y. In particular if $f: X \to Y$ is of finite type and V is any affine set in Y, then $f^{-1}(V)$ is the finite union of affine open sets U_i such that every $A(U_i)$ is of finite type over $A(V)$ (Grothendieck, "Eléments," I, 6.3.2).

We define similarly the notion of a morphism $f: X \to Y$ local over X. There exist very few properties of morphisms which are local neither over X nor over Y (e.g., projective morphisms, see p. 313), most properties are local over Y.

In the above definition of morphism of finite type it is essential to require that for each V_α there may exist more than one $U_{\alpha i}$. For instance the projective space $\mathbf{P}_n(k)$ is of finite type over $\mathrm{Spec}(k)$ but it is not affine, it can only be covered by finitely many affine open sets (p. 240).

If the morphism $f: X \to Y$ is of finite type and Y is noetherian (resp. locally noetherian) then X is noetherian (resp. locally noetherian) (Grothendieck, "Eléments," I, 6.3.7). This is a consequence of the fact that the notion of morphism of finite type is local over Y.

To establish further properties of morphisms of finite type we first consider the following general situation (Grothendieck, "Eléments," I, 5.5.12). Let $\boldsymbol{P}$ be a property of morphisms of preschemata and consider the following propositions:

(i) *Every closed immersion has property* $\boldsymbol{P}$.

(ii) *If* $f: X \to Y$ *and* $g: Y \to Z$ *have property* $\boldsymbol{P}$, *then* $g \circ f: X \to Z$ *has property* $\boldsymbol{P}$.

(iii) *If* $f: X \to Y$ *is an* S*-morphism having property* $\boldsymbol{P}$, *then for every change of basis* $S' \to S$ *the morphism* $f_{(S')}: X_{(S')} \to Y_{(S')}$ *has property* $\boldsymbol{P}$.

(iv) *If* $f: X \to Y$ *and* $f': X' \to Y'$ *are* S*-morphisms having property* $\boldsymbol{P}$ *then* $f \times_S f': X \times_S X' \to Y \times_S Y'$ *has property* $\boldsymbol{P}$.

(v) *If the composition* $g \circ f$ *of the morphisms* $f: X \to V$, $g: Y \to Z$ *has property* $\boldsymbol{P}$ *and* g *is a separated morphism then* f *has property* $\boldsymbol{P}$.

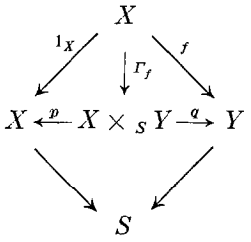
(vi) *If* f *has property* $\boldsymbol{P}$ *then so has* f_{red}.

If we suppose that propositions (i) and (ii) hold then (iii) and (iv) are equivalent. Furthermore (v) and (vi) are consequences of (i), (ii) and (iii).

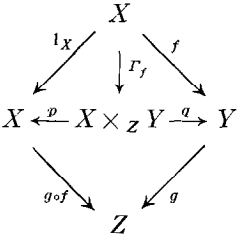
Indeed, let us assume that (ii) and (iii) are true. The morphism $f \times_S f': X \times_S X' \to Y \times_S Y'$ is the composition of the two morphisms $f \times_S 1_{X'}: X \times_S X' \to Y \times_S X'$ and $1_Y \times_S f': Y \times_S X' \to Y \times_S Y'$. Now by (iii) the morphisms $f_{(X')} = f \times_S 1_{X'}$ and $f'_{(Y)} = 1_Y \times_S f'$ have property $\boldsymbol{P}$, hence by (ii) also $f \times_S f'$ has property $\boldsymbol{P}$.

Conversely suppose that (i) and (iv) are true. Since $1_{S'}$ is a closed immersion it has property $\boldsymbol{P}$ by (i). Thus if f has property $\boldsymbol{P}$, by (iv) so does $f_{(S')} = f \times_S 1_{S'}$. Hence (iii) is true.

Let us suppose next that (i), (ii) and (iii) are true. The graph $\Gamma_f : X \to X \times_S Y$ of an S-morphism $f : X \to Y$ is defined as $\Gamma_f : (1_X, f)_S$, i.e., as the unique morphism for which the diagram



is commutative (Grothendieck, "Eléments," I, 3.3.14). Now let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. Both X and Y can be considered as Z-preschemata with structural morphisms $g \circ f$ and g; furthermore f is a Z-morphism. If Γ_f is the graph of f we have the commutative diagram



and in particular we have the factorization

$$f : X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{q} Y.$$

Let us now assume that $g \circ f$ has Property ***P*** and that g is separated. Then Γ_f is a closed immersion (Grothendieck, "Eléments," I, 5.4.3) and by (i) has property ***P***. On the other hand $q : Y \times_Z Y \to Y$ can be identified with the morphism $(g \circ f) \times_Z 1_Y : X \times_Z Y \to Z \times_Z Y$, since the projection $Z \times_Z Y \to Y$ is a functorial isomorphism (Grothendieck, "Eléments," I, 3.3.3). Since 1_Y is a closed immersion and (iv) holds, it follows that q has property ***P*** and thus by (ii) also $f = q \circ \Gamma_f$ has property ***P***, i.e., (v) is true.

Finally let us prove that (vi) is a consequence of (i), (ii) and (iii). We have the commutative diagram (p. 256)

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & Y \end{array}$$

where g and h are closed immersions and thus by (i) have property $\boldsymbol{P}$. Assume that f has property $\boldsymbol{P}$, then by (ii), $f \circ g = h \circ f_{\text{red}}$ has property $\boldsymbol{P}$. But h being a closed immersion is separated (Grothendieck, "Eléments," I, 5.5.1) and thus by (v) f_{red} has property $\boldsymbol{P}$.

Observe that if we replace Proposition (i) by the following:

(i′) *Every immersion has property* $\boldsymbol{P}$,

then (i′), (ii) and (iii) imply

(v′) *If the composition* $g \circ f$ *of the morphisms* $f : X \to Y$ *and* $g : Y \to Z$ *has property* $\boldsymbol{P}$, *then* f *has property* $\boldsymbol{P}$.

Indeed, the graph Γ_f of a morphism is an immersion (Grothendieck, "Eléments," I, 5.3.11) and thus the above proof applies.

Let us now consider the case when the property $\boldsymbol{P}$ is "to be of finite type" and establish the following

Propositions. (i) *Every closed immersion is of finite type.*

(ii) *If* $f : X \to Y$ *and* $g : Y \to Z$ *are of finite type, then* $g \circ f : X \to Z$ *is of finite type.*

(iii) *If* $f : X \to Y$ *is an* S*-morphism of finite type, then for every change of basis* $S' \to S$ *the morphism* $f_{(S')} : X_{(S')} \to Y_{(S')}$ *is of finite type.*

(iv) *If* $f : X \to X'$ *and* $g : Y \to Y'$ *are* S*-morphisms of finite type, then* $f \times_S g$ *is of finite type.*

(v) *If* $g \circ f$ *is of finite type and* g *is separated, then* f *is of finite type.*

(vi) *If* f *is of finite type, so is* f_{red}. (Grothendieck, "Eléments," I, 6.3.4).

By the preceding discussion it is sufficient to prove (i), (ii), and (iii).

(i) Let X be a closed preschema of Y and let $f : X \to Y$ be the canonical injection. It is enough to prove that f is of finite type. Let U be an affine open set in Y, then $f^{-1}(U) = U \cap X$ is closed in U. If A is the ring of U, then $U \cap X = \text{Spec}(A/\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of A. Clearly $A/\mathfrak{a}$ is of finite type over A. In this case even more is true : for

every affine open set U the set $f^{-1}(U)$ is an affine open set, i.e., the morphism is also affine (see below p. 271, or Grothendieck, "Eléments," II, 1).

(ii) Let U be an affine open set in Z. Then $g^{-1}(U)$ has a finite covering (V_i) by affine open sets, such that each $A(V_i)$ is an algebra of finite type over $A(U)$. For every i there is a finite covering (W_{ij}) of $f^{-1}(V_i)$ by affine open sets such that $A(W_{ij})$ is an algebra of finite type over $A(V_i)$. But then the W_{ij} form a finite covering of $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ and each $A(W_{ij})$ is of finite type over $A(U)$.

(iii) It is sufficient to consider the case $Y = S$, since $f : X \to Y$ defines X as a Y-preschema,

$$X_{(S')} = X_{(Y_{(S')})} \qquad \text{and} \qquad f_{(S')} = f_{(Y_{(S')})}$$

(Grothendieck, "Eléments," I, 3.3.9).

Suppose therefore that we have the diagram

$$\begin{array}{ccc} X & \xleftarrow{h} & X_{(S')} = X \times_S S' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & \xleftarrow{g} & S' \end{array}$$

and assume that f is of finite type. Let V be an affine open set in S; then $f^{-1}(V)$ is the finite union of affine open sets U_i such that each $A(U_i)$ is an algebra of finite type over $A(V)$. Let V' be an affine open set contained in $g^{-1}(V)$. Since $g \circ f' = f \circ h$, the set $f'^{-1}(V')$ is contained in the union of the sets $h^{-1}(U_i)$. Now the intersection $h^{-1}(U_i) \cap f'^{-1}(V')$ can be identified with $U_i \times_V V'$ (Grothendieck, "Eléments," I, 3.2.7). But $U_i \times_V V'$ is an affine schema whose ring is $A(U_i) \otimes_{A(V)} A(V')$ and this is a finitely generated algebra over $A(V')$.

Let $j : X \to Y$ be an immersion morphism. If X is noetherian or Y locally noetherian, then j is of finite type (Grothendieck, "Eléments," I, 6.3.5). If X is a preschema of finite type over a locally noetherian preschema S, then every S-morphism $X \to Y$ is of finite type (Grothendieck, "Eléments," I, 6.3.9).

Let k be a field. An *algebraic preschema over k* is a preschema of finite type over Spec(k) (Grothendieck, "Eléments," I, 6.4.1). This is quite close to the classical notion of variety over a field, except that we do not exclude nilpotent elements, do not restrict ourselves to irreducible varieties, consider prime ideals besides maximal ones and furthermore

do not assume that k is algebraically closed. Explicitly, X is an algebraic preschema over k if there exists a finite covering (U_i) of X by affine open sets such that each $A(U_i)$ is a finitely generated k-algebra.

Theorem (Grothendieck, "Eléments," I, 6.4.2). *Let X be an algebraic preschema over k. A point $x \in X$ is closed if and only if $\kappa(x)$ is finite over k.*

For the proof we may assume that X is affine. Its ring A will then be a k-algebra of finite type. Suppose first that x is a closed point of X. Then $\mathfrak{j}_x$ is a maximal ideal in A and thus $A/\mathfrak{j}_x$ is a field. Since A is of finite type over k, $A/\mathfrak{j}_x$ is also of finite type over k, but then (p. 260) $A/\mathfrak{j}_x$ is a finite algebraic extension of k. In general $\kappa(x)$ is the field of fractions of $A/\mathfrak{j}_x$, so that here we have $\kappa(x) = A/\mathfrak{j}_x$. Conversely, assume that $\kappa(x)$ is a finite algebraic extension of k. The ring $A/\mathfrak{j}_x$ is integral and such that $k \subset A/\mathfrak{j}_x \subset \kappa(x)$, whence it follows that $A/\mathfrak{j}_x$ is a field (Bourbaki, "Algèbre," Chap. V, § 3, No. 2, Prop. 3). Thus $\mathfrak{j}_x$ is a maximal ideal in A, i.e., x is a closed point.

Corollary 1. *If X is an algebraic preschema over a field k, then the set of closed points is dense in X* (which explains why Serre could work with only the closed points of preschemata).

Since we know that the closedness of a point x depends only on the behavior of $\kappa(x)$, a point will be closed in X if it is closed in some open set. The quasi-compact open sets form a basis for the topology of X. But a quasi-compact Kolmogoroff space always contains a closed point (Grothendieck, "Eléments," 0_{I}, 2.1.3), which proves the assertion.

The spectrum of a valuation ring shows that the closed points are not always dense in a preschema.

Corollary 2. *If X and Y are algebraic preschemata over a field k and $f: X \to Y$ is a morphism, then the image $f(x)$ of a closed point x is closed. Indeed, we have a k-monomorphism $\kappa(f(x)) \to \kappa(x)$ and since $\kappa(x)$ is a finite algebraic extension of k, so is $\kappa(f(x))$.*

This last property also fails to hold in general. Thus if X is again $\mathrm{Spec}(V)$ (p. 261), then to the natural injection $V \to K$ of V into its field of fractions K there corresponds a morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(V)$, which maps the unique point $\mathrm{Spec}(K)$ onto the open point a and not the closed point b.

A ring is said to be a *Jacobson ring* if every prime ideal is the intersection of maximal ideals. Any field and the ring $\mathbf{Z}$ are examples of

Jacobson rings. On the other hand a local ring of strictly positive dimension is never a Jacobson ring.

Theorem (Grothendieck, "Eléments," I, 6.4.4). *Let X be an algebraic preschema over a field k. Then the number of closed points of X is finite if and only if X is artinian.*

We may suppose that $X = \mathrm{Spec}(A)$ and A is a k-algebra of finite type. We shall prove below that a k-algebra of finite type is a Jacobson ring. Now if there are only finitely many closed points in X, then there are only finitely many maximal ideals in A. If a prime ideal is the intersection of finitely many prime ideals it is necessarily one of them, hence in A every prime ideal is maximal. Furthermore A is noetherian, hence (p. 259) A is artinian.

It remains to prove that a k-algebra A of finite type is a Jacobson ring. Let $\mathfrak{p}$ be a prime ideal in A and x an element not in $\mathfrak{p}$. We have to prove that there exists a maximal ideal $\mathfrak{m}$ which contains $\mathfrak{p}$ but does not contain x. The image $\bar{x}$ of x is not zero in the integral ring $A/\mathfrak{p}$ and $(A/\mathfrak{p})[1/x]$ is a finitely generated k-algebra different from $\{0\}$. By (1) of p. 260 there exists a k-homomorphism $\varphi : (A/\mathfrak{p})[1/\bar{x}] \to \bar{k}$. Let $\mathfrak{m}' = \mathrm{Ker}(\varphi)$ and $\mathfrak{m}'' = \mathfrak{m}' \cap (A/\mathfrak{p})$, then $\mathfrak{m}''$ is a maximal ideal in $A/\mathfrak{p}$. Indeed, if ψ is the restriction of φ to $A/\mathfrak{p}$, then $k \subset \mathrm{Im}(\psi) \subset \bar{k}$ and thus $\mathrm{Im}(\psi)$ is a field. We have $\bar{x} \notin \mathfrak{m}''$ since $\bar{x} \in \mathfrak{m}''$ would imply $\bar{x} \in \mathfrak{m}'$, i.e., $\varphi(\bar{x}) = 0$, which contradicts $\varphi(\bar{x}) \cdot \varphi(1/\bar{x}) = 1$. Let $\mathfrak{m}$ be the inverse image of $\mathfrak{m}''$ in A. Then $\mathfrak{m}$ is a maximal ideal such that $\mathfrak{p} \subset \mathfrak{m}$ and $x \notin \mathfrak{m}$.

Let us observe finally that there are two important constructions which lead to algebras which are not necessarily of finite type. If A is a ring and S a multiplicative system, then the ring of fractions $S^{-1}A$ is in general not of finite type over A as the example $A = \mathbf{Z}$, $S = \mathbf{C}\{0\}$, $S^{-1}A = \mathbf{Q}$ shows. If A is a ring and $\mathfrak{m}$ an ideal in A, then the completion $\hat{A}$ of A for the $\mathfrak{m}$-adic topology is in general not of finite type over A. Take for instance the ring $\mathbf{Z}$ and consider on it the p-adic topology defines by the ideal (p), where p is prime. The completion of $\mathbf{Z}$ for this topology is the ring $\mathbf{Z}_p$ of p-adic integers which is certainly not of finite type over $\mathbf{Z}$ because $\mathbf{Z}$ is countable and $\mathbf{Z}_p$ has the power of the continuum. The situation is saved, however, by the fact that $S^{-1}A$ is flat over A (Grothendieck, "Eléments," 0_{I}, 6.3.1) and also $\hat{A}$ is flat over A (Grothendieck, "Eléments," 0_{I}, 7.3.3). This leads to the definition of flat morphisms of ringed spaces (Grothendieck, "Eléments," 0_{I}, 6.7.1) and it turns out that in many respects flat morphisms behave just as well as those of finite type.

Direct Image of a Quasi-Coherent Sheaf

(See Grothendieck, "Eléments," I, 9.2.) Let X and Y be two preschemata, $f: X \to Y$ a morphism. We want to show by an example that if $\mathscr{F}$ is a quasi-coherent sheaf on X, then its direct image $f_*(\mathscr{F})$ need not be a quasi-coherent sheaf on Y. Let $Y = \mathrm{Spec}(\mathbf{Z})$, p a prime number and $X = \coprod \mathrm{Spec}(\mathbf{Z}/p^n\mathbf{Z})$, where the sum is taken over all integers $n \geqslant 1$. For every n we have an epimorphism $\varphi_n : \mathbf{Z} \to \mathbf{Z}/p^n\mathbf{Z}$, hence a morphism $f_n : \mathrm{Spec}(\mathbf{Z}/p^n\mathbf{Z}) \to Y$ and finally a morphism $f: X \to Y$. Since $\mathbf{Z}/p^n\mathbf{Z}$ has exactly one prime ideal, namely the ideal generated by the image of p, each spectrum $\mathrm{Spec}(\mathbf{Z}/p^n\mathbf{Z})$ consists of a single point. Furthermore f_n maps the single point of $\mathrm{Spec}(\mathbf{Z}/p^n\mathbf{Z})$ into the same point of Y, namely the point p which corresponds to the prime ideal (p) of $\mathbf{Z}$. Let $\mathscr{O}_X$ be the structure sheaf of X. Since X is the disjoint sum of the $\mathrm{Spec}(\mathbf{Z}/p^n\mathbf{Z})$, we have $\Gamma(X, \mathscr{O}_X) = \prod_{n=1}^{\infty}(\mathbf{Z}/p^n\mathbf{Z})$ i.e., a section of $\mathscr{O}_X$ over X is an arbitrary sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$, where $\alpha_n \in \mathbf{Z}/p^n\mathbf{Z}$. Let U be an open set in Y not containing p. Then $f^{-1}(U) = \varnothing$ and thus setting $\mathscr{G} = f_*(\mathscr{O}_X)$ we have $\Gamma(U, \mathscr{G}) = \Gamma(f^{-1}(U), \mathscr{O}_X) = 0$ (Godement, "Théorie des faisceaux," II.1.1, p. 109). It follows that for every point $y \neq p$ of Y the stalk $\mathscr{G}_y$ is $\{0\}$. Assume now that $\mathscr{G}$ is quasi-coherent. Then $\mathscr{G} = \tilde{M}$, where M is the $\mathbf{Z}$-module $\Gamma(Y, \mathscr{G}) = \Gamma(X, \mathscr{O}_X) = \prod_{n=1}^{\infty}(\mathbf{Z}/p^n\mathbf{Z})$ (Grothendieck, "Eléments," I, 1.4.1). The generic point η of Y corresponds to the minimal prime ideal (0) of $\mathbf{Z}$ and thus the local ring at η is the field of fractions $\mathbf{Q}$ of $\mathbf{Z}$. The stalk $\mathscr{G}_\eta = M_\eta$ is then canonically isomorphic to $\mathbf{Q} \otimes_{\mathbf{Z}} M$ (Grothendieck, "Eléments," 0_{I}, 1.2.5) and this last tensor product is not zero since M contains elements which are free over $\mathbf{Z}$ (Bourbaki, "Algèbre," Chap. III, § 1, Exerc. (3)b). Indeed, take $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$, where α_n is the generator of $\mathbf{Z}/p^n\mathbf{Z}$. If $m\alpha = 0$ for $m \in \mathbf{Z}$, then $m\alpha_n = 0$ for all n, i.e., $m \equiv 0\ (p^n)$ for all n, which is clearly impossible unless $m = 0$. Since $\eta \neq p$ we have arrived at a contradiction, which proves that $\mathscr{G}$ is not quasi-coherent.

If $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are affine schemata and $f: X \to Y$ a morphism, then the direct image $f_*(\mathscr{F})$ of a quasi-coherent sheaf $\mathscr{F}$ on X is quasi-coherent. Indeed, $\mathscr{F} = \tilde{M}$ for some A-module M. The homomorphism $\varphi : B \to A$ defines a structure of B-module $M_{[\varphi]}$ on M and we have $f_*(\mathscr{F}) = (M_{[\varphi]})^{\sim}$ (p. 249) which shows that $f_*(\mathscr{F})$ is quasi-coherent. In the above counterexample X was too far from being affine. More generally we have the following

Theorem (Grothendieck, "Eléments," I, 9.2.1). *Let $f: X \to Y$ be a morphism of preschemata and suppose that there exists a covering (Y_α) of Y by affine open sets such that*

(1) *$f^{-1}(Y_\alpha)$ is the finite union of affine open sets $X_{\alpha i}$;*

(2) *each intersection $X_{\alpha i} \cap X_{\alpha j}$ is quasi-compact.*

Then, if $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-Module, its direct image $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_Y$-Module.

Since the question is local over Y (p. 262) we may assume that $Y = Y_\alpha$ and drop the indices α.

(a) Suppose first that the intersections $X_i \cap X_j$ are affine open sets. This is stronger then condition (2) which requires only that $X_i \cap X_j$ can be covered by a finite number of affine open sets. Thus for instance the complement in the plane of the origin is quasi-compact but not affine.

We shall embed $f_*(\mathscr{F})$ in an exact sequence $0 \to f_*(\mathscr{F}) \to \mathscr{G} \to \mathscr{H}$, where the $\mathscr{O}_Y$-Modules $\mathscr{G}$ and $\mathscr{H}$ are quasi-coherent. It will then follow that $f_*(\mathscr{F})$ is quasi-coherent. Set $\mathscr{F}_i = \mathscr{F} \mid X_i$, $\mathscr{F}_{ij} = \mathscr{F} \mid X_i \cap X_j$, $\mathscr{F}'_i = f_*(\mathscr{F}_i)$, $\mathscr{F}'_{ij} = f_*(\mathscr{F}_{ij})$. Then, by the remark just made $\mathscr{F}'_i$ and $\mathscr{F}'_{ij}$ are quasi-coherent. Define $\mathscr{G} = \bigoplus_i \mathscr{F}'_i$, $\mathscr{H} = \bigoplus_{i,j} \mathscr{F}'_{ij}$. These sheaves are quasi-coherent as direct sums of quasi-coherent sheaves. To define a homomorphism $u: \mathscr{G} \to \mathscr{H}$ let W be an open set in Y. We have to define a homomorphism $\bigoplus_i \Gamma(W, \mathscr{F}'_i) \to \bigoplus_{i,j} \Gamma(W, \mathscr{F}'_{ij})$, i.e., a homomorphism $u_W : \bigoplus_i \Gamma(f^{-1}(W), \mathscr{F}_i) \to \bigoplus_{i,j} \Gamma(f^{-1}(W), \mathscr{F}_{ij})$. Let $s_i \in \Gamma(f^{-1}(W), \mathscr{F}_i)$ and denote by $s_{i|j}$ the restriction of s_i to $f^{-1}(W) \cap X_i \cap X_j$. Then we set $u_W((s_i)) = (s_{i|j} - s_{j|i})$. Clearly the u_W define a homomorphism u. We write $\mathscr{R} = \mathrm{Ker}(u)$ and define the homomorphism $v: f_*(\mathscr{F}) \to \mathscr{R}$ as follows. Given an open set W in Y, let $s \in \Gamma(W, f_*(\mathscr{F})) = \Gamma(f^{-1}(W), \mathscr{F})$. Let $s_i = s \mid f^{-1}(W) \cap X_i$; then $v(s) = (s_i) \in \bigoplus_i \Gamma(W, \mathscr{F}'_i)$. Clearly $v(s) \in \mathscr{R}$, since $s_{i|j} - s_{j|i} = s_{ij} - s_{ji} = 0$, where $s_{ij} = s \mid f^{-1}(W) \cap X_i \cap X_j$. To prove that v is an isomorphism one has to check that it is both injective and surjective, but this is an immediate consequence of the two sheaf axioms (F1) and (F2) (Godement "Théorie des faisceaux," II, 1.1, p. 109).

(b) In the general case, if we want to repeat the construction of part (a), it will not be clear a priori that $\mathscr{H}$ is quasi-coherent. But $X_i \cap X_j = \bigcup_k X_{ijk}$, where the X_{ijk} are affine open sets and since we are in a schema, their intersections will also affine open sets (Grothendieck, "Eléments," I, 5.5.6). Applying part (a) of the proof to the sets $X_i \cap X_j$, we see that $\mathscr{H}$ is quasi-coherent.

Affine Morphisms

(See Grothendieck, "Eléments," II, 1). The notion of affine morphism is a relativization of that of affine schema. A morphism $f: X \to Y$ is *affine* if for every affine open set V in Y the open set $f^{-1}(V)$ in X is affine (Grothendieck, "Eléments," II, 1.6.1). If X and Y are affine schemata, then every morphism $f: X \to Y$ is affine. If the morphism $f: X \to Y$ is affine, we also say that X is an affine preschema over Y (Grothendieck, "Eléments," II, 1.2.1). An affine preschema over Y is separated over Y, i.e., a Y-schema (Grothendieck, "Eléments," II, 1.2.4).

Let us observe that both the concept of an affine morphism and that of morphism of finite type are strengthenings of that of a *quasi-compact* morphism (Grothendieck, "Eléments," I, 6.6.1). Indeed, $f: X \to Y$ is quasi-compact if for every affine open set V in Y, the set $f^{-1}(V)$ can be covered by finitely many affine open sets U_i. It is of finite type if furthermore each $A(U_i)$ is of finite type over $A(V)$. It is affine, if $f^{-1}(V)$ itself is already affine.

If property $\boldsymbol{P}$ is taken to mean that f is an affine morphism, then Propositions (i)–(vi) of p. 263 hold (Grothendieck, "Eléments," II, 1.6.2).

If $Y = \mathrm{Spec}(A)$ is an affine schema and $f: X \to Y$ an affine morphism, then $X = \mathrm{Spec}(B)$ for some A-algebra B. More generally, if Y is a preschema and $f: X \to Y$ an affine morphism, we can cover Y by affine open sets Y_α; if $Y_\alpha = \mathrm{Spec}(A_\alpha)$ then $f^{-1}(Y_\alpha) = \mathrm{Spec}(B_\alpha)$, where B_α is an A_α-algebra. Gluing together the sheaves $\tilde{A}_\alpha$ we obtain $\mathcal{O}_Y$ and gluing together the sheaves $\tilde{B}_\alpha$ we obtain an $\mathcal{O}_Y$-Algebra $\mathscr{B}$. Thus we are led to the following

Theorem (Grothendieck, "Eléments," II, 1.2.6 and 1.3.1). *Let Y be a preschema. There is a one-to-one correspondence between the quasi-coherent $\mathcal{O}_Y$-Algebras and the affine preschemata over Y.*

Indeed, if X is affine over Y, then the conditions of the theorem of p. 270 are satisfied and thus $f_*(\mathcal{O}_X)$ is a quasi-coherent $\mathcal{O}_Y$-Algebra. Conversely, given a quasi-coherent $\mathcal{O}_Y$-Algebra $\mathscr{B}$, consider a covering (Y_α) of Y by affine open sets and take $A_\alpha = \Gamma(Y_\alpha, \mathcal{O}_Y)$, $B_\alpha = \Gamma(Y_\alpha, \mathscr{B})$, $X_\alpha = \mathrm{Spec}(B_\alpha)$. The homomorphism $A_\alpha \to B_\alpha$ gives a morphism $f_\alpha: X_\alpha \to Y_\alpha$. There is a canonical isomorphism between the ringed spaces $f_\alpha^{-1}(Y_\alpha \cap Y_\beta)$ and $f_\beta^{-1}(Y_\alpha \cap Y_\beta)$ and these isomorphisms satisfy the cocycle condition (Grothendieck, "Eléments," 0_{I}, 3.3.1). Thus the affine schemata X_α can be glued together (Grothendieck, "Eléments,"

I, 2.3.1) to give an affine preschema X over Y such that $f_*(\mathcal{O}_X) = \mathscr{B}$.

The Y-schema X just constructed will be called the Y-schema associated to the $\mathcal{O}_Y$-Algebra $\mathscr{B}$ and denoted by $X = \mathrm{Spec}(\mathscr{B})$.

If $\mathscr{B}$ is a quasi-coherent $\mathcal{O}_Y$-Algebra and $\mathscr{M}$ a quasi-coherent $\mathscr{B}$-Module, then on $X = \mathrm{Spec}(\mathscr{B})$ there exists a unique $\mathcal{O}_X$-Module denoted by $\tilde{\mathscr{M}}$, for which $f_*(\tilde{\mathscr{M}}) = \mathscr{M}$, where $f : X \to Y$ (Grothendieck, "Eléments," II, 1.4.3).

Let Z be a preschema and $X = \mathrm{Spec}(B)$ an affine schema. Then the morphisms $Z \to X$ are in one-to-one correspondence with the homomorphisms $B \to \Gamma(Z, \mathcal{O}_Z)$ (Grothendieck, "Eléments," I, 2.2.4). This generalizes as follows. Let Z be a Y-preschema with structural morphism $g : Z \to Y$. Let $X = \mathrm{Spec}(\mathscr{B})$ be an affine Y-schema. Then the morphisms $Z \to X$ are in one-to-one correspondence with the $\mathcal{O}_Y$-homomorphisms of Algebras $\mathscr{B} \to g_*(\mathcal{O}_Z)$ (Grothendieck, "Eléments," II, 1.2.7).

Let X be an affine preschema over Y. Then X is of finite type over Y (p. 262) if and only if $f_*(\mathcal{O}_X) = \mathscr{B}$ is an $\mathcal{O}_Y$-Algebra of finite type, i.e., for every $y \in Y$ there exists an affine open neighborhood U of y such that $\Gamma(U, \mathscr{B})$ is an algebra of finite type over $\Gamma(U, \mathcal{O}_X)$ (Grothendieck, "Eléments," I, 9.6.2 and II, 1.3.7).

As an important example of affine morphisms we shall discuss *vector bundles* over a preschema (Grothendieck, "Eléments," II, 1.7). Let k be a field and $k[X_1, \ldots, X_n]$ the ring of polynomials in n indeterminates over k. The affine schema $\mathrm{Spec}(k[X_1, \ldots, X_n])$ is called the affine n-space or the n-dimensional vector space over k. Of course $\mathrm{Spec}(k[X_1, \ldots, X_n])$ has enormously more points than the usual affine n-space consisting of n-tuples of elements of k. If k is algebraically closed, then the points of the usual affine n-space correspond to the maximal ideals of $k[X_1, \ldots, X_n]$, but if k is not algebraically closed, even this is not true. Thus the maximal ideal generated by $X^2 + 1$ in $\mathbf{R}[X]$ does not correspond to a point on $\mathbf{R}$ but to the two points $\pm i$ in $\mathbf{C}$.

Let Y be a preschema. We want to define a morphism $f : X \to Y$ such that for each $y \in Y$ the fiber $f^{-1}(y)$ should be an affine n-space over $\kappa(y)$ (see p. 254). This will be a generalization of the usual notion of vector bundle (Hirzebruch, "Neue topologische Methoden in der algebraischen Geometrie," § 3.5, p. 48).

We first recall the notion of a *symmetric algebra* over a module (Bourbaki, "Algèbre," Chap. IV, § 1, Exerc. 2, or Chap. III (2nd ed.), § 8). Let A be a ring and M a module over A. Intuitively, we want to define an algebra whose elements are linear combinations of monomials

$x_1x_2 \cdots x_n$ in elements of M, where the multiplication is assumed to be commutative. To obtain a correct definition, consider the tensor algebra $\mathsf{T}(M) = \oplus_{n=0}^{\infty} T_n(M)$ over M generated by the tensor products $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ (Bourbaki, "Algèbre," Chap. III, § 4, No. 6 or Chap. III (2nd ed.), § 4). Let $\mathfrak{I}$ be the ideal in $\mathsf{T}(M)$ generated by the elements of the form $x \otimes y - y \otimes x$ for x, y in M. The quotient algebra $\mathsf{T}(M)/\mathfrak{I}$ will be called the symmetric algebra over M and denoted by $\mathsf{S}(M)$. The elements of the form $x_1 \otimes \cdots \otimes (x_i \otimes y_i - y_i \otimes x_i) \otimes \cdots \otimes x_{n-1}$ span the subspace $\mathfrak{I}_n = \mathfrak{I} \cap T_n(M)$ of $T_n(M)$. It follows that $\mathsf{S}(M)$ is a graded algebra, $\mathsf{S}(M) = \oplus_{n=0}^{\infty} S_n(M)$, where $S_n(M) = T_n(M)/\mathfrak{I}_n$. Since $\mathfrak{I}_n = (0)$ for $n = 0$ and 1, $S_0(M)$ can be identified with A and $S_1(M)$ with M. It can happen that all $S_n(M) = (0)$ for $n \geqslant 2$, but if M is a free module with a basis having p elements, then $\mathsf{S}(M)$ is isomorphic to $A[X_1, \ldots, X_p]$.

If $M \to N$ is a A-linear map, then we have a homomorphism of algebras $\mathsf{S}(M) \to \mathsf{S}(N)$ such that the diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ \mathsf{S}(M) & \longrightarrow & \mathsf{S}(N) \end{array}$$

is commutative, i.e., $M \rightsquigarrow \mathsf{S}(M)$ is a covariant functor. The symmetric algebra has a universal property similar to that of the tensor algebra. Indeed, if $\varphi : M \to \mathsf{T}(M)$ is the canonical injection, then for any linear map $u : M \to B$ into a A-algebra B, there exists a unique homomorphism $v : \mathsf{T}(M) \to B$ of algebras such that the diagram

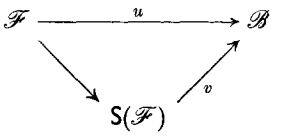
$$\begin{array}{ccc} M & \xrightarrow{\quad u \quad} & B \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle v} & \\ & \mathsf{T}(M) & \end{array}$$

is commutative and this property characterizes the tensor algebra. Similarly if $\sigma : M \to \mathsf{S}(M)$ is the canonical injection, then for any linear map $u : M \to B$ into a *commutative* A-algebra B there exists a unique homomorphism $v : \mathsf{S}(M) \to B$ of algebras such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad u \quad} & B \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle v} & \\ & \mathsf{S}(M) & \end{array}$$

is commutative and this property characterizes the symmetric algebra.

A similar definition can be given for sheaves of modules. Let $(Y, \mathcal{O}_Y)$ be a ringed space and let $\mathscr{F}$ be an $\mathcal{O}_Y$-Module on Y. We define a graded $\mathcal{O}_Y$-Algebra $\mathsf{S}(\mathscr{F})$ by associating to every open set $U \subset Y$ the symmetric algebra $\mathsf{S}(\Gamma(U, \mathscr{F}))$ of the $\Gamma(U, \mathcal{O}_Y)$-module $\Gamma(U, \mathscr{F})$. Since $\mathsf{S}(M)$ is a covariant functor of M, these algebras define a presheaf and $\mathsf{S}(\mathscr{F})$ will be the associated sheaf. $\mathsf{S}(\mathscr{F})$ has a universal property analogous to that of the symmetric algebra. For any homomorphism $u : \mathscr{F} \to \mathscr{B}$ of $\mathcal{O}_Y$-Modules into a commutative $\mathcal{O}_Y$-Algebra $\mathscr{B}$ there exists a unique homomorphism of $\mathcal{O}_Y$-Algebras $v : \mathsf{S}(\mathscr{F}) \to \mathscr{B}$ such that the diagram



is commutative.

Let now $(Y, \mathcal{O}_Y)$ be a preschema and $\mathscr{E}$ a quasi-coherent $\mathcal{O}_Y$-Module. Then $\mathsf{S}(\mathscr{E})$ is a quasi-coherent $\mathcal{O}_Y$-Algebra (Grothendieck, "Eléments," II, 1.7.7), to which (p. 271) there corresponds an affine preschema $\mathrm{Spec}(\mathsf{S}(\mathscr{E}))$ over Y. We shall denote this Y-schema by $\mathbf{V}(\mathscr{E})$ and call it the vector bundle on Y defined by the sheaf $\mathscr{E}$. Let us list a few properties of $\mathbf{V}(\mathscr{E})$.

(1) If $\mathscr{E}$ is an $\mathcal{O}_Y$-Module of finite type, then $\mathbf{V}(\mathscr{E})$ is of finite type over Y.

(2) If $\mathscr{E}$ and $\mathscr{F}$ are quasi-coherent $\mathcal{O}_Y$-Modules then $\mathbf{V}(\mathscr{E} \oplus \mathscr{F})$ can be identified canonically with $\mathbf{V}(\mathscr{E}) \times_Y \mathbf{V}(\mathscr{F})$.

(3) If $g : Y' \to Y$ is an extension of basis and $\mathscr{E}$ a quasi-coherent $\mathcal{O}_Y$-Module, then $\mathbf{V}(g^*(\mathscr{E}))$ can be identified canonically with $\mathbf{V}(\mathscr{E})_{(Y')} = \mathbf{V}(\mathscr{E}) \times_Y Y'$. In particular we have the commutative diagram

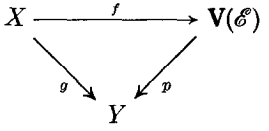
$$\begin{array}{ccc} \mathbf{V}(\mathscr{E}) & \longleftarrow & \mathbf{V}(g^*(\mathscr{E})) \\ \downarrow & & \downarrow \\ Y & \xleftarrow{g} & Y'. \end{array}$$

This corresponds to the familiar operation of pulling back a fiber space over a new basis by a continuous map (Steenrod, "The Topology of Fibre Bundles," § 10, p. 47).

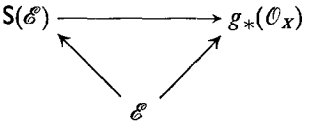
(4) A surjective homomorphism of quasi-coherent $\mathcal{O}_Y$-Modules $\mathcal{E} \to \mathcal{F}$ defines a closed immersion $\mathbf{V}(\mathcal{F}) \to \mathbf{V}(\mathcal{E})$. This follows from the fact that the corresponding homomorphism of $\mathcal{O}_Y$-Algebras $\mathsf{S}(\mathcal{E}) \to \mathsf{S}(\mathcal{F})$ is then also surjective and its kernel defines an Ideal in $\mathsf{S}(\mathcal{E})$.

(5) Let Y be a noetherian preschema. Then any affine preschema over Y which is of finite type over Y is isomorphic to a closed subpreschema of some $\mathbf{V}_Y^n$, where $\mathbf{V}_Y^n = \mathrm{Spec}(\mathsf{S}(\mathcal{O}_Y^n))$.

We want to stress the important fact that the object $\mathbf{V}(\mathcal{E})$ represents a representable functor. Let X be a preschema over Y with structural morphism g and consider all Y-morphisms $X \to \mathbf{V}(\mathcal{E})$, i.e., all morphisms f for which the diagram



is commutative. The morphisms f correspond, in a one-to-one way, to homomorphisms $\mathsf{S}(\mathcal{E}) \to g_*(\mathcal{O}_X)$ of $\mathcal{O}_Y$-Algebras (p. 272) and to each such homomorphism corresponds uniquely a homomorphism of $\mathcal{O}_Y$-Modules $\mathcal{E} \to g_*(\mathcal{O}_X)$ so that the diagram



is commutative. Fixing Y and $\mathcal{E}$ we obtain thus a functor $F : X \rightsquigarrow F(X) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}, g_*(\mathcal{O}_X))$ from the category of Y-preschemata into the category ***Ens*** of sets (cf. p. 240). Of course one should also define for every Y-morphism $u : X' \to X$ a map $F(u) : F(X) \to F(X')$ and check that it satisfies the requirements, but this is trivial.

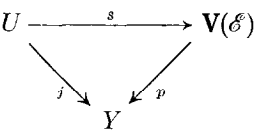
Let now in general F be a contravariant functor from an arbitrary category $\boldsymbol{C}$ into the category ***Ens***. We say that F is *representable* if there exists a fixed object $Y \in \boldsymbol{C}$ such that $F(X) = \mathrm{Mor}(X, Y)$ for every $X \in \boldsymbol{C}$. Of course to the map $F(u) : F(X) \to F(X')$ there should correspond the natural map $\mathrm{Mor}(X, Y) \to \mathrm{Mor}(X', Y)$ defined by $u : X' \to X$. We say that Y represents the functor F. It is unique up to isomorphism.

Let us go back to our functor $F : X \rightsquigarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}, g_*(\mathcal{O}_X))$. By our

previous discussion $\mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{E}, g_*(\mathscr{O}_X)) = \mathrm{Hom}_Y(X, \mathbf{V}(\mathscr{E}))$, which expresses the fact that $\mathbf{V}(\mathscr{E})$ represents the functor F. This property of $\mathbf{V}(\mathscr{E})$ is shared by many other important objects we shall introduce later. Projective schemata, the schemata corresponding to Chow coordinates (Hilbert schemata), Picard schemata all represent representable functors.

In the classical theory of fiber bundles X over a topological space Y one considers the sections over an open set U of Y, i.e., maps $s : U \to X$ which composed with the projection $p : X \to Y$ give the identity. The various sets $\Gamma(U, X)$ of sections of X over open subsets U of Y define the sheaf of germs of sections of X, which plays a very important role in the theory of fiber bundles.

In the case of a vector bundle $\mathbf{V}(\mathscr{E})$ on a preschema Y defined by a quasi-coherent $\mathscr{O}_Y$-Algebra $\mathscr{E}$, a section over an open set U of Y is simply a Y-morphism $s : U \to \mathbf{V}(\mathscr{E})$. If j is the canonical injection, then the diagram



will be commutative and the maps of the underlying spaces satisfy the condition $p(s(y)) = y$ for all $y \in U$. Thus the set of sections over U is $\mathrm{Hom}_Y(U, \mathbf{V}(\mathscr{E}))$ and, by what we have seen before, this is equal to $\mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{E}, \mathscr{O}_Y \mid U) = \Gamma(U, \mathscr{E}^\vee)$ (p. 245), since $j_*(\mathscr{O}_U) = \mathscr{O}_Y \mid U$. Thus the sheaf of germs of sections of $\mathbf{V}(\mathscr{E})$ is $\mathscr{E}^\vee$ (and not $\mathscr{E}$ itself, as in the classical theory). If $\mathscr{E} = \mathscr{O}_Y^n$, then of course $\mathscr{E}$ and $\mathscr{E}^\vee$ can be identified, however not canonically. In general the sheaf $\mathscr{E}$ need not be free or even locally free, i.e., we do not impose a local triviality condition on our vector bundles.

By definition (p. 254) the fiber of $\mathbf{V}(\mathscr{E})$ above a point $y \in Y$ is $f^{-1}(y) = \mathbf{V}(\mathscr{E}) \times_Y \mathrm{Spec}(\kappa(y)) = \mathbf{V}(g^*(\mathscr{E}))$, where g is the morphism $\mathrm{Spec}(\kappa(y)) \to Y$ defined on p. 254. Now $g^*(\mathscr{E}) = \mathscr{E}_y/\mathfrak{m}_y\mathscr{E}_y$ is a vector space over $\kappa(y)$ and the fiber $f^{-1}(y)$ is $\mathbf{V}(\mathscr{E}_y/\mathfrak{m}_y\mathscr{E}_y)$. In particular if $\mathscr{E}_y/\mathfrak{m}_y\mathscr{E}_y$ is finite dimensional then it is of the form $\mathscr{O}_y^n$ and $\mathsf{S}(\mathscr{O}_y^n)$ is isomorphic to a ring of polynomials. In this case the fiber $\mathbf{V}(\mathscr{O}_y^n) = \mathrm{Spec}(\mathsf{S}(\mathscr{O}_y^n))$ is an affine n-space in the sense defined at the beginning.

As the next important examples of affine morphisms we introduce *integral* and *finite morphisms* (Grothendieck, "Eléments," II, 6). The notion of an algebra B integral or finite over a ring A was introduced on

p. 257 and we proved that if B is integral and of finite type over A then it is finite (p. 257). Conversely, if B is finite over A, then it is integral over A. Indeed, every element $x \in B$ can be written in the form $x = \sum_{i=1}^{n} \alpha_i c_i$, $c_i \in B$ and, in particular, we have $xc_j = \sum_{i=1}^{n} \alpha_{ji} c_i$, i.e., $\sum_{i=1}^{n} (\delta_{ji} x - \alpha_{ji}) c_i = 0$, where δ_{ji} is the Kronecker symbol. Writing $\Delta = \det(\delta_{ji} x - \alpha_{ji})$, we obtain $\Delta c_i = 0$ for $1 \leqslant i \leqslant n$ and thus $\Delta y = 0$ for all $y \in B$ and in particular $\Delta = \Delta \cdot 1 = 0$. But $\Delta = 0$ is clearly an equation of integral dependence for x.

Let $f : X \to Y$ be a morphism of preschemata. We say that f is *integral* if for every affine open set U in Y, the set $f^{-1}(U)$ is affine and the ring $\Gamma(f^{-1}(U), \mathcal{O}_X)$ is integral over $\Gamma(U, \mathcal{O}_Y)$. In this case we also say that X is an integral preschema over Y. Similarly f is *finite* if for every affine open set U in Y, the set $f^{-1}(U)$ is affine and $\Gamma(f^{-1}(U), \mathcal{O}_X)$ is finite over $\Gamma(Y, \mathcal{O}_Y)$. In this case X is said to be a finite preschema over Y (Grothendieck, "Eléments," II, 6.1.1).

Clearly a finite morphism is integral and an integral morphism is affine. The notion of finite morphism is much stronger than that of finite type (p. 262): we require that $V = f^{-1}(U)$ itself be affine and not only the union of affine open sets and furthermore we require that $A(V)$ be finite and not only of finite type over $A(U)$. There is an intermediate concept, where for each open set U in Y the set $f^{-1}(U)$ is the finite union of affine open sets V_i such that each ring $A(V_i)$ is finite over $A(U)$. To construct an example let us consider the polynomial rings $K[s_1, t_1]$, $K[s_2, t_2]$ over the field K, and the affine planes $X_1 = \mathrm{Spec}(K[s_1, t_1])$, $X_2 = \mathrm{Spec}(K[s_2, t_2])$ (p. 272). Set $U_{12} = D(s_1) \cup D(t_1) \subset X_1$ and $U_{21} = D(s_2) \cup D(t_2) \subset X_2$. The closed points of U_{12} or of U_{21} correspond to the complement of the origin of the usual plane. Now let $u_{12} : U_{21} \to U_{12}$ be the restriction to U_{21} of the isomorphism $X_2 \to X_1$ corresponding to the isomorphism of rings which to the polynomial $P(s_1, t_1)$ associates the polynomial $P(s_2, t_2)$. Since the cocycle condition is vacuously satisfied, we can glue together X_1 and X_2 along U_{12} and U_{21} and obtain a preschema X. Intuitively this corresponds to the gluing together two planes along the complements of their origins and therefore we refer to X as the affine plane over K with double point at the origin (Grothendieck, "Eléments," I, 5.5.11). X is not an affine schema and is not separated over K, but it is the union of two affine schemata corresponding to X_1 and X_2. If $f : X \to Y$ is the natural morphism of X onto the affine plane Y over K, then $f^{-1}(Y) = X$ is not affine but the union of two affine open sets whose rings are identical with that of Y.

An affine morphism $f : X \to Y$ is integral (resp. finite) if and only

if the Algebra $\mathscr{B} = f_*(\mathscr{O}_X)$ is integral (resp. finite) over $\mathscr{O}_Y$, i.e., if for every affine open set $U \subset Y$ the algebra $\Gamma(U, \mathscr{B})$ is integral (resp. finite) over $\Gamma(U, \mathscr{O}_Y)$ (Grothendieck, "Eléments," II, 6.1.2). Propositions (i)–(vi) of p. 263 are true for the properties of being integral or finite (Grothendieck, "Eléments," II, 6.1.5).

If $f : X \to Y$ is an integral morphism then f is *closed*, i.e. the image of every closed set is closed (Grothendieck, "Eléments," II, 6.1.10). In this theorem a ring theoretical property implies a topological one. We shall prove it with the help of several reductions and using Propositions (i), (ii), (v), and (vi) of p. 263.

(a) It is enough to prove that $f(X)$ is closed in Y. Indeed, suppose that this is true and let Z be a closed subset of X. There exists a unique reduced subpreschema of X whose underlying space is Z (p. 256). If $j : Z \to X$ is the injection morphism, then by (i) it is integral and by (ii) the morphism $f \circ j : Z \to Y$ is integral. Hence by our assumption $f(Z)$ is closed in Y.

(b) We may assume that X and Y are both reduced. Indeed, if f is integral, then by (vi) so is f_{red}. But as continuous maps of topological spaces f and f_{red} are the same (p. 256).

(c) It is enough to prove the theorem for the case when $f(X)$ is dense in Y, i.e., $\overline{f(X)} = Y$. Indeed, set $Z = \overline{f(X)}$. There exists a unique reduced subpreschema of Y whose underlying space is Z. Since X is reduced and $f(X) \subset Z$, we have the factorization $f : X \xrightarrow{g} Z \xrightarrow{j} Y$, where j is the injection morphism (p. 256). Since $j \circ g = f$ is integral and j a closed immersion, by (v) g is also integral. Hence $f(X)$ is closed in Z and thus also closed in Y.

(d) Finally we may assume that X and Y are affine. Indeed, since a set in Y is closed if its intersections with affine open sets are closed, we may suppose that Y is an affine schema. But then, by the definition of an integral morphism, $X = f^{-1}(Y)$ is also affine.

Thus we have to prove that if $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$, where A and B are reduced rings, and $f : X \to Y$ is such that $\overline{f(X)} = Y$, then $f(X) = Y$. Let $\varphi : A \to B$ be the ring homomorphism corresponding to f, i.e., such that $f = {}^a\varphi$. In general $\overline{f(X)} = Y$ means that every element of $\mathrm{Ker}(\varphi)$ is nilpotent (Grothendieck, "Eléments," I, 1.2.7). Since A is reduced, we have $\mathrm{Ker}(\varphi) = (0)$, i.e. φ is injective. By the Cohen–Seidenberg going-up theorem (p. 259) f is surjective.

We say that a morphism $f : X \to Y$ is *universally closed*, if for any

extension of basis $Y' \to Y$ the morphism $f_{(Y')} : X_{(Y')} = X \times_Y Y' \to Y'$ is closed. An integral morphism is universally closed. This is an immediate consequence of proposition (iii) of p. 263 and of the previous theorem.

Let $f : X \to Y$ be a finite morphism. Then for every $y \in Y$ the fiber $f^{-1}(y)$ is discrete and finite (Grothendieck, "Eléments," II, 6.1.7), i.e., X is a covering (revêtement) of Y in the classical sense. Indeed, as a preschema $f^{-1}(y)$ is identified with $X \times_Y \mathrm{Spec}(\kappa(y))$ (p. 254) and by (iii) this is finite over $\mathrm{Spec}(\kappa(y))$. It follows that $f^{-1}(y)$ is an affine schema whose ring is a finitely generated module over the field $\kappa(y)$ and is therefore artinian. The conclusion follows from the definition of an artinian preschema (p. 262).

The converse is trivially false. Let K be an extenson of infinite degree of transcendence of the field k. Then the fiber of $\mathrm{Spec}(K)$ over the unique point of $\mathrm{Spec}(k)$ is the unique point of $\mathrm{Spec}(K)$ and is therefore discrete and finite. The morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ is, however, not finite, since K is not finite over k. But even if we make the additional assumption that $f : X \to Y$ is of finite type it does not follow from the fact that the fibers are discrete and finite that f is finite. Indeed, if X is the affine plane over k with double point at the origin (p. 277), Y the affine plane, then the natural morphism $X \to Y$ is of finite type and for every $y \in Y$ the fiber $f^{-1}(y)$ is discrete with one or two points, but f is not finite, since $f^{-1}(Y) = X$ is not affine.

The notion of an integral morphism is closely related to the important *normalization* process. If A and B are two rings such that $A \subset B$, the *integral closure* of A in B is the set of all elements of B which are integral over A. It is the greatest ring intermediate between A and B which is integral over A (p. 257). If A is an integral ring, then its integral closure in its field of fractions K is simply referred to as the *integral closure* A' of A. If $A = A'$ then A is said to be *integrally closed.*

Similarly let X be a ringed space and $\mathscr{B}$ an $\mathscr{O}_X$-Algebra. A section $f \in \Gamma(X, \mathscr{B})$ is said to be *integral* over $\mathscr{O}_X$ if the following two equivalent conditions are satisfied:

(a) For every $x \in X$ there exists a neighborhood U such that the restriction of f to U is an element of the ring $\Gamma(U, \mathscr{B})$ which is integral over $\Gamma(U, \mathscr{O}_X)$.

(b) For every $x \in X$ the element $f_x \in \mathscr{B}_x$ is integral over $\mathscr{O}_x$ (Grothendieck, "Eléments," II,6.3.1). Clearly (a) implies (b). Conversely if we have an equation of integral dependence $f_x^n + (a_1)_x f_x^{n-1} + \cdots + (a_n)_x = 0$, where the a_i $(1 \leqslant i \leqslant n)$ are sections of $\mathscr{O}_X$ defined in a neighborhood V of

x, then by a fundamental property of sheaves, there exists a neighborhood $U \subset V$ of x such that $(f \mid U)^n + (a_1 \mid U)(f \mid U)^{n-1} + \cdots + (a_n \mid U) = 0$.

Given an $\mathcal{O}_X$-Algebra $\mathscr{B}$ we define a sub-$\mathcal{O}_X$-Algebra $\mathscr{A}'$ of $\mathscr{B}$ by taking for stalks $\mathscr{A}'_x$ the integral closure of $\mathcal{O}_x$ in $\mathscr{B}_x$. Equivalently we can define $\mathscr{A}'$ by the presheaf formed by the sets $\Gamma(U, \mathscr{A}')$ of sections of $\mathscr{B}$ over U integral over $\mathcal{O}_X \mid U$. The $\mathcal{O}_X$-Algebra $\mathscr{A}'$ is called the *integral closure* of $\mathcal{O}_X$ in $\mathscr{B}$ (Grothendieck, "Eléments," II, 6.3.2). If $X = \operatorname{Spec}(A)$, $\mathscr{B} = \tilde{B}$, then the integral closure of $\tilde{A}$ in $\mathscr{B}$ is $\tilde{A}'$, where A' is the integral closure of A in B (Grothendieck, "Eléments," II, 6.3.4).

Let now Y be a preschema, $\mathscr{B}$ a quasi-coherent $\mathcal{O}_Y$-Algebra and $X = \operatorname{Spec}(\mathscr{B})$. The preschema $Y' = \operatorname{Spec}(\mathscr{A}')$, where $\mathscr{A}'$ is the integral closure of $\mathcal{O}_Y$ in $\mathscr{B}$, is called the *integral closure* of Y with respect to $\mathscr{B}$ (or to X) (Grothendieck, "Eléments," II, 6.3.4). To the injection homomorphisms $\mathcal{O}_Y \to \mathscr{A}' \to \mathscr{B}$ there correspond morphisms $X \to Y' \to Y$.

If X is an arbitrary integral preschema then for every affine open set U the ring $\Gamma(U, \mathcal{O}_X)$ is integral (p. 255) and we can consider the integral closure of $\mathcal{O}_X$ in the sheaf of rational fractions $\mathscr{R}_X$ on X (p. 244). The integral closure X' of X with respect to $\mathscr{R}_X$ is called the *normalization* of X. More generally, if X is a reduced preschema (p. 254) the normalization X' of X corresponds to the integral closure of $\mathcal{O}_X$ in its total ring of fractions. If X is integral, then X' is integral and the morphism $X' \to X$ is birational (Grothendieck, "Eléments," I, 2.2.9), i.e., the generic points ξ' of X' and ξ of X correspond to each other and the residual fields $\kappa(\xi)$ and $\kappa(\xi')$ are the same (Grothendieck, "Eléments," II, 6.3.8).

Normalization is a desingularization process. Some bad features of X disappear if we pass from X to its normalized preschema X'. For curves the normalization process makes the singularities disappear completely, in general the singularities of codimension 1 will disappear. In the normalized variety X' the irreducible components of X may split.

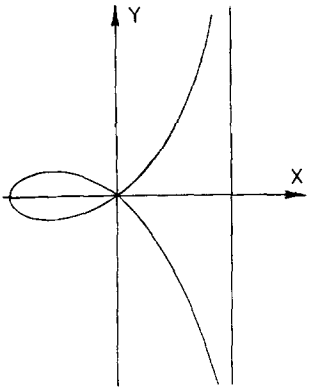
As an example consider the cubic $X(X^2 + Y^2) + (X^2 - Y^2) = 0$ which has a double point at the origin. The ring associated to this curve is $A = K[X, Y]/(X(X^2 + Y^2) + (X^2 - Y^2))$ (p. 236) and the maximal ideal corresponding to the point $(0, 0)$ is $\mathfrak{m} = (X) + (Y)$ taken modulo the ideal generated by $X(X^2 + Y^2) + (X^2 - Y^2)$. If ξ and η denote the images of X and Y in A, then we have $A = K[\xi, \eta]$, where ξ and η satisfy the relation

$$(*) \qquad \xi(\xi^2 + \eta^2) + (\xi^2 - \eta^2) = 0,$$

and $\mathfrak{m} = (\xi) + (\eta)$. To make computations easier we consider the local

ring $A_{\mathfrak{m}}$, i.e., instead of the original curve we consider the local schema at the origin. Furthermore we suppose that K is algebraically closed. $A_{\mathfrak{m}}$ is the ring formed by all expressions $P(\xi, \eta)/Q(\xi, \eta)$, where P, Q are polynomials and $Q(0, 0) \neq 0$. The ring $A_{\mathfrak{m}}$ is not integrally closed. Indeed, let E be the field of fractions of $A_{\mathfrak{m}}$, then $t = \eta/\xi \in E$, $t \notin A_{\mathfrak{m}}$ but t is integral over $A_{\mathfrak{m}}$ since dividing through $(*)$ by ξ^2 we get after some simplifications the equation of integral dependence

$$t^2 + \frac{\xi + 1}{\xi - 1} = 0.$$



The integral closure of $A_{\mathfrak{m}}$ is formed by all elements $R(t)/S(t)$, where R, S are polynomials and $S(\pm 1) \neq 0$. Indeed, let A' be the integral closure of A and let A'' be the ring formed by all elements $R(t)/S(t)$, where $S(\pm 1) \neq 0$. Since

$$\xi = \frac{t^2 - 1}{t^2 + 1}, \qquad \eta = t\xi,$$

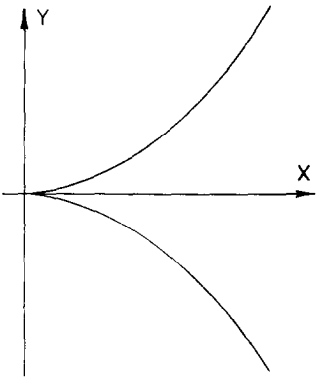
we have $\xi, \eta \in A''$ and thus $A_{\mathfrak{m}} \subset A''$. Furthermore A'' is the ring of fractions of the polynomial ring $K[t]$ with respect to the multiplicative system of polynomials $S(t)$ such that $S(\pm 1) \neq 0$. Hence A'' is integrally closed (Samuel–Zariski, "Commutative Algebra," Vol. I, Chap. V, § 3, Examples 1 and 2, p. 261) and we have $A' \subset A''$. But $t - a$ is invertible in A' for $a \in K$, $a \neq \pm 1$ since

$$(t - a)(t + a) = t^2 - a^2 = \frac{\xi + 1}{1 - \xi} - a^2 = \frac{1 - a^2 + \xi(1 + a^2)}{1 - \xi}.$$

Hence $(t-a)^{-1} \in A'$ and since K is supposed to be algebraically closed we have $R(t)/S(t) \in A'$, i.e., $A'' \subset A'$. Thus $A' = A''$.

The elements $t-1$ and $t+1$ are not invertible in A' and thus they generate two maximal ideals $\mathfrak{m}' = (t-1)A'$ and $\mathfrak{m}'' = (t+1)A'$ lying above $\mathfrak{m}$, i.e., $\mathfrak{m}' \cap A_{\mathfrak{m}} = \mathfrak{m}$ and $\mathfrak{m}'' \cap A_{\mathfrak{m}} = \mathfrak{m}$. In other words, the schema $\mathrm{Spec}(A')$ has two points lying above the point $\xi = 0$, $\eta = 0$ of our original curve and these two points correspond to the "parameter values" $t = \pm 1$. If we had applied the same procedure to the ring A itself instead of the local ring $A_{\mathfrak{m}}$, we would have obtained a twisted cubic in 3-space lying above the original curve and having no double points.

As another example, consider the curve $Y^2 - X^3 = 0$ having a cusp at (0, 0). The ring associated to the curve is $A = K[X, Y]/(X^3 - Y^2) = K[\xi, \eta]$, where $\xi^3 = \eta^2$. The element $t = \eta/\xi$ of the field of fractions of A does not belong to A but is integral over A since it satisfies the equation of integral dependence $t^2 = \xi$. The integral closure of A is the polynomial ring $A' = K(t)$. Indeed, it contains A since $\xi = t^2$, $\eta = t^3$ and is integrally closed as a polynomial ring. Thus the normalized curve $\mathrm{Spec}(A')$ is a straight line.



We have an injection $A = K[\xi, \eta] \to A' = K[t]$ defined by $\xi \to t^2$, $\eta \to t^3$ to which there corresponds a morphism $f : X' = \mathrm{Spec}(A') \to X = \mathrm{Spec}(A)$. The point $x \in X$ corresponding to the point $\xi = 0$, $\eta = 0$ of the curve is determined by the maximal ideal $\mathfrak{m}_x = (\xi) + (\eta)$ of A. The fiber $x' = f^{-1}(x)$ above the point x is a one point preschema corresponding to the maximal ideal $\mathfrak{m}_{x'} = (t)$ of A', which satisfies $\mathfrak{m}_x = \mathfrak{m}_{x'} \cap A$. Since $\mathfrak{m}_x A' = (t^2) + (t^3)$, we have $\mathfrak{m}_{x'}^2 \subset \mathfrak{m}_x A'$. Thus

the local ring $\mathcal{O}_{x'}/\mathfrak{m}_x\mathcal{O}_{x'}$ at the unique point x' of the fiber (p. 254) has nilpotent elements since $(\mathfrak{m}_{x'}/\mathfrak{m}_x\mathcal{O}_{x'})^2 = (0)$. This singular behavior of the morphism reflects the singularity of the original curve at $\xi = \eta = 0$.

Cohomology of Affine Schemata

(See Grothendieck, "Eléments," III, 1). Let X be a preschema and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-Module. We pose the problem to compute the Grothendieck cohomology groups $H^p(X, \mathcal{F})$ using, if possible, the Čech cohomology groups $\check{H}^p(X, \mathcal{F})$ (Hirzebruch, "Neue topologische Methoden in der algebraischen Geometrie," § 2.6, p. 27).

Let $\mathfrak{U} = (U_i)_{1 \leqslant i \leqslant r}$ be a finite family of open sets in X. We write $A = \Gamma(X, \mathcal{O}_X)$, $M = \Gamma(X, \mathcal{F})$ and assume that the sets U_i are of the form X_{f_i}, where $f_i \in A$ $(1 \leqslant i \leqslant r)$. Recall that for $f \in A$ the set X_f consists of those points of X for which $f(x) \neq 0$, where $f(x)$ is the image of f in $\mathcal{O}_x/\mathfrak{m}_x$ (Grothendieck, "Eléments," 0_1, 5.5.2). Intuitively, X_f can be thought of as the complement of a hypersurface. We have

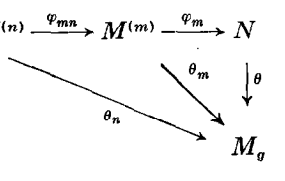
$$U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p} = U_{i_0 i_1 \cdots i_p} = X_{f_{i_0} f_{i_1} \cdots f_{i_p}}$$

since $\mathcal{O}_x/\mathfrak{m}_x$ is a field. Furthermore we set $M_{i_0 i_1 \ldots i_p} = \Gamma(U_{i_0 i_1 \ldots i_p}, \mathcal{F})$.

We shall consider $M_{i_0 i_1 \ldots i_p}$ as the direct limit of an inductive sequence $M^{(n)}_{i_0 i_1 \ldots i_p}$, where each module $M^{(n)}_{i_0 i_1 \ldots i_p}$ is the same. Indeed (Grothendieck "Eléments," 0_1, 1.6.1), suppose that M is an A-module and consider a sequence $(M^{(n)})$ of A-modules, where each module $M^{(n)}$ is equal to M. Let g be an element of A and for $n \leqslant m$ define the homomorphism $\varphi_{mn} : M^{(n)} \to M^{(m)}$ by $\varphi_{mn}(z) = g^{m-n}z$. We see that $\varphi_{pm} \circ \varphi_{mn} = \varphi_{pn}$ for $n \leqslant m \leqslant p$. Let

$$N = \varinjlim_n M^{(n)}, \qquad \varphi_n : M^{(n)} \to N$$

the canonical homomorphism and define the homomorphism $\theta_n : M^{(n)} \to M_g$ by $\theta_n(z) = z/g^n$. Clearly $\theta_m \circ \varphi_{mn} = \theta_n$ for $n \geqslant m$ and by the universal property of the direct limit there exists a homomorphism $\theta : N \to M_g$ such that the diagram



is commutative. Each element of M_g is of the form z/g^n for some n, and $z/g^n = \theta_n(z) = \theta(\varphi_n(z))$, which shows that θ is surjective. Furthermore if $\theta(\varphi_n(z)) = 0$, then $\theta_n(z) = z/g^n = 0$ and there exists an integer $k > 0$ such that $g^k z = 0$, i.e., $\varphi_{n+k,n}(z) = 0$. Thus $\varphi_n(z) = \varphi_{n+k} \circ \varphi_{n+k,n}(z) = 0$ which proves that θ is injective. It follows that θ is an isomorphism and thus we can identify $\varinjlim_n M^{(n)}$ and M_g.

Returning to our case, let us assume that either the underlying space X is noetherian or that X is a schema and the underlying space is quasi-compact. Under these hypotheses we have

$$M_{i_0 i_1 \cdots i_p} = M_{f_{i_0} f_{i_1} \cdots f_{i_p}}$$

(Grothendieck, "Eléments," I, 9.3.3). We set $M^{(n)}_{i_0 i_1 \ldots i_p} = M$ for all $n \in \mathbf{N}$ and define the homomorphisms φ_{mn} as the multiplication by $(f_{i_0} f_{i_1} \cdots f_{i_p})^{m-n}$ for $n \leqslant m$. Then $M_{i_0 i_1 \ldots i_p}$ is the direct limit $\varinjlim_n M^{(n)}_{i_0 i_1 \ldots i_p}$ and the canonical homomorphism

$$\varphi_n : M^{(n)}_{i_0 i_1 \cdots i_p} \to M_{i_0 i_1 \cdots i_p} \qquad \text{is given by} \qquad \varphi_n(x) = \frac{x}{(f_{i_0} f_{i_1} \cdots f_{i_p})^n}.$$

Let $C^p_n(M)$ be the set of alternating mappings of $[1, r]^{p+1}$ into the module M. An element $g \in C^p_n(M)$ is a map which to each $(p+1)$-tuple $(i_0, i_1, \ldots i_p)$ of integers satisfying $1 \leqslant i_k \leqslant r$ associates an element $g(i_0, i_1, \ldots i_p) \in M$. The set $C^p_n(M)$ is the same for all $n \in \mathbf{N}$. Define the coboundary operator $d : C^p_n(M) \to C^{p+1}_n(M)$ by

$$dg(i_0, i_1, \ldots, i_{p+1}) = \textstyle\sum_{k=0}^{p+1} (-1)^k f^n_{i_k} g(i_0, \ldots, \hat{i}_k, \ldots, i_{p+1}).$$

We also define maps $\psi_{mn} : C^p_n(M) \to C^p_m(M)$ for $n \leqslant m$ by setting $(\psi_{mn} g)(i_0, i_1, \ldots, i_p) = (f_{i_0} f_{i_1} \cdots f_{i_p})^{m-n} g(i_0, i_1, \ldots, i_p)$. Clearly, for each p the $C^p_n(M)$ form a direct system of groups and it is easy to check that the diagram

$$\begin{array}{ccc} C^p_n(M) & \xrightarrow{d} & C^{p+1}_n(M) \\ {\scriptstyle \psi_{n+1,n}}\downarrow & & \downarrow{\scriptstyle \psi_{n+1,n}} \\ C^p_{n+1}(M) & \xrightarrow{d} & C^{p+1}_{n+1}(M) \end{array}$$

is commutative.

The group $C^p(\mathfrak{U}, \mathscr{F})$ of p-cochains of the covering $\mathfrak{U}$ of $U = \bigcup_{i=1}^r U_i$ with values in $\mathscr{F}$ is defined as the set of maps s which to each

$$(i_0, i_1, \ldots i_p) \in [1, r]^{p+1}$$

associates a section $s(i_0, i_1, \ldots, i_p) \in \Gamma(U_{i_0 i_1 \ldots i_p}, \mathscr{F}) = M_{i_0 i_1 \ldots i_p}$ (Hirzebruch, *loc. cit.* or Godement, "Théorie des faisceaux," II. 5.1, p. 201). The coboundary operator $d : C^p(\mathfrak{U}, \mathscr{F}) \to C^{p+1}(\mathfrak{U}, \mathscr{F})$ is defined by

$$ds(i_0, i_1, \ldots, i_{p+1}) = \sum_{k=0}^{p+1} (-1)^k \rho_k(s(i_0, \ldots, \hat{i}_k, \ldots, i_{p+1})),$$

where ρ_k is the restriction map $\Gamma(U_{i_0 \ldots i_k \ldots i_{p+1}}, \mathscr{F}) \to \Gamma(U_{i_0 i_1 \ldots i_{p+1}}, \mathscr{F})$. We want to show that $C^p(\mathfrak{U}, \mathscr{F}) = \varinjlim_n C_n^p(M)$ and also that the coboundary operator $d : C^p(\mathfrak{U}, \mathscr{F}) \to C^{p+1}(\mathfrak{U}, \mathscr{F})$ is the direct limit of the operators $d : C_n^p(M) \to C_n^{p+1}(M)$.

Define the maps $\eta_n : C_n^p(M) \to C^p(\mathfrak{U}, \mathscr{F})$ as follows: for $g \in C_n^p(M)$ set

$$(\eta_n g)(i_0, \ldots, i_p) = \varphi_n(g(i_0, \ldots, i_p)) = \frac{g(i_0, \ldots, i_p)}{(f_{i_0} f_{i_1} \cdots f_{i_p})^n} \in M_{i_0 i_1 \ldots i_p}.$$

We have $\eta_m \circ \psi_{mn} = \eta_n$ for $m \geqslant n$, since

$$(\eta_m(\psi_{mn}(g)))(i_0, \ldots, i_p) =$$
$$= \frac{(\psi_{mn}(g))(i_0, \ldots, i_p)}{(f_{i_0} f_{i_1} \cdots f_{i_p})^m} = \frac{(f_{i_0} f_{i_1} \cdots f_{i_p})^{m-n} g(i_0, \ldots, i_p)}{(f_{i_0} f_{i_1} \cdots f_{i_p})^m} = \frac{g(i_0, \ldots, i_p)}{(f_{i_0} f_{i_1} \cdots f_{i_p})^n}.$$

Thus there exists a map $\eta : C_\infty^p(M) \to C^p(\mathfrak{U}, \mathscr{F})$ for which the diagram

$$\begin{array}{ccccc} C_n^p(M) & \xrightarrow{\psi_{mn}} & C_m^p(M) & \xrightarrow{\psi_m} & C_\infty^p(M) \\ & \searrow^{\eta_n} & & \swarrow_{\eta_m} \quad \downarrow \eta & \\ & & & C^p(\mathfrak{U}, \mathscr{F}) & \end{array}$$

is commutative. Here we have written $C_\infty^p(M)$ for $\varinjlim_n C_n{}^p(M)$. The map η is surjective. Indeed, let $s \in C^p(\mathfrak{U}, \mathscr{F})$. For $(i_0, i_1, \ldots, i_p) \in [1, r]^{p+1}$ we have

$$s(i_0, i_1, \ldots, i_p) = \frac{g(i_0, i_1, \ldots, i_p)}{(f_{i_0} f_{i_1} \cdots f_{i_p})^n} \in M_{i_0 i_1 \ldots i_p},$$

where $g(i_0, i_1, \ldots, i_p) \in M$. Then $g \in C_n^p(M)$ and $s = \eta(\psi_n(g))$. It is easy to see that η is also injective and thus it is an isomorphism. We identify $C_\infty^p(M)$ and $C^p(\mathfrak{U}, \mathscr{F})$ with the help of this isomorphism.

To prove the assertion concerning the coboundary operators d we have to prove that the diagram

$$\begin{array}{ccc} C_n^p(M) & \xrightarrow{d} & C_n^{p+1}(M) \\ {\scriptstyle \eta_n}\downarrow & & \downarrow{\scriptstyle \eta_n} \\ C^p(\mathfrak{U}, \mathscr{F}) & \xrightarrow{d} & C^{p+1}(\mathfrak{U}, \mathscr{F}) \end{array}$$

is commutative. Let us observe first that if $f, g \in A$, then $D(gh) \subset D(g)$ and the restriction map $M_g \to M_{gh}$ is given by $x/g^k \rightsquigarrow h^k x/(gh)^k$. Taking this into account, an easy calculation shows that both $\eta(d(g))(i_0, i_1, \ldots, i_{p+1})$ and $d(\eta(g))(i_0, i_1, \ldots, i_{p+1})$ are equal to

$$\sum_{k=0}^{p+1} (-1)^k \frac{f_{i_k}^n g(i_0, \ldots, \hat{i}_k, \ldots, i_{p+1})}{(f_{i_0} f_{i_1} \cdots f_{i_{p+1}})^n}.$$

We shall now give a very important alternative interpretation of the cochain complex $C_n^p(M)$ as a Koszul complex or exterior algebra complex. Let A be a ring, $\boldsymbol{f} = (f_1, \ldots, f_r)$ a system of elements of A. We want to attach a chain complex $K_\bullet(\boldsymbol{f})$ to $\boldsymbol{f}$. First let f be a single element of A and define the chain complex $K_\bullet(f)$ by $K_0(f) = K_1(f) = A$, $K_n(f) = 0$ for $n \neq 0, 1$. The boundary operators $0 \leftarrow K_0(f) \leftarrow K_1(f) \leftarrow 0 \leftarrow \cdots$, are all trivial except $d_1 : K_1(f) \to K_0(f)$ which is defined by $x \rightsquigarrow fx$. The complex $K_\bullet(\boldsymbol{f})$ is then defined as the tensor product $K_\bullet(f_1) \otimes K_\bullet(f_2) \otimes \cdots \otimes K_\bullet(f_r)$. The complex $K_\bullet(\boldsymbol{f})$ is graded in the usual manner by

$$K_n(\boldsymbol{f}) = \coprod_{q_1 + \cdots + q_r = n} K_{q_1}(f_1) \otimes \cdots \otimes K_{q_r}(f_r)$$

and the boundary operator $\partial : K_n(\boldsymbol{f}) \to K_{n-1}(\boldsymbol{f})$ is defined by

$$\partial(x_1 \otimes \cdots \otimes x_r) = \sum_{k=1}^{r} (-1)^{q_1 + \cdots + q_{k-1}} x_1 \otimes \cdots \otimes d_k x_k \otimes \cdots \otimes x_r,$$

where $x_k \in K_{q_k}(f_k)$ and d_k is the boundary operator in the elementary complex $K_\bullet(f_k)$.

If M is an A-module, we define $K_\bullet(\boldsymbol{f}, M) = K_\bullet(\boldsymbol{f}) \otimes_A M$ and $K^\bullet(\boldsymbol{f}, M) = \mathrm{Hom}_A(K_\bullet(\boldsymbol{f}), M)$. These are respectively chain and cochain complexes and their homology resp. cohomology groups will be denoted by $H_\bullet(\boldsymbol{f}, M)$ and $H^\bullet(\boldsymbol{f}, M)$. We have an isomorphism

$$H_i(\boldsymbol{f}, M) \simeq H^{r-i}(\boldsymbol{f}, M)$$

(Grothendieck, "Eléments," III, 1.1.3; see p. 306 below).

Returning to the covering $\mathfrak{U} = (X_{f_i})_{1 \leqslant i \leqslant r}$ and setting $f^n = (f_1^n, f_2^n, \ldots, f_r^n)$, the module $C_n^p(M)$ can be identified with $K^{p+1}(f^n, M)$. Indeed, let $(i_0, i_1, \ldots, i_p) \in [1, r]^{p+1}$ and define the element

$$y_{i_0 i_1 \cdots i_p} \in K_{p+1}(f^n) \quad \text{by} \quad y_{i_0 i_1 \cdots i_p} = y_1 \otimes y_2 \otimes \cdots \otimes y_r,$$

where y_i is the unit element in $K_1(f_i^n)$ if $i = i_k$ for some k and y_i is the unit element in $K_0(f_i^n)$ if $i \neq i_k$ for all k. Clearly the $y_{i_0 i_1 \cdots i_p}$ form a basis for $K_{p+1}(f^n)$. Let $\gamma \in K^{p+1}(f^n, M) = \mathrm{Hom}_A(K_{p+1}(f^n), M)$. If the elements of $(i_0, \ldots, i_p)$ are all different we set $g(i_0, i_1, \ldots, i_p) = (-1)^\nu \gamma(y_{i_0 i_1 \cdots i_p})$, where ν is the number of pairs (i_j, i_k) with $j < k$, $i_j > i_k$. If two or more elements i_k are equal, we set $g(i_0, i_1, \ldots, i_p) = 0$. Thus we obtain an element $g \in C_n^p(M)$ and $\gamma \rightsquigarrow g$ is clearly an isomorphism. A simple computation shows that this isomorphism is compatible with the coboundary operators.

The name exterior algebra complex can be explained as follows. The exterior algebra $\wedge A$ of the ring A is isomorphic to $A \oplus A = K_\bullet(f)$, $f \in A$. The exterior algebra $\wedge(A^r)$ is isomorphic to the twisted tensor product $(\wedge A)^{\otimes r}$. For the sake of simplicity we exhibit this isomorphism only in the case $r = 2$. Let E and F be two A-modules; then the isomorphism

$$\bigwedge^{p+q}(E \oplus F) \xrightarrow{\sim} \bigoplus_{p,q} \left(\bigwedge^p E\right) \otimes \left(\bigwedge^q F\right)$$

is given by assigning to the element $x_1 \wedge \cdots \wedge x_p \wedge y_1 \wedge \cdots \wedge y_q$ the element $(x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \wedge \cdots \wedge y_q)$. The product of two elements is given by

$$\begin{aligned}&((x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \wedge \cdots \wedge y_q)) \otimes ((x_{p+1} \wedge \cdots \wedge x_r) \otimes (y_{q+1} \wedge \cdots \wedge y_s)) \\ &\quad = (-1)^{q(r-p)} (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \wedge \cdots \wedge y_s).\end{aligned}$$

Thus for $f = (f_1, f_2, \ldots, f_r)$ the graded module $K_\bullet(f)$ is isomorphic to $\wedge(A^r)$. To define the derivation in $K_\bullet(f)$ let us recall the operation of interior multiplication (Bourbaki, "Algèbre," Chap. III, § 8, No. 4, Déf. 1). Let E be a free module over A with basis $e_1, \ldots, e_r$ and let z' be an element of the dual $\check{E}$ of E. The map $i_{z'} : \wedge E \to \wedge E$ is defined by $i_{z'}(e_i) = z_i'$, where z_i' is the i-th component of z' with respect to the dual basis $e_1', \ldots, e_r'$ of $\check{E}$. For $z = \sum z_i e_i \in E$ we have then by linearity $i_{z'}(z) = \sum_{i=1}^r z_i z_i' = \langle z, z' \rangle$ and $i_{z'}$ can be extended by induction to elements of any degree of $\wedge E$ so that it becomes an antiderivation of degree -1. (A linear map $d : \wedge E \to \wedge E$ of degree -1 is an antiderivation if it satisfies $d(u \wedge v) = du \wedge v + (-1)^p u \wedge dv$ for $u \in \wedge^p E$).

In the Koszul complex $K_\bullet(\boldsymbol{f}) = \wedge(A^r)$ we define the boundary operator by $\partial = i_f$, where $\boldsymbol{f}$ is considered to be an element to the dual $(A_r)^\vee$. In this interpretation the isomorphism between $K^{p+1}(\boldsymbol{f}^n, M)$ and $C_n^p(M)$ is given by $g(i_0, i_1, \ldots, i_p) = g(e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge e_{i_p})$.

The homomorphisms $\psi_{mn} : C_n^p(M) \to C_n^p(M)$ (p. 284) can also be interpreted in terms of the Koszul complex. In general, let $\boldsymbol{f} = (f_1, \ldots, f_r)$ and $\boldsymbol{g} = (g_1, \ldots, g_r)$ be two sequences of r elements of A and set $\boldsymbol{fg} = (f_1 g_1, \ldots, f_r g_r)$. Then there exists a natural homomorphism of complexes $K_\bullet(\boldsymbol{fg}) \to K_\bullet(\boldsymbol{f})$. To define it, it is enough to consider elementary Koszul complexes $K_\bullet(\boldsymbol{fg})$ and $K_\bullet(\boldsymbol{f})$, where $f, g \in A$. We let $K_0(\boldsymbol{fg}) \to K_0(\boldsymbol{f})$ be the identity map and $K_1(\boldsymbol{f}) \to K_1(\boldsymbol{fg})$ the multiplication by g. Then the diagram

$$\begin{array}{ccc} K_0(\boldsymbol{fg}) & \xleftarrow{d_1} & K_1(\boldsymbol{fg}) \\ \downarrow & & \downarrow \\ K_0(\boldsymbol{f}) & \xleftarrow{d_1} & K_1(\boldsymbol{f}) \end{array}$$

is commutative since the image of $x \in K_1(\boldsymbol{fg})$ under both composite maps is $\boldsymbol{fgx}$. Hence $K_\bullet(\boldsymbol{fg}) \to K_\bullet(\boldsymbol{f})$ is a homomorphism of complexes and by taking tensor products we obtain the homomorphism $K_\bullet(\boldsymbol{fg}) \to K_\bullet(\boldsymbol{f})$. If we consider $K_\bullet(\boldsymbol{f})$ as the exterior algebra $\wedge(A^r)$ then this homomorphism is obtained by extension to $\wedge(A^r)$ of the linear map defined by $e_i \to g e_i$ $(1 \leqslant i \leqslant r)$. In particular we have a homomorphism $K_\bullet(\boldsymbol{f}^m) \to K_\bullet(\boldsymbol{f}^n)$ for $m \leqslant n$ and thus a homomorphism $K^\bullet(\boldsymbol{f}^n, M) \to K^\bullet(\boldsymbol{f}^m, M)$, which corresponds to the ψ_{mn} in the isomorphism between $K^{p+1}(\boldsymbol{f}^n, M)$ and $C_n^p(M)$.

Let us write

$$C^\bullet((\boldsymbol{f}), M) = \varinjlim_n K^\bullet(\boldsymbol{f}^n, M)$$

and

$$H^\bullet((\boldsymbol{f}), M) = H^\bullet(C^\bullet((\boldsymbol{f}), M)) = \varinjlim_n H^\bullet(\boldsymbol{f}^n, M),$$

where the last equality is due to the fact that the direct limit of exact sequences is exact. By our preceding discussions we have the canonical isomorphisms $C^p(\mathfrak{U}, \mathscr{F}) \simeq C^{p+1}((\boldsymbol{f}), M)$ for $p \geqslant 0$ and $H^p(\mathfrak{U}, \mathscr{F}) \simeq H^{p+1}((\boldsymbol{f}), M)$ for $p \geqslant 1$. Furthermore we have the exact sequence $0 \to H^0((\boldsymbol{f}), M) \to M \to H^0(\mathfrak{U}, \mathscr{F}) \to H^1((\boldsymbol{f}), M) \to 0$. Indeed, $H^0(\mathfrak{U}, \mathscr{F})$ is the group of cocycles of $C^0(\mathfrak{U}, \mathscr{F})$ and this is identified to the group of

cocycles of $C^1((f), M)$ which we can denote by $Z^1((f), M)$. Also $M = C^0((f), M)$ and thus the sequence goes into

$$0 \to H^0((f), M) \to C^0((f), M) \to Z^1((f), M) \to H^1((f), M) \to 0.$$

Taking into account that $C^{-1}((f), M) = 0$, this sequence is exact by the definition of the cohomology groups $H^0((f), M)$ and $H^1((f), M)$.

Suppose that the elements $f_1, f_2, \ldots, f_r$ generate the ideal A, i.e. that there exist elements $g_1, g_2, \ldots, g_r$ such that $g_1 f_1 + \cdots + g_r f_r = 1$. Then $H^i((f), M) = 0$ for all i. Indeed, we shall prove that in this case the complex $K_\cdot(f)$ is homotopically trivial. If this is true, then $K_\cdot(f^n)$ is also homotopically trivial, since the f_i^n also generate A. Thus $H^\cdot(f^n, M) = 0$ for all n and finally $H^\cdot((f), M) = 0$. We have to construct a homotopy operator $h : K_p(f) \to K_{p+1}(f)$ satisfying $d \circ h + h \circ d = 1$. This is best done if we consider the interpretation of $K_\cdot(f)$ as $\Lambda(A^r)$. We consider then $f = (f_1, \ldots, f_r)$ as an element of the dual $(A^r)^\vee$ and $d = i_f$. Take $g = (g_1, \ldots, g_r)$, whose existence is assumed, as an element of A^r and consider the exterior multiplication e_g defined by $x \rightsquigarrow x \wedge g$. Clearly e_g raises the degree by 1 and a simple computation shows that $i_f \circ e_g + e_g \circ i_f = \langle g, f \rangle 1$, i.e., in our case $i_f \circ e_g + e_g \circ i_f = 1$. Thus e_g is the required homotopy operator.

The group $H^0(\mathfrak{U}, \mathscr{F})$ is isomorphic to the group of sections $\Gamma(U, \mathscr{F})$ and will be different from 0 in general. If the sections $f_i \in A = \Gamma(X, \mathcal{O}_X)$ are such that there exist sections $g_i \in \Gamma(U, \mathcal{O}_X)$ such that $\sum_{i=1}^r g_i(f_i \mid U) = 1$ then by the previous results $H^p(\mathfrak{U}, \mathscr{F}) = 0$ for $p \geqslant 1$.

Theorem (Grothendieck, "Eléments," III, 1.3.1). *Let X be an affine schema and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-Module. Then $H^p(X, \mathscr{F}) = 0$ for all $p \geqslant 1$.*

Proof. If $\mathfrak{U} = (U_i)_{1 \leqslant i \leqslant r}$ is a covering of $X = \mathrm{Spec}(A)$ by affine open sets of the form $U_i = X_{f_i} = D(f_i)$, where $f_i \in A$, then f_i generate the ideal A. It follows by the last remark $H^p(\mathfrak{U}, \mathscr{F}) = 0$ for $p \geqslant 1$. Now there exist arbitrarily fine coverings of X by sets X_f and since X is quasi-compact, there exist finite coverings of this kind. It follows by the definition of the Čech cohomology groups that $\check{H}^p(X, \mathscr{F}) = 0$ for $p \geqslant 1$. The same is true for every preschema X_g, where $g \in A$, i.e., we have $\check{H}^p(X_g, \mathscr{F}) = 0$ for $p \geqslant 1$. Furthermore we have $X_g \cap X_h = X_{gh}$ for $g, h \in A$. The fact that $H^p(X, \mathscr{F}) = 0$ for $p \geqslant 1$ now follows from the following theorem of Henri Cartan (Godement, "Théorie des faisceaux," II, Th. 5.9.2, p. 227). Let X be a topological space and $\mathscr{A}$ a sheaf of abelian groups on X. Suppose that there exists a covering $\mathfrak{U}$

of X by open sets such that $\mathfrak{U}$ contains arbitrarily small sets and if $U' \in \mathfrak{U}$, $U'' \in \mathfrak{U}$, then $U' \cap U'' \in \mathfrak{U}$. Under these conditions, if $\check{H}^p(U, \mathscr{A}) = 0$ for all $p \geqslant 1$ and $U \in \mathfrak{U}$, then $\check{H}^p(X, \mathscr{A})$ and $H^p(X, \mathscr{A})$ are isomorphic.

Let us now see some consequences of the above theorem.

(1) Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be two ringed spaces and $f = (\psi, \theta)$ a morphism from X into Y. Let $\mathscr{F}$ be an $\mathscr{O}_X$-Module and consider on Y the presheaf defined by $V \rightsquigarrow H^p(f^{-1}(V), \mathscr{F})$. This presheaf defines a sheaf which is in a natural way an $\mathscr{O}_Y$-Module and will be denoted by $R^p f_*(\mathscr{F})$ (Grothendieck, "Eléments," 0_{III}, 12.2.1). The functor $R^p f_*$ from the category of $\mathscr{O}_X$-Modules into the category of $\mathscr{O}_Y$-Modules can be considered as the p-th right derived functor of f_*.

Theorem (Grothendieck, "Eléments," III, 1.3.2). *Let $f: X \to Y$ be an affine morphism* (p. 271), *then for every quasi-coherent $\mathscr{O}_X$-Module $\mathscr{F}$ we have $R^p f_*(\mathscr{F}) = 0$ for $p \geqslant 1$.*

Indeed, for every affine open set V in Y, the set $f^{-1}(V)$ is affine and thus $H^p(f^{-1}(V), \mathscr{F}) = 0$.

(2) **Theorem** (Grothendieck, "Eléments," III, 1.3.3). *Let $f: X \to Y$ be an affine morphism and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-Module. Then $H^p(Y, f_*(\mathscr{F}))$ and $H^p(X, \mathscr{F})$ are canonically isomorphic for $p \geqslant 0$.*

We have seen that on an affine schema X, for every quasi-coherent $\mathscr{O}_X$-Module $\mathscr{F}$, we have $H^p(X, \mathscr{F}) = 0$ for $p \geqslant 1$. As a converse we have the following result:

Serre's Criterion (Grothendieck, "Eléments," II, 5.2.1). *Let X be a quasi-compact schema or a preschema with noetherian underlying space. Then the following conditions are equivalent.*

(a) *X is affine.*

(b) *There exists a covering (X_{f_α}) of X, where $f_\alpha \in A = \Gamma(X, \mathscr{O}_X)$ such that the X_{f_α} are affine and the ideal generated by the f_α in A is A.*

(c) *The functor $\mathscr{F} \rightsquigarrow \Gamma(X, \mathscr{F})$ is exact in the category of quasi-coherent $\mathscr{O}_X$-Modules*, i.e., *if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of quasi-coherent $\mathscr{O}_X$-Modules, then the sequence*

$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \to 0$$

is exact.

(c′) *If* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence of quasi-coherent* $\mathcal{O}_X$*-Modules where* $\mathscr{F}$ *is isomorphic to a sub-*$\mathcal{O}_X$*-Module of some* $\mathcal{O}_X^n$*, then the sequence*

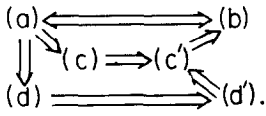
$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \to 0$$

is exact.

(d) $H^1(X, \mathscr{F}) = 0$ *for every quasi-coherent* $\mathcal{O}_X$*-Module* $\mathscr{F}$.

(d′) $H^1(X, \mathscr{J}) = 0$ *for every quasi-coherent Ideal* $\mathscr{J}$ *of* $\mathcal{O}_X$.

We shall prove the theorem according to the following pattern:



(a) ⇒ (b). It is sufficient to take the covering consisting of the one element $X = X_f$, where $f = 1$.

(b) ⇒ (a). In general if A is a ring and $(Y, \mathcal{O}_Y)$ a preschema, then to each homomorphism $A \to \Gamma(Y, \mathcal{O}_Y)$ there corresponds canonically a unique morphism $Y \to \mathrm{Spec}(A)$ (Grothendieck, "Eléments," I, 2.2.4). Thus in our case to the identity map $A \to \Gamma(X, \mathcal{O}_X)$ there corresponds a morphism $\chi : X \to \mathrm{Spec}(A) = S$. We want to prove that χ is an isomorphism. To each $f_\alpha \in A$ we associate the open set $D(f_\alpha)$ in S. By hypothesis (b) the $D(f_\alpha)$ form a covering of S. We have $\chi^{-1}(D(f_\alpha)) = X_{f_\alpha}$ and the X_{f_α} form a covering of X. Now by the hypothesis that X is either quasi-compact and separated or that its underlying space is noetherian, the restriction of χ to X_{f_α} is an isomorphism from X_{f_α} onto $D(f_\alpha)$ since both can be identified canonically with $\mathrm{Spec}(\Gamma(X_{f_\alpha}, \mathcal{O}_X))$ (Grothendieck, "Eléments," I, 9.3.3). Thus χ is an isomorphism.

(a) ⇒ (c) is known (Grothendieck, "Eléments," I, 1.3.11).

(c) ⇒ (c′) is clear.

(c′) ⇒ (b). First we prove the following lemma. Suppose that condition (c′) is satisfied, let x be a closed point of X and U an open neighborhood of x. Then there exists an element $f \in A = \Gamma(X, \mathcal{O}_X)$ such that $x \in X_f \subset U$. Indeed, let $\mathscr{J}$ be the quasi-coherent Ideal of $\mathcal{O}_X$ which defines the unique reduced subpreschema of X on the closed set $X - U$ (p. 256). Similarly, let $\mathscr{J}'$ be the Ideal which defines the unique reduced subpreschema of X on the closed set $(X - U) \cap \{x\}$.

Then $\mathscr{J}' \subset \mathscr{J} \subset \mathscr{O}_X$ and the quotient sheaf $\mathscr{J}'' = \mathscr{J}/\mathscr{J}'$ has a unique nonzero stalk at the point x, where $\mathscr{J}''_x = \mathscr{O}_x/\mathfrak{m}_x = \kappa(x)$. The sequence $0 \to \mathscr{J}' \to \mathscr{J} \to \mathscr{J}'' \to 0$ is exact and thus by c') the map $\Gamma(X, \mathscr{J}) \to \Gamma(X, \mathscr{J}'')$ is surjective. Let $s \in \Gamma(X, \mathscr{J}'')$ be the section whose germ s_x at x is the unit element 1_x of $\kappa(x)$ and let $f \in \Gamma(X, \mathscr{J}) \subset \Gamma(X, \mathscr{O}_X)$ be a section which maps into s. Then $f(x) = s_x = 1_x$, hence $x \in X_f$ and for $y \in X - U$ we have $f(y) = 0$, i.e., $X_f \subset U$.

If we take U affine in the previous lemma, then X_f will also be affine. Attach now to every closed point $x \in X$ an affine open neighborhood U and in each neighborhood take an affine open set X_f such that $x \in X_f \subset U$. The union Z of all the sets X_f is an open set which contains all the closed points. But $\complement Z$ is a closed set and thus a quasi-compact Kolmogoroff space. If $\complement Z$ were nonempty, it would have to contain a closed point (p. 267). Thus $Z = X$ and the X_f form a covering of X.

Let us now choose a finite covering $(X_{f_i})_{1 \leqslant i \leqslant n}$ from the given covering (X_f) by affine open sets. The sections $f_i \in \Gamma(X, \mathscr{O}_X)$, $1 \leqslant i \leqslant n$ define a homomorphism $\mathscr{O}_X^n \to \mathscr{O}_X$. Indeed, given a family of sections $(s_i)_{1 \leqslant i \leqslant n}$ of $\mathscr{O}_X$ over an open set U we associate with it the section $\sum_{i=1}^n s_i(f_i \mid U)$ of $\mathscr{O}_X$. Now for every $x \in X$ at least one germ $(f_i)_x$ is invertible since $x \in X_{f_i}$ for some i. Thus $\mathscr{O}_X^n \to \mathscr{O}_X$ is surjective and we have an exact sequence $0 \to \mathscr{R} \to \mathscr{O}_X^n \to \mathscr{O}_X \to 0$, where $\mathscr{R}$ is a quasi-coherent sub-$\mathscr{O}_X$-Module of $\mathscr{O}_X^n$. Thus by (c') the homomorphism $\Gamma(X, \mathscr{O}_X^n) \to \Gamma(X, \mathscr{O}_X)$ is surjective and in particular there exist sections $s_i \in (X, \mathscr{O}_X)$, $1 \leqslant i \leqslant n$, such that $\sum_{i=1}^n s_i f_i = 1$. This proves (b).

(a) $\Rightarrow$ (d) is a special case of the theorem proved above (p. 289). For $p = 1$ a more elementary proof can be given (Grothendieck, "Eléments," I, 5.1.9.2).

(d) $\Rightarrow$ (d') is clear.

(d') $\Rightarrow$ (c'). If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence then the sequence $0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \to H^1(X, \mathscr{F}')$ is also exact, thus it is enough to prove that for every quasi-coherent sub-$\mathscr{O}_X$-Module $\mathscr{F}$ of $\mathscr{O}_X^n$ we have $H^1(X, \mathscr{F}) = 0$.

The filtration $\mathscr{O}_X^n \supset \mathscr{O}_X^{n-1} \supset \cdots \supset \mathscr{O}_X \supset 0$ of $\mathscr{O}_X^n$ defines a filtration $\mathscr{F} = \mathscr{F}_n \supset \mathscr{F}_{n-1} \supset \cdots \supset \mathscr{F}_1 \supset 0$ of $\mathscr{F}$ if we set $\mathscr{F}_k = \mathscr{O}_X^k \cap \mathscr{F}$. Now $\mathscr{F}_{k+1}/\mathscr{F}_k$ is isomorphic to a quasi-coherent sub-$\mathscr{O}_X$-Module of $\mathscr{O}_X^{k+1}/\mathscr{O}_X^k = \mathscr{O}_X$, i.e., to a quasi-coherent Ideal of $\mathscr{O}_X$. By (d') we have $H^1(X, \mathscr{F}_{k+1}/\mathscr{F}_k) = 0$ $(0 \leqslant k \leqslant n-1)$. The exact sequence $0 \to \mathscr{F}_k \to \mathscr{F}_{k+1} \to \mathscr{F}_{k+1}/\mathscr{F}_k \to 0$ yields the exact sequence

$$H^1(X, \mathscr{F}_k) \to H^1(X, \mathscr{F}_{k+1}) \to H^1(X, \mathscr{F}_{k+1}/\mathscr{F}_k).$$

Since $H^1(X, \mathscr{F}_1) = 0$ by (d'), we obtain by induction on k that $H^1(X, \mathscr{F}_k) = 0$ for all k.

Let us list some consequences of Serre's criterion.

(1) Chevalley's theorem (Grothendieck, "Eléments," II, 6.7.1). If X is an affine schema, Y a noetherian preschema (p. 261), $f: X \to Y$ a finite (p. 277), surjective morphism, then Y is an affine schema. The proof consists in showing that $H^1(Y, \mathscr{F}) = 0$ for every quasi-coherent $\mathscr{O}_Y$-Module $\mathscr{F}$.

(2) It is clear that if X is an affine schema, then X_{red} (p. 255) is also affine. Here we prove the converse result.

Theorem (Grothendieck, "Eléments," I, 5.1.10). *If X is a noetherian preschema and X_{red} is affine, then X is also affine.*

Proof. The structure sheaf of X_{red} is by definition $\mathscr{O}_X/\mathscr{N}_X$, where $\mathscr{N}_X$ is the Nilradical of $\mathscr{O}_X$ (p. 255). Since X is noetherian, there exists an integer k such that $\mathscr{N}_X^k = 0$ (p. 262). We shall make use of a standard reduction to the case $k = 2$. Indeed, consider the sequence of Ideals $0 \subset \mathscr{N}_X^{k-1} \subset \mathscr{N}_X^{k-2} \subset \cdots \subset \mathscr{N}_X$, to which there corresponds a sequence of sheaves of rings $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{N}_X^{k-1} \to \mathscr{O}_X/\mathscr{N}_X^{k-2} \to \cdots \to \mathscr{O}_X/\mathscr{N}_X$ and a sequence of preschemata $X_1 = X_{\text{red}} \to \cdots \to X_{k-2} \to X_{k-1} \to X = X_k$ which have all the same underlying topological space. Now $\mathscr{O}_{X_i} = \mathscr{O}_X/\mathscr{N}_X^i = (\mathscr{O}_X/\mathscr{N}_X^{i+1})/(\mathscr{N}_X^i/\mathscr{N}_X^{i+1}) = \mathscr{O}_{X_{i+1}}/\mathscr{J}$, where $\mathscr{J}^2 = 0$. Thus we are reduced to prove the following. Let X be a noetherian preschema, $\mathscr{J}$ a coherent Ideal in $\mathscr{O}_X$ such that $\mathscr{J}^2 = 0$. If the subpreschema X' defined by $\mathscr{O}_X/\mathscr{J}$ is affine then X is affine.

By virtue of Serre's criterion we have to prove that for every quasi-coherent $\mathscr{O}_X$-Module $\mathscr{F}$ the condition $H^1(X, \mathscr{F}) = 0$ is satisfied. Consider the exact sequence $0 \to \mathscr{J}\mathscr{F} \to \mathscr{F} \to \mathscr{F}/\mathscr{J}\mathscr{F} \to 0$. Now $\mathscr{F}/\mathscr{J}\mathscr{F}$ is a quasi-coherent $(\mathscr{O}_X/\mathscr{J})$-Module, and since X' is affine, by Serre's criterion we have $H^1(X', \mathscr{F}/\mathscr{J}\mathscr{F}) = 0$. But as topological spaces X' and X are the same, hence also $H^1(X, \mathscr{F}/\mathscr{J}\mathscr{F}) = 0$. Now the condition $\mathscr{J}^2 = 0$ implies that $\mathscr{J}\mathscr{F}$ can also be considered as an $(\mathscr{O}_X/\mathscr{J})$-Module, hence $H^1(X', \mathscr{J}\mathscr{F}) = H^1(X, \mathscr{J}\mathscr{F}) = 0$. The exact cohomology sequence $H^1(X, \mathscr{J}\mathscr{F}) \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{F}/\mathscr{J}\mathscr{F})$ gives $H^1(X, \mathscr{F}) = 0$. Q.E.D.

The fact that $H^p(X, \mathscr{F}) = 0$ for an affine schema X gives means to compute these cohomology groups for an arbitrary preschema using only coverings.

Theorem (Grothendieck, "Eléments," III, 1.4.1 and 1.4.2). *Let X be a preschema and suppose that $\mathfrak{U} = (U_\alpha)$ is a covering by affine open sets such that the finite intersections of the U_α are also affine (this will always be the case if X is a schema*; Grothendieck, "Eléments," I, 5.5.6). *Then for every quasi-coherent $\mathcal{O}_X$-Module $\mathscr{F}$, the $\Gamma(X, \mathcal{O}_X)$-modules $H^\bullet(\mathfrak{U}, \mathscr{F})$ and $H^\bullet(X, \mathscr{F})$ are canonically isomorphic.*

Indeed, we have $H^q(V, \mathscr{F}) = 0$ for all finite intersections V of the U_α and then the result follows from Leray's theorem (Godement, "Théorie des faisceaux," II, Th. 5.4.1, p. 212).

Theorem (Grothendieck, "Eléments," III, 1.4.10 and 1.4.11). *Let $f : X \to Y$ be a separated and quasi-compact morphism. Then for every quasi-coherent $\mathcal{O}_X$-Module $\mathscr{F}$ the $R^q f_*(\mathscr{F})$ are quasi-coherent $\mathcal{O}_Y$-Modules* (p. 270). *Furthermore, for every affine open set V in Y we have a canonical isomorphism $\Gamma(V, R^q f_*(\mathscr{F})) \simeq H^q(f^{-1}(V), \mathscr{F})$ for $q \geqslant 0$,* i.e., *$V \rightsquigarrow H^q(f^{-1}(V), \mathscr{F})$ is a sheaf and not only a presheaf.*

Projective Schemata

We propose to define "projective spaces" or "projective schemata". Recall that in Grothendieck's theory $\mathrm{Spec}(k[X_1, \ldots, X_n])$ is considered as affine n-space (p. 272) and some of its closed points correspond to the points of affine n-space in the usual sense.

In classical geometry projective space $\mathbf{P}_n(k)$ is formed by all lines through the origin of k^{n+1} and projective subvarieties of $\mathbf{P}_n(k)$ are simply algebraic cones in k^{n+1}, i.e., varieties defined by systems of algebraic equations $P_\alpha(T_0, T_1, \ldots, T_n) = 0$, where the P_α are homogeneous polynomials. The P_α generate a graded ideal (Bourbaki, "Algèbre," Chap. II (3rd ed.), § 11) $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k \oplus \cdots$ in the graded algebra $S = k[T_0, \ldots, T_n] = S_0 \oplus S_1 \oplus \cdots \oplus S_k \oplus \cdots$ of polynomials in $n + 1$ indeterminates, where $\mathfrak{a}_k$ is a submodule of S_k.

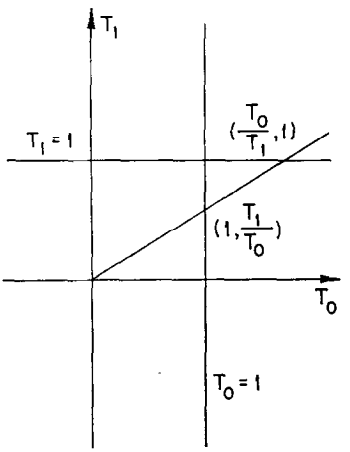
Thus we are led to the following definition. Let

$$S = S_0 \oplus S_1 \oplus \cdots \oplus S_k \oplus \cdots$$

be a graded ring and denote by $\mathrm{Proj}(S)$ the set of all prime graded ideals of S (Grothendieck, "Eléments," II, 2.3). Let us recall that an ideal $\mathfrak{a} \subset S$ is graded if $\mathfrak{a}$ is the direct sum of the modules $\mathfrak{a} \cap S_k$, or, in other words, if $f \in \mathfrak{a}$ and f_k is the component of f in S_k, then $f_k \in \mathfrak{a}$. Clearly $\mathrm{Proj}(S) \subset \mathrm{Spec}(S)$ and on $\mathrm{Proj}(S)$ we put the topology induced

by Spec(S). More precisely, let E be a subset of S and denote by $V_+(E)$ the set of prime graded ideals of S which contain E, i.e., $V_+(E) = V(E) \cap \mathrm{Proj}(S)$. Then the sets $V_+(E)$ are the closed sets of Proj(S). For $f \in S$ we write $D_+(f) = D(f) \cap \mathrm{Proj}(S)$, i.e., $D_+(f)$ is the set of all prime graded ideals of S which do not contain f. The sets $D_+(f)$, where f runs through the *homogeneous* elements of S, form a basis of open sets for the topology of Proj(S). Indeed, if $\sum f_k$ is the decomposition of f into homogeneous elements and $\mathfrak{p}$ is a graded prime ideal in S, then $f \notin \mathfrak{p}$ if and only if $f_k \notin \mathfrak{p}$ for some k, i.e., $D_+(f) = \bigcup_k D_+(f_k)$.

Next we want to define a structure sheaf on Proj(S). To do this, we define a structure of affine schema on each set $D_+(f)$ and obtain a sheaf on Proj(S) by gluing together the sheaves on the sets $D_+(f)$. The idea how to define the affine schema $D_+(f)$ is suggested by the classical case.



Indeed, the n-dimensional projective space $\mathbf{P}_n(k)$ is obtained by associating each open subset $T_i \neq 0$ $(0 \leqslant i \leqslant n)$ of $\mathbf{P}_n(k)$ with an n-dimensional euclidean space and gluing these spaces together. Thus the point $(T_0, T_1, \ldots, T_n)$ of $\mathbf{P}_n(k)$ with $T_i \neq 0$ and $T_j \neq 0$ corresponds to the point

$$\left(\frac{T_0}{T_i}, \ldots, \frac{T_{i-1}}{T_i}, \frac{T_{i+1}}{T_i}, \ldots, \frac{T_n}{T_i}\right)$$

of one such space and also to the point

$$\left(\frac{T_0}{T_j}, \ldots, \frac{T_{j-1}}{T_j}, \frac{T_{j+1}}{T_j}, \ldots, \frac{T_n}{T_j}\right)$$

of another such space. We can consider the set $T_i \neq 0$ in $\mathbf{P}_n(k)$ as the set $D_+(T_i)$ and also as the n-dimensional vector space

$$\mathrm{Spec}(k[T_0, \ldots, \hat{T}_i, \ldots, T_n]).$$

To any polynomial $Q(T_0, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n)$ there corresponds the rational fraction of degree zero

$$Q\left(\frac{T_0}{T_i}, \ldots, \frac{T_{i-1}}{T_i}, \frac{T_{i+1}}{T_i}, \ldots, \frac{T_n}{T_i}\right),$$

which is of the form

$$\frac{P(T_0, \ldots, T_n)}{T_i^k},$$

where k is the degree of the polynomial $P(T_0, \ldots, T_n)$. Conversely, if in a rational fraction of degree zero of the form

$$\frac{P(T_0, \ldots, T_n)}{T_i^k}$$

we set $T_i = 1$, then we obtain a polynomial $Q(T_0, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n)$. Now the ring of fractions $k[T_0, \ldots, T_n]_{T_i}$ consists of all rational fractions of the form

$$\frac{P(T_0, \ldots, T_n)}{T_i^k}$$

and it is graded by the total degree. If we denote its component of degree zero by $k[T_0, \ldots, T_n]_{(T_i)}$ then we have proved that $k[T_0, \ldots, \hat{T}_i, \ldots, T_n]$ and $k[T_0, \ldots, T_n]_{(T_i)}$ are isomorphic, i.e., that $D_+(T_i)$ is also

$$\mathrm{Spec}(k[T_0, \ldots, T_n]_{(T_i)}).$$

Going back to the general case, let $S = S_0 \oplus S_1 \oplus \cdots \oplus S_k \oplus \cdots$ and consider $X = \mathrm{Proj}(S)$. Let $f \in S_d$ be a homogeneous element of degree $d > 0$, then the ring of fractions $S_f = \{x/f^k \mid x \in S, k \geqslant 0\}$ is a graded ring if we take for $(S_f)_n$ the set of those elements x/f^k for which $x \in S_{n+kd}$, where $n \in \mathbf{Z}$. We have now

$$S_f = \cdots \oplus (S_f)_{-2} \oplus (S_f)_{-1} \oplus (S_f)_0 \oplus (S_f)_1 \oplus (S_f)_2 \oplus \cdots.$$

We write $S_{(f)}$ for the component of degree zero $(S_f)_0$ of S_f. We shall show that $D_+(f)$ is homeomorphic to the underlying space of $\mathrm{Spec}(S_{(f)})$ and thus we can transport the structure of affine schema of $\mathrm{Spec}(S_{(f)})$ onto $D_+(f)$. In the first place let $g \in S_e$, then $D_+(fg) \subset D_+(f)$ and $x/f^k \rightsquigarrow xg^k/(fg)^k$ defines a map $S_f \to S_{fg}$ which preserves the degree and in particular maps $S_{(f)}$ into $S_{(fg)}$. We obtain therefore a morphism $\mathrm{Spec}(S_{(fg)}) \to \mathrm{Spec}(S_{(f)})$.

Next we define a map $\psi_f : D_+(f) \to \mathrm{Spec}(S_{(f)})$ as follows. Let $\mathfrak{p} \in D_+(f)$ be a graded prime ideal of S which does not contain f. Then

$$\mathfrak{p}_f = \{x/f^k \mid x \in \mathfrak{p},\ k \geqslant 0\}$$

is a graded prime ideal of S_f and $\mathfrak{p}_f \cap S_{(f)}$ is a prime ideal of $S_{(f)}$. We set $\psi_f(\mathfrak{p}) = \mathfrak{p}_f \cap S_{(f)}$. It is easy to check that the diagram

$$\begin{array}{ccc} D_+(fg) & \xrightarrow{\psi_{fg}} & \mathrm{Spec}(S_{(fg)}) \\ \downarrow & & \downarrow \\ D_+(f) & \xrightarrow{\psi_f} & \mathrm{Spec}(S_{(f)}) \end{array}$$

is commutative. It is also easy to see that ψ_f is continuous and injective and that the inverse map $\psi_f(D_+(f)) \to D_+(f)$ is continuous (Grothendieck, "Eléments," II, 2.3.6). Finally ψ_f is surjective, since if $\mathfrak{q}_0$ is a prime ideal in $S_{(f)}$, let $\mathfrak{p}_n$ be the set of all elements $x \in S_n$ such that $x^d/f_n \in \mathfrak{q}_0$ and set $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k \oplus \cdots$. Then $\mathfrak{p}$ is a graded prime ideal in S and $\psi_f(\mathfrak{p}) = \mathfrak{q}_0$ (Grothendieck, "Eléments," II, 2.1.9; Bourbaki, "Algèbre commutative," Chap. III, § 1, No. 4, Prop. 5).

If we now transport the structure of affine schema of $\mathrm{Spec}(S_{(f)})$ onto $D_+(f)$, then by the above commutative diagram the affine schema $D_+(fg)$ coincides with the affine schema induced by $D_+(f)$ on the open set $D_+(fg)$ and thus we obtain on $\mathrm{Proj}(S)$ a structure of preschema. The preschema $\mathrm{Proj}(S)$ is actually a schema (Grothendieck, "Eléments," II, 2.4.2). This is almost equivalent to the relation $D_+(f) \cap D_+(g) = D_+(fg)$ (Grothendieck, "Eléments," I, 5.5.6).

A nonzero homogeneous element f of S can give an empty set $D_+(f)$. More precisely $D_+(f) = \varnothing$ if and only if f is nilpotent (Grothendieck, "Eléments," II, 2.3.7).

We denote by S_+ the ideal $S_1 \oplus S_2 \oplus \cdots \oplus S_k \oplus \cdots$ of S. Let (f_α) be a family of homogeneous elements of S_+. Then $\mathrm{Proj}(S) = \bigcup_\alpha D_+(f_\alpha)$ if and only if each element of S_+ has a power in the ideal generated by the f_α (Grothendieck, "Eléments," II, 2.3.14).

If A is a ring, then $\mathrm{Proj}(A[T]) = \mathrm{Spec}(A)$. In particular a projective space of dimension 0 over a field k consists of a single point. We have

$$\mathrm{Proj}(A[T]) = D_+(T)$$

and

$$S_T = A[T, T^{-1}] = \cdots \oplus A\frac{1}{T^k} \oplus \cdots \oplus A \oplus AT \oplus \cdots .$$

Thus $S_{(T)} = (S_T)_0 = A$ and $\mathrm{Spec}(S_{(T)}) = \mathrm{Spec}(A)$.

A graded S-module M is a direct sum of subgroups

$$\cdots \oplus M_{-n} \oplus \cdots \oplus M_0 \oplus M_1 \oplus \cdots \oplus M_n \oplus \cdots$$

such that for $a \in S_m$, $x \in M_p$ we have $ax \in M_{p+m}$. We want to associate with M a quasi-coherent $\mathcal{O}_X$-Module $\tilde{M}$ on $\mathrm{Proj}(S)$. We obtain $\tilde{M}$ by gluing together from pieces. Let $f \in S_d$, $d > 0$, and consider $D_+(f) = \mathrm{Spec}(S_{(f)})$. The module of fractions $M_f = \{x/f^k \mid x \in M\}$ is a graded S_f-module

$$M_f = \cdots \oplus (M_f)_{-n} \oplus \cdots \oplus (M_f)_0 \oplus (M_f)_1 \oplus \cdots,$$

where $(M_f)_n$ consists of those elements x/f^k for which $x \in M_{n+kd}$. We denote by $M_{(f)}$ the component of degree zero $(M_f)_0$ of M_f. Then $M_{(f)}$ is an $S_{(f)}$-module since if $a/f^k \in S_{(f)}$ and $x/f^l \in M_{(f)}$, i.e., $a \in S_{kd}$, $x \in M_{ld}$, then $ax \in M_{(k+l)d}$ and $ax/f^{k+l} \in M_{(f)}$. To $M_{(f)}$ there corresponds a quasi-coherent sheaf $(M_{(f)})^\sim$ on $\mathrm{Spec}(S_{(f)}) = D_+(f)$. It is easily seen (Grothendieck, "Eléments," II, 2.5.2) that there exists a unique quasi-coherent $\mathcal{O}_X$-Module $\tilde{M}$ on $X = \mathrm{Proj}(S)$ such that for every homogeneous f in S_+ we have $\tilde{M} \mid D_+(f) = (M_{(f)})^\sim$. In particular we have $\tilde{S} = \mathcal{O}_X$.

In the affine case the correspondence $A \rightsquigarrow \mathrm{Spec}(A)$ between rings and affine schemata and the correspondence $M \rightsquigarrow \tilde{M}$ between A-modules and quasi-coherent $\mathcal{O}_X$-Modules is one-to-one. In the projective case the correspondences $S \rightsquigarrow \mathrm{Proj}(S)$, $M \rightsquigarrow \tilde{M}$ are not so nice. Thus two different graded rings can have the same homogeneous prime spectrum. For instance if we change the component S_0 of S, the schema $\mathrm{Proj}(S)$ will remain unchanged, since $S_{(f)} = \{x/f^k \mid \deg x = k \deg f\}$ only involves elements x of degree > 0. Also if we replace

$$S = S_0 \oplus S_1 \oplus \cdots \oplus S_k \oplus \cdots$$

by $S_0 \oplus S_m \oplus S_{2m} \oplus \cdots \oplus S_{km} \oplus \cdots$, the schema $\mathrm{Proj}(S)$ will remain unchanged. If in a graded S-module M we drop everything below a certain degree, the associated sheaf $\tilde{M}$ will still be the same since in the expressions x/f^k which form the $M_{(f)}$, the elements x can be taken as far to the right as we wish.

Let $S = S_0 \oplus S_1 \oplus \cdots \oplus S_k \oplus \cdots$ be a graded ring and $n \in \mathbf{Z}$. Setting $(S(n))_k = S_{n+k}$ we obtain a graded S-module

$$S(n) = \cdots \oplus (S(n))_{-n} \oplus (S(n))_{-n+1} \oplus \cdots \oplus (S(n))_0 \oplus \cdots.$$

The associated sheaf $(S(n))^\sim$ will be denoted by $\mathcal{O}_X(n)$. The sheaf $\mathcal{O}_X(1)$ is called the fundamental sheaf of $X = \mathrm{Proj}(S)$. If we set $M = S(n)$, then $M_{(f)}$ is the set of elements z/f^k, where $z \in (S(n))_{kd} = S_{n+kd}$, $d = \deg f$. Thus we see that $M_{(f)} = (S_f)_n$. In the classical example of

$S = k[T_0, \ldots, T_r]$ and $f = T_i$ the module $M_{(f)}$ is the set of all rational fractions

$$\frac{P(T_0, \ldots, T_r)}{T_i^k},$$

where P is of degree $n + k$.

Similarly, if $M = \oplus_n M_n$ is a graded S-module, we define the graded S-module $M(n)$ by $(M(n))_k = M_{n+k}$.

If $d > 0$, $f \in S_d$, then $\mathcal{O}_X(nd) \mid D_+(f)$ is canonically isomorphic to $\mathcal{O}_X \mid D_+(f)$. Indeed, the element $f = f/1$ is invertible in S_f and therefore $x \rightsquigarrow f^n x$ is a bijection from $S_{(f)} = (S_f)_0$ onto $S_{(f)nd} = (S(nd))_{(f)}$.

We shall now make the assumption that the graded ring S is generated by S_1. Then $X = \bigcup_{f \in S_1} D_+(f)$ (p. 297) and $\mathcal{O}_X(n)$ is an invertible $\mathcal{O}_X$-Module for each $n \in \mathbf{Z}$ (p. 299). We know that to each invertible $\mathcal{O}_X$-Module there corresponds a well-determined divisor class on X (p. 15). Which is the divisor class associated with the sheaf $\mathcal{O}_X(n)$? In the classical case $X = \mathrm{Proj}(k[X_0, \ldots, X_n]) = \mathbf{P}_n(k)$ we have $X = \bigcup_{0 \leqslant i \leqslant n} D_+(X_i)$ and S_1 indeed generates $S = k[X_0, \ldots, X_n]$. Then $\mathcal{O}_X(n) \mid D_+(X_i)$ is the $(\mathcal{O}_X \mid D_+(X_i))$-Module generated by $(X_i/1)^n \simeq X_i^n$ and in $D_+(X_i) \cap D_+(X_j)$ the element $(X_j/X_i)^n$ is invertible. Thus in the case $n > 0$ the divisor class associated to $\mathcal{O}_X(n)$ contains the divisor which is n times the divisor of all hyperplanes. In the case $n < 0$ we obtained the so-called virtual divisors of the Italian school, which have useful and important properties.

If $M = \oplus_{i=-\infty}^{\infty} M_i$ is a graded S-module and $\tilde{M}$ the associated $\mathcal{O}_X$-Module, then $\tilde{M} = 0$ if and only if for every $z \in M$ and homogeneous $f \in S$ there exists an integer $k > 0$ such that $f^k z = 0$ (Grothendieck, "Eléments," II, 2.5.6). The functor $M \rightsquigarrow \tilde{M}$ is exact and commutes with $\oplus$ and $\varinjlim$ (Grothendieck, "Eléments," II, 2.5.4). Exactness follows from the fact that $M \rightsquigarrow \tilde{M}$ is composed of the three exact functors $M \rightsquigarrow M_f \rightsquigarrow M_{(f)} \rightsquigarrow (M_{(f)})^{\sim}$.

Let M and N be two graded S-modules. We want to define $M \otimes_S N$ as a graded S-module (Grothendieck, "Eléments," II, 2.1.2). Consider first the $\mathbf{Z}$-module $M \otimes_{\mathbf{Z}} N$ and graduate it by

$$(M \otimes_{\mathbf{Z}} N)_n = \oplus_{i+j=n} M_i \otimes_{\mathbf{Z}} N_j .$$

Then define $M \otimes_S N = (M \otimes_{\mathbf{Z}} N)/P$, where P is the sub-$\mathbf{Z}$-module of $M \otimes_{\mathbf{Z}} N$ generated by the elements of the form

$$xa \otimes y - x \otimes ay, \text{ where } x \in M, y \in N, a \in S.$$

There exists a natural homomorphism $\tilde{M} \otimes_{\mathscr{O}_X} \tilde{N} \to (M \otimes_S N)^{\sim}$ (Grothendieck, "Eléments," II, 2.5.11.2) which by virtue of our assumption that S_1 generates S is an isomorphism (Grothendieck, "Eléments," II, 2.5.13).

We say that a homomorphism $u : M \to N$ is a graded homomorphism of degree k if $u(M_n) \subset N_{n+k}$ for every $n \in \mathbf{Z}$. We denote by $\mathrm{Hom}_S(M, N)$ the S-module of all *graded* homomorphisms $u : M \to N$ and grade it by taking for $(\mathrm{Hom}_S(M, N)_k$ the set of all homomorphisms of degree k. Then we have $\mathrm{Hom}_S(M, N) = \oplus_k (\mathrm{Hom}_S(M, N))_k$. In general $\mathrm{Hom}_S(M, N)$ is distinct from the S-module of all homomorphisms of M into N but coincides with it if M is finitely generated (Grothendieck, "Eléments," II, 2.1.2). There exists a natural homomorphism $(\mathrm{Hom}_S(M,N))^{\sim} \to \mathscr{H}om_{\mathscr{O}_X}(\tilde{M}, \tilde{N})$ (Grothendieck, "Eléments," II, 2.5.12.2) which is an isomorphism if S is generated by S_1 and M is of finite presentation (Grothendieck, "Eléments," II, 2.5.13). Let us recall what this last notion means in the case of graded modules. If M is a graded S-module and $a \in M_k$, then $s \rightsquigarrow s.a$ is a homomorphism of degree k from the S-module S into M and a homomorphism of degree 0 (i.e., a morphism in the category of graded S-modules) from $S(-k)$ into M, since $(S(-k))_{n+k} = S_n$ and $s \in S_n$ implies $sa \in M_{n+k}$. According to general definitions a graded S-module M is of finite type if there exists an exact sequence $\oplus_{1 \leqslant j \leqslant n} S(k_j) \to M \to 0$ of graded homomorphisms of degree 0. Now we say that it is of finite presentation if there exists an exact sequence $\oplus_{1 \leqslant i \leqslant m} S(h_i) \to \oplus_{1 \leqslant j \leqslant n} S(k_j) \to M \to 0$ of graded homomorphisms of degree 0.

Under our blanket assumption that S is generated by S_1 we have $\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) = \mathscr{O}_X(m + n)$ and $\mathscr{O}_X(n) = (\mathscr{O}_X(1))^{\otimes n}$ for all n, $m \in \mathbf{Z}$ (Grothendieck, "Eléments," II, 2.5.14). If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-Module, then we define the "twisted Module" $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$. We have then $\tilde{M}(n) = (M(n))^{\sim}$ for every $n \in \mathbf{Z}$ (Grothendieck, "Eléments," II, 2.5.15).

Let M be a graded S-module. We want to define a homomorphism of graded abelian groups $\alpha : M \to \oplus_{n \in \mathbf{Z}} \Gamma(X, \tilde{M}(n))$, first introduced by Serre. Take $f \in S_1$ and define a homomorphism $M_0 \to M_{(f)}$ of abelian groups by $x \rightsquigarrow x/1$. Now $M_{(f)} = \Gamma(D_+(f), \tilde{M})$ and since for $f \in S_1$, $g \in S_1$ the restrictions of $x/1$ to $D_+(f) \cap D_+(g) = D_+(fg)$ coincide, we have defined a homomorphism $\alpha_0 : M_0 \to \Gamma(X, \tilde{M})$. Since $M_n = (M(n))_0$ we have also homomorphisms $\alpha_n : M_n \to \Gamma(X, \tilde{M}(n))$ and thus a homomorphism of degree zero of graded abelian groups $\alpha : M \to \oplus_{n \in \mathbf{Z}} \Gamma(X, \tilde{M}(n))$ (Grothendieck, "Eléments," II, 2.6.2).

In particular if we take S for M we have a homomorphism $\alpha : S \to \oplus_{n\in\mathbf{Z}} \Gamma(X, \mathcal{O}_X(n))$.

Lemma (Grothendieck, "Eléments," II, 2.6.3). *For $f \in S_1$ the set $D_+(f)$ is the set of all $x \in X = \mathrm{Proj}(S)$ such that $\alpha_1(f)(x) \neq 0$.*

Intuitively, since $\alpha_1(f) \in \Gamma(X, \mathcal{O}_X(1))$, the set $D_+(f)$ can be thought of as the complement of a hypersurface of degree 1 (i.e. a projective hyperplane).

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-Module on $X = \mathrm{Proj}(S)$ and set $\Gamma_*(\mathcal{F}) = \oplus_{n\in\mathbf{Z}} \Gamma(X, \mathcal{F}(n))$. We want to define a natural mapping $\beta : (\Gamma_*(\mathcal{F}))^\sim \to \mathcal{F}$, also introduced by Serre. Let $f \in S_1$. By the lemma $\alpha_1(f) \mid D_+(f)$ is never zero. Write $M = \Gamma_*(\mathcal{F})$; then if $z \in M_k = \Gamma(X, \mathcal{F}(k))$, the map $z/f^k \rightsquigarrow (z \mid D_+(f))(\alpha_1(f) \mid D_+(f))^{-k}$ is a homomorphism from $M_{(f)}$ into $\Gamma(D_+(f), \mathcal{F})$. This map is compatible with the restriction $D_+(f) \to D_+(fg)$. Since $M_{(f)} \simeq \Gamma(D_+(f), \tilde{M})$ and the $D_+(f)$ form a basis for the topology of X, we obtain the required homomorphism of $\mathcal{O}_X$-Modules $\beta : (\Gamma_*(\mathcal{F}))^\sim \to \mathcal{F}$.

The composed homomorphisms

$$\tilde{M} \xrightarrow{\tilde{\alpha}} (\Gamma_*(\tilde{M}))^\sim \xrightarrow{\beta} \tilde{M}$$

and

$$\Gamma_*(\mathcal{F}) \xrightarrow{\alpha} \Gamma_*((\Gamma_*(\mathcal{F}))^\sim) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathcal{F})$$

are identical isomorphisms (Grothendieck, "Eléments," II, 2.6.5).

If S is a noetherian graded ring, then the schema $X = \mathrm{Proj}(S)$ is noetherian (Grothendieck, "Eléments," II, 2.7.1).

The following two finiteness conditions were introduced by Serre. Let S be a graded ring and $M = \oplus_{n\in\mathbf{Z}} M_n$ a graded S-module.

(TF) There exists an integer k such that $\oplus_{n\geqslant k} M_n$ is a finitely generated S-module.

(TN) There exists an integer k such that $M_n = 0$ for $n \geqslant k$.

If M satisfies condition (TN) then $\tilde{M} = 0$. If M satisfies condition (TF) then $\tilde{M}$ is a finitely generated $\mathcal{O}_X$-Module. If M satisfies condition (TF) and $\tilde{M} = 0$, then M satisfies condition (TN) (Grothendieck, "Eléments," II, 2.7.3).

Theorem (Grothendieck, "Eléments," II, 2.7.5). *If S_1 is a finitely generated S_0-module, then for every quasi-coherent $\mathcal{O}_X$-Module $\mathcal{F}$ the homomorphism $\beta : (\Gamma_*(\mathcal{F}))^\sim \to \mathcal{F}$ is an isomorphism.*

We want to define an inverse isomorphism $\mathscr{F} \to (\Gamma_*(\mathscr{F}))^\sim$, i.e., for every $f \in S_1$ a map $\Gamma(D_+(f), \mathscr{F}) \to M_{(f)}$. We have the following more general result (Grothendieck, "Eléments," I, 9.3.2):

Let X be a schema whose underlying space is quasi-compact. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-Module (p. 246) and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-Module. Let us write

$$S = \Gamma_*(\mathscr{L}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathscr{L}^{\otimes n})$$

and

$$M = \Gamma_*(\mathscr{L}, \mathscr{F}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathscr{F} \underset{\mathscr{O}_X}{\otimes} \mathscr{L}^{\otimes n})$$

(Grothendieck, "Eléments," 0_I, 5.4.6). S is a graded ring since $\mathscr{L}^{\otimes(m+n)} = \mathscr{L}^{\otimes m} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$ (p. 246). Let $f \in \Gamma_1(\mathscr{L}) = \Gamma(X, \mathscr{L})$ and consider the open set $X_f = \{x \mid f(x) \neq 0\}$ in X. Then there exists a canonical isomorphism

$$\Gamma(X_f, \mathscr{F}) \xrightarrow{\sim} M_{(f)} = \left\{ \frac{z}{f^k} \mid z \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes k}), k > 0 \right\}.$$

We have a homomorphism $M_{(f)} \to \Gamma(X_f, \mathscr{F})$ which in the special case of $\mathscr{L} = \mathscr{O}_X(1)$ reduces to Serre's homomorphism β and which with $z/f^k \in M_{(f)}$ associates $(z \mid X_f) \otimes (f \mid X_f)^{\otimes(-k)} \in \Gamma(X_f, \mathscr{F})$. To prove that this homomorphism is an isomorphism we have to establish the following two propositions (Grothendieck, "Eléments," I, 9.3.1).

Propositions. (i) *Given a section $s \in \Gamma(X, \mathscr{F})$ such that $s \mid X_f = 0$, there exists an integer $n > 0$ such that $s \otimes f^{\otimes n} = 0$.*

(ii) *Given $t \in \Gamma(X_f, \mathscr{F})$, there exists an $n > 0$ and a section $s \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$ such that $t \otimes f^{\otimes n} = s \mid X_f$.*

Proof of (i). Since X is quasi-compact, there exists a finite covering $(U_i)_{1 \leqslant i \leqslant r}$ by affine open sets such that $\mathscr{L} \mid U_i \simeq \mathscr{O}_X \mid U_i$ for $1 \leqslant i \leqslant r$. Thus it is enough to prove (i) for affine X and $\mathscr{L} = \mathscr{O}_X$. Indeed, if we find for each index i an exponent n_i such that $(s \otimes f^{\otimes n_i}) \mid U_i = 0$, then setting $n = \max_{1 \leqslant i \leqslant r} n_i$ we have $s \otimes f^{\otimes n} = 0$. For an affine X the section f can be identified with an element of the ring of X and $X_f = D(f)$. Writing $N = \Gamma(X, \mathscr{F})$ we have $\Gamma(X_f, \mathscr{F}) = N_f$, thus $s \mid X_f = 0$ means by the definition of a ring of fractions that $f^n s = 0$ for some $n > 0$.

Proof of (ii). Again let $x = \bigcup_{1 \leqslant i \leqslant r} U_i$, where the U_i are affine open sets and $\mathscr{L} \mid U_i \simeq \mathscr{O}_X \mid U_i$. Set $t_i = t \mid (X_f \cap U_i)$, then by affine theory

(Grothendieck, "Eléments," I, 1.4.1) there exists an integer n_i and $s_i \in \Gamma(U_i, \mathscr{F} \otimes \mathscr{L}^{\otimes n_i})$ such that $t_i \otimes f^{\otimes n_i} = s_i \mid (X_f \cap U_i)$. We can assume that all indices n_i $(1 \leqslant i \leqslant r)$ are the same. Let $s_{i|j} = s_i \mid (U_i \cap U_j)$. The restrictions of $s_{i|j}$ and $s_{j|i}$ to $X_f \cap U_i \cap U_j$ coincide since $t_i \otimes f^{\otimes n} = t_j \otimes f^{\otimes n} = t \otimes f^{\otimes n}$ over $X_f \cap U_i \cap U_j$. By assumption X is a schema and thus $U_i \cap U_j$ is an affine open set and in particular quasi-compact. Hence by (i) there exists an integer m independent of i and j such that $(s_{i|j} - s_{j|i}) \otimes f^{\otimes m} = 0$. Then $s_{i|j} \otimes f^{\otimes m} = s_{j|i} \otimes f^{\otimes m}$ in $U_i \cap U_j$, hence there exists a section $s \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes(m+n)})$ such that $s \mid U_i = s_i \otimes f^{\otimes m}$. Furthermore, $s \mid X_f \cap U_i = s_i \otimes f^{\otimes m} \mid X_f \cap U_i = t_i \otimes f^{\otimes(m+n)}$, hence $s \mid X_f = t \otimes f^{\otimes(m+n)}$.

Under the hypotheses of the previous theorem every quasi-coherent $\mathcal{O}_X$-Module $\mathscr{F}$ has the form $\tilde{M}$, where M is a graded S-module (Grothendieck, "Eléments," II, 2.7.7). Indeed, we can take $M = \Gamma_*(\mathscr{F})$. The module M is not unique, as observed on p. 298.

Still under the same hypotheses if $\mathscr{F}$ is a quasi-coherent $\mathcal{O}_X$-Module of finite type, then $\mathscr{F} = \tilde{M}$, where M is a graded S-module of finite type (Grothendieck, "Eléments," II, 2.7.8). Indeed, we have $\mathscr{F} = \tilde{M}$, where $M = \varinjlim M_\lambda$ and the M_λ are of finite type. Now it follows from a general sheaf theoretic fact (Grothendieck, "Eléments," 0_I, 5.2.3) that if $\mathscr{F} = \varinjlim \tilde{M}_\lambda$, X is a quasi-compact ringed space and $\mathscr{F}$ of finite type, then there exists an index μ such that $\mathscr{F} \simeq \tilde{M}_\mu$.

If X is an affine schema and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-Module of finite type, then there exists an integer $n > 0$ such that the sequence $\mathcal{O}_X^n \to \mathscr{F} \to 0$ is exact. This follows immediately from the corresponding exact sequence of modules $A^n \to M \to 0$. The analogous statement for projective schemata is not true as shown by the example $\mathscr{F} = \mathcal{O}_X(1)$, whose only global sections are in general constants. However, if we twist enough, things become nice, and we have the following

Theorem (Grothendieck, "Eléments," II, 2.7.9). *Suppose that S_1 is a finitely generated S_0-module and let $X = \operatorname{Proj}(S)$. Let $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-Module of finite type. There exists an integer n_0 such that for every integer $n \geqslant n_0$ we have an exact sequence $\mathcal{O}_X^{k_n} \to \mathscr{F}(n) \to 0$. Thus $\mathscr{F}(n)$ is generated by a finite number of its sections over X.*

Proof. We are reduced to the case $\mathscr{F} = \mathcal{O}_X(n)$. Indeed, by the last result we have $\mathscr{F} = \tilde{M}$, where M is of finite type, i.e. there exists an exact sequence $\bigoplus_{1 \leqslant i \leqslant m} S(q_i) \to M \to 0$. By the exactness of the functor

$M \rightsquigarrow \tilde{M}$ we have the exact sequences $\oplus_{1\leqslant i\leqslant m} \mathscr{O}_X(q_i) \to \mathscr{F} \to 0$ and $\oplus_{1\leqslant i\leqslant m} \mathscr{O}_X(q_i + n) \to \mathscr{F}(n) \to 0$. Thus we have to show that for $n \geqslant 0$ there exists an exact sequence $\mathscr{O}_X^{k_n} \to \mathscr{O}_X(n) \to 0$.

To do this we have to show that there exists a graded homomorphism *of degree* 0 from S^k into $S(n)$ which is (TN)-surjective, i.e., there exists an m_0 such that $(S^k)_m \to (S(n))_m$ is surjective for $m \geqslant m_0$. Since S_1 generates S, we have $SS_n = S_n \oplus S_{n+1} \oplus \cdots$. On the other hand $S(n) = (S(n))_{-n} \oplus (S(n))_{-n+1} \oplus \cdots \oplus (S(n))_0 \oplus (S(n))_1 \oplus \cdots$, where now $SS_n = (S(n))_0 \oplus (S(n))_1 \oplus \cdots$. Let $a_1, \ldots, a_k$ be a system of generators of the S_0-module S_n. Then $(x_1, \ldots, x_k) \rightsquigarrow x_1a_1 + \cdots + x_ka_k$ defines an epimorphism from S^k onto SS_n.

The last result can also be expressed saying that there exists an exact sequence $(\mathscr{O}_X(-n))^{k_n} \to \mathscr{F} \to 0$ (Grothendieck, "Eléments," II, 2.7.10).

Cohomology of Projective Schemata

(See Grothendieck, "Eléments," III, 2). Let A be an arbitrary ring, $S = A[T_0, \ldots, T_r]$, $X = \mathrm{Proj}(S) = \mathbf{P}_A^r$. We set ourselves the task of computing the homology groups $H^q(X, \mathscr{F})$, where $\mathscr{F}$ is a coherent $\mathscr{O}_X$-Module. It turns out that it is more convenient to compute the groups $H^q(X, \mathscr{F}(n))$ for all $n \in \mathbf{Z}$ at once.

We return to our earlier notations (p. 283) with slight modifications. Let S be a graded ring, $X = \mathrm{Proj}(S)$. Assume, as always, that S is generated by S_1. We consider a finite family $\mathfrak{U} = (U_i)_{1\leqslant i\leqslant r}$ of open subsets of X of the form $X_i = X_{f_i}$, where $f_i \in S_{d_i}$, $d_i > 0$ $(1 \leqslant i \leqslant r)$. We write

$$U_{i_0 i_1 \cdots i_p} = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p} = X_{f_{i_0} f_{i_1} \cdots f_{i_p}}$$

and $U = \bigcup U_i$. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-Module. We set $M = \Gamma_*(X, \mathscr{F}) = \oplus_{n\in\mathbf{Z}} \Gamma(X, \mathscr{F}(n))$. If S_1 is a finitely generated S_0-module, then the sets $U_{i_0 i_1 \cdots i_p}$ are quasi-compact and separated and thus

$$\Gamma_*(U_{i_0 \cdots i_p}, \mathscr{F}) \simeq M_{f_{i_0} \cdots f_{i_p}}$$

(Grothendieck, "Eléments," I, 9.3.3). In general for a homogeneous element f in S we have $\Gamma(X_f, \mathscr{F}) \simeq M_{(f)} = (M_f)_0$, $\Gamma(X_f, \mathscr{F}(n)) = (M(n))_{(f)} = ((M(n))_f)_0 = (M_f)_n$ and thus $\oplus_{n\in\mathbf{Z}} \Gamma(X_f, \mathscr{F}(n)) \simeq M_f$.

We consider the graded S-module $C^p(\mathfrak{U}, \mathscr{F}(*)) = \oplus_{n\in\mathbf{Z}} C^p(\mathfrak{U}, \mathscr{F}(n))$. Similarly as before (p. 285) we have

$$C^p(\mathfrak{U}, \mathscr{F}(*)) = \varinjlim_n C_n^p(M),$$

where $C_n^p(M)$ is the set of alternating maps from $[1, r]^{p+1}$ into the graded S-module M. The map $\psi_{mn} : C_n^p(M) \to C_m^p(M)$ $(m \geqslant n)$ defined by $(\psi_{mn}g)(i_0, \ldots, i_p) = (f_{i_0} \cdots f_{i_p})^{m-n} g(i_0, \ldots, i_p)$ does not disturb homogeneity but increases the degree by $(m - n)(d_{i_0} + \cdots + d_{i_p})$. We also have the relation

$$C^p(\mathfrak{U}, \mathscr{F}(*)) \simeq C^{p+1}((f), M) = \varinjlim_n K^{p+1}(f^n, M)$$

(p. 287), where the isomorphism preserves the degree.

We have, just as in the previous case,

$$M_{f_{i_0} f_{i_1} \cdots f_{i_p}} \simeq \varinjlim_n M^{(n)}_{i_0 i_1 \cdots i_p},$$

where $M^{(n)}_{i_0 \cdots i_p} = M$ for all n. This is an isomorphism of graded S-modules, if we define the degree of $z \in \varinjlim M^{(n)}_{i_0 \cdots i_p}$ as follows. Suppose that z is the image of a homogeneous element $x \in M^{(n)}_h$, then the degree of z is $h - n(d_{i_0} + \cdots + d_{i_p})$. If $y = \varphi_{mn}(x) \in M^{(m)}_k$, then $k - h = (m - n)(d_{i_0} + \cdots + d_{i_p})$ since $y = (f_{i_0} \cdots f_{i_p})^{m-n} x$. Thus $k - m(d_{i_0} + \cdots + d_{i_p}) = h - n(d_{i_0} + \cdots + d_{i_p})$ and the degree of z is independent of the choice of x.

Just as on p. 288 we conclude that for $p \geqslant 1$ we have canonical isomorphisms $H^p(\mathfrak{U}, \mathscr{F}(*)) = \oplus_n H^p(\mathfrak{U}, \mathscr{F}(n)) \simeq H^{p+1}((f), M) = H^{p+1}(C^\bullet((f), M))$ and furthermore we have the exact sequence $0 \to H^0((f), M) \to M \to H^0(\mathfrak{U}, \mathscr{F}(*)) \to H^1((f), M) \to 0$. The new features are only that we consider all the values of n at the same time, the elements f_i are homogeneous and the homomorphisms are homomorphisms of degree 0 of graded S-modules (Grothendieck, "Eléments," III, 2.1.3).

Now let A be an arbitrary ring, S the polynomial ring $A[T_0, \ldots, T_r]$ in the $r + 1$ indeterminates $T_i (0 \leqslant i \leqslant r)$ and $X = \mathrm{Proj}(S) = \mathbf{P}^r_A$ the r-dimensional projective space in the language of Grothendieck. We take $\mathscr{O}_X$ for $\mathscr{F}$, i.e., $M = S$ and $f_i = T_i$, i.e., $X_{f_i} = D_+(T_i)$. The intersection of two sets X_{f_i} is affine and thus by the theorem on p. 294 we have $H^q(X, \mathscr{F}(*)) = H^q(\mathfrak{U}, \mathscr{F}(*))$. For $p \geqslant 1$ we have the

isomorphism $H^p(\mathfrak{U}, \mathcal{O}_X(*)) \simeq H^{p+1}((\boldsymbol{T}), S)$, where in general $\boldsymbol{T}^n = (T_0^n, \ldots, T_r^n)$. We also have the exact sequence

$$0 \to H^0((\boldsymbol{T}), S) \to S \to H^0(X, \mathcal{O}_X(*)) \to H^1((\boldsymbol{T}), S) \to 0.$$

Finally $H^p((\boldsymbol{T}), S) = \varinjlim_n H^p(\boldsymbol{T}^n, S)$, so that we are reduced to compute $H^p(\boldsymbol{T}^n, S)$.

Lemma on Koszul Complexes (Grothendieck, "Eléments," III, 1.1.4). *Let A be a ring, M an A-module, $(f_i)_{1 \leqslant i \leqslant m}$ a finite family of elements of A. Suppose that for $1 \leqslant i \leqslant m$ the homothecy $z \mapsto f_i z$ in $M/(f_1 M + \cdots + f_{i-1}M)$ is injective. Then $H^i(\boldsymbol{f}, M) = 0$ for $i = m$.*

If the hypothesis of the lemma is satisfied, we say that the family (f_i) is regular with respect to M. This notion plays an important role in local algebra. Let us give a few indications of the proof. In the first place we have an isomorphism $H^i(\boldsymbol{f}, M) \simeq H_{m-i}(\boldsymbol{f}, M)$ $(0 \leqslant i \leqslant m)$ (cf. p. 286). Indeed, $K_\cdot(\boldsymbol{f}, M) = K_\cdot(\boldsymbol{f}) \otimes_A M$ and $K^\cdot(\boldsymbol{f}, M) = \mathrm{Hom}_A(K_\cdot(\boldsymbol{f}), M)$ (p. 286). With every chain $z = \sum(e_{i_1} \wedge \cdots \wedge e_{i_p}) \otimes z_{i_1 \ldots i_p}$ we associate the cochain g_z such that $g_z(j_1, \ldots, j_{m-p}) = \varepsilon z_{i_1 \ldots i_p}$, where $(j_h)_{1 \leqslant h \leqslant m-p}$ is the strictly increasing sequence complementary to the strictly increasing sequence $(i_h)_{1 \leqslant h \leqslant p}$ in $[1, m]$ and $\varepsilon = (-1)^\nu$, where ν is the number of inversions in the permutation $i_1, \ldots, i_p, j_1, \ldots, j_{m-p}$ of $[1, m]$. Then $g_{dz} = d(g_z)$, from where the isomorphism follows.

Thus it is sufficient to prove that $H_i(\boldsymbol{f}, M) = 0$ for $i > 0$. The proof goes by induction on m. Set $\boldsymbol{f}' = (f_i)_{1 \leqslant i \leqslant m-1}$. By the definition of $K_\cdot(\boldsymbol{f})$ (p. 286) we have $K_\cdot(\boldsymbol{f}, M) = K_\cdot(f_m) \otimes L_\cdot$, where $L_\cdot = K(\boldsymbol{f}', M)$. Since $\boldsymbol{f}'$ satisfies the conditions of the lemma, we have $H_i(\boldsymbol{f}', M) = 0$ for $i > 0$ and $H_0(\boldsymbol{f}', M) = M/(f_1 M + \cdots + f_{m-1}M)$. We now use the following

Auxiliary Lemma (Grothendieck, "Eléments," III, 1.1.4.1). *Let $K_\cdot$ be a chain complex formed by free A-modules, which are zero except in dimensions 0 and 1. Then for every chain complex $L_\cdot$ of A-modules we have the exact sequence*

$$0 \to H_0(K_\cdot \otimes H_p(L_\cdot)) \to H_p(K_\cdot \otimes L_\cdot) \to H_1(K_\cdot \otimes H_{p-1}(L_\cdot)) \to 0$$

for every index p.

Let us apply the auxiliary lemma to $K_\cdot = K_\cdot(f_m)$. We have $H_p(L_\cdot) = 0$ for $p \geqslant 1$, hence it follows from the exact sequence that $H_p(K_\cdot \otimes L_\cdot) =$

$H_p(f, M) = 0$ for $p \geqslant 2$. Let us show that also $H_1(K_\cdot \otimes H_0(L_\cdot)) = 0$, from where $H_1(f, M) = 0$ will follow. We have

$$H_0(L_\cdot) = M/(f_1M + \cdots + f_{m-1}M) = M'$$

and thus we have to compute the H_1 of the complex

$$0 \longleftarrow M' \xleftarrow{d_1} M' \longleftarrow 0 \longleftarrow \cdots,$$

where d_1 is the multiplication by f_m (p. 286). Thus $H_1(K_\cdot \otimes M')$ is the kernel of the map $z \rightsquigarrow f_m z$ of M' into itself which by hypothesis is injective. Hence $H_1(K_\cdot \otimes M') = 0$.

We have used in the previous proof that $H_0(f, M) = H^m(f, M) = M/(f_1M + \cdots + f_mM)$. This is always true and can be seen as follows. A cochain $g \in K^p(f, M)$ can be identified with an alternating mapping from $[1, m]^p$ into M (p. 288; Grothendieck, "Eléments," III, 1.1.2). In particular every m-cochain is a cocycle and the group of cocycles can be identified with M. Furthermore

$$dg(i_1, \ldots, i_m) = \sum_{k=1}^{m} (-1)^{k-1} f_{i_k} g(i_1, \ldots, \hat{i}_k, \ldots, i_m)$$

and the fact that $g(i_1, \ldots, \hat{i}_k, \ldots, i_m)$ can take an arbitrary value in M implies that the group of coboundaries in $K^m(f, M)$ can be identified with $f_1M + \cdots + f_mM$.

By virtue of the previous lemma on Koszul complexes we have $H^i(T^n, S) = 0$ for $i \neq r + 1$. Indeed, the homothecy $z \rightsquigarrow T_k z$ in $S/(T_0S + \cdots + T_{k-1}S)$ is injective, since this quotient ring is canonically isomorphic to the polynomial ring $A[T_k, \ldots, T_r]$. Furthermore $H^{r+1}(T^n, S) = S/(T^n)$, where (T^n) is the ideal in S generated by $T_0^n, \ldots, T_r^n$. The A-module $H^{r+1}(T^n, S)$ has a basis formed by the classes $\mathrm{mod}(T^n)$ of the monomials $T_0^{n-p_0} T_1^{n-p_1} \cdots T_r^{n-p_r}$, where $n \geqslant p_i > 0$ for $0 \leqslant i \leqslant r$. Passing to the direct limit as $n \to \infty$ we obtain $H^i((T), S) = 0$ for $i \neq r + 1$ and $H^{r+1}((T), S)$ is a free A-module whose basis is formed by elements of the form $\xi_{p_0 p_1 \ldots p_r}$, where $p_i > 0$ for $0 \leqslant i \leqslant r$. The element $\xi_{p_0 p_1 \ldots p_r}$ is defined as the image of the element represented by $T_0^{n-p_0} \cdots T_r^{n-p_r}$ in $H^{r+1}(T^n, S)$ and by our convention of p. 305 has degree $(n - p_0) + \cdots + (n - p_r) - (r + 1)n = -(p_0 + p_1 + \cdots + p_r)$. This degree is independent of the choice of n since the homomorphism $H^{r+1}(T^n, S) \to H^{r+1}(T^m, S)$ for $n \leqslant m$ is given by multiplication by $T_0^{m-n} \cdots T_r^{m-n}$ and transforms $T_0^{n-p_0} \cdots T_r^{n-p_r}$ into $T_0^{m-p_0} \cdots T_r^{m-p_r}$. We have thus arrived at the following

Theorem (Grothendieck, "Eléments," III, 2.1.12). *Let A be a ring, $S = A[T_0, ..., T_r]$ $(r \geqslant 1)$, $X = \mathrm{Proj}(S) = \mathbf{P}^r_A$. Then*

(i) *$H^i(X, \mathscr{O}_X(n)) = 0$ for $i \neq 0$, r and all $n \in \mathbf{Z}$.*

(ii) *The Serre homomorphism $\alpha : S \to H^0(X, \mathscr{O}_X(*))$ (p. 300) is bijective.*

(iii) *$H^r(X, \mathscr{O}_X(*))$ is a free A-module with basis $(\xi_{p_0 \ldots p_s})$, where $p_i > 0$ for $0 \leqslant i \leqslant r$. The degree of $\xi_{p_0 \ldots p_r}$ is $-(p_0 + \cdots + p_r)$ and the product $T_i \xi_{p_0 \ldots p_r}$ is equal to $\xi_{p_0, \ldots, p_i - 1, \ldots p_r}$ if $p_i > 1$ and to 0 if $p_i = 1$.*

Proof. We have $H^i(X, \mathscr{O}_X(*)) \simeq H^{i+1}((\boldsymbol{T}), S)$ for $i \geqslant 1$ (p. 305) and the exact sequence $0 \to H^0((\boldsymbol{T}), S) \to S \to H^0(X, \mathscr{O}_X(*)) \to H^1((\boldsymbol{T}), S) \to 0$ yields an isomorphism $S \to H^0(X, \mathscr{O}_X(*))$ because $H^0((\boldsymbol{T}), S) = H^1((\boldsymbol{T}), S) = 0$. But this map is Serre's homomorphism α as can be seen from the identification of $H^0(\mathfrak{U}, \mathscr{O}_X(*))$ and $H^0(X, \mathscr{O}_X(*))$.

Corollary (Grothendieck, "Eléments," III, 2.1.13). *The only values of (i, n) for which we can have $H^i(X, \mathscr{O}_X(n)) \neq 0$ are $i = 0$ and $n \geqslant 0$, $i = r$ and $n \leqslant -(r + 1)$.*

We shall now study the cohomology of a closed subschema of a projective space with values in a coherent sheaf.

Theorem. (Serre; Grothendieck, "Eléments," III, 2.2.2). *Let A be a noetherian ring, $S = A[T_0, ..., T_r]$ and X a closed subschema of $\mathbf{P}^r_A = \mathrm{Proj}(S)$. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-Module. Then:*

(i) *the $H^q(X, \mathscr{F})$ are finitely generated A-modules;*

(ii) *$H^q(X, \mathscr{F}) = 0$ for $q > r$;*

(iii) *there exists an integer N such that for $n \geqslant N$ we have $H^q(X, \mathscr{F}(n)) = 0$ for all $q > 0$;*

(iv) *there exists an integer N such that for $n \geqslant N$ the sheaf $\mathscr{F}(n)$ is generated by its sections over X.*

Observe that (iv) has been proved already without the assumption that A is noetherian (p. 303). Let us also remark that $X = \mathrm{Proj}(S')$, where $S' = S/\mathfrak{I}$ and $\mathfrak{I}$ is a graded ideal in S (Grothendieck, "Eléments," II, 2.9.2).

To prove the remaining assertions we can reduce ourselves to the case $X = \mathbf{P}^r_A$. Indeed, if $j : X \to P = \mathbf{P}^r_A$ is the canonical injection, then $j_*(\mathscr{F})$ is a coherent $\mathscr{O}_P$-Module and $j_*(\mathscr{F}(n)) = (j_*(\mathscr{F}))(n)$. Furthermore $H^q(X, \mathscr{F}(n)) = H^q(P, (j_*(\mathscr{F}))(n))$ (Godement, "Théorie des faisceaux," II, Corr. of Th. 4.9.1, p. 188).

We first prove (ii). The space $X = \mathbf{P}_A^r$ is covered by the affine open sets $U_i = D_+(T_i)$, $0 \leqslant i \leqslant r$. Hence we have $H^q(X, \mathscr{F}) \simeq H^q(\mathfrak{U}, \mathscr{F})$ (p. 305). On the other hand $H^q(\mathfrak{U}, \mathscr{F}) \simeq H^{q+1}((T), M)$, where M is the S-module such that $\mathscr{F} = \tilde{M}$. Now the cochains of $K^{q+1}((T), M)$ are alternating mappings, hence they are all zero for $q > r$.

We prove (i) and (iii) simultaneously. The extremely remarkable proof proceeds by descending induction on q and its idea is due to Kodaira. By the previous theorem these assertions are true in the special case $\mathscr{F} = \mathscr{O}_X(m)$ since $H^r(X, \mathscr{O}_X(n)) = 0$ for $r \geqslant -n$, i.e. for $n \geqslant -r$. Hence they are also true for a coherent sheaf $\mathscr{F}$ of the form $\bigoplus_{1 \leqslant i \leqslant l} \mathscr{O}_X(m_i)$. On the other hand the assertions are trivially true for $q > r$ by (ii). We know that there exists an exact sequence $\mathscr{E} \to \mathscr{F} \to 0$, where $\mathscr{E} = \bigoplus_i \mathscr{O}_X(m_i)$ and the m_i are in general large negative integers (p. 304). Let $\mathscr{R}$ be the kernel of the epimorphism $\mathscr{E} \to \mathscr{F}$, then we have the exact sequence $0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0$, where $\mathscr{R}$ is coherent. We also have an exact sequence $0 \to \mathscr{R}(n) \to \mathscr{E}(n) \to \mathscr{F}(n) \to 0$ for every $n \in \mathbf{Z}$ and an exact cohomology sequence

$$H^{q-1}(X, \mathscr{E}(n)) \to H^{q-1}(X, \mathscr{F}(n)) \to H^q(X, \mathscr{R}(n)).$$

By the induction hypothesis $H^q(X, \mathscr{R}(n))$ is a finitely generated A-module. Also $H^{q-1}(X, \mathscr{E}(n))$ is a finitely generated A-module, since $\mathscr{E}(n)$ is of the form $\bigoplus_i \mathscr{O}_X(m_i + n)$. Now if $L \to M \to N$ is an exact sequence of modules over a noetherian ring, where L and N are finitely generated, then $M/\mathrm{Im}(L)$ is a submodule of N and thus finitely generated (p. 258) and consequently M itself is finitely generated. It follows that $H^{q-1}(X, \mathscr{F}(n))$ is a finitely generated A-module. This proves (i).

By the induction hypothesis there exists an N such that $H^q(X, \mathscr{R}(n)) = 0$ for $n \geqslant N$. Since $\mathscr{E}$ satisfies condition (iii) we can choose N so large that also $H^{q-1}(X, \mathscr{E}(n)) = 0$ for $n \geqslant N$. But then $H^{q-1}(X, \mathscr{F}(n)) = 0$ for $n \geqslant N$. At each step we have to take a new number N, but there are only finitely many steps by (ii). Thus (iii) is also proved.

The previous theorem is the generalization announced on p. 248 of the fact that the vector spaces $L(D)$ are finite-dimensional if X is a projective variety over a field.

We now deduce some corollaries of the previous theorem.

(1) Let $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ be an exact sequence of coherent $\mathscr{O}_X$-Modules. Then the sequence $\Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{H})$ is not necessarily exact, but under the hypotheses of the theorem there exists an integer N

such that for $n \geqslant N$ the sequence $\Gamma(X, \mathscr{F}(n)) \to \Gamma(X, \mathscr{G}(n)) \to \Gamma(X, \mathscr{H}(n))$ is exact (Grothendieck, "Eléments," III, 2.2.3).

(2) Under the hypotheses of the previous theorem there exists an integer N such that for $n \geqslant N$ the Serre homomorphism

$$\alpha_n : M_n \to \Gamma(X, \tilde{M}(n))$$

is bijective (Grothendieck, "Eléments," III, 2.3.1). First consider the special case when $X = \mathbf{P}^r_A = P$. There exists an exact sequence $L' \to L \to M \to 0$ where L' and L have the form of finite sums $\oplus_k S(n_k)$. We then have a commutative diagram

$$\begin{array}{ccccccc} L'_n & \longrightarrow & L_n & \longrightarrow & M_n & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_n} & & \\ \Gamma(P, \tilde{L}'(n)) & \longrightarrow & \Gamma(P, \tilde{L}(n)) & \longrightarrow & \Gamma(P, \tilde{M}(n)) & \longrightarrow & 0. \end{array}$$

For large enough n the bottom row is exact. On the other hand the first two vertical arrows are isomorphisms by part (ii) of the theorem on p. 308. It follows from the five lemma that the third is also an isomorphism.

In the general case $X = \mathrm{Proj}(S')$, where $S' = S/\mathfrak{I}$ and $\mathfrak{I}$ is an ideal of $S = A[T_0, \ldots, T_r]$. The preceding reasoning shows that it is enough to prove the assertion for $M = S'$. If $j : X \to P$ is the canonical injection, $\alpha_n : S'_n \to \Gamma(P, j_*(\tilde{S}'(n)))$ is bijective for large n by the first part of the proof and $\Gamma(P, j_*(\tilde{S}'(n))) = \Gamma(X, \tilde{S}'(n))$.

Euler-Poincaré Characteristic and Hilbert Polynomial

(See Grothendieck, "Eléments," III, 2.5.) Let A be an artinian ring, which most of the time will be a field, but we need to consider rings because if we extend the basis we get a ring and not necessarily a field. Let $S = \oplus_{0 \leqslant i < \infty} S_i$ be a graded A-algebra, where S is generated by S_1 and S_1 is a finitely generated A-module. Let $X = \mathrm{Proj}(S)$ and $\mathscr{F}$ a coherent $\mathscr{O}_X$-Module. By Serre's theorem (p. 308) the $H^i(X, \mathscr{F})$ are finitely generated A-modules and they reduce to zero for large values of i. In particular the $H^i(X, \mathscr{F})$ are of finite length (p. 258 of the present paper; Bourbaki, "Algèbre," Chap. I, § 6, No. 14, Scholium) and thus the number $\chi_A(\mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i \log(H^i(X, \mathscr{F}))$, where $\log(H^i(X, \mathscr{F}))$ denotes the length of the A-module $H^i(X, \mathscr{F})$, is a well defined integer. The number $\chi_A(\mathscr{F})$ is called the *Euler–Poincaré characteristic* of $\mathscr{F}$ with

respect to the ring A. In the special case $\mathscr{F} = \mathscr{O}_X$ the number $\chi_A(\mathscr{O}_X)$ is called the *arithmetic genus* of X with respect to A.

Proposition (Grothendieck, "Eléments," III, 2.5.2). *Let*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

be an exact sequence of coherent $\mathscr{O}_X$-Modules. Then

$$\chi_A(\mathscr{F}) = \chi_A(\mathscr{F}') + \chi_A(\mathscr{F}'').$$

Proof. Since the cohomology modules of $\mathscr{F}$, $\mathscr{F}'$, and $\mathscr{F}''$ are 0 except for finitely many, there exists an integer $r > 0$ such that the exact cohomology sequence takes the form

$$0 \to H^0(X, \mathscr{F}') \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}'') \to H^1(X, \mathscr{F}') \to H^1(X, \mathscr{F})$$
$$\to \cdots \to H^r(X, \mathscr{F}') \to H^r(X, \mathscr{F}) \to H^r(X, \mathscr{F}'') \to 0.$$

Now if $0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$ is an exact sequence of A-modules, let $\mu_i = \operatorname{long} M_i$ and $\nu_i = \operatorname{long}(\operatorname{Im}(M_{i-1} \to M_i)) = \operatorname{long}(\operatorname{Ker}(M_i \to M_{i+1}))$, $M_0 = M_{n-1} = 0$. Then $\mu_i = \nu_i + \nu_{i-1}$ $(1 \leqslant i \leqslant n)$ and $\sum_{i=1}^{n} (-1)^i \mu_i = \nu_1 + (-1)^n \nu_{n+1} = 0$, since $\nu_1 = \nu_{n+1} = 0$. Applying this to the above exact sequence we get

$$\sum_{i=0}^{r} (-1)^i \operatorname{long}(H, \mathscr{F}')) - \sum_{i=0}^{r} (-1)^i \operatorname{long}(H^i(X, \mathscr{F}))$$
$$+ \sum_{i=0}^{r} (-1)^i \operatorname{long}(H^i(X, \mathscr{F}'')) = 0,$$

from where the proposition follows.

Theorem (Hilbert–Serre; Grothendieck, "Eléments," III, 2.5.3). *Let A be an artinian local ring. Then with the notations introduced above, we get the following*:

(i) *There exists a polynomial $P \in \mathbf{Q}[T]$ such that $\chi_A(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbf{Z}$. This polynomial P is called the Hilbert polynomial of $\mathscr{F}$ with respect to A.*

(ii) $P(n) = \operatorname{long}_A \Gamma(X, \mathscr{F}(n))$ *for large n.*

(iii) *The leading coefficient in P is $\geqslant 0$.*

(iv) *The degree of P is the dimension of the support of $\mathscr{F}$.*

Proof. We shall not prove (iv) here or even define the dimension of $\operatorname{Supp}(\mathscr{F})$ (Grothendieck's "Eléments," IV, 5.3.1).

(ii) follows from (i) since for large values of n we have $H^i(X, \mathscr{F}(n)) = 0$ for $i > 0$ (p. 308). It follows from (ii) that $P(n) \geqslant 0$ for large n, hence P must have a positive leading coefficient.

Thus we have to prove (i). This will be done after several reductions.

(1) We can assume that $\mathfrak{m}\mathscr{F} = 0$, where $\mathfrak{m}$ is the maximal ideal of A. Indeed, suppose that (i) holds in this case. We have $\mathfrak{m}^h = 0$ for some $h > 0$ (p. 258) and hence a finite filtration

$$\mathscr{F} \supset \mathfrak{m}\mathscr{F} \supset \mathfrak{m}^2\mathscr{F} \supset \cdots \supset \mathfrak{m}^{h-1}\mathscr{F} \supset 0.$$

Set $\mathscr{F}_j = \mathfrak{m}^j\mathscr{F}/\mathfrak{m}^{j+1}\mathscr{F}$, then we have the exact sequence $0 \to \mathfrak{m}^{j+1}\mathscr{F} \to \mathfrak{m}^j\mathscr{F} \to \mathscr{F}_j \to 0$ and also

$$0 \to \mathfrak{m}^{j+1}\mathscr{F}(n) \to \mathfrak{m}^j\mathscr{F}(n) \to \mathscr{F}_j(n) \to 0.$$

By the previous proposition

$$\chi(\mathscr{F}_j(n)) + \chi(\mathfrak{m}^{j+1}\mathscr{F}(n)) = \chi(\mathfrak{m}^j\mathscr{F}(n))$$

and thus $\chi(\mathscr{F}(n)) = \sum_{j=0}^{h-1} \chi(\mathscr{F}_j(n))$. On the other hand $\mathfrak{m}\mathscr{F}_j = 0$ for $0 \leqslant j \leqslant h-1$, hence (i) holds in general.

(2) We may suppose that A is a field k. Indeed, let $k = A/\mathfrak{m}$. To the epimorphism $A \to k$ there corresponds an extension of the basis (p. 252)

$$\begin{array}{ccc} X & \longleftarrow & X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

and $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k)$ as a topological space is the same as X since $\mathrm{Spec}(k)$ has one point. Furthermore $\mathfrak{m}\mathscr{F} = 0$ means that $A/\mathfrak{m} = k$ operates on $\mathscr{F}$. Thus the $H^i(X, \mathscr{F})$ are k-vector spaces and their dimensions are equal to their lengths as A-modules.

(3) X can be considered as a closed subpreschema of some $\mathbf{P}_k^r$. Indeed, S is the quotient of a polynomial ring $k[T_0, \ldots, T_r]$ modulo a graded ideal, hence there is an injection $j : X \to P = \mathbf{P}_k^r$. We have then $H^i(X, \mathscr{F}(n)) = H^i(P, j_*(\mathscr{F}(n))$ (Godement, "Théorie des faisceaux," II, Cor. of Th. 4.9.1, p. 188).

Thus we are reduced to the case when A is a field k, $S = k[T_0, \ldots, T_r]$ and $X = \mathrm{Proj}(S) = \mathbf{P}_k^r$. We have $\mathscr{F} = \tilde{M}$, where M is a graded S-module of finite type (p. 303). By Hilbert's theorem on "chains of syzygies" (Hilbert, *Math. Ann.*, **36** (1890), p. 473–534; Cartan–Eilenberg,

"Homological Algebra," Chap. VIII, Th. 6.5, p. 157) there exists a finite resolution of M by graded free modules:

$$0 \to L_q \to L_{q-1} \to \cdots \to L_0 \to M \to 0.$$

Hence we have an exact sequence

$$0 \to \tilde{L}_q \to \tilde{L}_{q-1} \to \cdots \to \tilde{L}_0 \to \tilde{M} \to 0$$

and also

$$0 \to \tilde{L}_q(n) \to \tilde{L}_{q-1}(n) \to \cdots \to \tilde{L}_0(n) \to \tilde{M}(n) \to 0$$

for any $n \in \mathbf{Z}$. It follows by induction on q from the above proposition (p. 310) that

$$\chi(\tilde{M}(n)) = \chi(\tilde{L}_0(n)) - \chi(\tilde{L}_1(n)) + \cdots + (-1)^q \chi(\tilde{L}_q(n)).$$

Thus to prove that $\chi(\mathscr{F}(n))$ is a polynomial in n, it is sufficient to consider the case $\mathscr{F} = \tilde{L}$, where L is free, i.e., $L = \bigoplus_{i=1}^{k} S(m_i)$ and finally the case $L = S(m)$. But it follows from our earlier results (p. 307) that $\chi_A(\mathscr{O}_X(n)) = \binom{n+r}{r}$ for all n. Indeed, if $n \geqslant 0$, then $\chi_A(\mathscr{O}_X(n)) = \operatorname{long} \Gamma(X, \mathscr{O}_X(n))$, since $H^i(X, \mathscr{O}_X(n)) = 0$ for $i \geqslant 1$. But $\Gamma(X, \mathscr{O}_X(n)) = S_n$ and the dimension of S_n is the number of monomials in the $T_0, \ldots, T_r$ of degree n, i.e., $\dim S_n = \binom{n+r}{r}$. If $-r \leqslant n < 0$, then $\binom{n+r}{r} = 0$ and all $H^i(X, \mathscr{O}_X(n)) = 0$ for $i \geqslant 0$. Finally, if $n \leqslant -r - 1$, then $\chi_A(\mathscr{O}_X(n)) = (-1)^r \operatorname{long}(H^r(X, \mathscr{O}_X(n)))$. If $n = -r - h$, then the dimension of $H^r(X, \mathscr{O}_X(n))$ is the number of sequences $(p_i)_{0 \leqslant i \leqslant r}$ of integers $p_i > 0$ such that $\sum_{i=0}^{r} p_i = r + h$, or setting $q_i = p_i - 1$, the number of sequences $(q_i)_{0 \leqslant i \leqslant r}$ of integers $q_i \geqslant 0$ such that $\sum_{i=0}^{r} q_i = h - 1$. This number is equal to

$$\binom{h + r - 1}{r} = \binom{-n - 1}{r} = (-1)^r \binom{n + r}{r}.$$

As a corollary (Grothendieck, "Eléments," III, 2.5.4) we obtain that if A is an artinian local ring, S a graded A-algebra of finite type generated by S_1 and M a graded S-module of finite type, then on $X = \operatorname{Proj}(S)$ we have $\chi_A(\tilde{M}(n)) = \operatorname{long} M_n$ for large n. Indeed, then the Serre homomorphism $\alpha_n : M_n \to \Gamma(X, \tilde{M}(n))$ is bijective for large n (p. 309).

Projective and Proper Morphisms

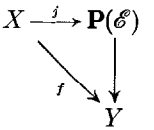
(Grothendieck, "Eléments," II, 5.) Let Y be a preschema, $\mathscr{S}$ a quasi-coherent graded $\mathscr{O}_Y$-Algebra. Let U be an affine open set in Y and

$A = \Gamma(U, \mathcal{O}_Y)$ its ring. Then $\Gamma(U, \mathcal{S})$ is an A-algebra and we can consider the schema $\mathrm{Proj}(\Gamma(U, \mathcal{S}))$. If U' is an affine open set contained in U we have an open immersion $\mathrm{Proj}(\Gamma(U', \mathcal{S})) \to \mathrm{Proj}(\Gamma(U, \mathcal{S}))$ satisfying the usual compatibility requirements. Thus we can glue the schemata $\mathrm{Proj}(\Gamma(U, \mathcal{S}))$ together and obtain a schema over Y denoted $\mathrm{Proj}(\mathcal{S})$ and called the *homogeneous spectrum* of the $\mathcal{O}_Y$-Algebra $\mathcal{S}$ (Grothendieck, "Eléments," II, 3.1.3).

Definition (Grothendieck, "Eléments," II, 5.5.2). *Let X and Y be two preschemata and $f: X \to Y$ a morphism. We say that f is a projective morphism if there exists a quasi-coherent graded $\mathcal{O}_Y$-Algebra $\mathcal{S}$, such that $\mathcal{S}_1$ is of finite type and generates $\mathcal{S}$ and such that X is Y-isomorphic to* $\mathrm{Proj}(\mathcal{S})$.

Observe that a finiteness condition is built into this definition. If $f: X \to Y$ is a projective morphism we say that X is projective over Y. A projective morphism is always separated and of finite type (p. 262). We can give an alternate definition of projective morphisms, but first we have to introduce the notion or projective fiber bundles. Let us first consider the affine case. Let E be a finitely generated A-module and $\mathsf{S}_A(E)$ the symmetric algebra of E (p. 273). Of course $\mathsf{S}_A(E)$ is a graded algebra, $\mathsf{S}_A(E) = S_0 \oplus S_1 \oplus \cdots$. The corresponding projective schema $\mathrm{Proj}(\mathsf{S}_A(E))$ will be called the projective fiber bundle over $\mathrm{Spec}(A)$ defined by E and denoted by $\mathbf{P}(E)$. At a point $x \in \mathrm{Spec}(A)$ the fiber $f^{-1}(x)$ over x is the projective space $\mathrm{Proj}(\mathsf{S}_A(E_x))$ over the field $\kappa(x)$. In the general case, let Y be a preschema, $\mathscr{E}$ a quasi-coherent $\mathcal{O}_Y$-Module, $\mathsf{S}_{\mathcal{O}_Y}(\mathscr{E})$ the symmetric $\mathcal{O}_Y$-Algebra of $\mathscr{E}$ (p. 274). Then the Y-schema $\mathrm{Proj}(\mathsf{S}_{\mathcal{O}_Y}(\mathscr{E}))$ will be called the *projective fiber bundle* over Y defined by $\mathscr{E}$ and denoted by $\mathbf{P}(\mathscr{E})$ (Grothendieck, "Eléments," II, 4.1.1).

A morphism $f: X \to Y$ of preschemata is projective if and only if X is Y-isomorphic to a closed subpreschema of a projective fiber bundle $\mathbf{P}(\mathscr{E})$ over Y defined by some quasi-coherent $\mathcal{O}_Y$-Module $\mathscr{E}$ of finite type (Grothendieck, "Eléments," II, 5.5.1). In other words, f can be factored in the form



where j is a closed immersion.

The handling of projective morphisms is not easy due to the fact that the property of being a projective morphism is not local over Y

(p. 262). This follows from examples given by Nagata (*Illinois J. Math.* 2 (1958), 490–498) and Hironaka.

The propositions (i)–(vi) of p. 263 hold for projective morphisms, with the sole exception that in (ii) we must suppose that Z is either quasi-compact and separated or that the underlying space is noetherian (Grothendieck, "Eléments," II, 5.5.5).

Let $f: X \to Y$ be a projective morphism. Since f is then separated and of finite type, hence in particular quasi-compact (p. 271), the theorem of p. 294 applies, i.e., for every quasi-coherent $\mathcal{O}_X$-Module $\mathcal{F}$ the $\mathcal{O}_Y$-Modules $R^q f_*(\mathcal{F})$ are quasi-coherent and furthermore $U \rightsquigarrow H^q(f^{-1}(U), \mathcal{F})$ is already the sheaf $R^q f_*(\mathcal{F})$, i.e. $H^q(f^{-1}(U), \mathcal{F}) = \Gamma(U, R^q f_*(\mathcal{F}))$. For projective morphisms we can complement Serre's theorem (p. 308) in the following way:

Theorem (Grothendieck, "Eléments," III, 2.2.1 and 2.2.5). *Let Y be a locally noetherian preschema, $f: X \to Y$ a projective morphism and $\mathcal{F}$ a coherent $\mathcal{O}_X$-Module. Then:*

(i) *The $\mathcal{O}_Y$-Modules $R^q f_*(\mathcal{F})$ are coherent.*

(ii) *If Y is noetherian, then there exists an integer N such that $R^q f_*(\mathcal{F}(n)) = 0$ for $n \geqslant N$ and $q > 0$.*

(iii) *If Y is noetherian, then there exists an integer N such that for $n \geqslant N$ the canonical homomorphism $f^*(f_*(\mathcal{F}(n))) \to \mathcal{F}(n)$ is surjective.*

The proof is almost trivial. Property (i) is local and for affine open sets U it follows from Serre's theorem. Properties (ii) and (iii) are also true for each U with an N which depends on U. The noetherian condition ensures that we can take a finite covering by sets U.

Since projective morphisms are so difficult to work with, as a substitute one introduces the much simpler proper morphisms. Their definition is almost topological and they are in close analogy to the proper maps of general topology (Bourbaki, "Topologie générale," Chap. I, 3rd ed., § 10)

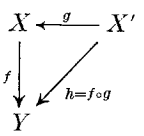
Definition (Grothendieck, "Eléments," II, 5.4.1). *A morphism $f: X \to Y$ of preschemata is proper if it is universally closed* (p. 278), *of finite type* (p. 262) *and separated. If this is the case we also say that X is proper over Y.*

In topology if we have a proper map $f: X \to Y$, then the fiber $f^{-1}(y)$ over a point $y \in Y$ is quasi-compact (Bourbaki, "Top. gén.," Chap. I,

3rd ed., § 10, No. 2, Th. 1). If $f: X \to Y$ is a proper morphism of preschemata, then the fiber $f^{-1}(y)$ will have the analogous property of being "complete". We say that a preschema X over a field k is complete if the morphism $X \to \mathrm{Spec}(k)$ is proper. This generalizes the classical notion introduced by Chevalley in 1957 and the former notion of a complete abstract variety in the sense of André Weil (see Lang, "Introduction to Algebraic Geometry," Chap. IV, § 6, p. 112). If $k = \mathbf{C}$, then a complete variety is compact.

Proper morphisms have the properties listed in propositions (i)–(vi) of p. 262 (Grothendieck, "Eléments," II, 5.4.2, 5.4.3 and 5.4.6). The property of being proper is local over Y (p. 263). A projective morphism is proper (Grothendieck, "Eléments," II, 5.3.3) but the converse is not true as follows from examples given by Nagata (*loc. cit.*).

Chow's Lemma (Grothendieck, "Eléments," II, 5.6.1 and 5.6.2). *Let Y be a noetherian preschema and $f: X \to Y$ a proper morphism. Then there exists a preschema X' and a commutative diagram*



such that:

(i) *g is projective and surjective*;

(ii) *h is projective*;

(iii) *there exists an open set U in X which is dense in X, such that the restriction of g to $g^{-1}(U)$ is an isomorphism from $g^{-1}(U)$ onto U and $g^{-1}(U)$ is dense in X'.*

If, furthermore, X is integral (p. 277) (resp. *reduced or irreducible*) *then X' is integral* (resp. *reduced or irreducible*). *If X is irreducible, g maps the generic point of X' onto the generic point of X.*

The Finiteness Theorem

We know (p. 315) that if Y is a locally noetherian preschema and $f: X \to Y$ a projective morphism then for every coherent $\mathcal{O}_X$-Module $\mathcal{F}$ the $\mathcal{O}_Y$-Modules $R^q f_*(\mathcal{F})$ are coherent. Serre conjectured that the same conclusion holds if we suppose only that f is proper and proved

it for $q = 0$. The proof for general q is Grothendieck's first main contribution to the cohomology theory of coherent sheaves.

Theorem (Grothendieck, "Eléments," III, 3.2.1). *Let Y be a locally noetherian preschema and $f : X \to Y$ a proper morphism. If $\mathscr{F}$ is a coherent $\mathcal{O}_X$-Module then all the $\mathcal{O}_Y$-Modules $R^q f_*(\mathscr{F})$ are coherent.*

A similar result in analytic geometry was proved by Grauert (Publications IHES, No. 5, 1960). The proof is very typical of the way how Grothendieck's mind works and therefore we reproduce it here. In the first place we may suppose that Y is noetherian since the question is local. Then X is also noetherian because f is of finite type (p. 263).

(A) We first consider a very special case of the theorem. Let X be integral (p. 244) and let x be the generic point of X. We prove that there exists *one* coherent sheaf $\mathscr{F}$ on X such that $\mathscr{F}_x \neq 0$ and such that $R^q f_*(\mathscr{F})$ is coherent for all $q \geqslant 0$. In the proof we use Chow's lemma, as has been done already by Serre (*J. Math. Pures Appl.* **36** (1957), 1–16), and spectral sequences, which is new.

By Chow's lemma (p. 316) we have a projective, surjective morphism $g : X' \to X$ such that $h = f \circ g : X' \to Y$ is projective. We show that $\mathscr{F} = g_*(\mathcal{O}_{X'}(n))$ for large n has the required properties.

(1) Since g is projective we can apply Serre's theorem (p. 315) and so $\mathscr{F}$ is coherent for large n.

(2) Again by Serre's theorem the homomorphism

$$g^*(g_*(\mathcal{O}_{X'}(n))) \to \mathcal{O}_{X'}(n)$$

is surjective for large n. Since the generic point of X' is mapped by g onto x, we have $\mathscr{F}_x \neq 0$.

(3) Since h is projective we have, still by Serre's theorem, that $R^q h_*(\mathcal{O}_{X'}(n))$ is coherent for $q \geqslant 0$. Now there exists a spectral sequence whose initial term is $E_2^{pq} = R^p f_*(R^q g_*(\mathcal{O}_{X'}(n)))$ and whose terminal term is $R^q h_*(\mathcal{O}_{X'}(n))$ (Grothendieck, Sur quelques points d'algèbre homologique, *Tohoku Math. J.* **9** (1957), 119–221; see Th. 2.4.1, p. 148). On the other hand, $R^q g_*(\mathcal{O}_{X'}(n)) = 0$ for $q > 0$ hence $E_2^{pq} = 0$ for $q \neq 0$, i.e., the spectral sequence is degenerate. We know that in this case $R^p h_*(\mathcal{O}_{X'}(n)) = E_2^{p0} = R^p f_*(g_*(\mathcal{O}_{X'}(n))) = R^p f_*(\mathscr{F})$ (Grothendieck "Eléments," 0_{III}, 11.1.6).

(B) Now we prove the general result with the help of a remarkable reduction, the "unscrewing lemma" (lemme de dévissage).

Let $\boldsymbol{K}$ be the category of all coherent $\mathcal{O}_X$-Modules $\mathscr{F}$ and $\boldsymbol{K}'$ the subset formed by those $\mathscr{F}$ for which the conclusion of the theorem holds, i.e., for which $\mathcal{O}_Y$-Modules $R^q f_*(\mathscr{F})$ are coherent for all $q \geqslant 0$.

(1) Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of coherent $\mathcal{O}_X$-Modules. If two of the Modules $\mathscr{F}$, $\mathscr{F}'$, $\mathscr{F}''$ belong to $\boldsymbol{K}'$ then so does the third one. Suppose for instance that $\mathscr{F}$ and $\mathscr{F}'$ are in $\boldsymbol{K}'$. We have the exact cohomology sequence

$$R^q f_*(\mathscr{F}') \to R^q f_*(\mathscr{F}) \to R^q f_*(\mathscr{F}'') \to R^{q+1} f_*(\mathscr{F}') \to R^{q+1} f_*(\mathscr{F}).$$

Now by assumption the four extreme terms are coherent, hence the middle one is also (Grothendieck, "Eléments," 0_{I}, 5.3.4). We could also use a three-term exact sequence considering affine open sets U in Y and using the fact that if $\tilde{A} \to \tilde{B} \to \tilde{C}$ is an exact sequence of quasi-coherent $\mathcal{O}_U$-Modules and A and C are of finite type, then B is of finite type, i.e., $\tilde{B}$ is coherent (Grothendieck, "Eléments," I, 1.5.1). Observe that in particular any direct sum of objects in $\boldsymbol{K}'$ belongs to $\boldsymbol{K}'$.

(2) Any coherent direct factor $\mathscr{F}'$ of an object $\mathscr{F}$ in $\boldsymbol{K}'$ belongs to $\boldsymbol{K}'$. Indeed, if $\mathscr{F} = \mathscr{F}' \oplus \mathscr{F}''$, then since the direct limit preserves direct sums we have $R^q f_*(\mathscr{F}) = R^q f_*(\mathscr{F}') \oplus R^q f_*(\mathscr{F}'')$ (Godement, "Théorie des faisceaux," II, Th. 4.4.4, p. 175). Hence $R^q f_*(\mathscr{F}')$ if of finite type and since it is quasi-coherent (p. 294) and Y is noetherian, it is coherent (Grothendieck, "Eléments," I, 1.5.1).

The theorem will now follow from the following

Lemma (Grothendieck, "Eléments," III, 3.1.3). *Let X be a noetherian preschema and X' a closed subset of the underlying space of X. Let $\boldsymbol{K}$ be the abelian category of all coherent $\mathcal{O}_X$-Modules and $\boldsymbol{K}'$ a subset of $\boldsymbol{K}$ having the properties* (1) *and* (2). *Suppose that for every closed, irreducible subset Z of X' with generic point z there exists an $\mathcal{O}_X$-Module $\mathscr{G} \in \boldsymbol{K}'$ which has its support in Z and such that $\mathscr{G}_z \neq 0$. Then every coherent $\mathcal{O}_X$-Module whose support is contained in X' belongs to $\boldsymbol{K}'$. In particular, if $X' = X$ then $\boldsymbol{K} = \boldsymbol{K}'$.*

Assume that the lemma is proved. To prove the theorem we have only to show that for every closed, irreducible subset Z of X an $\mathcal{O}_X$-Module $\mathscr{G} \in \boldsymbol{K}'$ with the required property exists. Let $j : Z \to X$ be the canonical injection, then $f \circ j : Z \to Y$ is proper (p. 316). Applying part (A) of the proof to Z, there exists a coherent $\mathcal{O}_Z$-Module $\mathscr{F}$ on Z such that $\mathscr{F}_z \neq 0$ and such that $R^q(f \circ j)_*(\mathscr{F})$ is a coherent $\mathcal{O}_Y$-Module for all $q \geqslant 0$.

Then $\mathscr{G} = j_*(\mathscr{F})$ satisfies the requirements. Indeed, $\mathscr{G}$ is a coherent $\mathscr{O}_X$-Module since j is a closed immersion (Grothendieck, "Eléments," 0_I, 5.3.12). Clearly $\mathscr{G}_z \neq 0$ and $R^q f_*(\mathscr{G}) = R^q(f \circ j)_*(\mathscr{F})$ is coherent (Godement, "Théorie des faisceaux," II, Th. 4.9.1, p. 187). Hence by the lemma $\boldsymbol{K}' = \boldsymbol{K}$.

It remains to prove the lemma. This will be done by noetherian induction (Bourbaki, "Alg. comm.," Chap. II, § 4, No. 2, Lemma 1; Grothendieck, "Eléments," 0_I, 2.2.2): Let E be a noetherian space and $\boldsymbol{P}$ a property which has a meaning for all closed subsets of E. Assume that if F is a closed subset of E such that for every closed proper subset G of F the property $\boldsymbol{P}(G)$ is true then $\boldsymbol{P}(F)$ is true. Under this assumption $\boldsymbol{P}(F)$ is true for all closed subsets F of E and in particular $\boldsymbol{P}(E)$ is true. Indeed, let $\mathfrak{M}$ be the set of all closed subsets of E for which $\boldsymbol{P}$ is false. If $\mathfrak{M}$ is not empty, then by the noetherian condition there exists a minimal element F_0 in $\mathfrak{M}$. Thus for every proper closed subset G of F_0 the property $\boldsymbol{P}(G)$ is true but $\boldsymbol{P}(F_0)$ is false, which is a contradiction. Hence $\mathfrak{M}$ must be empty.

Let Y be a closed subset of X' and let $\boldsymbol{P}(Y)$ be the following property: every coherent $\mathscr{O}_X$-Module whose support is contained in Y belongs to $\boldsymbol{K}'$. Then we have to show that if $\boldsymbol{P}(Y')$ is true for every closed proper subset Y' of Y, then $\boldsymbol{P}(Y)$ is true.

If Y is a closed subset of X', then there exists a unique reduced closed subpreschema of X whose underlying space is Y (p. 256) defined by a coherent Ideal $\mathscr{J}$ of $\mathscr{O}_X$. Consider an $\mathscr{F}$ in $\boldsymbol{K}$ with support in Y and let us prove that $\mathscr{F}$ belongs to $\boldsymbol{K}'$. We can make two reductions.

(1) We can suppose that $\mathscr{J}\mathscr{F} = 0$. Indeed, there exists an integer n such that $\mathscr{J}^n\mathscr{F} = 0$ (Grothendieck, "Eléments," I, 9.3.4). We thus obtain the usual filtration

$$\mathscr{F} \supset \mathscr{J}\mathscr{F} \supset \mathscr{J}^2\mathscr{F} \supset \cdots \supset \mathscr{J}^{n-1}\mathscr{F} \supset 0$$

and if we set $\mathscr{F}_j = \mathscr{J}^j\mathscr{F}/\mathscr{J}^{j+1}\mathscr{F}$, then we have the exact sequences $0 \to \mathscr{J}^{j+1}\mathscr{F} \to \mathscr{J}^j\mathscr{F} \to \mathscr{F}_j \to 0$. But $\mathscr{J}\mathscr{F}_j = 0$, thus by induction and using property (1) of p. 317 we obtain that $\mathscr{F} \in \boldsymbol{K}'$ if we know that this is true for all $\mathscr{F}$ with the additional hypothesis $\mathscr{J}\mathscr{F} = 0$.

(2) We can write $\mathscr{F} = j_*(\mathscr{F}_1)$, where $\mathscr{F}_1$ is an $\mathscr{O}_Y$-Module on Y and $j : Y \to X$ is the canonical injection and $\mathscr{O}_Y = (\mathscr{O}_X/\mathscr{J}) \mid Y$ (Grothendieck, "Eléments," I, 4.1.2). Indeed, we can take $\mathscr{F}_1 = j^*(\mathscr{F})$ (Grothendieck, "Eléments," 0_I, 5.3.11) and $\mathscr{F}_1$ is an $\mathscr{O}_Y$-Module, since $\mathscr{J}\mathscr{F} = 0$.

Now we distinguish two cases.

(a) Y is reducible. Then $Y = Y' \cup Y''$, where Y' and Y'' are two closed, proper subsets of Y. On Y' and Y'' we have structures of reduced closed subpreschemata of X defined respectively by the coherent Ideals $\mathscr{J}'$ and $\mathscr{J}''$ of $\mathscr{O}_X$. Let $j' : Y' \to X$ and $j'' : Y'' \to X$ be the canonical injections. Write $\mathscr{F}' = j'^*(\mathscr{F}) = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}')$, $\mathscr{F}'' = j''^*(\mathscr{F}) = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}'')$ (Grothendieck, "Eléments," I, 9.1.2). The canonical homomorphisms $\mathscr{F} \to \mathscr{F}'$ and $\mathscr{F} \to \mathscr{F}''$ define a natural map $u : \mathscr{F} \to \mathscr{F}' \oplus \mathscr{F}''$. The homomorphism $u_z : \mathscr{F}_z \to \mathscr{F}'_z \oplus \mathscr{F}''_z$ is bijective for every $z \notin Y' \cap Y''$. If for instance $z \in Y'$ and $z \notin Y''$, then $\mathscr{F}''_z = 0$ and $\mathscr{J}'_z = \mathscr{J}_z$, hence $\mathscr{F}'_z = \mathscr{F}_z$ and u_z is the identity mapping. Thus both $\mathrm{Ker}(u)$ and $\mathrm{Coker}(u)$, which we know belong to $\boldsymbol{K}$, have their supports in $Y' \cap Y''$. Since $Y' \cap Y'' \neq Y$, it follows from the induction hypothesis that $\mathrm{Ker}(u) \in \boldsymbol{K}'$ and $\mathrm{Coker}(u) \in \boldsymbol{K}'$. Furthermore the supports of $\mathscr{F}'$ and $\mathscr{F}''$ are contained respectively in Y' and Y'', hence $\mathscr{F}' \in \boldsymbol{K}'$ and $\mathscr{F}'' \in \boldsymbol{K}'$, and by the observation made at the end of property (1) of p. 317, $\mathscr{F}' \oplus \mathscr{F}'' \in \boldsymbol{K}'$. We have the exact sequences

$$0 \to \mathrm{Ker}(u) \to \mathscr{F} \to \mathrm{Im}(u) \to 0$$

and

$$0 \to \mathrm{Im}(u) \to \mathscr{F}' \oplus \mathscr{F}'' \to \mathrm{Coker}(u) \to 0.$$

From the second exact sequence it follows using property (1) of p. 317 that $\mathrm{Im}(u) \in \boldsymbol{K}'$ and then from the first one we can conclude that $\mathscr{F} \in \boldsymbol{K}'$.

(b) Y is irreducible. Since Y is reduced it is integral (p. 255). Let y be its generic point, then $\kappa(y) = (\mathscr{O}_Y)_y$ since (0) is the unique minimal prime ideal of the integral ring $\Gamma(U, \mathscr{O}_Y)$, where U is an affine open neighborhood of y. Since $\mathscr{F}_1 = j^*(\mathscr{F})$ is a coherent $\mathscr{O}_Y$-Module, it follows that $\mathscr{F}_y = (\mathscr{F}_1)_y$ is a finite-dimensional vector space over $\kappa(y)$. Let m be the dimension of $\mathscr{F}_y$. Let $\mathscr{G}$ be the $\mathscr{O}_X$-Module in $\boldsymbol{K}'$ which has its support in Y and for which $\mathscr{G}_y \neq 0$. Let q be the dimension of $\mathscr{G}_y$ over $\kappa(y)$. Then $\mathscr{F}_y^q \simeq \mathscr{G}_y^m$ since both are isomorphic to $(\kappa(y))^{mq}$. It is enough to prove that $\mathscr{F}^q \in \boldsymbol{K}'$ since then, by property (2) of p. 317, the direct factor $\mathscr{F}$ will also belong to $\boldsymbol{K}'$.

There exists an open neighborhood W of y in X and an isomorphism $v : \mathscr{G}^m \mid W \to \mathscr{F}^q \mid W$ (Grothendieck, "Eléments," 0_I, 5.2.7). If we could extend this isomorphism to the whole space X then the proof would be finished. This can, however, not be done and the difficulty is circumvented as follows. Let $\mathscr{H}$ be the graph of the isomorphism v. Then $\mathscr{H}$ is a coherent sub-$(\mathscr{O}_X \mid W)$-Module of $(\mathscr{G}^m \oplus \mathscr{F}^q) \mid W$ and the

projections $\mathscr{H} \to \mathscr{G}^m \mid W$ and $\mathscr{H} \to \mathscr{F}^q \mid W$ are isomorphisms. By a difficult theorem (Grothendieck, "Eléments," I, 9.4.7) there exists a coherent sub-$\mathcal{O}_X$-Module $\mathscr{H}_0$ of $\mathscr{G}^m \oplus \mathscr{F}^q$ such that $\mathscr{H}_0 \mid W = \mathscr{H}$. Consider the restrictions $w_1 : \mathscr{H}_0 \to \mathscr{G}^m$ and $w_2 : \mathscr{H}_0 \to \mathscr{F}^q$ of the canonical projections of $\mathscr{G}^m \oplus \mathscr{F}^q$. Then $w_i \mid W$ $(i = 1, 2)$ is an isomorphism by what we have just said and also $w_i \mid (X - Y)$ is an isomorphism since on $X - Y$ both $\mathscr{G}^m$ and $\mathscr{F}^q$ are zero. In other words $\mathrm{Ker}(w_1)$ and $\mathrm{Coker}(w_1)$ have their supports contained in $Y - (Y \cap W)$ which is a closed, proper subset of Y. Thus by our induction assumption both belong to $\boldsymbol{K}'$. We have the exact sequences

$$0 \to \mathrm{Ker}(w_1) \to \mathscr{H}_0 \to \mathrm{Im}(w_1) \to 0$$

and

$$0 \to \mathrm{Im}(w_1) \to \mathscr{G}^m \to \mathrm{Coker}(w_1) \to 0.$$

From the second exact sequence it follows by property (1) of p. 317 that $\mathrm{Im}(w_1) \in \boldsymbol{K}'$ and then from the first one that $\mathscr{H}_0 \in \boldsymbol{K}'$. Similarly $\mathrm{Ker}(w_2)$ and $\mathrm{Coker}(w_2)$ have their supports in $Y - (Y \cap W)$ and so $\mathrm{Ker}(w_2) \in \boldsymbol{K}'$, $\mathrm{Coker}(w_2) \in \boldsymbol{K}'$. We have now the exact sequences $0 \to \mathrm{Ker}(w_2) \to \mathscr{H}_0 \to \mathrm{Im}(w_2) \to 0$ and $0 \to \mathrm{Im}(w_2) \to \mathscr{F}^q \to \mathrm{Coker}(w_2) \to 0$. From the first one it follows that $\mathrm{Im}(w_2) \in \boldsymbol{K}'$ and from the second one that $\mathscr{F}^q \in \boldsymbol{K}'$. This concludes the proof of the lemma.