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Developments of the theory of generalized functions or distributions – A vision of Paul Dirac

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> "Strictly, of course, $\delta(x)$ is not a proper function of x, but can be regarded as the limit of a certain sequence of functions. All the same, one can use $\delta(x)$ as though it were a proper function for practically all the purpose of quantum mechanics without getting incorrect results. One can also use the differential coefficients of $\delta(x)$, namely, $\delta'(x), \delta''(x), \ldots$, which are even more discontinuous and less proper than $\delta(x)$ itself."
>
> P. A. M. Dirac

> "To **Paul Dirac** who saw that it must be true, **Laurent Schwartz** who proved it, and **Geoge Temple** who showed how simple it could be made."
>
> James Lighthill

Summary: This paper deals with six major developments of the theory of generalized functions or distributions as a whole new branch of modern analysis. These include (i) The Dirac delta function as the limit of sequences of ordinary functions, (ii) Schwartz's new theory of distributions based on test functions, (iii) Temple–Lighthill's simpler analytical approach to generalized functions based on good functions, (iv) Mikusinski's algebraic approach to generalized functions, (v) Cauchy's representation of distributions by analytic functions, and (vi) Sato's less abstract and computationally more effective approach to generalized functions (or hyperfunctions) based on complex variables. Included are basic features, properties and examples of generalized functions including the Dirac delta function and the Heaviside function. Some additional properties of convolution are discussed in the end.

1 Origin of the Dirac delta function in the works of Kirchhoff and Heaviside

In mathematics, one of the most fundamental concepts is *function* (or *mapping*) which is defined as an element $f : X \to Y$ that assigns to each point x in a set X, a unique point $y = f(x)$ in the set Y. Clearly, this set theoretic definition of a function provides

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the fundamental operation of evaluation of $y = f(x)$ of y for a given element x of X. Thus, the domain X of f is a set of points and the range Y is also a set of points. In the standard notation, $y = f(x)$ represents the value of the function f at x, and hence, f and $f(x)$ are often used to represent function interchangeably without confusion.

In particular, the idea of an impulsive function which is very large in a vanishingly small region and very small outside of it such that the integral over the small region is finite, had been known in applied mathematics and theoretical mechanics with great importance for a long period of time. Indeed, there are many physical problems in which the idealized feature such as a force concentrated at a point, or an impulsive force $f(t)$ acting instantaneously for a very short time τ so that it satisfies the relation

$$\int_0^\tau f(t)\,dt = I, \tag{1.1}$$

where I is called the *magnitude of the impulsive force* (or *impulse*) $f(t)$ at time $t = 0$. If $f(t)$ has a constant value f_0 in (1.1), then $I = \tau f_0$. Such an impulsive force can be visualized in the form $f(t) = I\delta(t)$, where $\delta(t)$ is the limit of $\delta_\varepsilon(t)$ as $\varepsilon \to 0$ which has the following properties:

$$\delta_\varepsilon(t) = 0 \quad \text{for} \quad t \notin (0,\varepsilon) \quad \text{and} \quad \int_{-\infty}^{\infty} \delta_\varepsilon(t)\,dt = 1. \tag{1.2a,b}$$

In the limit as $\varepsilon \to 0$, $\delta(t)$ is expected to have the corresponding properties:

$$\delta(t) = 0, \quad t \neq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1. \tag{1.3a,b}$$

According to the above definition of an ordinary function, $\delta(t)$ is not an ordinary function, it is referred to as the *generalized function* (or a *distribution*).

Historically, the idea of generalized functions or distributions arose in applied mathematics and mathematical physics in order to use the derivatives of nondifferentiable functions. In continuum mechanics or in electromagnetic theory, all phenomena are governed by partial differential equations where the physical variables such as displacement or electric field are not only continuous functions of space and time, but also have sufficient partial derivatives so that governing equations remain valid. However, there was a remarkable difference between static problems, usually described by elliptic equations, and dynamical problems, normally described by hyperbolic or parabolic equations. In static problems, the physical variables are not only continuous and differentiable, but often infinitely differentiable. On the other hand, the physical variables in the dynamical problems often exhibit discontinuities which are regarded as their essential features. For example, potential functions are infinitely differentiable, and wave functions can exhibit a discontinuity at a front moving into undisturbed region. In particular, an elastic string of finite length plucked its interior point oscillates according to the dynamical law of the form

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)], \tag{1.4}$$

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where $c^2 = (T/\rho)$, T is the tension and ρ is the line density of the string. The displacement function $u(x,t)$ is discontinuous at $x = \pm ct$.

Gustav Kirchhoff (1824–1887), a renowned German engineering scientist, first applied an impulsive electromagnetic force to electric circuit problems so that an instantaneous change in the current in the circuit can be calculated by electric analog of dynamical theory of impulsive motion. In his famous 1882 and 1883 work on the equation of wave propagation governed by the partial differential equation

$$u_{tt} = c^2\left(u_{xx} + u_{yy} + u_{zz}\right), \tag{1.5}$$

where $u = u(x,y,z,t)$, Kirchhoff first obtained the fundamental solution in term of a generalized function δ in the form

$$u(r,t) = \frac{1}{4\pi r}\,\delta(r-ct), \tag{1.6}$$

where $r = \left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{\frac{1}{2}} > 0$ and $\delta(x)$ is defined by

$$\delta(x) = 0, \quad x \neq 0, \quad \text{and} \quad \int_a^b \delta(x)\,dx = 1, \quad a < 0 < b. \tag{1.7a,b}$$

In order to show that such a function exists, he introduced a sequence of functions, $\delta_n(x)$ defined by

$$\delta_n(x) = \frac{n}{\sqrt{\pi}}\exp\left(-n^2x^2\right), \qquad n = 1,2,3,\ldots, \tag{1.8}$$

such that it becomes extremely small for large n, when $x \neq 0$ and

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1. \tag{1.9}$$

Without any formal justification, he regarded $\delta(x)$ is the limit of the sequence $\delta_n(x)$ as $n \to \infty$. Subsequently, many other sequences with similar properties to Kirchhoff's sequence (1.8) have been discovered. We shall discuss more examples of other sequences later on.

In his study of the dynamical problem (1.5), Joseph Liouville (1809–1882) a renowned French mathematician, introduced spherical wave solutions of (1.5) in the form

$$u(r,t) = \frac{1}{4\pi r}\,f(r-ct), \tag{1.10}$$

On the other hand, in static problem, $u(r,t)$ is independent of time t and so, u satisfies the Laplace equation with the fundamental solution

$$u(r) = \frac{1}{4\pi r}. \tag{1.11}$$

In 1827, Augustine Louis Cauchy (1789–1857), a renowned French mathematician, first made an explicit use of the impulsive function as an infinitesimal version of the Cauchy

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distribution in partial differential equations, analysis, and probability theory. At about the same time, another French mathematician, Siméon Denis Poisson (1871–1840) also used the impulsive function in his study of wave propagation. Both, Hermann von Helmholtz (1821–1894), an outstanding German mathematician, and Lord Kelvin (1824–1907), a famous British mathematician, recognized the great importance of the impulsive function and employed it for representing a point source/sink in fluid mechanics.

It was Oliver Heaviside (1850–1925), an eminant British engineering scientist, who was very successful for his informal uses and applications of the impulsive function in both space and time variables for the development of operational calculus and electromagnetic theory. In his study of the response of a mechanical system to a constant force of unit magnitude acting at $t = 0$, Heaviside observed that if this force is represented by a Fourier integral in the form

$$f(t) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin xt}{x}\,dx, \tag{1.12}$$

then the problem is reduced to the response of the dynamical system to a sinusoidal force. The derivative of $f(t)$ represents the unit impulse given by

$$f'(t) = \frac{1}{\pi}\int_0^\infty \cos xt\,dx, \tag{1.13}$$

acting at time $t = 0$. This led Heaviside to obtain the sequence of functions, $\delta_n(t)$ as

$$\delta_n(t) = \frac{1}{\pi}\int_0^n \cos xt\,dx = \frac{\sin nt}{\pi t}, \qquad n = 1,2,3,\ldots \tag{1.14}$$

Thus, the Heaviside sequence (1.14) converges to Kirchhoff's delta function $\delta(t)$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} \delta_n(t) = \lim_{n\to\infty} \frac{\sin nt}{\pi t} = \delta(t). \tag{1.15}$$

Or, equivalently,

$$\delta(t) = \lim_{n\to\infty} \frac{\sin nt}{\pi t} = \lim_{n\to\infty} \frac{1}{2\pi}\int_{-n}^{n} e^{ikt}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikt}\,dk. \tag{1.16}$$

Heaviside also introduced another generalized limit for the delta function given by

$$\delta_n(x) = \lim_{n\to -1} \frac{x^n}{\Gamma(n+1)}, \tag{1.17}$$

where $\Gamma(n+1)$ is the Gamma function (see Debnath (2007)). He also gave a wide variety of results for divergent, alternating series and for generalized derivative of fractional order. His creative imagination led him to introduce the Heaviside unit step function $H(x)$ with a jump discontinuity at $x = 0$ which is defined by

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases} \tag{1.18}$$

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so that its derivative $H'(x)$ is the delta function $\delta(x)$ which is zero if $x > 0$ or $x < 0$ and infinite at $x = 0$. Clearly, $H(x)$ is not an ordinary function, but can be treated as a generalized function with a jump discontinuity at $x = 0$. Obviously, the Heaviside function $H(x-a)$ is given by

$$H(x-a) = \begin{cases} 0, & x < a, \\ 1, & x > a, \end{cases} \tag{1.19}$$

with a jump discontinuity at $x = a$. Thus, the graph of $H(x-a)$ is shown in Figure 1.1.

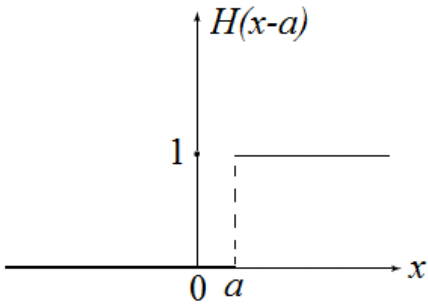


Figure 1.1 The graph of the Heaviside function $H(x-a)$.

The Heaviside function $H(x)$ is very useful in the study of a continuous function $f(x)$ with a jump discontinuity at $x = a$ so that it can be written as

$$f(x) = \begin{cases} f_1(x), & x < a, \\ f_2(x), & x > a. \end{cases} \tag{1.20}$$

This can be written in terms of the Heaviside function as

$$f(x) = f_1(x) H(a-x) + f_2(x) H(x-a). \tag{1.21}$$

This simple idea can be extended for any function $f(x)$ which has several jump discontinuities. For example, if $g(x)$ has jump discontinuities at $x = a$ and $x = b$

$$g(x) = \begin{cases} g_1(x), & 0 < x < a, \\ g_2(x) & a < x < b, \\ 0, & x > b, \end{cases} \tag{1.22}$$

then $g(x)$ can be written in terms of the Heaviside function

$$g(x) = g_1(x) H(x) + [g_2(x) - g_1(x)] H(x-a) - g_2(x) H(x-b). \tag{1.23}$$

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In ignorance of the 1832 work of Liouville and the 1847 work of the outstanding German mathematician, Bernhard Riemann (1826–1866), Heaviside also developed the theory of fractional integrals of order α defined by

$$ {}_aD_x^{-\alpha} f(x) = {}_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{1.24} $$

where ${}_aD_x^{-\alpha} = {}_aI_x^{\alpha}$ is the *Riemann–Liouville integral operator*. When $a = 0$, (1.24) is the Riemann definition of fractional integral and if $a = -\infty$, (1.24) represents the Liouville definition. Integrals (1.24) were also found to arise in the theory of linear ordinary differential equations where they are known as Euler's (1707–1783) transform of the first kind. Obviously, when $\alpha = 1$, (1.24) represents the integral of $f(x)$

$$ {}_aI_x f(x) = \int_a^x f(t)\, dt. \tag{1.25} $$

However, for all positive values of α and β

$$ I^{\alpha} I^{\beta} f(x) = I^{\beta} I^{\alpha} f(x), \qquad \frac{d}{dx}\left[I^{\alpha+1} f(x)\right] = I^{\alpha} f(x). \tag{1.26a,b} $$

Although the fractional integrals (1.24) of order α is valid for all positive values of α, Heaviside was particularly interested in the case when $\alpha = \frac{1}{2}$. Subsequently, Marcel Riesz (1886–1969) generalized the integral operator (1.24) for complex order α with $Re(\alpha) > 0$. This generalization became useful for applications to potential theory, wave propagation problems and electron theory. Riesz also proved the existence theorems for a large variety of linear partial differential equations with constant or variable coefficients.

2 The Dirac delta function and its basic properties

Paul Dirac (1901–1985), a great British mathematical physicists and the 1933 Nobel Prize winner in Physics, fully recognized the deep physical significance of both Kirchhoff's and Heaviside's works, even though their work was not mathematically rigorous. In his mathematical formulation of quantum mechanics in 1926 based on his creative imagination, Dirac introduced the famous Dirac delta function, $\delta(x)$, $x \in \mathbb{R} = (-\infty, \infty)$ defined by

$$ \delta(x) = 0 \quad x \neq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{2.1a,b} $$

The delta function is shown in Figure 2.1.

Indeed, he rediscovered Kirchhoff's delta function and Heaviside's unit impulse function and then made its formal and systematic use in quantum mechanics where the state of an atomic system is described by a complex function ψ of the spatial coordinates of its constituent proton and electrons. The first fundamental quantum-mechanical commutation relation $\mathbf{pq} - \mathbf{qp} = -i\hbar\mathbf{1}$, where $h = 2\pi\hbar$ is the celebrated Planck constant, $\mathbf{q}$, $\mathbf{p}$ and $\mathbf{1}$ are the position, momentum and the identity matrices respectively, due to

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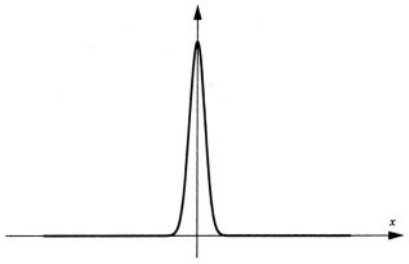


Figure 2.1 The delta function.

Born–Heisenberg–Jordan (1926) was also confirmed independently by Dirac based on the classical Poisson brackets. This commutator relation plays a crucial role in the modern mathematical development of quantum mechanics due to Heisenberg and Dirac. The probabilities that the system may be in eigenstate corresponding to an energy E are derived from an expansion of ψ in the form

$$\psi = \sum_n a_n \psi_n + \int a_p \psi_p dp, \tag{2.2}$$

where ψ_n corresponds to energy levels E_n in the discrete spectrum, and ψ_p refers to energies in the continuous spectrum. The eigenvectors ψ_n of the discrete spectrum may be considered as vectors in a Hilbert space which form an orthonormal system such that

$$(\psi_m, \psi_n) = \delta_{mn} = \begin{cases} 0, & m \neq n, \\ 1, & m = n, \end{cases} \tag{2.3}$$

where the symbol (,) represents the inner (or scalar) product in the Hilbert space, and δ_{mn} is the delta symbol due to Leopold Kronecker (1823–1891), a renowned German mathematician, who made outstanding contributions to different areas of pure mathematics. At a professional meeting in Berlin in 1886, he made a delightful quotation: "God Himself made the integers, everything else is the work of man". This is one of the most famous quotations in all history of mathematics.

The eigenvectors ψ_p of the continuous spectrum satisfy the fundamental relation

$$(\psi_p, \psi_q) = \delta(p-q), \tag{2.4}$$

where $\delta(x)$ is the Dirac delta function which may be thought as a generalization of the Kronecker delta symbol, δ_{mn} in the case of a continuous variable.

In spite of the name delta function and its unusual definition (2.1a,b), it is mathematically *not* an ordinary function. Like the value $f(x)$ of an ordinary function f at x, $\delta(x)$ does not represent the value of δ at x. But it can be visualized as zero everwhere except at the origin ($x = 0$), where it is infinite so that the infinitely tall and thin rectangle under the graph has area one. However, the delta functions can be added and subtracted, but the ordinary product $\delta(x) \cdot \delta(x) = \delta^2(x)$, $\sqrt{\delta(x)}$, $\log \delta(x)$, etc. are *not* defined, and the convolution product $\delta(x) * \delta(x)$ is well defined. The delta function is also the derivative

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of the discontinuous Heaviside function $H(x)$ so that $H'(x) = \delta(x)$, and hence, $\delta(x)$ is considered as a *generalized function*. Dirac recognized that $\delta(x)$ still works like an ordinary function and gives useful and correct results which led him to believe that it must be *true*. Subsequently, the Dirac delta function became an effective and extremely useful tool not only in quantum mechanics, but also in mathematical physics and applied mathematics. Thus, it is necessary to provide a mathematical justification of the Dirac definition of the delta function (2.1a,b) and to generalize the whole concept of ordinary functions and their derivatives which led to the introduction of generlized functions or distributions.

Many physical and engineering problems can usually be solved through approximation of some physically realizable quantity by an idealization – a delta function or one of its derivatives. Impressed by Heaviside's fundamental nonrigorous work and convinced by his own deep knowledge in electrical engineering, Dirac once said: "All electrical engineers are familiar with the idea of a pulse and δ-function is just a way of expressing a pulse mathematically". For example, a short strong current pulse, varying with time t and centered at $t = 0$, is the charge given by the pulse and is equal to the area (integral) of the original pulse. According to Dirac's original idea, integral (2.1b) can be interpreted as the mass or charge distribution at $x = 0$ so that

$$\int_{-\infty}^{\infty} \rho(x)\,dx = \int_{-\infty}^{\infty} m\delta(x)\,dx = m, \tag{2.5}$$

where $\rho(x) = m\delta(x)$ represents the mass or charge density at $x = 0$. The Dirac delta function can also be used to represent an impulse given to a dynamical system, or a pulse generated by a sudden electric impulse in an electrical circuit.

It may be noted that equations (1.2a,b) do not specify the variation of $\delta_\varepsilon(x)$ in $0 < x < \varepsilon$. If $\delta_\varepsilon(x)$ is defined by the step function in the form

$$\delta_\varepsilon(x) = \begin{cases} 0, & x < \varepsilon, \\ \frac{1}{\varepsilon}, & 0 < x < \varepsilon, \\ 1, & x > \varepsilon, \end{cases} \tag{2.6}$$

and if $f(x)$ is any continuous function in an interval containing the origin at $x = 0$, then

$$\int_{-\infty}^{\infty} \delta_\varepsilon(x) f(x)\,dx = \frac{1}{\varepsilon}\int_0^{\varepsilon} f(x)\,dx. \tag{2.7}$$

Using the mean value theorem of integral calculus, the right hand side of (2.7) gives $\frac{1}{\varepsilon}\cdot\varepsilon f(\varepsilon\theta)$, $0 < \theta < 1$ so that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \delta_\varepsilon(x) f(x)\,dx = \int_{-\infty}^{\infty} \delta(x) f(x)\,dx = \lim_{\varepsilon\to 0} f(\varepsilon\theta) = f(0). \tag{2.8}$$

This is consistent with (2.1b) for $f(x) = 1$, and is called the *shifting property* of $\delta(x)$ by Balthasar van der Pol (1889–1959), a well-known Dutch physicist. This property can be interpreted as the *action* of the delta function on the continuous function f that produces

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the number $f(0)$. Thus, the *action* is denoted by the symbol $\delta(f)$ or $\langle \delta, f \rangle$ so that (2.8) can be written as

$$\delta(f) \equiv \langle \delta, f \rangle = \int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0) = \int_{-\infty}^{\infty} \delta(x) f(0) dx. \qquad (2.9)$$

Thus, the symbol $\delta(f)$ or $\langle \delta, f \rangle$ means that δ is a *function of a function* or a *functional*. The basic idea of the functional approach is to study a function f not by looking at its point values, $f(x)$, but rather by looking at its actions $\delta(f) \equiv \langle \delta, f \rangle$. This reveals the fact that, unlike ordinary functions, the domain of the delta function is a *set of functions*, but the range is still a set of points.

Further, (2.9) leads to two results:

$$\delta(x) f(x) = \delta(x) f(0) \quad \text{and} \quad x\delta(x) = 0. \qquad (2.10a,b)$$

Using an argument similar to (2.9) gives more general properties

$$\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a), \qquad (2.11)$$

for any continuous function $f(x)$, and

$$f(x) \delta(x-a) = f(a) \delta(x-a). \qquad (2.12)$$

Thus, the shifting property (2.9) or (2.11) is often used as a definition of the delta function. It also follows from (2.11) that $\delta(x)$ is an even function of its argument, that is

$$\delta(-x) = \delta(x) \quad \text{and} \quad \delta(x-a) = \delta(a-x). \qquad (2.13a,b)$$

Another proof that $\delta(x)$ is the derivative of $H(x)$ follows from the following integral and integration by parts

$$\int_{-\infty}^{\infty} H'(x) f(x) dx = [H(x) f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(x) f'(x) dx,$$

which is, assuming $f(x) \to 0$ as $x \to \infty$ and using (2.9),

$$\int_{-\infty}^{\infty} H'(x) f(x) dx = -\int_{-\infty}^{\infty} f'(x) dx = f(0) = \int_{-\infty}^{\infty} \delta(x) f(x) dx. \qquad (2.14)$$

This shows that $H'(x) = \delta(x)$.

It can be shown that

$$\int_{-\infty}^{\infty} \delta(a-x) \delta(x-b) dx = \delta(a-b). \qquad (2.15)$$

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This follows from the result

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(a)\,da \int_{-\infty}^{\infty} \delta(a-x)\,\delta(x-b)\,dx \\
&\quad = \int_{-\infty}^{\infty} \delta(x-b)\,dx \int_{-\infty}^{\infty} f(a)\,\delta(a-x)\,da \\
&\quad = \int_{-\infty}^{\infty} f(x)\,\delta(x-b)\,dx = f(b) \\
&\quad = \int_{-\infty}^{\infty} f(a)\,\delta(a-b)\,da.
\end{aligned}$$

In the similar way, for $a > 0$, a result

$$\delta\left(x^2-a^2\right) = \frac{1}{2a}\left[\delta(x-a)+\delta(x+a)\right], \tag{2.16}$$

can be proved.

The next question is whether $\delta(x)$ is differentiable, and what is the meaning of $\delta'(x)$? To answer this question, we assume a continuous function f whose first derivative is continuous at $x = 0$. Thus, we have

$$\frac{d}{dx}\left[f(x)\,\delta(x)\right] = f'(x)\,\delta(x)+\delta'(x)\,f(x)$$

so that, for $a < 0 < b$,

$$\int_a^b \frac{d}{dx}\left[f(x)\,\delta(x)\right] = \int_a^b f'(x)\,\delta(x)\,dx + \int_a^b \delta'(x)\,f(x)\,dx. \tag{2.17}$$

Evaluating the left hand side and using (2.1a) yields zero as the value of the left hand side of (2.17) so that

$$\int_a^b f(x)\,\delta'(x)\,dx = -\int_a^b \delta(x)\,f'(x)\,dx = -f'(0) = -\int_a^b f'(0)\,\delta(x)\,dx. \tag{2.18}$$

Or equivalently,

$$f(x)\,\delta'(x) = -f'(0)\,\delta(x). \tag{2.19}$$

In particular, when $f(x) = x$, (2.19) yields

$$x\,\delta'(x) = -\delta(x), \qquad \delta'(x) = -x^{-1}\,\delta(x). \tag{2.20a,b}$$

Similarly, (2.17) can be generalized for the nth derivative of $\delta(x)$ as

$$f(x)\,\delta^{(n)}(x) = (-1)^n\,f^{(n)}(0)\,\delta(x). \tag{2.21}$$

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If $f(x) = x^n$, then

$$x^n \delta^{(n)}(x) = (-1)^n n! \delta(x), \qquad \delta^{(n)}(x) = (-1)^n n! x^{-n} \delta(x), \tag{2.22a,b}$$

$$x^n \delta^{(m)}(x) = \begin{cases} (-1)^n \frac{m!}{(m-n)!} \delta^{(m-n)}(x), & n \le m, \\ 0, & n > m. \end{cases} \tag{2.23}$$

It then follows from (2.10a) and (2.23) that

$$x^n \delta^{(m)}(x) = \sum_{n=0}^{m} (-1)^n \frac{m!}{n!(m-n)!} f^{(n)}(0) \delta^{(m-n)}(x). \tag{2.24}$$

It also follows from the above that (2.20a,b) and (2.22a,b) can be generalized to obtain

$$f(x)\delta'(x-a) = -f'(a)\delta(x), \quad f(x)\delta^{(n)}(x-a) = (-1)^n f^{(n)}(a)\delta(x). \tag{2.25a,b}$$

All these results reveal that $\delta'(x), \delta''(x), \cdots, \delta^{(n)}(x)$ are also generalized functions.

It also follows from (2.22a) that

$$\int_{-\infty}^{\infty} \frac{x^m}{m!} (-1)^n \delta^{(n)}(x)\, dx = \delta_{mn}, \tag{2.26}$$

where δ_{mn} is the Kronecker delta symbol.

This shows that the two sequences

$$\left\{\frac{x^m}{m!}\right\} \text{ and } \left\{(-1)^n \delta^{(n)}(x)\right\} \tag{2.27a,b}$$

form a *biorthogonal system*.

We close this section by considering two geometric delta sequences depending on a particular parameter which also converge to a Dirac delta function. If $V_n(r)$ and $S_n(r)$ represent respectively the volume and surface area sequences of the n-dimensional sphere (or hypersphere) of radius r (see Debnath (2007)). To obtain a formula for $V_n(r)$, we first relate it to the volume, $V_{n-1}(r)$ of the $(n-1)$-dimensional hypersphere. If z denote the coordinate of an axis through the origin of the coordinate system, and if a is a constant less than r $(a < r)$, then the intersection of the hypersurface $z = a$ and the n-dimensional sphere is an $(n-1)$-dimensional hypersphere of radius $\sqrt{r^2 - z^2}$. Thus,

$$V_n(r) = \int_{-r}^{r} V_{n-1}\left(\sqrt{r^2 - z^2}\right) dz. \tag{2.28}$$

We next write some particular elements of $\{V_n(r)\}$ as

$$V_1(r) = \int_{-r}^{r} dz = 2r \qquad \text{when} \qquad n = 1$$

$$V_2(r) = \int_{-r}^{r} 2\left(\sqrt{r^2 - z^2}\right) dz = 4r^2 \int_0^1 \left(1 - t^2\right)^{\frac{1}{2}} dt = \pi r^2, \qquad \text{when} \qquad n = 2$$

$$V_3(r) = \int_{-r}^{r} \pi\left(r^2 - z^2\right) dz = 2\pi r^3 \int_0^1 \left(1 - t^2\right)^{\frac{1}{2}} dt = \frac{4}{3}\pi r^3, \qquad \text{when} \qquad n = 3.$$

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Repeating this process leads to the result $V_n(r) = A_n r^n$, where A_n is defined by

$$A_{n+1} = 2A_n \int_0^1 \left(1-t^2\right)^{n/2} dt. \tag{2.29}$$

The integral in (2.29) can be calculated using the standard result involving the beta and the gamma functions,

$$\int_0^1 \left(1-t^2\right)^{n/2} dt = \frac{1}{2} B\left(\frac{1}{2}, 1+\frac{n}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{n}{2}\right)}, \tag{2.30}$$

where $B(x, y)$ and $\Gamma(x)$ denote the beta and gamma functions respectively. Thus, it turns out that

$$A_n = \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^n}{\Gamma\left(1+\frac{n}{2}\right)} = \frac{\pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)}. \tag{2.31}$$

Consequently, when n is an integer,

$$V_n(r) = A_n r^n = \frac{\pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)} r^n = \frac{2\pi^{n/2}}{n\,\Gamma\left(\frac{n}{2}\right)} r^n. \tag{2.32}$$

This is the exact formula for the volume of the n-dimensional sphere of radius r.

We next consider the volume of a thin spherical shell to obtain the surface area $S_n(r)$ of the n-dimensional hypershere of radius r so that to the first order of thickness S_r, this is given by

$$\begin{aligned} S_n(r)\,\delta r &= V_n(r) - V_n(r-\delta r) = \frac{\pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)}\left[r^n - (r-\delta r)^n\right] \\ &= \frac{\pi^{n/2} r^n}{\Gamma\left(1+\frac{n}{2}\right)}\left(\frac{n\,\delta r}{r}\right) + O(\delta r)^2. \end{aligned} \tag{2.33}$$

Consequently, the surface area of the hypersphere is given by

$$S_n(r) = \frac{\pi^{n/2} n\, r^{n-1}}{\Gamma\left(1+\frac{n}{2}\right)} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \cdot r^{n-1} = S_n(1)\, r^{n-1}, \tag{2.34}$$

where

$$S_n(1) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \tag{2.35}$$

is the surface area of the n-dimensional hypersphere of radius $r = 1$.

Clearly, it follows from (2.32) and (2.34) that

$$S_n(r) = \frac{d}{dr} V_n(r). \tag{2.36}$$

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When we deal with a function $f(\mathbf{x})$ which depends on r only (say $\phi(r)$), then

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,d\mathbf{x} = \int_0^\infty \phi(r)\,\frac{dV_n}{dr}\,dr = \int_0^\infty \phi(r)\,S_n(r)\,dr = S_n(1)\int_0^\infty \phi(r)\,r^{n-1}dr. \tag{2.37}$$

If $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ is a nonnegative locally integrable function satisfying the condition

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,d\mathbf{x} = 1, \tag{2.38}$$

and if $f(\mathbf{x}) = f_\varepsilon(\mathbf{x}) = \varepsilon^{-n} f\left(\frac{\mathbf{x}}{\varepsilon}\right)$, $\varepsilon > 0$, then

$$\lim_{\varepsilon\to 0} f_\varepsilon(\mathbf{x}) = \delta(\mathbf{x}). \tag{2.39}$$

In particular, when $f(\mathbf{x})$ is a function of $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, that is, $f(\mathbf{x}) = \phi(r)$, then (2.38) becomes

$$\int_0^\infty r^{n-1}\phi(r)\,dr = \frac{1}{S_n(1)}. \tag{2.40}$$

In particular, when $n = 2$, then

$$\int_0^\infty r\phi(r)\,dr = \frac{1}{S_2(1)} = \frac{1}{2\pi} \tag{2.41}$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\phi\left(\frac{r}{\varepsilon}\right) = \delta(x_1, x_2), \tag{2.42}$$

where $\phi(r) = \frac{1}{2\pi}\frac{1}{(r^2+1)^{\frac{1}{2}}}$ so that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\phi\left(\frac{r}{\varepsilon}\right) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{2\pi\left(r^2+\varepsilon^2\right)^{3/2}} = \delta(x_1, x_2). \tag{2.43}$$

Similarly, when $n = 3$,

$$\int_0^\infty r^2\phi(r)\,dr = \frac{1}{S_3(1)} = \frac{1}{4\pi}, \tag{2.44}$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^3}\phi\left(\frac{r}{\varepsilon}\right) = \delta(x_1, x_2, x_3), \tag{2.45}$$

where $\phi(r) = \frac{1}{\pi^2}\left(r^2+1\right)^2$ so that

$$\lim_{\varepsilon\to 0} \frac{1}{\pi^3\varepsilon^3}\frac{\varepsilon^4}{\left(r^3+\varepsilon^2\right)^2} = \lim_{\varepsilon\to 0} \frac{\varepsilon}{\pi^2\left(r^2+\varepsilon^3\right)^2} = \delta(x_1, x_2, x_3). \tag{2.46}$$

It also follows from the above analysis that

$$\lim_{\varepsilon\to 0} \phi_\varepsilon(r) = \lim_{\varepsilon\to 0} \left(\frac{1}{\sqrt{\pi}\,\varepsilon}\right)^n \exp\left(-\frac{r^2}{\varepsilon^2}\right) = \delta(r) = \delta(x_1, x_2, \ldots, x_n). \tag{2.47}$$

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3 Different modern approaches to generalized functions or distributions

With the development of the theory of generalized functions, it became clear that the class of generalized functions or distributions is quite large and distributions have many interesting algebraic and analytic properties and applications to applied mathematics and mathematical physics.

There are several major approaches to generalized functions or distributions which readily reproduce the essential features of the Dirac delta function. These approaches include: 3.1. The Dirac delta function as a weak limit of a sequence of ordinary functions, 3.2. The Schwartz theory of distributions based on test functions, 3.3. The Temple–Lighthill theory of generalized functions based on good functions, 3.4. Mikusinski's algebraic approach to generalized functions, 3.5. Cauchy's representation of distributions by analytic functions of one variable, and 3.6. Sato's generalized functions or *hyperfunctions* based on complex functions.

Dirac recognized that the formal use of his delta function simplified many calculations to produce correct results, even though the rules of calculations do not follow from the methods of classical analysis. In order to provide rigorous justifications of the formal use of the delta function and its derivatives, it has been necessary to generalize the whole concept of ordinary functions which led to the introduction of generalized functions in the works of a famous German born American mathematician, Solomon Bochner (1899–1982) in 1932 and a renowned Soviet mathematician, Sergei Llowitsch Sobolev (1908–1989) in 1936. Bochner also made some significant contributions to harmonic analysis, differential geometry, probability theory, and other areas of pure and applied mathematics. Almost simultaneously, Sobolev introduced generalized functions in his outstanding work in mathematical analysis and modern partial differential equations.

About twenty years after Dirac's discovery of delta function, Laurent Schwartz (1915–2002), a celebrated French mathematician and the 1950 Fields Prize Winner of Mathematics, began publishing a series of new and original papers on generalized functions or distributions from 1945 which led subsequently to a full development of his two-volume treatise on *Théorie des Distributions* in 1950–1951. Making use of functional analysis, his brilliant work was not only provided a rigorous justification of the Dirac delta function and its derivatives based on a continuous linear functional on the space of test functions with compact support, but also nicely developed the new study of topological space of distributions. This also led to a topological definition of derivatives and a proof that any distribution in the Euclidean space corresponds to the generalized derivatives of a continuous function in a given bounded domain.

Furthermore, Schwartz discovered a beautiful theory of Fourier series and Fourier transforms using the Schwartz space of tempered distributions. All these led to new applications of distributions to Fourier analysis and partial differential equations. It may be appropriate to quote his famous quotation: "The theory of Fourier Series and Integrals has always had major difficulties and necessitated a large mathematical apparatus in dealing with questions of convergence. It engineered the development of methods of summation, although these did not lead to a completely satisfactory so-

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lution of the problem … For the Fourier transform, the introduction of distribution (hence the space S) is inevitable either in an explicit or hidden form … As a result one may obtain all that is desired from the point of view of the continuity and inversion of the Fourier transform." His discovery of a beautiful and systematic theory of distributions, among other things, provided a new and modern approach to the problems of Fourier analysis, and partial differential equations and to many topics including fundamental solutions, weak solutions, and Shannon's sampling theorem. According to George Temple (1901–1992), a celebrated British mathematician; "In analysis invention of distributions, or generalized functions by Laurent Schwartz has enormously increased the range of the subject and has greatly extended the domain of soluble differential equation."

Stimulated by the Schwartz theory of distributions and its profound usefulness, a famous Polish mathematician, Jan Mikusinski (1913–1987) formulated a less abstract but rigorous algebraic approach to generalized functions during 1949–1960. During the mid 1950's, another simple but rigorous analytic approach to generalized functions was developed by George Temple and then further simplified by another renowned British applied mathematician, Sir James Lighthill (1924–1998) based on the idea of a class of good functions. In 1959, a famous Japanese mathematician, Mikio Sato (1928–) used functions of a complex variable to develop the theory of generalized functions or *hyperfunctions*. His new and original approach was less abstract and computationally more effective than the Schwartz theory of distributions. So, hyperfunctions seem to be a useful in analysis as distributions. Finally, Cauchy's representation of distributions was another approach based on functions of a complex variable and test functions.

3.1 The Dirac delta function as a weak limit of a sequence of ordinary functions

As stated earlier, the Dirac delta function $\delta(x)$ is not a function in the ordinary sense of analysis. However, its fundamental property (2.9) can be rigorously developed as the *limit* of a sequence of ordinary functions. So, $\delta(x)$ can be defined as the weak limit of a sequence $\{\delta_n(x)\}$ of ordinary functions, that is,

$$\lim_{n\to\infty} \delta_n(x) = \delta(x) \tag{3.1}$$

if the sequence of integrals, $\langle \delta_n(x), f(x)\rangle$ has the limit $\langle \delta(x), f(x)\rangle$ as $n \to \infty$ for any continuous (or well behaved) function $f(x)$, that is,

$$\lim_{n\to\infty} \langle \delta_n(x), f(x)\rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \langle \delta(x), f(x)\rangle = f(0). \tag{3.2}$$

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There are many different delta sequences, $\{\delta_n(x)\}$ of ordinary functions. For example,

$$\delta_n(x) = \begin{cases} n, & |x| < \frac{1}{2n}, \\ 0, & |x| > \frac{1}{2n}, \end{cases} \qquad \text{(Dirac)} \tag{3.3}$$

$$\delta_n(x) = \sqrt{\frac{n}{\pi}} \exp\left(-nx^2\right), \qquad \text{(Kirchhoff)} \tag{3.4}$$

$$\delta_n(x) = \left(\frac{n}{\pi}\right)\left(1+n^2x^2\right)^{-1}, \qquad \text{(Cauchy)} \tag{3.5}$$

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} e^{ikx}\,dk. \qquad \text{(Dirichlet)} \tag{3.6}$$

The graph of each $\delta_n(x)$ defined by (3.3)–(3.6) are given in Figure 3.1(a)–(d).

To verify (3.3), we consider

$$\lim_{n\to\infty}\langle \delta_n(x), f(x)\rangle = \lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)\,f(x)\,dx = \lim_{n\to\infty} n\int_{-\frac{1}{2n}}^{\frac{1}{2n}} f(x)\,dx,$$

which is, by the mean value theorem of integral calculus,

$$= \lim_{n\to\infty} n\left(\frac{1}{2n}+\frac{1}{2n}\right) f\left(\frac{2\theta-1}{2n}\right), \qquad -1 < 2\theta - 1 < 1$$

$$= \lim_{n\to\infty} f\left(\frac{2\theta-1}{2n}\right) = f(0) = \langle \delta(x), f(x)\rangle.$$

This shows that $\lim_{n\to\infty}\delta(x) = \delta(x)$.

It follows from (3.4) that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)\,dx = \lim_{n\to\infty}\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-nx^2}dx$$

$$= \lim_{n\to\infty}\sqrt{\frac{n}{\pi}}\cdot\sqrt{\frac{\pi}{n}} = 1 = \int_{-\infty}^{\infty}\delta(x)\,dx.$$

This shows that the delta function $\delta(x)$ is the limit of the Gaussian sequence $\delta_n(x)$ given by (3.4), that is,

$$\lim_{n\to\infty}\sqrt{\frac{n}{\pi}}\,e^{-nx^2} = \delta(x). \tag{3.7}$$

The Cauchy delta sequence defined by (3.5) satisfies the following properties:

$$\lim_{n\to\infty}\delta_n(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0, \end{cases} \tag{3.8}$$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)\,dx = \lim_{n\to\infty}\left(\frac{n}{\pi}\right)\int_{-\infty}^{\infty}\frac{dx}{1+n^2x^2} = \lim_{n\to\infty}\frac{1}{\pi}\left[\tan^{-1}nx\right]_{-\infty}^{\infty}$$

$$= \frac{1}{\pi}\cdot\pi = 1 = \int_{-\infty}^{\infty}\delta(x)\,dx.$$

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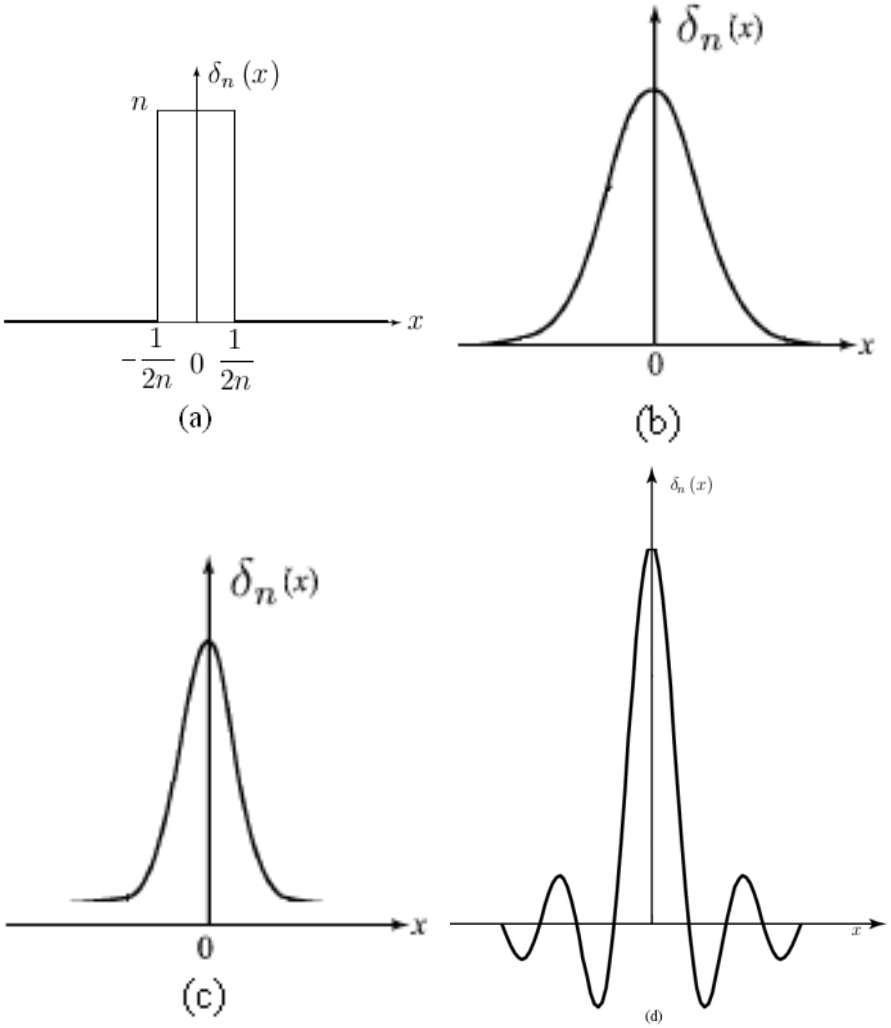


Figure 3.1 (a) Rectangular (or Dirac) delta sequence, (b) The Kirchhoff or Gaussian δ-sequence, (c) Cauchy's δ-sequence, (d) Dirichlet's δ-sequence.

This means that

$$\lim_{n\to\infty} \delta_n(x) = \delta(x). \tag{3.9}$$

We may view the derivative of $\delta(x)$ as the limit of the derivative of $\delta_n'(x)$ so that

$$\delta_n'(x) = \lim_{n\to\infty} \delta_n'(x) = \lim_{n\to\infty} -\frac{n}{\pi}\frac{2n^2x}{(1+n^2x^2)^2} = \lim_{n\to\infty} \frac{n}{\pi}\frac{d}{dx}\frac{1}{(1+n^2x^2)}. \tag{3.10}$$

The graphs of these rational functions become more and more concentrated spiked *doublets* as shown in Figure 3.2.

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Figure 3.2 The derivative of the delta function as the limit of doublets.

We next determine the effect of $\delta'(x)$ on any continuous (or well behaved) function $f(x)$ as follows:

$$\begin{aligned}\left\langle \delta'(x), f(x)\right\rangle &= \int_{-\infty}^{\infty} \delta'(x) f(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n'(x) f(x)\, dx,\\ &= \lim_{n\to\infty} \frac{n}{\pi} \int_{-\infty}^{\infty} \left[\frac{d}{dx}\frac{1}{(1+n^2x^2)}\right] f(x)\, dx\\ &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n'(x) f(x)\, dx,\end{aligned}$$

which is, integrating by parts and noting the first term vanishes as $|x| \to \infty$,

$$= -\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx = -\int_{-\infty}^{\infty} \delta(x) f'(x)\, dx = -f'(0), \qquad (3.11)$$

so that

$$\left\langle \delta'(x), f(x)\right\rangle = -\left\langle \delta(x), f'(x)\right\rangle = (-1) f'(0), \qquad (3.12)$$

$$\left\langle \delta''(x), f(x)\right\rangle = (-1)^2 \left\langle \delta(x), f''(x)\right\rangle = (-1)^2 f''(0), \qquad (3.13)$$

$$\left\langle \delta^{(n)}(x), f(x)\right\rangle = (-1)^n \left\langle \delta(x), f^{(n)}(x)\right\rangle = (-1)^n f^{(n)}(0). \qquad (3.14)$$

Thus, all derivatives of $\delta(x)$ exist and are generalized functions.

Finally, we show that $\delta_n(x)$ defined by (3.6) converges to $\delta(x)$ as $n \to \infty$. First, it follows from Figure 3.1(d) that fixed n, $\delta_n(x)$ becomes small as $|x| \to \infty$.

Using the formula

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$$

and changing x to nx gives

$$\int_{-\infty}^{\infty} \frac{\sin nx}{\pi x}\, dx = \int_{-\infty}^{\infty} \delta_n(x)\, dx = 1 = \int_{-\infty}^{\infty} \delta(x)\, dx. \qquad (3.15)$$

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This shows that $\delta_n(x) \to \delta(x)$ as $n \to \infty$. We next examine the shifting property and the well-known Dirichlet integral formula

$$\lim_{n\to\infty} \langle \delta_n(x), f(x) \rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{\sin nx}{\pi x} f(x)\, dx = f(0) = \langle \delta(x), f(x) \rangle .$$

We note that for any $0 < a < b$,

$$\int_a^b \delta_n(x)\, dx = \frac{1}{\pi} \int_a^b \frac{\sin nx}{x}\, dx = \frac{1}{\pi} \int_{na}^{nb} \frac{\sin t}{t}\, dt \to 0 \quad \text{as} \quad n \to \infty. \tag{3.16}$$

Similarly, for $a < b < 0$.

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{an}^{bn} \frac{\sin t}{t}\, dt = 0. \tag{3.17}$$

Thus, for $f'(x)$ exists, continuous and bounded,

$$\lim_{n\to\infty} \langle \delta_n(x), f(x) \rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx$$

which is, using (3.15) and the mean value theorem of integral calculus,

$$= f(0) \left[\lim_{n\to\infty} \int_{-\varepsilon}^{\varepsilon} \frac{\sin nx}{\pi x} dx \right] = f(0) \left[\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{\sin nx}{\pi x}\, dx \right] = f(0).$$

It also follows from (3.15) that

$$\delta(x) = \lim_{n\to\infty} \delta_n(x) = \lim_{n\to\infty} \frac{\sin nx}{\pi x} = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{n} e^{ikx} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx. \tag{3.18}$$

This is the celebrated integral representation (1.16) of the delta function which was widely used in quantum mechanics.

It is noted that the limit of $\delta_n(x)$ as $n \to \infty$ does not exist when $x = 0$ which can be verified by all four examples (3.3)–(3.6). In fact, the limit of $\delta_n(x)$ in each of the above examples diverges at $x = 0$. Evidently, all $\delta_n(x)$ defined by (3.3)–(3.6) may be interpreted as a sequence of normalized functions because

$$\int_{-\infty}^{\infty} \delta_n(x)\, dx = 1. \tag{3.19}$$

We observe that the above delta sequences (3.3)–(3.6) have varying degrees of usefulness for most physical problems and they can easily be used to derive the fundamental integral property (3.2). The Gaussian delta sequence (3.4) is easy to differentiate. The delta sequence (3.6) is particularly useful in Fourier analysis and quantum mechanics.

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In addition to Heaviside's delta sequence (1.17), we include some more examples:

$$\delta_n(x) = \left\{ \begin{array}{ll} \frac{1}{n} x^{\frac{1}{n}-1}, & x \geq 0 \\ 0, & x < 0 \end{array} \right\}, \tag{3.20}$$

$$\delta_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(ikx - \frac{|k|}{n} \right) dk, \tag{3.21}$$

$$\delta_n(x) = \frac{1}{2} n \operatorname{sech}^2 nx, \tag{3.22}$$

$$\delta_n(x) = \frac{1}{2} \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\frac{x}{2}} \qquad \text{(Dirichlet kernel)}, \tag{3.23}$$

$$\delta_n(x) = \frac{1}{2n\pi} \left[\frac{\sin\left(\frac{nx}{2}\right)}{\sin\frac{x}{2}} \right]^2 \qquad \text{(Fejer kernel)}. \tag{3.24}$$

The Wigner semi-circle delta distribution is given by

$$\delta_\varepsilon(x) = \left(\frac{2}{\pi\varepsilon^2} \right) \left(\sqrt{\varepsilon^2 - x^2} \right) H(\varepsilon - |x|), \qquad \lim_{\varepsilon \to 0} \delta_\varepsilon(x) = \delta(x). \tag{3.25a,b}$$

The Dirichlet kernel $D_n(x) \equiv \delta_n(x)$ given by (3.23) arose in the theory of Fourier series. On the other hand, the Fejer kernel, $F_n(x) \equiv \delta_n(x)$ given by (3.24) is naturally associated Fejer's method of summability of series. The upshot is that all delta sequences (3.20)–(3.25a,b) represent the delta function in the sense of (3.1).

3.2 The Schwartz approach to distributions based on test functions

Schwartz first discovered a distribution as a continuous linear functionals $F(\phi) \equiv \langle F, \phi \rangle$ on the space $\mathcal{D}(\mathbb{R})$ of test functions $\phi(x)$ which are infinitely differentiable functions with compact support in $\mathbb{R}$. More explicitly, a linear functional $F : \mathcal{D} \to \mathbb{C}$ is called a *distribution* if the following two conditions hold:

$$(i) \qquad F(a\phi + b\psi) = aF(\phi) + bF(\psi), \tag{3.26}$$

for any two constants a, b and ϕ, $\psi \in \mathcal{D}(\mathbb{R})$,

$$(ii) \qquad \lim_{n \to \infty} F(\phi_n) = \lim_{n \to \infty} \langle F, \phi_n \rangle = \langle F, \phi \rangle, \tag{3.27}$$

whenever $\phi_n \to \phi$ as $n \to \infty$ in $\mathcal{D}(\mathbb{R})$. Thus, $\langle F, \phi \rangle$ is a bilinear form.

Thus, the domain $\mathcal{D}$ of a distribution is a *set of test functions* in contrast to a set of points as domain of an ordinary function. Clearly, an ordinary function on $\mathbb{R}$ is *not* a distribution because its domain is *not* a set $\mathcal{D}$ of test functions. However, any locally integrable function f on $\mathbb{R}$ can be identified with a distribution $F \equiv f$ in the following way:

$$\langle F, \phi \rangle = \langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x) \phi(x)\, dx = \int_a^b f(x) \phi(x)\, dx, \tag{3.28}$$

where $[a, b]$ is the support of $\phi(x)$.

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Thus, any locally integrable function f generates a *regular distribution*, denoted by f such that

$$f(\phi) = \langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx = \int_{a}^{b} f(x)\,\phi(x)\,dx, \tag{3.29}$$

for every test function $\phi \in \mathcal{D}$.

Several mathematicians including Halperin (1952), König (1953) and Sauer (1958) developed the theory of generalized derivatives of locally integrable functions. Their work was found to be very useful for the study of boundary value problems for partial differential equations than that of linear integral transforms.

A distribution which is not regular is called a *singular distribution*.

One of the most common examples of regular distribution is the Heaviside distribution H which is a piecewise continuous function in $\mathbb{R}$ such that

$$H(\phi) = \langle H, \phi \rangle = \int_{-\infty}^{\infty} H(x)\,\phi(x)\,dx = \int_{0}^{\infty} \phi(x)\,dx, \tag{3.30}$$

since $H(x) = 1$, $x > 0$ and $H(x) = 0$, $x < 0$.

On the other hand, the Dirac delta function is one of the most common and important examples of singular distribution because $\delta(x)$ is *not* a locally integrable function and is defined by

$$\delta(\phi) = \langle \delta, \phi \rangle = \int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx = \phi(0). \tag{3.31a}$$

More generally,

$$\langle \delta(x-a), \phi \rangle = \int_{-\infty}^{\infty} \delta(x-a)\,\phi(x)\,dx = \phi(a). \tag{3.31b}$$

In the definition of a regular distribution (3.29), there is *no* problem with integrability of the product $f\phi$ because it vanishes outside a bounded set. Thus, the integral (3.29) can be interpreted, for some test function ϕ, as the average value of f. This interpretation of distributions is very natural from a physical point of view because when a quantity is measured, the result is *not* the exact value at a single point.

It is possible to extend the definition of derivatives of a function to derivatives of a distribution so that every distribution possesses derivatives of all orders which are also distributions. To explain this idea, we consider the derivatives F' of a continuously differentiable function F on $\mathbb{R}$ using integration by parts in classical analysis so that

$$F'(\phi) = \langle F', \phi \rangle = \int_{\mathbb{R}} F'(x)\,\phi(x)\,dx = [F(x)\,\phi(x)]_{-\infty}^{\infty} - \int_{\mathbb{R}} F(x)\,\phi'(x)\,dx,$$

where ϕ is a test function $\in C^{\infty}(\mathbb{R})$ and $\phi(x) \to 0$ as $|x| \to \infty$.

Consequently, the derivative of F exists and is given by the formula

$$\langle F', \phi \rangle = \langle F, -\phi' \rangle = -\langle F, \phi' \rangle. \tag{3.32}$$

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This formula is valid not only for a differentiable function F, but also for an arbitrary distribution. Thus, (3.32) can be used as the simple definition of the derivatives of the distribution F which is obtained from integration by parts in classical analysis. Moreover, (3.32) remains valid for derivatives of all orders of a distribution F so that

$$\left\langle F^{(n)}, \phi \right\rangle = (-1)^n \left\langle F, \phi^{(n)} \right\rangle. \tag{3.33}$$

Similarly, it is possible to define the partial derivatives of all orders of a distribution on $\mathbb{R}^n$. The existence of derivatives or partial derivatives of all orders of a distribution is one of the striking features of distributions so that they can be used to study differential equations.

It is known that Heaviside function $H(x)$ is not differentiable in the ordinary sense, but the distributional derivative of H exists and is equal to $\delta(x)$. It follows from (3.32) that

$$\begin{aligned} \left\langle H', \phi \right\rangle = -\left\langle H, \phi' \right\rangle &= -\int_{-\infty}^{\infty} H(x)\,\phi'(x)\,dx \\ &= -\int_{0}^{\infty} \phi'(x)\,dx = \phi(0) = \langle \delta, \phi \rangle, \end{aligned} \tag{3.34}$$

so that $H'(x) = \delta(x)$.

Similarly, the Dirac delta function has derivatives of all orders and they are all generalized functions. Thus, it follows from (3.32) that

$$\left\langle \delta', \phi \right\rangle = -\left\langle \delta, \phi' \right\rangle = -\int_{-\infty}^{\infty} \delta(x)\,\phi'(x)\,dx = -\phi'(0). \tag{3.35}$$

This represents the negative of the *dipole distribution*.

Similarly, we can derive

$$\left\langle \delta^{(n)}, \phi \right\rangle = (-1)^n \left\langle \delta, \phi^{(n)} \right\rangle = (-1)^n \phi^{(n)}(0). \tag{3.36}$$

Thus, all derivatives of $\delta(x)$ exist and are distributions.

In addition, many other basic operations in analysis can be extended from ordinary functions to generalized functions. For example, if f is an ordinary function, then the translation property is

$$\begin{aligned} \langle f(x-a), \phi(x) \rangle &= \int_{-\infty}^{\infty} f(x-a)\,\phi(x)\,dx \\ &= \int_{-\infty}^{\infty} f(x)\,\phi(x+a)\,dx = \langle f(x), \phi(x+a) \rangle. \end{aligned} \tag{3.37}$$

This result can be used to define the translation property for distribution F as follows:

$$\langle F(x-a), \phi(x) \rangle = \langle F(x), \phi(x+a) \rangle. \tag{3.38}$$

In particular, if $F = \delta$, then

$$\langle \delta(x-a), \phi(x) \rangle = \langle \delta(x), \phi(x+a) \rangle = \int_{-\infty}^{\infty} \delta(x)\,\phi(x+a)\,dx = \phi(a). \tag{3.39}$$

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This is consistent with result (2.11) and is often use to define the Dirac delta function.

Another important property of a distribution F is called the *scaling property* given by

$$\langle F(ax), \phi(x)\rangle = \frac{1}{|a|}\left\langle F(x), \phi\left(\frac{x}{a}\right)\right\rangle, \qquad a \neq 0. \tag{3.40}$$

This follows from that, for $a > 0$,

$$\begin{aligned}\langle F(ax), \phi(x)\rangle &= \int_{-\infty}^{\infty} F(ax)\,\phi(x)\,dx \\ &= \frac{1}{a}\int_{-\infty}^{\infty} F(y)\,\phi\left(\frac{y}{a}\right)dy = \frac{1}{a}\left\langle F(x), \phi\left(\frac{x}{a}\right)\right\rangle.\end{aligned} \tag{3.41}$$

Similarly, for $a < 0$, we obtain

$$\langle F(ax), \phi(x)\rangle = -\frac{1}{a}\left\langle F(x), \phi\left(\frac{x}{a}\right)\right\rangle. \tag{3.42}$$

Combining (3.41) and (3.42) gives (3.40).

In particular, if $F = \delta$, then (3.42) leads to the famous result

$$\delta(ax) = \frac{1}{|a|}\delta(x), \qquad a \neq 0. \tag{3.43}$$

We next consider the product fF of an infinitely differentiable function f and a distribution F so that

$$\langle fF, \phi\rangle = \int_{-\infty}^{\infty} f(x)\,F(x)\,\phi(x)\,dx = \int_{-\infty}^{\infty} F(x)\,(f\phi)(x)\,dx = \langle F, f\phi\rangle, \tag{3.44}$$

where $(f\phi)(x)$ is a test function on $\mathbb{R}$.

In particular, if $f(x) = x$ and $F = \delta$, then it follows from (3.44) that

$$x\,\delta(x) = 0, \qquad x\,\delta'(x) = -\delta(x). \tag{3.45a,b}$$

We next prove term wise differentiation formula for distributions. More explicitly, if $\{F_n\}$ is a sequence of distributions and F is a distribution so that $F_n \to F$ as $n \to \infty$, then

$$\lim_{n\to\infty} F_n' = F' \quad \text{as} \quad n \to \infty. \tag{3.46}$$

Proof. It follows from (3.32) that

$$\left\langle F_n', \phi\right\rangle = -\left\langle F_n, \phi'\right\rangle \to -\left\langle F, \phi'\right\rangle = \left\langle F', \phi\right\rangle \quad \text{as} \quad n \to \infty.$$

This proves the result (3.46).

This shows a striking difference between (3.46) and the corresponding result for ordinary functions. In case of ordinary functions f_n and f, result (3.46) is also valid provided both $\{f_n\}$ and $\{f_n'\}$ are uniformly convergent as $n \to \infty$. Thus, the stringent restriction of uniform convergent is *not* required for the case of sequence of distributions.

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Any convergent sequence of distributions can be differentiated term by term without any restriction on the differentiated sequence.

As stated before, Schwartz's discovery of distributions became extremely useful for the study of Fourier series and Fourier transforms. He also introduced the idea of the test function space of slow growth, called the *Schwartz space* $S(\mathbb{R})$, and defined a distribution of slow growth (also called a *tempered distribution*) as a continuous linear functional over $S(\mathbb{R})$. He also defined the *Fourier transforms* $(\mathcal{F}f)(k)$ of a tempered distribution $f(x)$ is defined by the Parseval formula

$$\langle \mathcal{F}f, \phi \rangle = \langle f, \mathcal{F}\phi \rangle, \qquad \phi \in S, \tag{3.47}$$

where the functional on the right hand side of (3.47) is well defined because $(\mathcal{F}\phi)(k) \in S$. Evidently, the Fourier transform of a tempered distribution f is a continuous linear functional in $S'(\mathbb{R})$, and is a tempered distribution. It assigns the same number to $\phi \in S(\mathbb{R})$ as f assigns to $(\mathcal{F}\phi)(k)$. We can then define the *inverse Fourier transform*

$$\left\langle \mathcal{F}^{-1}f, \phi \right\rangle = \left\langle f, \mathcal{F}^{-1}\phi \right\rangle, \qquad \phi \in S, \tag{3.48}$$

for all $f \in S'(\mathbb{R})$. Evidently, $\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = I$, and hence $\mathcal{F} : S'(\mathbb{R}) \to S'(\mathbb{R})$.

It is essential that definition (3.47) is consistent with the classical definition of the Fourier transform of a function $f(x)$:

$$\begin{aligned}
\langle \mathcal{F}f, \phi \rangle &= \langle f, \mathcal{F}\phi \rangle \\
&= \int_{-\infty}^{\infty} f(x)\,dx \int_{-\infty}^{\infty} e^{-ikx}\phi(k)\,dk, \\
&= \int_{-\infty}^{\infty} \phi(k)\,dk \int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx, \\
&= \int_{-\infty}^{\infty} \left\langle f(x), e^{-ikx} \right\rangle \phi(k)\,dk, \\
&= \int_{-\infty}^{\infty} (\mathcal{F}f(x))(k)\,\phi(k)\,dk,
\end{aligned} \tag{3.49}$$

where

$$(\mathcal{F}f(x))(k) = \left\langle f(x), e^{-ikx} \right\rangle = \int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx, \tag{3.50}$$

is the classical Fourier transform of $f(x)$.

For example, the Fourier transform of the delta function δ is given by

$$\begin{aligned}
\langle \mathcal{F}\delta, \phi \rangle &= \langle \delta, \mathcal{F}\phi \rangle \\
&= \left\langle \delta(x), \int_{-\infty}^{\infty} e^{-i\xi x}\phi(\xi)\,d\xi \right\rangle \\
&= \int_{-\infty}^{\infty} \phi(\xi)\,d\xi = \langle 1, \phi \rangle.
\end{aligned} \tag{3.51}$$

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Thus,

$$\mathcal{F}\{\delta(x)\} = 1. \tag{3.52}$$

Similarly, the inverse Fourier transform of δ is

$$\begin{aligned}\left\langle \mathcal{F}^{-1}\delta, \phi \right\rangle &= \left\langle \delta, \mathcal{F}^{-1}\phi \right\rangle = \left\langle \delta, \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)\, e^{itx}\, dt \right\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)\, dt = \left\langle \frac{1}{2\pi}, \phi \right\rangle \end{aligned} \tag{3.53}$$

so that

$$\mathcal{F}^{-1}\delta = \frac{1}{2\pi}. \tag{3.54}$$

Consequently,

$$\mathcal{F}\left\{\frac{1}{2\pi}\right\} = \delta, \qquad \mathcal{F}\{1\} = 2\pi\delta. \tag{3.55a,b}$$

It can be verified that

$$\left(\mathcal{F}\delta'(x)\right)(k) = ik, \qquad \left(\mathcal{F}\delta^{(n)}(x)\right)(k) = (ik)^n. \tag{3.56a,b}$$

Another fundamental idea is the *convolution of two distributions* $f, g \in D'(\mathbb{R})$ which can be extended from the convolution of two functions f and g defined on $\mathbb{R}$ by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy \tag{3.57}$$

provided the integral exists.

It is possible to extend the definition (3.57) to distributions so that (3.57) remains valid for the convolution of two regular distributions. The idea of the *direct* (or *tensor*) *product* is required for the development of the convolution of distributions. If f and g are two distributions in $\mathcal{D}'(\mathbb{R})$, then *tensor* (or *direct*) product $f \times g$ is defined as the distribution in $\mathcal{D}'\left(\mathbb{R}^2\right)$ by

$$\langle f \times g, \phi(x,y) \rangle = \langle f(x), \langle g(y), \phi(x,y) \rangle \rangle \tag{3.58}$$

for all the test functions $\phi(x,y) \in \mathcal{D}\left(\mathbb{R}^2\right)$.

We now show that

$$\delta(x) \times \delta(y) = \delta(x,y). \tag{3.59}$$

It follows from (3.58) that

$$\begin{aligned}\langle \delta(x) \times \delta(y), \phi(x,y) \rangle &= \langle \delta(x), \langle \delta(y), \phi(x,y) \rangle \rangle \\ &= \langle \delta(x), \phi(x,0) \rangle = \phi(0,0) = \langle \delta(x,y), \phi(x,y) \rangle .\end{aligned}$$

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This proves result (3.59).

In particular, it follows from (3.58) that

$$\begin{aligned}\langle f(x)\times 1(y),\phi(x,y)\rangle &= \left\langle f(x),\int_{-\infty}^{\infty}\phi(x,y)\,dy\right\rangle\\ &= \int_{-\infty}^{\infty}\langle f(x),\phi(x,y)\rangle\,dy\\ &= \langle 1(y)\times f(x),\phi(x,y)\rangle.\end{aligned}$$

In other words,

$$\left\langle f(x),\int_{-\infty}^{\infty}\phi(x,y)\,dy\right\rangle = \int_{-\infty}^{\infty}\langle f(x),\phi(x,y)\rangle\,dy.$$

It can be shown that the direct product of two distributions is commutative. If f and g are two locally integrable functions and f and g are the corresponding regular distributions, then

$$\begin{aligned}\langle f\times g,\phi(x,y)\rangle &= \langle f(x),\langle g(y),\phi(x,y)\rangle\rangle\\ &= \left\langle f(x),\int_{-\infty}^{\infty}g(y)\phi(x,y)\,dy\right\rangle\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\,f(x)\,g(y)\,dy\,dx\end{aligned}$$

which is, due to the fact that the support of $\phi(x,y)\,f(x)\,g(y)$ is bounded, and hence, the order of integration can be interchanged,

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\,f(x)\,g(y)\,dx\,dy = \langle g\times f,\phi(x,y)\rangle.$$

This shows that

$$f\times g = g\times f. \tag{3.60}$$

If f and g are two locally integrable functions in $\mathbb{R}$, then $f*g$ is also locally integrable, and hence, defines a regular distribution

$$\begin{aligned}\langle f*g,\phi\rangle &= \int_{-\infty}^{\infty}(f*g)(z)\,\phi(z)\,dz = \int_{-\infty}^{\infty}\phi(z)\left[\int_{-\infty}^{\infty}g(y)\,f(z-y)\,dy\right]dz\\ &= \int_{-\infty}^{\infty}g(y)\left[\int_{-\infty}^{\infty}f(z-y)\,\phi(z)\,dz\right]dy\\ &= \int_{-\infty}^{\infty}g(y)\left[\int_{-\infty}^{\infty}f(x)\,\phi(x+y)\,dx\right]dy\\ &= \int_{-\infty}^{\infty}f(x)\left[\int_{-\infty}^{\infty}g(y)\,\phi(x+y)\,dy\right]dx\\ &= \langle f(x),\langle g(y),\phi(x+y)\rangle\rangle\\ &= \langle f\times g,\phi(x+y)\rangle \quad \text{by} \quad (3.58) \quad \text{and} \quad \phi\in\mathcal{D}.\end{aligned} \tag{3.61}$$

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This shows an interesting link between the convolution product and direct product of two distributions, and hence, can be used to define the *convolution* of f and g by (3.61) provided the intersection of the support of $f \times g$ and the support of $\phi(x+y)$ is bounded.

It follows from the commutative property (3.60) of the direct product that the convolution is commutative and associative.

There is a convolution theorem of the Fourier transform of distributions. If f and g are two distributions with compact supports whose Fourier transforms are denoted respectively by $\hat{f}(k)$ and $\hat{g}(k)$, then

$$\begin{aligned}\mathcal{F}\{(f*g)(x)\} &= \left\langle (f*g)(x), e^{-ikx} \right\rangle \\ &= \left\langle f(x), \left\langle g(y) e^{-ik(x+y)} \right\rangle \right\rangle \\ &= \left\langle f(x), e^{-ikx} \right\rangle \left\langle g(y), e^{-iky} \right\rangle \\ &= \hat{f}(k) \cdot \hat{g}(k). \end{aligned} \tag{3.62}$$

This completes the proof of the convolution theorem of two distributions.

If $\psi \in \mathcal{D}$ has a compact support, then convolution of a distribution f with the test function ψ exists and is given by

$$f * \psi = \langle f(y), \psi(x-y) \rangle \in C^{\infty}(\mathbb{R}). \tag{3.63}$$

We have

$$\begin{aligned}\langle f*\psi, \phi \rangle &= \langle f(y) \times \psi(z), \phi(y+z) \rangle \\ &= \langle f(y), \langle \psi(z), \phi(y+z) \rangle \rangle \\ &= \left\langle f(y), \int_{-\infty}^{\infty} \psi(z)\phi(y+z)\,dz \right\rangle, \qquad (y+z=x) \\ &= \left\langle f(y), \int_{-\infty}^{\infty} \phi(x)\psi(x-y)\,dx \right\rangle \\ &= \int_{-\infty}^{\infty} \phi(x) \langle f(y), \psi(x-y) \rangle\,dx \\ &= \langle \langle f(y), \psi(x-y) \rangle, \phi \rangle. \end{aligned}$$

This proves the result.

We give some examples of convolution involving the delta function and an arbitrary distribution f.

(a) $$\langle f*\delta, \phi \rangle = \langle f(x), \langle \delta(y), \phi(x+y) \rangle \rangle = \langle f(x), \phi(x) \rangle.$$

Or,

$$f*\delta = \delta*f = f. \tag{3.64}$$

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(b)

$$f(x-a) = f(x) * \delta(x-a) = (f * \delta_a)(x), \tag{3.65}$$

where $\delta_a(x) \equiv \delta(x-a)$.

This follows from the fact that

$$\begin{aligned}\langle \delta_a * f, \phi \rangle &= \langle f(y), \langle \delta_a(x), \phi(x+y) \rangle \rangle \\ &= \langle f(y), \phi(a+y) \rangle = \langle f(x-a), \phi(x) \rangle .\end{aligned}$$

In particular, we obtain

$$(\delta_a * \delta_b)(x) = \delta_{a+b}(x). \tag{3.66}$$

This follows easily from the convolution theorem (3.62), that is,

$$\mathcal{F}\{(\delta_a * \delta_b)(x)\} = e^{-ik(a+b)} = \mathcal{F}\{(\delta_{a+b})(x)\}.$$

Taking the inverse Fourier transform gives the desired result (3.66)

(c) We can write the integral as a convolution:

$$\begin{aligned}f(x) = \int_{-\infty}^{x} \phi(\xi)\, d\xi &= \int_{0}^{\infty} \phi(x-z)\, dz \\ &= \int_{-\infty}^{\infty} H(z)\, \phi(x-z)\, dz = (H * \phi)(x). \end{aligned} \tag{3.67}$$

Making references to Debnath and Mikusinski (2005), if L is a differential operator with constant coefficients, it is easy to verify that

$$L(f * g) = (Lf) * g = f * (Lg), \tag{3.68}$$

where f and g are distributions.

In particular,

$$(L\delta) * f = \delta * (Lf) = Lf. \tag{3.69}$$

Thus, every differential operator with constant coefficients can be represented as a convolution.

One of the major applications of convolution is to find the particular solution of a differential equation

$$Lu = f, \tag{3.70}$$

where u and f are distributions. If f in (3.70) is a singular distribution, then the equation cannot have a classical solution u as a function. The remarkable fact is that including distributions generate new solution of classical equations with u and f as functions. Some classical equations may *not even have* a classical solution, but can have distributional solutions.

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If G is distribution which satisfies the equation

$$LG = \delta, \tag{3.71}$$

then G is called *fundamental solution*. Thus, if G is a solution of (3.71), then $u = f * G$ is a solution of (3.70) because

$$Lu = L(f * G) = f * LG = f * \delta = f.$$

3.3 The Temple–Lighthill approach to generalized functions based on good functions

A *good function*, $g(x)$ in $C^{\infty}(\mathbb{R})$ (the space of infinitely differentiable (or smooth) functions on $\mathbb{R}$) that decays sufficiently rapidly so that $g(x)$ and all of its derivatives decay to zero faster than $|x|^{-m}$ as $|x| \to \infty$ for some positive integer m. For example, $\exp(-x^2)$and $\operatorname{sech}^2 x$ are good functions. A sequence of good functions, $\{g_n(x)\}$ is called *regular*, if for any good function $g(x)$,

$$\lim_{n\to\infty} \langle g_n(x), g(x)\rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x), g(x)\, dx \tag{3.72}$$

exist.

According to Temple and Lighthill's idea, a *generalized function*, $f(x)$ is a regular sequence of good functions, $\{f_n(x)\}$ and is defined in terms of its *action* on the sequence of integrals of good functions, if

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \lim_{n\to\infty} \langle f_n, g\rangle, \tag{3.73}$$

for any good function $g(x)$, where $f(g) \equiv \langle f, g\rangle$ is the value of f at g, or the *action* of the generalized function on the good function $g(x)$. In other words, $f(g) \equiv \langle f, g\rangle$ represents the number that f associates with g.

If $f(x)$ is an ordinary functions such that $\left(1 + x^2\right)^{-m} f(x)$ is integrable on $\mathbb{R}$ for some m, then the *generalized function* $f(x)$ equivalent to the ordinary function is defined as a sequence of good functions $\{f_n(x)\}$ such that for any good functions $g(x)$

$$\lim_{n\to\infty} \langle f_n, g\rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx = \langle f, g\rangle. \tag{3.74}$$

The *Dirac delta function*, $\delta(x)$ is defined in terms of any good function $g(x)$ by

$$\delta(g) = \langle \delta, g\rangle = \int_{-\infty}^{\infty} \delta(x)\, g(x)\, dx = g(0). \tag{3.75}$$

There is *no* ordinary functions equivalent to the delta function $\delta(x)$. It follows from (3.74) that any generalized function is defined only through the *action* on the integral of good functions. The discontinuous Heaviside function with a jump discontinuity $H(x)$ which is 0 for $x < 0$ and 1 for $x > 0$, is a generalized function. For any good function $g(x)$,

$$\int_{-\infty}^{\infty} H(x)\, g(x)\, dx = \int_{0}^{\infty} g(x)\, dx. \tag{3.76}$$

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The derivatives of generalized functions are defined by derivatives of any of the equivalent sequences of good functions. We denote the derivative of a generalized function $f(x)$ by $\frac{df}{dx} \equiv f'(x)$ and integrate the product $f'(x)\,g(x)$ from $-\infty$ to ∞ by parts and use the decay property of the good function $g(x)$ as $|x| \to \infty$ so that the derivative of $f(x)$ is

$$\begin{aligned}\langle f', g\rangle &= \int_{-\infty}^{\infty} f'(x)\,g(x)\,dx = [f(x)\,g(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\,g'(x)\,dx \\ &= -\langle f, g'\rangle. \end{aligned} \tag{3.77}$$

This is usually used as a definition of the derivative $f'(x)$ of the distribution $f(x)$.

Repeating this process of differentiation leads to the nth derivative of the generalized function $f(x)$ so that

$$\left\langle f^{(n)}, g\right\rangle = (-1)^n \left\langle f, g^{(n)}\right\rangle. \tag{3.78}$$

It follows from (3.77) and (3.78) that generalized functions have derivatives of all orders that are also generalized functions. Using not only ordinary functions such that (3.74) holds and differentiating generalized functions by (3.78), we can greatly increase the number of generalized functions. This confirms that there is a large class of generalized functions.

According to definition (3.77) of a derivative of a generalized function,

$$\langle H', g\rangle = -\langle H, g'\rangle = -\int_0^{\infty} g'(x)\,dx = g(0) = \langle \delta, g\rangle. \tag{3.79}$$

This shows that the derivative of the Heaviside function is the Dirac delta function.

The derivatives of the Dirac delta functions are given by

$$\left\langle \delta^{(n)}, g\right\rangle = (-1)^n \left\langle \delta, g^{(n)}\right\rangle = (-1)^n\, g^{(n)}(0). \tag{3.80}$$

The Dirac delta function $\delta(x)$ has derivatives of all orders and they are all generalized functions.

We next consider the *modulus function*, $|x| = x \operatorname{sgn} x$ and its derivative, where discontinuous function $\operatorname{sgn} x$, which is 1 for $x > 0$ and -1 for $x < 0$, is a generalized function, and

$$(\operatorname{sgn} x)' = 2\delta(x). \tag{3.81}$$

A simple observation reveals that

$$\operatorname{sgn} x = H(x) - H(-x) = 2H(x) - I(x). \tag{3.82}$$

$$(\operatorname{sgn} x)' = 2H'(x) = 2\delta(x) \tag{3.83}$$

$$\begin{aligned}\frac{d}{dx}|x| &= \frac{d}{dx}(x \operatorname{sgn} x) = x\,(\operatorname{sgn} x)' + \operatorname{sgn} x \\ &= 2x\delta(x) + \operatorname{sgn} x = \operatorname{sgn} x, \qquad \text{since} \qquad x\delta(x) = 0. \end{aligned} \tag{3.84}$$

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The ordinary functions which can be treated as generalized functions include $|x|^{\alpha}$ for $\alpha > -1$. For values of m the integral

$$\int_{-\infty}^{\infty} \left(1+x^2\right)^{-m} |x|^{\alpha}\, dx$$

converges only if $\alpha > -1$. If $\alpha > -1$, this integral is convergent for all $m > \frac{1}{2}(\alpha+1)$. Taking derivatives both sides of the result

$$\log |x|^{\alpha} = \alpha \log |x| \tag{3.85}$$

and using the fact that

$$\frac{d}{dx} \log |x| = \frac{1}{|x|} \operatorname{sgn} x,$$

we obtain

$$\frac{d}{dx} \log |x|^{\alpha} = \alpha \frac{d}{dx} \log |x|,$$

or

$$\frac{1}{|x|^{\alpha}} \frac{d}{dx} |x|^{\alpha} = \frac{\alpha}{|x|} \operatorname{sgn} x.$$

Thus,

$$\frac{d}{dx} |x|^{\alpha} = \alpha\, |x|^{\alpha-1} \operatorname{sgn} x, \qquad \alpha > -1. \tag{3.86}$$

Following Lighthill (1958), we can define three more new generalized functions for $\alpha < -1$ and is not an integer as follows:

$$\left.\begin{array}{l} |x|^{\alpha} \\ |x|^{\alpha} \operatorname{sgn} x \\ x^{\alpha} H(x) \end{array}\right\} = \frac{1}{(\alpha+1)_n} \frac{d^n}{dx^n} \left\{\begin{array}{l} |x|^{\alpha+n} (\operatorname{sgn} x)^n, \\ |x|^{\alpha+n} (\operatorname{sgn} x)^{n+1}, \\ |x|^{\alpha+n} H(x), \end{array}\right. \tag{3.87}$$

where $(a)_n$ denotes the Pochhammer symbol $(a)_n = a(a+1)\cdots(a+n-1)$, n is an integer such that $\alpha + n > -1$.

From the property of the delta function

$$\delta(ax-b) = \frac{1}{|a|} \delta\left(x - \frac{b}{a}\right), \qquad a \neq 0,$$

an interesting question is whether this formula can be generalized to $\delta(f(x))$ where $f(x)$ is an arbitrary ordinary function. The answer follows from the proof of the result

$$\delta(f(x)) = \frac{\delta(x-x_1)}{|f'(x_1)|}, \qquad \left|f'(x_1)\right| \neq 0, \tag{3.88}$$

where x_1 is a *simple zero* of $f(x)$.

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We assume that $f'(x_1) > 0$ and substitute $y = f(x)$ in the following functional

$$\langle \delta(f(x)), \phi(x) \rangle = \int_{-\infty}^{\infty} \delta(f(x))\,\phi(x)\,dx = \int_{-\infty}^{\infty} \frac{\delta(y)\,\phi(x(y))}{f'(x)}\,dy$$

$$= \frac{\phi(x_1)}{f'(x_1)} = \int_{-\infty}^{\infty} \frac{\delta(x-x_1)}{f'(x_1)}\,\phi(x)\,dx, \tag{3.89}$$

$$= \left\langle \frac{\delta(x-x_1)}{f'(x_1)}, \phi(x) \right\rangle. \tag{3.90}$$

When $f'(x_1) < 0$, a similar calculation reveals the same result (3.89) with $\frac{\phi(x_1)}{f'(x_1)}$ replaced by $-\frac{\phi(x_1)}{f'(x_1)}$. Thus, combining the above two cases of positive and negative $f'(x_1)$ gives the result (3.88). However, the assumption of simple zero of $f(x)$ is crucial for the representation of (3.88). For example, $\delta(x^2)$ has no meaning at all because it does not satisfy (3.88).

Formula (3.88) can be generalized to the case of n simple zeros at $x = x_1, x_2, \ldots, x_n$ so that n terms similar to (3.88) can be found to obtain the formula

$$\delta(f(x)) = \sum_{m=1}^{n} \frac{\delta(x-x_m)}{|f'(x_m)|}, \qquad f'(x_m) \neq 0. \tag{3.91}$$

This result can be applied to prove

$$\delta[(x-a)(x-b)] = \frac{1}{(b-a)}[\delta(x-a) + \delta(x-b)], \qquad a > b. \tag{3.92}$$

In this case, $f(x) = (x-a)(x-b)$ which has two simple zeros at $x = x_1 = a$ and $x = x_2 = b$. Consequently, the use of (3.88) leads to the result

$$\delta[(x-a)(x-b)] = \frac{\delta(x-a)}{|a-b|} + \frac{\delta(x-b)}{|b-a|} = \frac{1}{a-b}[\delta(x-a) + \delta(x-b)].$$

In particular, if $b = -a < 0$, then

$$\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x-a) + \delta(x+a)], \qquad a \neq 0. \tag{3.93}$$

Once again, when $a = 0$, $\delta(x^2)$ loses its meaning.

For example, $f(x) = \cos x$, $0 < x < \pi$ so that $x = \frac{\pi}{2}$ is a simple zero of $f(x)$, $f'(x) = -\sin x$ and $f'\left(\frac{\pi}{2}\right) = -1$. Thus, application of (3.88) gives

$$\delta(\cos x) = \delta\left(x - \frac{\pi}{2}\right).$$

Result (3.91) remains valid for an infinite number of simple zeros of $f(x)$. For instance, $f(x) = \sin x$ which has infinite number of simple zeros at $x = n\pi$ so that

$$\delta(\sin x) = \sum_{n=-\infty}^{\infty} \frac{\delta(x-n\pi)}{|\cos n\pi|} = \sum_{n=-\infty}^{\infty} \delta(x-n\pi), \tag{3.94}$$

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where the right hand side represents an infinite sequence of impulses (or an infinite row of delta functions). Such representation is useful in electrical and electronic engineering.

It has been observed earlier that $\delta(x)$ is *not* locally integrable and hence, it is a singular distribution. Similarly, the function $f(x) = \frac{1}{x}$ is not locally integrable in any neighborhood of the origin and hence, does not define a regular distribution because

$$\int_{-\infty}^{\infty} \frac{\phi(x)}{x}\, dx \tag{3.95}$$

is not convergent for all test functions $\phi(x)$. However, it is possible to use the idea of the Cauchy principal value to define

$$\left\langle \mathcal{P}\left(\frac{1}{x}\right), \phi \right\rangle = \int_{-\infty}^{\infty} \frac{\phi(x)}{x}\, dx = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\phi(x)}{x}\, dx, \tag{3.96}$$

where $\mathcal{P}$ stands for the Cauchy principal value.

Since $\phi(x)$ is differentiable at $x = 0$, there exists a function $\psi(x)$ continuous at $x = 0$ such that $\phi(x) = \phi(0) + x\,\psi(x)$. If $[-a, a]$ is the support of $\phi(x)$ so that for $\varepsilon > 0$

$$\int_{|x| \geq \varepsilon} \frac{\phi(x)}{x}\, dx = \int_{a \geq |x| \geq \varepsilon} \psi(x)\, dx \to \int_{-a}^{a} \psi(x)\, dx \quad \text{as} \quad \varepsilon \to 0. \tag{3.97}$$

It turns out that $\left(\mathcal{P}\left(\frac{1}{x}\right), \phi\right)$ is a continuous linear functional, and hence, it defines a distribution. Such a distribution is also called a *pseudofunction* which is denoted by $\mathcal{P}\left(\frac{1}{x}\right)$.

The functions

$$\delta^{\pm} = \frac{1}{2}\delta(x) \mp \frac{1}{2\pi i}\mathcal{P}\left(\frac{1}{x}\right). \tag{3.98}$$

represents singular distributions and are called the *Heisenberg distributions*. It can be shown that

$$\left\langle \delta^{\pm}(x), \phi(x) \right\rangle = \mp \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(x)}{x \pm i\varepsilon}\, dx. \tag{3.99}$$

In other words,

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(x)\, dx}{x + i\varepsilon} = -\frac{1}{2}\phi(0) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(x)}{x}\, dx, \tag{3.100}$$

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(x)\, dx}{x - i\varepsilon} = \frac{1}{2}\phi(0) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(x)}{x}\, dx. \tag{3.101}$$

Results (3.100) and (3.127) are called *Plemelj formulas*, and they can be expressed as

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(x)}{x \pm i\varepsilon}\, dx = \mp \frac{1}{2}\langle \delta, \phi \rangle + \frac{1}{2\pi i} \left\langle \mathcal{P}\left(\frac{1}{x}\right), \phi \right\rangle, \tag{3.102}$$

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so that (3.99) follows from (3.102). Symbolically, (3.99) can be written as

$$\delta^{\pm}(x) = \mp \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\,\frac{1}{(x\pm i\varepsilon)} = \mp \frac{1}{2\pi i}\,\frac{1}{x\pm i0}, \tag{3.103}$$

where

$$\delta(x) = \delta^{+}(x)+\delta^{-}(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\,\frac{\varepsilon}{(x^2+\varepsilon^2)}, \tag{3.104}$$

$$\frac{1}{\pi i}\,\mathcal{P}\left(\frac{1}{x}\right) = \delta^{-}(x)-\delta^{+}(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi i}\,\frac{x}{(x^2+\varepsilon^2)}. \tag{3.105}$$

Writing $\varepsilon = \frac{1}{n}$ in (3.104) gives the delta sequence (3.5). Formulas (3.98), are often written as

$$\frac{1}{x+i0} = -\pi i\,\delta(x)+\mathcal{P}\left(\frac{1}{x}\right), \tag{3.106}$$

$$\frac{1}{x-i0} = \pi i\,\delta(x)+\mathcal{P}\left(\frac{1}{x}\right), \tag{3.107}$$

are known as *Sokhotski–Plemelj formulas*.

Dirac first solved the equation for the distribution

$$(x-a)\,f(x-a) = g(x). \tag{3.108}$$

Obviously, the corresponding homogenous equation

$$(x-a)\,f(x-a) = 0 \tag{3.109}$$

has the solution $\delta(x-a)$ because

$$\int_{-\infty}^{\infty}(x-a)\,\delta(x-a)\,\phi(x)\,dx = 0$$

for any test function ϕ.

The particular generalized solution of (3.108) is given by

$$g(x)\,\mathcal{P}\left(\frac{1}{x-a}\right), \tag{3.110}$$

so that the solution of (3.108) becomes

$$f(x-a) = \delta(x-a)+g(x)\,\mathcal{P}\left(\frac{1}{x-a}\right). \tag{3.111}$$

This Dirac solution (3.111) has useful applications in transport phenomena.

It turns out that locally integrable function on $\mathbb{R}$

$$f(x) = \left\{\begin{array}{ll}\frac{1}{\sqrt{x}}, & x>0\\ 0, & x<0\end{array}\right\} = \frac{H(x)}{\sqrt{x}} \tag{3.112}$$

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produces the distribution

$$\langle f,\phi\rangle = \int_{-\infty}^{\infty} H(x)\,\frac{\phi(x)}{\sqrt{x}}\,dx, \qquad \phi(x)\in\mathcal{D}. \tag{3.113}$$

Thus, the distributional derivative of $f(x)$ is given by

$$\langle f',\phi\rangle = -\langle f,\phi'\rangle = -\int_{-\infty}^{\infty} \frac{H(x)\phi'(x)}{\sqrt{x}}\,dx = -\int_{0}^{\infty} \frac{\phi'(x)}{\sqrt{x}}\,dx. \tag{3.114}$$

The function $f(x)$ has the classical derivative except at $x=0$ given by

$$f'(x) = \left[\frac{\delta(x)}{\sqrt{x}} - \frac{1}{2}\,\frac{H(x)}{x^{3/2}}\right]. \tag{3.115}$$

Thus, the ordinary derivative (3.115) generates a distribution identical to the distributional derivative (3.114). However, $f'(x)$ given by (3.115) is *not* locally integrable and hence, it does not produce a regular distribution. Every locally integrable function has a regular distribution and therefore, has a distributional derivative, but the classical derivative of a locally integrable function may *not* be the same with the distributional derivative.

The function $f(x)$ has the classical derivative except at $x=0$ given by

$$f'(x) = \left\{\begin{matrix} -\frac{1}{2}x^{-3/2}, & x>0 \\ 0, & x<0 \end{matrix}\right\} = -\frac{1}{2}\,\frac{H(x)}{x^{3/2}}. \tag{3.116}$$

Consequently,

$$\frac{d}{dx}\left[H(x)\,x^{-\frac{1}{2}}\right] = -\frac{1}{2}\mathcal{P}\left(H(x)\,x^{-3/2}\right). \tag{3.117}$$

This analysis can be generalized to the distributional derivative of $H(x)\,x^{-\alpha}$ as

$$\frac{d}{dx}\left[H(x)\,x^{-\alpha}\right] = -\alpha\,\mathcal{P}\left(H(x)\,x^{-\alpha-1}\right), \qquad 0<\alpha<1. \tag{3.118}$$

3.4 Mikusinski's algebraic approach to generalized functions

In 1948–1959, Jan Mikusinski proposed a new approach to the theory of generalized functions. His less abstract formulation based on commutative convolution ring is equivalent to the continuous linear functional of Schwartz on the test function space. He introduced the class of all ordered pairs of continuous function (a,b) in $[0,\infty)$, denoted by (a/b) which is called a *convolution quotient* and then defined operations of addition, scalar multiplication, and multiplication in the set D of all convolution quotients. More importantly, he introduced the convolution quotient $(a*f)/a$ as f for any pair of continuous functions a and f, and $(\lambda a)/a$ as scalar λ in D with the unit element as the Dirac delta function $\delta = a/a$ so that multiplication by δ produces f. Thus, the elements of D consists of numbers, functions, and the delta function and its derivatives so that they can be considered as generalized functions. These elements may be thought as operators

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describing a convenient and simple approach to Heaviside operational calculus. Subsequently, Weston (1957–1959) provided an alternative approach to generalized functions as operators acting on a class of perfect functions rather than convolution quotients of elements in $\mathbb{R} = [0, \infty)$.

Mikusinski also described a viable alternative to the method of Laplace transforms as a rigorous justification of the Heaviside operational calculus. His new and original approach because became very successful with functions defined on the positive real axis and with functions of several real variables. However, the major weakness of his approach was that it could not deal with functions of unrestricted real variables, or functions defined on an arbitrary domain of a space n dimensions where $n \geq 2$. Finally, Mikusinski also considered the weak convergence of a sequence $\{f_n\}$ so that $\{f_n\}$ converges weakly to the limit f if the sequence $\{f_n \phi\}$ converges in $\mathbb{C}$ for all test functions ϕ in Φ. The weak limit of a sequence $\{f_n \phi\}$ is a linear functional in the space Φ that forms the Hadamard representation of the functional $\{f\phi\}$ so that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,\phi(x)\,dx = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx. \tag{3.119}$$

3.5 The Cauchy representation of distributions by analytic functions

We denote the space $\mathcal{D}$ of test functions $\phi(x)$ which are continuous and have continuous derivatives of all orders. If z is a complex number located in the upper or lower half-plane but *not* on the x-axis, then the function $[2\pi i\,(x-z)]^{-1}$ is continuous and has continuous derivatives of all orders for all x, and hence, it belongs to the space $\mathcal{D}$. We denote $\mathcal{D}'$ the space of all linear functionals f on $\mathcal{D}$ that are continuous so that $\mathcal{D}'$ is the dual space of $\mathcal{D}$. A special feature of the distribution of $f \in \mathcal{D}'$ is that they have compact support. For a distribution $f(x) \in \mathcal{D}'$, the operation

$$\frac{1}{2\pi i}\left\langle f(x), \frac{1}{x-z}\right\rangle = f_c(z) \tag{3.120}$$

produces a function $f_c(z)$ of the complex variable z for all z outside the real axis. This function $f_c(z)$ is called the *Cauchy representation* of the distribution $f(x)$ in the following sense: $f_c(z+i\varepsilon) - f_c(z-i\varepsilon)$ converges to $f(x)$ as $\varepsilon \to 0+$ in the topology of $\mathcal{D}'$. However, $f(x) \in \mathcal{D}'$ the function $(x-z)^{-1}$ is *not* a test function, $f_c(z)$, in general, does not exist, but the following result is true: there exists a pair of functions $f_+(z)$ and $f_-(z)$ analytic in the upper and lower half-plane, respectively such that $f_+(x+i\varepsilon) - f_-(x-i\varepsilon)$ tends to $f(x)$ as $\varepsilon \to 0+$ in the topology in $\mathcal{D}'$. If a pair of analytic functions represents $f(x)$, then their complex derivatives represents $f'(x)$.

It turns out that

$$\frac{d}{dz} f_c(z) = \frac{1}{2\pi i}\left\langle f(x), \frac{1}{(x-z)^2}\right\rangle \tag{3.121}$$

so that the complex derivatives of $f_c(z)$ exist for z with $\operatorname{Im} z \neq 0$, and hence, it is analytic in that region. Result (3.121) also shows that

$$\frac{d}{dz} f_c(z) = f_c'(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{(x-z)^2}. \tag{3.122}$$

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In general,

$$f_c^{(n)}(z) = \frac{n!}{2\pi i}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{(x-z)^{n+1}}. \tag{3.123}$$

If $f(x)$ is a locally integrable function which vanishes outside of some finite interval on this real axis, and this restriction leads to the fact that $f(z)$ vanishes for large values of z. Thus, for any f,

$$\langle f,\phi\rangle = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx \tag{3.124}$$

is a distribution of bounded support. The Cauchy representation of this distribution assumes the form of the regular Cauchy integral

$$f_c(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-z} = \left\{\begin{array}{ll} f(z), & \mathrm{Im}\,z>0 \\ 0, & \mathrm{Im}\,z<0 \end{array}\right\}. \tag{3.125}$$

This Cauchy representation $f_c(z)$ of a distribution $f(x)$ on $\mathcal{D}$ has far-field behavior $f_c(z) = O\left(\frac{1}{z}\right)$ as $z\to\infty$.

If $f(x)=\delta(x)$, then

$$\delta_c(z) = \frac{1}{2\pi i}\left\langle f(x), \frac{1}{(x-z)}\right\rangle = -\frac{1}{2\pi i z}, \quad \text{and} \quad \delta_e'(z) = \frac{1}{2\pi i z^2}. \tag{3.126a,b}$$

This shows that $\delta_c(z) = O\left(\frac{1}{z}\right)$ and $\delta_c'(z) = O\left(\frac{1}{z^2}\right)$ as $z\to\infty$.

3.6 Sato's hyperfunctions based on the theory of complex functions

In 1959, Sato discovered a hyperfunction $f(x)$ of a real variable x as the difference $F(x+i0)-F(x-i0)$ of the boundary values of a complex function $F(z)$ of a complex variable z which is analytic (or holomorphic) in the upper-half plane ($\mathrm{Re}\,z>0$) and the lower-half plane ($\mathrm{Re}\,z<0$). If $F(z)$ is holomorphic for real x, that is, real analytic at x, the $F(x+i0)-F(x-i0)=0$, so that singularities of $F(z)$ on the real axis play an important role to produce a nontrivial hyperfunction $f(x)$. Making reference to Graf's (2000) recent book, we give some elementary examples of hyperfunctions.

One of the most common and well-known hyperfunctions is the Dirac delta function $\delta(x)$ defined by

$$\delta(x) = [F(z)] = \left[-\frac{1}{2\pi i}\,\frac{1}{z}\right], \tag{3.127}$$

where $F(z)$ is meromorphic, that is, it is defined on the whole complex plane and has a polar singularity at $z=0$. For every real $x\neq 0$,

$$\lim_{\varepsilon\to 0} F(x+i\varepsilon) - F(x-i\varepsilon) = 0 \tag{3.128}$$

so that $\delta(x)=0$ for all $x\neq 0$. The limit does not exist at $x=0$ as expected.

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Another related example of hyperfunction is the unit-step hyperfunction, $H(x)$ defined by

$$H(x) = \left[-\frac{1}{2\pi i}\log(-z)\right], \tag{3.129}$$

where $\log z$ is a multiple valued function with infinity of branches defined by

$$\log z = \log|z| + i\arg z + 2\pi i n, \qquad n \in \mathbb{Z}, \tag{3.130}$$

where the principle branch is used to obtain a well-defined single-valued function of z. The function $H(x)$ in (3.129) is identified with the Heaviside unit step function. Clearly, $\log(-z)$ has the value $\log|z| - \pi i$ on the upper side, and $\log|z| + \pi i$ on the lower side of the positive x-axis. So, it is real analytic on the negative real axis. Thus, it turns out that

$$H(x) = \lim_{\varepsilon\to 0}[F(x+i\varepsilon) - F(x-i\varepsilon)] = \left[\begin{matrix} 0, & x > 0 \\ 1, & x < 0 \end{matrix}\right]. \tag{3.131}$$

At $x = 0$, this hyperfunction has no value.

Similarly, the unit-step hyperfunction $H(-x)$ is defined by

$$H(-x) = \left[\frac{1}{2\pi i}\log z\right], \tag{3.132}$$

which has the value 1 for all negative x, and 0 for all positive real x. It can be interpreted as the unit-step Heaviside function $H(-x)$.

Thus, the hyperfunction

$$\begin{aligned} H(x) + H(-x) &= \left[-\frac{1}{2\pi i}\log(-z) + \frac{1}{2\pi i}\log(z)\right] \\ &= \left[\frac{1}{2\pi i}\{\log(z) + \log(-z)\}\right], \end{aligned} \tag{3.133}$$

which can be interpreted as ordinary constant hyperfunctions; $f(x) = 1$ for $x \neq 0$.

Another hyperfunction is called $\operatorname{sgn} x$ defined by

$$\operatorname{sgn}(x) = H(x) - H(-x) = -\frac{1}{2\pi i}[\log(z) + \log(-z)], \tag{3.134}$$

which has the value 1 for $x > 0$ and -1 for $x < 0$, but it is not defined at $x = 0$. This can be identified with ordinary sgn function.

Although the ordinary $H(x)$ is not differentiable at $x = 0$, but the hyperfunction $H(x)$ defined by (3.129) has the generalized derivative which is the Dirac delta function, $\delta(x)$ for

$$H'(x) = \left[\frac{d}{dz}\left\{-\frac{1}{2\pi i}\log(-z)\right\}\right] = \left[-\frac{1}{2\pi i}\frac{1}{z}\right] = \delta(x). \tag{3.135}$$

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We can define the product of a hyperfunction $f(x)$ and a real analytic function $\phi(x)$ by

$$\phi(x) f(x) = [\phi(z) F(z)]. \tag{3.136}$$

Thus, hyperfunctions can be multiplied by polynomials, exponential and trigonometric function such as $\exp x$, $\sin x$, $\cos x$ and the result is a again a hyperfunction. For example,

$$x\delta(x) = 0, \qquad x^n \delta(x) = 0, \qquad \phi(x)\delta(x) = \phi(0)\delta(x).$$

We have

$$x\delta(x) = \left[-z \cdot \frac{1}{2\pi i z}\right] = \left[-\frac{1}{2\pi i}\right] = 0,$$
$$x^n \delta(x) = \left[-z^n \cdot \frac{1}{2\pi i z}\right] = \left[-\frac{z^{n-1}}{2\pi i}\right] = 0.$$

We can also define the hyperfunction

$$f(ax+b) = [(\operatorname{sgn} a) F(az+b)]. \tag{3.137}$$

For example

$$\delta(ax \pm b) = \left[-\operatorname{sgn}(a)\frac{1}{2\pi i z}\right] = \left(\frac{\operatorname{sgn} a}{a}\right)\left[-\frac{1}{2\pi i\left(z \pm \frac{b}{a}\right)}\right]$$
$$= -\frac{1}{|a|}\left[-\frac{1}{2\pi i\left(z \pm \frac{b}{a}\right)}\right] = \frac{1}{|a|}\delta\left(x \pm \frac{b}{a}\right), \qquad a \neq 0. \tag{3.138}$$

Finally, this result can be generalized for the hyperfunction $\delta(\phi(x))$ which has the value

$$\delta(\phi(x)) = \frac{\delta(x-x_1)}{|\phi'(x_1)|}, \qquad \phi'(x_1) \neq 0, \tag{3.139}$$

where $\phi(x)$ is a real analytic function which has only one simple zero at $x = x_1$, that is, $\phi(x_1) = 0$ and $\phi'(x_1) \neq 0$. This follows from the expansion of

$$\phi(z) = (z-x_1)\phi'(x_1)\left\{1 + \frac{1}{2}\phi''(x_1)/\phi'(x_1)(z-x_1) + \cdots\right\}$$

and the result

$$-\frac{1}{2\pi i}\frac{1}{\phi(z)} = \frac{1}{2\pi i}\frac{1}{((z-x_1)\phi'(x_1))} + \psi(z)$$

where $\psi(z)$ is a real analytic function.

Similarly, if $\phi(x)$ has n *simple* zeros at $x_1, x_2, \ldots, x_n$, then

$$\delta(\phi(x)) = \sum_{m=1}^{n} \frac{\delta(x-x_n)}{|\phi'(x_n)|}, \qquad \phi'(x_n) \neq 0. \tag{3.140}$$

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This also holds true if $\phi(x)$ has an infinite number of *simple zeros* at $x_1, x_2, \ldots, \infty$, so that

$$\delta(\phi(x)) = \sum_{n=1}^{\infty} \frac{\delta(x-x_n)}{|\phi'(x_n)|}, \qquad \phi'(x_n) \neq 0. \tag{3.141}$$

3.7 Some additional properties of convolution of distributions

It follows from (3.70) that

$$(D^n\delta) * f = \delta * D^n f = D^n f. \tag{3.142}$$

In particular, for any constant c

$$\left(\delta' * c\right) = D\delta * c = \delta * Dc = 0. \tag{3.143}$$

Thus,

$$\left(c * \delta'\right) * H = \left(\delta' * c\right) * H = 0 * H = 0. \tag{3.144}$$

On the other hand,

$$(D\delta) * H = \delta * DH = \delta * \delta = \delta, \tag{3.145}$$

$$c * \left(\delta' * H\right) = c * \delta = c. \tag{3.146}$$

This shows that

$$\left(c * \delta'\right) * H \neq c * \left(\delta' * H\right), \tag{3.147}$$

that is, the convolution is *not* associative.

Theorem 3.1 (Differentiation) *If the convolution* $(f * g)$ *of two distributions* f *and* g *exists, then* $(D^n f) * g$ *and* $f * (D^n g)$ *exist and*

$$(D^n f) * g = D^n (f * g) = f * (D^n g). \tag{3.148}$$

The proof is straight forward and hence, it is omitted.

However, the mere existence of the convolutions $(D^n f) * g$ and $f * (D^n g)$ is not sufficient for the existence of the convolution $(f * g)$. When

$$(D^n f) * g \neq f * (D^n g), \tag{3.149}$$

then $(f * g)$ does not exist. For example, if $n = 1$, $f = H$ and $g =$ constant $= c$, then $H * (Dc) = H * 0 = 0$, $(DH) * c = \delta * c = c$ and $H * Dc \neq (DH) * c$. Thus,

$$(H * c) = \int_{-\infty}^{\infty} H(\xi)\, c\, d\xi = c \int_{-\infty}^{\infty} d\xi \tag{3.150}$$

does not exist.

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Schwartz has proved that multiplication of $\mathcal{P}\left(\frac{1}{x}\right)$ (principal part of $\frac{1}{x}$), x, and δ is *not* associative because $\mathcal{P}\left(\frac{1}{x}\right)x = 1$ and $x\delta = 0$ so that

$$\left[\mathcal{P}\left(\frac{1}{x}\right)x\right]\delta = \delta \qquad \text{and} \qquad \mathcal{P}\left(\frac{1}{x}\right)[x\delta] = 0. \tag{3.151a,b}$$

Further, it can be shown that

$$\delta\cdot\delta' = \frac{1}{4\pi i}\delta'' + c_0\delta + c_1\delta', \tag{3.152}$$

where c_0 and c_1 are arbitrary constants.

Using the convolution theorem in Fourier transforms, it turns out

$$\mathcal{F}(f * g) = (\mathcal{F}f)\cdot(\mathcal{F}g) \qquad \text{and} \qquad \mathcal{F}(f\cdot g) = \mathcal{F}(f) * \mathcal{F}(g). \tag{3.153a,b}$$

In particular, if $f = \delta$ and $g = \delta'$, then

$$\mathcal{F}\left(\delta\cdot\delta'\right) = \mathcal{F}(\delta)\mathcal{F}\left(\delta'\right) = 1 * ik = \int_{-\infty}^{\infty} i\,(k-\xi)\,d\xi. \tag{3.154}$$

Differentiating both sides twice with respect to k gives

$$\frac{d^2}{dk^2}\mathcal{F}\left(\delta\cdot\delta'\right) = 0 \tag{3.155}$$

so that $\mathcal{F}(\delta\cdot\delta') = a + bk$, and

$$\delta\cdot\delta' = c_0\delta + c_1\delta',$$

where c_0 and c_1 are arbitrary constants. Thus, this result agrees with (3.152) except the first particular term in (3.152).

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References

[1] Bochner, S., *Vorlesungen über Fouriersche Integrale*, Akademie-Verlag, Berlin (1932).

[2] Debnath, L., The Genesis of Convolution in Fourier Analysis with Applications, *Internat. Math. Jour. Analysis and Its Applications*, **32** (2012) 39–66.

[3] Debnath, L. and Mikusinski, P., *Introduction to Hilbert Spaces With Applications* (Third Edition), Elsevier Academic Press, San Diego (2005).

[4] Debnath, L. and Bhatta, D., *Integral Transforms and Their Applications* (Second Edition), CRC Press and Chapman-Hall, Boca Raton, Florida (2007).

[5] Dirac, P. A. M., The Physical Interpretation of the Quantum Dynamics, *Proc. Roy. Soc. London*, **A113** (1927) 621–641.

[6] Dirac, P. A. M., *The Principles of Quantum Mechanics*, Claredon Press, Oxford (1947).

[7] Graf, U., *Introduction to Hyperfunctions and Their Integral Transforms*, Birkhäuser/Springer, Basel (2010).

[8] Halperin, I. I., *Introduction to the Theory of Distributions Based on Lectures by Laurent Schwartz*, University of Toronto Press, Toronto (1952).

[9] Heaviside, O., On operations in physical mathematics, *Proc. Roy. Soc. London*, **A52** (1892) 504–529.

[10] Kirchhoff, G., Zur Theorie der Lichtstrahlen, *Sber. Preuss. Akad. Wiss. II* (1882) 641–669.

[11] König, H., Neue Begründung der Theorie der Distributionen von L. Schwartz, *Math. Nachr.*, **9** (1953) 129–142.

[12] Lighthill, M. J., *Introduction to Fourier Analysis and Generalised Functions*, Cambridge University Press, Cambridge, (1958).

[13] Mikusinski, J., Sur la méthode de généralisation de Laurent Schwartz et sur la convergence faible, *Fundam. Math.*, **35** (1948) 235–239.

[14] Sauer, R., *Anfangswertprobleme bei partiellen Differentialgleichungen* (second edition) Springer-Verlag, Berlin, (1958).

[15] Sobolev, S. L., Méthode nouvelle à resoudre le problème de Cauchy pour les équations linéaries hyperboliques normales, *Mat. Shornik*, **1** (1936) 39–71.

[16] Schwartz, L., *Théorie des Distributions*, Vols I & II, Hermann and Cie, Paris (1950, 1951).

[17] Sato, M., *Theory of Hyperfunctions I*, J. Fac. Univ. Tokyo Sect. I8 (1959) 133–193.

[18] Temple, G., The Theory of Generalized Functions, *Proc. Roy. Soc. London*, **228** (1955) 175–190.

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[19] Temple, G., *100 Years of Mathematics*, Springer Verlag, New York (1981).

[20] Weston, J. D., An extension of the Laplace-transform calculus, *Rend. Circ. Mat. Palermo*, **6** (1957) 1–12.

[21] Weston, J. D., Operational calculus and generalized functions, *Proc. Roy. Soc. London*, **A250** (1959) 460–478.

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