ON CLOSED CATEGORIES OF FUNCTORS II *

by

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In many examples of monoidal biclosed categories the tensor product is constructed as an extension of a monoidal structure on a basic generating set of objects in the category. A simple example is the construction of the tensor product of two abelian groups from the free tensor product of two free abelian groups. We shall consider a formal generalisation of this construction by asking when a monoidal structure or, more generally, a promonoidal structure on a category A can be extended along a dense functor $N: A^{op} \to C$ to produce a monoidal biclosed structure on the category C.

We give an answer to this question by combining the two basic types of dense functor considered in [4] and [7]. The first is the Yoneda embedding of A^{op} into the category $[A,S]$ of all set-valued functors on A. To each promonoidal structure on a small category A there corresponds a monoidal biclosed structure on $[A,S]$. The second type is the reflection functor; that is, the left adjoint to a full embedding $C \subset B$. If B has a monoidal biclosed structure then the adjunction is monoidal if and only if C is closed under all exponentiation by the internal-hom functors of B.

The two results are combined in the following manner. If $N: A^{op} \to C$ is a dense functor then it can be decomposed into the Yoneda embedding of A^{op} in $[A,S]$ followed by a reflection from $[A,S]$ to

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C provided C is sufficiently complete with respect to A. However, this over-all completeness hypothesis on C is generally unnecessary in order to produce a monoidal biclosed structure on C, as it is in the case when N itself is a reflection and A^{op} is monoidal biclosed. Thus we answer the original question by first considering a completion C* of C and finding conditions under which this completion is monoidal biclosed. The structure we obtain on C is simply the trace of the completed structure C*.

The procedure produces many known constructions of tensor-product functors and internal-hom functors. In particular, it produces the tensor product of algebras for a monoidal monad on a monoidal biclosed category, and gives a conceptual explanation of constructions by Linton [18] and Kock [16]. It also produces the canonical "convolution" structure on a functor category [A,B] when A is a promonoidal category and B is a suitably complete monoidal biclosed category.

One advantage of this approach is that the coherence of the structure produced on C follows from the coherence of C* which, in turn, follows from the coherence results already established in [4] and [7]. Completions can also be used to provide a concept of "change of V-universe" in the case where all categorical algebra is based on a fixed symmetric monoidal closed category V. This enables us to put a "V-structure" on any category [A,B] of V-functors from a (possibly large) V-category A to a suitably enriched category B. This, in turn, makes the relative completion process available for large V-categories (and ultimately leads to a reduction in algebraic computation).

The completion process is used in sections 5,6, and 7 to examine monoidal biclosed structures on categories of functors from a promonoidal category to a monoidal biclosed category. In section 6 we discuss biclosed categories of continuous functors and relate this to the work of Bastiani-Ehresmann [2]. Finally, section 8 contains a

proof of the representation theorem for monads.

The unexplained notations and terminology used in this article are standard, and familiarity with the representation theorem is assumed (cf. [11] §1). This material is a development of results in [4], [7], [8], and [9] and is based on a doctoral thesis by the author ([5] and [6]). The thesis was supervised by Professor G.M. Kelly at the University of New South Wales. The author has also benefited from several discussions with A. Kock and R. Street.

The section-headings are as follows:

§1 Reflection in closed functor categories.
§2 The completion process.
§3 Monoidal closed completion.
§4 Monoidal monads.
§5 Biclosed functor categories.
§6 Biclosed categories of continuous functors.
§7 Completion of functor categories.
§8 Denseness presentations.

§1 REFLECTION IN CLOSED FUNCTOR CATEGORIES

The formulas needed for monoidal biclosed functor categories and reflection of monoidal biclosed structures are recalled from [4] and [7]. $\mathcal{S}$ denotes "the" cartesian closed category of small sets and $\mathcal{V} = (V,\otimes,I,[-,-],\ldots)$ is a fixed symmetric monoidal closed base-category with small limits and colimits. All concepts of categorical algebra are henceforth assumed to be relative to this $\mathcal{V}$ unless otherwise stipulated.

An adjunction $(\varepsilon,\eta)\colon S \dashv T\colon \mathcal{C} \to \mathcal{B}$ is called a monoidal reflection or normal reflection if T is a full embedding and the adjunction

data admits monoidal enrichment. If $\mathcal{B} = (\mathcal{B},\otimes,I,/,\backslash,\ldots)$ is a monoidal biclosed category then $S \dashv T$ is monoidal if and only if one of the following pairs of conditions is satisfied (where T is omitted from the notation) for all $B,B' \in \mathcal{B}$ and $C \in \mathcal{C}$:

$$\left.\begin{array}{r} \eta\colon C/B \cong S(C/B) \\ \eta\colon B\backslash C \cong S(B\backslash C) \end{array}\right\}$$

$$\left.\begin{array}{r} \eta\backslash 1\colon SB\backslash C \cong B\backslash C \\ 1/\eta\colon C/SB \cong C/B \end{array}\right\}$$

$$\left.\begin{array}{r} S(\eta\otimes 1)\colon S(B\otimes B') \cong S(SB\otimes B') \\ S(1\otimes\eta)\colon S(B'\otimes B) \cong S(B'\otimes SB) \end{array}\right\}$$

$$S(\eta\otimes\eta)\colon S(B\otimes B') \cong S(SB\otimes SB').\ \}$$

A subcategory $\mathcal{A} \subset \mathcal{B}$ is called strongly generating in $\mathcal{B}$ if the class of functors $\{\mathcal{B}(A,-);\ A \in \mathcal{A}\}$ jointly reflects isomorphisms. The reflection theorem for monoidal biclosed categories (cf. [7] Theorem 1.2) states that the above conditions are equivalent to either of the pairs

$$\left.\begin{array}{r} \eta\colon D/A \cong S(D/A) \\ \eta\colon A\backslash D \cong S(A\backslash D) \end{array}\right\} \qquad (1.1)$$

$$\left.\begin{array}{r} S(\eta\otimes 1)\colon S(B\otimes A) \cong S(SB\otimes A) \\ S(1\otimes\eta)\colon S(A\otimes B) \cong S(A\otimes SB) \end{array}\right\}$$

for all $B \in \mathcal{B}$, $A \in \mathcal{A}$, and $D \in \mathcal{D}$, where $\mathcal{A}$ strongly generates $\mathcal{B}$ and $\mathcal{D}$ strongly cogenerates $\mathcal{C}$.

If $\mathcal{B}$ is a monoidal biclosed category and $\mathcal{C} \subset \mathcal{B}$ is a full subcategory we say that $\mathcal{C}$ is closed under exponentiation in $\mathcal{B}$ if $B\backslash C$ and C/B have isomorphs in $\mathcal{C}$ for all $B \in \mathcal{B}$ and $C \in \mathcal{C}$. Note that his is generally a stronger condition than requiring that the internal-hom functors $-\backslash-$ and $-/-$ have restrictions to $\mathcal{C}$.

Given a small category $\mathcal{A}$, each functor $F \in [\mathcal{A},\mathcal{V}]$ has an expansion:

$$F \cong \int^{A} FA \otimes \mathcal{A}(A,-) : \mathcal{A} \to \mathcal{V}$$

Thus each monoidal biclosed structure on the functor category is isomorphic to a structure of the following form:

$$F \otimes G = \int^{AB} FA \otimes GB \otimes P(AB\,-) \qquad (1.2)$$

$$\begin{aligned} G/F &= \int_{AB} [P(-AB), [FA, GB]] \\ F\backslash G &= \int_{AB} [P(A-B), [FA, GB]] \end{aligned} \qquad (1.3)$$

where $P(AB-) = A(A,-) \otimes A(B,-)$ in $[A, V]$. The resulting functor $P\colon A^{op} \otimes A^{op} \otimes A \to V$ is the "structure functor" of $\otimes$ on $[A, V]$.

A monoidal biclosed structure on $[A, V]$ is determined by P and I and associativity and identity isomorphisms:

$$\alpha = \alpha_{ABC}\colon P(AX-) \circ P(BCX) \cong P(ABX) \circ P(XC-)$$

$$\lambda = \lambda_A \;\colon IX \circ P(XA-) \cong A(A,-)$$

$$\rho = \rho_A \;\colon IX \circ P(AX-) \cong A(A,-)$$

satisfying suitable coherence axioms ([4] §3), where "$\circ$" denotes profunctor composition. We call such a collection $(A, P, I, \alpha, \lambda, \rho)$ a promonoidal structure on A.

The extension of a promonoidal structure on A to $[A, V]$ is sometimes called the convolution of A with V and it is an internal hom of promonoidal categories, with:

$$P(FGH) = \int_{ABC} [P(ABC), [FA \otimes GB, HC]]$$

$$[A, V](F, G) = \int_{AB} [A(A,B), [FA, GB]]$$

$$I(F) = \int_{A} [IA, [I, FA]],$$

where P, Hom, and I are regarded as the 2-dimensional, 1-dimensional, and 0-dimensional components of promonoidal structure.

Two small promonoidal categories A and B have a promonoidal tensor product $A \otimes B$ whose (P,Hom,I) are the respective tensor products of those for A and B. The usual isomorphism of categories:

$$[A \otimes B, V] \simeq [A,[B,V]]$$

then becomes an isomorphism of monoidal biclosed categories.

To combine convolution with reflection, let $[A,V]$ have a monoidal biclosed structure and let $S \dashv T\colon C \to [A,V]$ be a full reflective embedding where C contains $\mathcal{D}$ as a strongly cogenerating subcategory. The Yoneda embedding $A^{op} \to [A,V]$ is strongly generating, thus we have:

Proposition 1.1. The adjunction $S \dashv T\colon C \to [A,V]$ is monoidal if and only if the functors

$$G/A(A,-) = \int_B [P(-AB),GB]$$

$$A(A,-)\backslash G = \int_B [P(A-B),GB]$$

have isomorphs in C for all $A \in A$ and $G \in \mathcal{D}$.

Proof. This is a reformulation of Condition (1.1) with $B = [A,V]$. For example,

$$G/A(A,-) = \int_{B'B} [P(-B'B),[A(A,B'),GB]] \quad \text{by (1.3)}$$

$$\cong \int_B [P(-AB),GB] \quad \text{by the representation theorem}$$

applied to $B' \in A$.

Remark. The concept of convolution can be extended to include bicategories in the sense of Bénabou [3]. If

$$A = \{A_{xy};\ x,y \in \mathrm{Obj}\ A\}$$

is a V-bicategory, with hom-categories A_{xy}, then

$$[A,V] = \{[A_{xy},V];\ x,y \in \mathrm{Obj}\ A\}$$

is a biclosed bicategory by formulas analogous to (1.2) and (1.3). This consideration, in turn, leads to the concept of a V-pro-bicategory. The extension of the results in this article to V-bicategories is a straightforward exercise once this conceptual framework is established. The reflection theorem for biclosed bicategories is an exact analogue of the "one-object" theorem above. A form of this theorem has been

introduced by J. Meisen [19] in the study of relations in categories.

Remark on symmetry. The concept of a symmetric promonoidal category is defined in [4] §3. This produces an (obvious) analogous result-with-symmetry for each result verified in the sequel.

§2 THE COMPLETION PROCESS

For this section the base category $\mathcal{V}$ will be assumed to have all intersections of monomorphisms as well as all small limits and colimits.

To each small category $\mathbf{C}$ we will assign a completion C^* which is taken to be the largest full reflective subcategory of $[C^{op},\mathcal{V}]$ which contains the representable functors as a full strongly cogenerating subcategory. This coincides with the left-adjoint factorisation of the conjugation functor

$$[C^{op},\mathcal{V}] \to [C,\mathcal{V}]^{op};\ F \mapsto \int_C [FC, C(C,-)],$$

through a reflection followed by an isomorphism-reflecting embedding, denoted:

$$\begin{array}{ccc} [C^{op},\mathcal{V}] & \twoheadrightarrow & C^* \\ & \searrow & \downarrow \\ & & [C,\mathcal{V}]^{op}, \end{array}$$

The resulting embedding of C into its completion will be denoted by:

$$E = E(C): C \to C^*.$$

The basic properties of C^* can be verified by examining the cotriple-tower construction of C^* (Appelgate and Tierney [1] and Dubuc [10]) or by using a direct method of adjoint-functor factorisation ([9], Example 4.1). There is a closely related theorem in Wolff [22], §5.6.

The completion C^* can be described explicitly as the full subcategory of $[C^{op},\mathcal{V}]$ of functors F such that $\int_C [s_C, FC]$ is an isomorphism

whenever s is a morphism in $[C^{op},V]$ inverted by conjugation. As such, C^* is equivalent to the category of V-fractions of $[C^{op},V]$ with respect to the class of morphisms inverted by conjugation.

Remark. The assignment $C \mapsto C^*$ is canonically functorial (to within isomorphism) on functors $S: C \to B$ which have a right adjoint, and the image $S^*: C^* \to B^*$, $S^*E = ES$, has a right adjoint.

Proposition 2.1. If $N: A \to C$ is a dense functor then $EN: A \to C^*$ is dense.

This fact is established in [8] Corollary 3.2. More precisely, C^* is equivalent to the full subcategory of $[A^{op},V]$ of functors F such that $\int_A [s_A, FA]$ is an isomorphism whenever s is a morphism in $[A^{op},V]$ inverted by the restricted-conjugation functor:

$$[A^{op},V] \to [C,V]^{op};\ F \mapsto \int_A [FA, C(NA,-)].$$

Thus the reflection from $[A^{op},V]$ to C^* at $F \in [A^{op},V]$ has the value:

$$EN(F) = \int^A FA \cdot ENA,$$

and this is isomorphic to $E(\int^A FA \cdot NA)$ whenever the reflection $\int^A FA \cdot NA$ of F exists in C.

The second main observation of this section is that the completion process provides a "structural change of V-universe". Consider the case $V = S$ and $C = V$ and let S^* be a larger cartesian closed category of sets containing S and V, and any other categories that we want to regard as "small", as internal category objects. Let $W = V^*$ be the completion of V with respect to S^*. The basic properties of V^* (verified in [8]) are as follows:

Property 1. Because V^{op} is a symmetric monoidal category, the functor category $[V^{op},S^*]$ is symmetric monoidal closed and this structure extends the original structure of V. Because V is monoidal closed, the reflective embedding $W \subset [V^{op},S^*]$ is monoidal, hence the embedding $E: V \to W$ preserves tensor product and internal hom. This

implies that the symmetric monoidal closed category $\mathcal{V}$-Cat is fully embedded in $\mathcal{W}$-Cat and the embedding preserves tensor product and internal hom.

Property 2. The embedding $E: \mathcal{V} \to \mathcal{W}$ is strongly cogenerating and dense, hence it preserves coends. Thus the biclosed bicategory $\mathcal{V}$-$Prof$ (cf. [3], Introduction), consisting of $\mathcal{V}$-categories and $\mathcal{V}$-profunctors between them, is embedded (but not fully on morphisms) into the biclosed bicategory $\mathcal{W}$-$Prof$ in such a way as to preserve profunctor composition.

Property 3. Because $E: \mathcal{V} \to \mathcal{W}$ is strongly cogenerating, the completion $[A,\mathcal{V}]^*$ with respect to $\mathcal{W}$ of a functor category $[A,\mathcal{V}]$ is equivalent to $[A,\mathcal{W}]$ (as verified in [8] §4). This ensures that each $\mathcal{V}$-promonoidal structure on a small $\mathcal{V}$-category A corresponds to an essentially unique $\mathcal{W}$-promonoidal structure on A when A is regarded as a $\mathcal{W}$-category.

Because $\mathcal{W}$ has all S^*-small limits and colimits and all intersections of monomorphisms, the given base category $\mathcal{V}$ can be assumed to be an arbitrary symmetric monoidal closed category (with no completeness hypotheses on it).

Remark 2.2. If $\mathcal{V}$ is a cartesian closed category then $\mathcal{W}$ is cartesian closed and the reflection from $[\mathcal{V}^{op},S^*]$ to $\mathcal{W}$ preserves finite products ([7] Corollary 2.1). Thus finite products commute with filtered colimits in any cartesian closed category, because this is so in $[\mathcal{V}^{op},S^*]$.

§3 MONOIDAL CLOSED COMPLETION

Let $N: A^{op} \to C$ be a dense functor between two $\mathcal{V}$-categories and let $\mathcal{W}$ be a completion of $\mathcal{V}$ with respect to a cartesian closed category S^* of sets which contains S, $\mathcal{V}$, A, and C. Let $A = (A,P,I,\ldots)$ be a $\mathcal{W}$-promonoidal category.

Proposition 3.1. The reflective embedding $\mathcal{C}^* \subset [A, \mathcal{V}]$ is monoidal if and only if the functors

$$\begin{aligned} &\int_B [P(-AB),\ \mathcal{C}(NB,C)] \\ &\int_B [P(A-B),\ \mathcal{C}(NB,C)] \end{aligned} \tag{3.1}$$

have isomorphs in $\mathcal{C}^*$ for all $A \in \mathcal{A}$ and $C \in \mathcal{C}$.

Proof. This is a restatement of Proposition 1.1, with the strongly cogenerating class $\mathcal{D} \subset \mathcal{C}^*$ being $\{\mathcal{C}(N-,C);\ C \in \mathcal{C}\}$.

Remark. The objects $C \in \mathcal{C}$ in (3.1) could be restricted further to lie in any subcategory $\mathcal{D} \subset \mathcal{C}$ for which $\{\mathcal{C}(N-,D);\ D \in \mathcal{D}\}$ strongly cogenerates $\mathcal{C}^*$. For example, if $\mathcal{D} \subset \mathcal{C}$ and each object $C \in \mathcal{C}$ is the limit of some functor (depending on C) with object values in $\mathcal{D}$, then the embedding $\mathcal{D} \subset \mathcal{C}$ followed by $E: \mathcal{C} \to \mathcal{C}^*$ strongly cogenerates $\mathcal{C}^*$.

We shall consider the special case where the functors (3.1) have isomorphs in $\mathcal{C}$, and let $\mathcal{C}(N-,H(AC))$ and $\mathcal{C}(N-,K(AC))$ be their representations. The internal-hom and tensor-product operations on $\mathcal{C}^*$ then provide functors $-/-: \mathcal{C} \otimes \mathcal{C}^{op} \to \mathcal{C}^*$, $-\backslash- \mathcal{C}^{op} \otimes \mathcal{C} \to \mathcal{C}^*$, and $-\otimes-: \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}^*$ with values:

$$\begin{aligned} D/C &= \int_A [\mathcal{C}(NA,C),\ \mathcal{C}(N-,H(AD))] \\ C\backslash D &= \int_A [\mathcal{C}(NA,C),\ \mathcal{C}(N-,K(AD))] \end{aligned} \tag{3.2}$$

and

$$C \otimes D = \int^{AB} (\mathcal{C}(NA,C) \otimes \mathcal{C}(NB,D)) \cdot Q(AB) \tag{3.3}$$

where

$$Q(XY) = \int^{A} P(XYA) \cdot ENA.$$

The identity object of $\mathcal{C}^*$ is:

$$I = \int^A IA \cdot ENA. \tag{3.4}$$

Remark. Note that, by the construction of the monoidal biclosed completion $\mathcal{C}^*$, the representing objects H(AD) and K(AD) in $\mathcal{C}$ are isomorphic in $\mathcal{C}^*$ to the exponentials ED/ENA and ENA\ED respectively.

§4 MONOIDAL MONADS

Let $T = (T,\mu,\eta)$ be a monoidal monad on a monoidal category $\mathcal{B}$; that is, a monad where $T\colon \mathcal{B} \to \mathcal{B}$ has a monoidal functor structure $(T,\tilde{T},T^0)$ with respect to which $\mu\colon T^2 \to T$ and $\eta\colon 1 \to T$ are monoidal natural transformations.

Let $\mathcal{B}(T)$ denote the category of T-algebras over $\mathcal{B}$. This category exists over the base category $\mathcal{V}$ when $\mathcal{V}$ has the equalisers

$$\begin{array}{ccccc} \mathcal{B}(T)(CD) & \longrightarrow & \mathcal{B}(CD) & \xrightarrow{\mathcal{B}(\xi,1)} & \mathcal{B}(TC,D) \\ & & {\scriptstyle T_{CD}}\searrow & & \nearrow{\scriptstyle \mathcal{B}(1,\xi)} \\ & & & \mathcal{B}(TC,TD) & \end{array}$$

for all T-algebras (C,ξ) and (D,ξ) (as constructed in [10] or [16]).

The monoidal axioms on (T,μ,η) imply that the Kleisli category $\mathcal{K}(T)$ has a monoidal structure; the tensor-product functor

$$\tilde{\otimes}\colon \mathcal{K}(T) \otimes \mathcal{K}(T) \to \mathcal{K}(T)$$

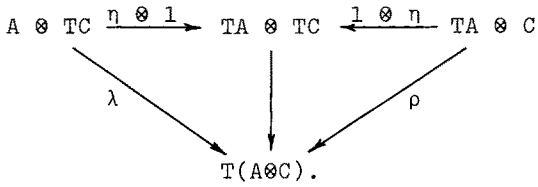
is defined on objects as it is in $\mathcal{B}$ and on morphism objects by the components:

$$\begin{array}{ccccc} \mathcal{K}(T)(B,C) & & \xrightarrow{A\ \tilde{\otimes}\ -} & & \mathcal{K}(T)(A\otimes B, A\otimes C) \\ \| & & & & \| \\ \mathcal{B}(B,TC) & \xrightarrow[A\otimes -]{} & \mathcal{B}(A\otimes B, A\otimes TC) & \xrightarrow[\mathcal{B}(1,\lambda)]{} & \mathcal{B}(A\otimes B, T(A\otimes C)) \end{array}$$

and

$$\begin{array}{ccccc} \mathcal{K}(T)(A,C) & & \xrightarrow{-\ \tilde{\otimes}\ B} & & \mathcal{K}(T)(A\otimes B, C\otimes B) \\ \| & & & & \| \\ \mathcal{B}(A,TC) & \xrightarrow[-\otimes B]{} & \mathcal{B}(A\otimes B, TC\otimes B) & \xrightarrow[\mathcal{B}(1,\rho)]{} & \mathcal{B}(A\otimes B, T(C\otimes B)), \end{array}$$

where λ and ρ are the left and right actions of $\mathcal{B}$ on T defined by commutativity of the triangles:



It is easily verified that the interchange law

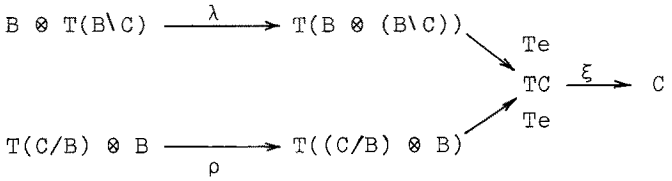
$$\tilde{T} = \mu \cdot T\lambda \cdot \rho = \mu \cdot T\rho \cdot \lambda$$

holds and this is equivalent to $\tilde{\otimes}$ being a V-bifunctor.

If T is a symmetric monoidal monad on a symmetric monoidal category $\mathcal{B}$ then the interchange law corresponds to the "commutativity law" of Kock [15]. The resulting monoidal structure on $K(T)$ is then symmetric.

Lemma 4.1. If a T-algebra is exponentiable in $\mathcal{B}$ then the exponents are evaluationwise T-algebras.

Proof. If (C,ξ) is a T-algebra and B is an object of $\mathcal{B}$ then the exponents C/B and $B\backslash C$ have algebra structures defined as the exponential transforms of the morphisms:



where e denotes the respective evaluation transformations. The algebra axioms for these structures follow directly from those for ξ.

Thus, if all T-algebras are exponentiable in $\mathcal{B}$ then we obtain adjoint actions of $\mathcal{B}^{op}$ on $\mathcal{B}(T)$.

Proposition 4.2. The completion $\mathcal{B}(T)^*$ is monoidal biclosed if each T-algebra is exponentiable in $\mathcal{B}$.

Proof. Let (C,ξ) be a T-algebra. Then the adjunctions:

$$\mathcal{B}(-\otimes B,C) \cong \mathcal{B}(-,C/B)$$

$$\mathcal{B}(B\otimes -,C) \cong \mathcal{B}(-,B\backslash C)$$

provide adjunctions:

$$\mathcal{B}(T)(F(-\otimes B),C) \cong \mathcal{B}(T)(F-,C/B)$$

$$\mathcal{B}(T)(F(B\otimes -),C) \cong \mathcal{B}(T)(F-,B\backslash C) \quad ,$$

where $F\colon \mathcal{B} \to \mathcal{B}(T)$ is the free-algebra functor. If these isomorphisms remain natural when F is extended to the dense comparison functor $N\colon K(T) \longrightarrow \mathcal{B}(T)$ (see §8) then $\mathcal{B}(T)^*$ is monoidal biclosed by Proposition 3.1. By Proposition 8.2, it suffices to show (working out one of the adjunctions) that the composites:

$$N(T(C/B)\ \tilde{\otimes}\ B) \underset{N(T\xi\tilde{\otimes}1)}{\overset{N(\mu\tilde{\otimes}1)}{\rightrightarrows}} N((C/B)\ \tilde{\otimes}\ B) \xrightarrow{Te} TC \xrightarrow{\xi} C$$

are equal. On composing both composites with η and filling in the definitions of $\tilde{\otimes}$ and $\xi_{C/B}$, we obtain a commuting diagram pair:

$$\begin{array}{ccccc}
T(C/B)\otimes B & & \xrightarrow{\quad\eta\quad} & & T(T(C/B)\otimes A) \\
{\scriptstyle \eta\otimes 1}\downarrow & & & & \\
T^2(C/B)\otimes B & & & & \downarrow{\scriptstyle N(\mu\tilde{\otimes}1)}\quad\downarrow{\scriptstyle N(T\xi\tilde{\otimes}1)} \\
{\scriptstyle \mu\otimes 1}\downarrow\ \downarrow{\scriptstyle T\xi\otimes 1} & & & & \\
T(C/B)\otimes B & & \xrightarrow{\quad\rho\quad} & & T((C/B)\otimes B) \\
{\scriptstyle \bar{\xi}_{C/B}}\downarrow & & & & \downarrow{\scriptstyle Te} \\
C & & \xrightarrow{\quad\xi\quad} & & TC
\end{array} \quad ,$$

where $\bar{\xi}_{C/B}$ is the adjoint-transform of $\xi_{C/B}$. Then the lefthand side is the adjoint-transform of

$$T(C/B) \xrightarrow{\eta} T^2(C/B) \underset{T\xi}{\overset{\mu}{\rightrightarrows}} T(C/B) \xrightarrow{\xi} C/B$$

and these composites are equal because $\xi_{C/B}$ is an algebra structure. This completes the proof.

Under the hypothesis of Proposition 4.2 we can determine when the trace of $\mathcal{B}(T)^*$ exists on $\mathcal{B}(T)$. First, for each T-algebra (C,ξ), there are natural transformations

$$\overline{Te\cdot\lambda}:\quad C/B \longrightarrow TC/TB$$

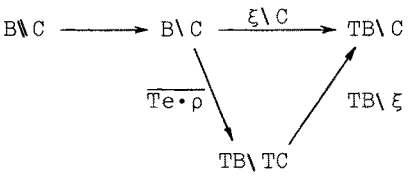
$$\overline{Te\cdot\rho}:\quad B\backslash C \longrightarrow TB\backslash TC,$$

corresponding to the transformations:

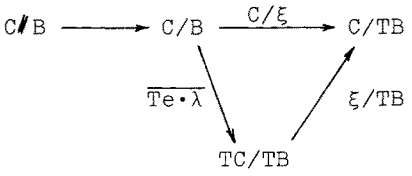
$$(C/B)\otimes TB \xrightarrow{\lambda} T((C/B)\otimes B) \xrightarrow{Te} TC$$

$$TB\otimes(B\backslash C) \xrightarrow{\rho} T(B\otimes(B\backslash C)) \xrightarrow{Te} TC.$$

Proposition 4.3. The category $\mathcal{B}(T)$ is biclosed in $\mathcal{B}(T)^*$ if and only if the equalisers:



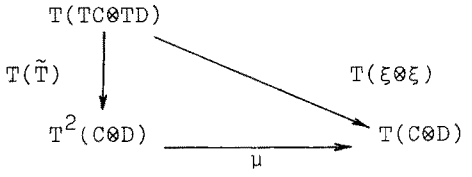
and



exist in $\mathcal{B}$ for all T-algebras (B,ξ) and (C,ξ).

Proof. Because $K(T)$ is monoidal, the unit object NI lies in $\mathcal{B}(T)$. The equalisers are precisely the ends (3.2) by Proposition 8.2.

Proposition 4.4. If the reflective embedding $\mathcal{B}(T)^* \subset [K(T)^{op},\mathcal{W}]$ is monoidal then the tensor product in $\mathcal{B}(T)^*$ of two algebras (C,ξ) and (D,ξ) lies in $\mathcal{B}(T)$ if and only if the coequaliser of the reflective pair:



exists in $\mathcal{B}(T)$.

Proof. The pair has a common right inverse $T(\eta\otimes\eta)$; that is, it

is reflective. Moreover, the coequaliser in $\mathcal{B}(T)$ is then the joint coequaliser of the pairs:

$$\begin{array}{ccccc} T(C\otimes TD) & \xrightarrow{T\lambda} & T^2(C\otimes D) & \xleftarrow{T\rho} & T(TC\otimes D) \\ & T(1\otimes\xi) \searrow & \downarrow \mu & \swarrow T(\xi\otimes 1) & \\ & & T(C\otimes D) & & \end{array} .$$

By Proposition 8.2, this coequaliser is the iterated coend (3.3);

$$C \otimes D = \int^{AB} (\mathcal{B}(T)(NA,C) \otimes \mathcal{B}(T)(NB,D)) \cdot N(A\tilde{\otimes}B)$$

in $\mathcal{B}(T)$. Because $E\colon \mathcal{B}(T) \to \mathcal{B}(T)^*$ preserves and reflects colimits, this completes the proof.

The preceding propositions provide an alternative approach to earlier work by Kock [16] and Linton [18]. The use of completion leads to a significant reduction in coherence computations.

Another result of Kock [17], Theorem 2.6, may be established using the completion method. Namely, if the base category $\mathcal{V}$ is cartesian closed and $\mathcal{B}$ is cartesian closed over $\mathcal{V}$ then the category $\mathcal{B}(T)$ is cartesian closed if the functor $T\colon \mathcal{B} \to \mathcal{B}$ preserves finite products. This follows immediately from the fact that $K(T)$ has the cartesian monoidal structure, hence the completion $\mathcal{B}(T)^* \subset [K(T)^{op}, \mathcal{W}]$ is cartesian closed.

As mentioned earlier, the Kleisli category $K(T)$ of a symmetric monoidal monad T has a symmetric monoidal structure, hence the completed structure on $\mathcal{B}(T)^*$ is symmetric. For a nonsymmetric monoidal monad on a symmetric monoidal closed category $\mathcal{B}$ the tensor product might exist on $\mathcal{B}(T)$ but have no symmetry.

Example. Let $\mathcal{V} = \mathcal{S}$ and let $\phi\colon \mathbb{N} \to \Delta$ be the monoidal inclusion of the discrete category $\mathbb{N}$ of finite integers into the simplical category Δ, both with the ordinal-sum monoidal structure. If $[\mathbb{N},\mathcal{S}]$ is given the convolution monoidal structure then the structure generated on $[\Delta^{op},\mathcal{S}]$ by the (monadic) adjoint pair

$$\phi \longrightarrow [\phi^{op},1] : [\mathbb{\Delta}^{op},S] \longrightarrow [\mathbb{N},S]$$

is isomorphic to the convolution monoidal structure on $[\mathbb{\Delta}^{op},S]$. This structure extends the non-symmetric ordinal-sum structure on $\mathbb{\Delta}$.

§5 BICLOSED FUNCTOR CATEGORIES

When A and B are two V-categories the category $[A,B]$ of all V-functors from A to B exists as a V-category if V is suitably complete; that is, if the end:

$$[A,B](F,G) = \int_A B(FA,GA)$$

exists in V for all $F,G \in [A,B]$.

Lemma 5.1. If the category B admits the V-tensor-copowers $A(A,-)\cdot B$ for all $A \in A$ and $B \in B$ then the Yoneda functor

$$N: A^{op} \otimes B \to [A,B]; \quad N(AB) = A(A,-)\cdot B,$$

is dense.

Proof.

$$\int_{A,B} [[A,B](N(AB),F), [A,B](N(AB),G)]$$

$$\cong \int_{A,B} [B(B,FA),B(B,GA)]$$

by definition of N,

$$\cong \int_A B(FA,GA)$$

by the representation theorem applied to $B \in B$,

$$= [A,B](F,G), \text{ as required.}$$

Remark. This result was established by F. Ulmer ([20], Theorem 1.33) for the case where V is the symmetric monoidal closed category Ab of abelian groups and group homomorphisms, with the usual tensor product of abelian groups. In fact, Ulmer considers a more general Yoneda functor $N: A^{op} \otimes \bar{B} \to [A,B]$, $N(AB) = A(A,-)\cdot MB$,

where $M: \bar{\mathcal{B}} \to \mathcal{B}$ is an Ab-dense functor.

In order to investigate the existence of a monoidal biclosed structure on $[A,\mathcal{B}]$, first take $\mathcal{W}$ to be the completion of $\mathcal{V}$ with respect to a cartesian closed category S^* of sets which contains A, $\mathcal{B}$, $\mathcal{V}$, and S, as "small" category objects. Note that the assumption that $A,\mathcal{B}$ exists as a $\mathcal{V}$-category can be avoided by the use of such a completion $\mathcal{W}$.

Let $A = (A,P,J,\ldots)$ be a $\mathcal{W}$-promonoidal structure on A and let $\mathcal{B}$ be a monoidal biclosed category relative to $\mathcal{V}$. Then the monoidal structure on $\mathcal{B}$ provides a promonoidal structure on $\mathcal{B}^{op}$ and consequently $A \otimes \mathcal{B}^{op}$ has the tensor-product of promonoidal structures given by the expressions:

$$P((X,B),(Y,C),(-,-)) = P(XY-) \otimes \mathcal{B}(-,B\otimes C)$$

$$I(X,B) = JX \otimes \mathcal{B}(B,I).$$

Proposition 5.2. The completion $[A,\mathcal{B}]^*$ of $[A,\mathcal{B}]$ with respect to $\mathcal{W}$ is monoidal biclosed if the functor

$$\int_A [P(XYA),\mathcal{B}(-,GA)]: \mathcal{B}^{op} \to \mathcal{W}$$

is representable for all $X,Y \in A$ and $G \in [A,\mathcal{B}]$.

Proof. The first functor of (3.1), with the object $(a,b) \in A^{op} \otimes \mathcal{B}$ marking the variable position, becomes:

$$\int_{XC} [P((a,b),(A,B),(X,C)),[A,\mathcal{B}](N(XC),G)]$$

$$= \int_{XC} [P(aAX) \otimes \mathcal{B}(C,b\otimes B), \int_Y \mathcal{B}(A(X,Y)\cdot C,GY)]$$

by the definitions of N and $A \otimes \mathcal{B}^{op}$,

$$\cong \int_X [P(aAX), \mathcal{B}(b\otimes B,GX)]$$

by the representation theorem applied $Y \in A$ and $C \in \mathcal{B}$,

$$\cong \int_X [P(aAX), \mathcal{B}(b,GX/B)]$$

because $\mathcal{B}$ is biclosed,

$$\cong \mathcal{B}(b, H(A,B,G)(a))$$

by the representability hypothesis,

$$\cong \int_X \mathcal{B}(A(a,X)\cdot b,\ H(A,B,G)(X))$$

by the representation theorem applied to $X \in \mathbf{A}$,

$$= [\mathbf{A},\mathcal{B}](N(ab),\ H(A,B,G)) \text{ by definition of } N,$$

as required. A similar computation reduces the second end in (3.1) to the required form.

Henceforth we assume that the hypothesis of Proposition 5.2 is satisfied, with natural isomorphisms:

$$\left.\begin{aligned} \int_A [P(XYA), \mathcal{B}(-, GA/B)] &\cong \mathcal{B}(-, H(Y,B,G)(X)) \\ \int_A [P(XYA), \mathcal{B}(-, B\backslash GA)] &\cong \mathcal{B}(-, K(X,B,G)(Y)) \end{aligned}\right\} \quad (5.1)$$

for chosen representations H and K.

Proposition 5.3. The category $[\mathbf{A},\mathcal{B}]$ is biclosed in $[\mathbf{A},\mathcal{B}]^*$ if and only if the ends:

$$\left.\begin{aligned} &\int_A H(A,FA,G) \\ &\int_A K(A,FA,G) \end{aligned}\right\}$$

exist in $[\mathbf{A},\mathcal{B}]$ for all $F,G \in [\mathbf{A},\mathcal{B}]$.

Proof. The internal-hom values (3.2) can be reduced as follows:

$$G/F = \int_{AC} [[\mathbf{A},\mathcal{B}](N(AC),F),\ [\mathbf{A},\mathcal{B}](N-, H(A,C,G))]$$

$$\int_{AC} [[\mathbf{A},\mathcal{B}](A(A,-)\cdot C, F),\ [\mathbf{A},\mathcal{B}](N-, H(A,C,G))]$$

by the definition of N,

$$\cong \int_{AC} [\mathcal{B}(C, FA),\ [\mathbf{A},\mathcal{B}](N-, H(A,C,G))]$$

by the representation theorem applied to $A \in \mathbf{A}$,

$$\cong \int_A [\mathbf{A},\mathcal{B}](N-, H(A,FA,G))$$

by the representation theorem applied to $C \in \mathcal{B}$,

$$\cong [A,\mathcal{B}](N-, \int_A H(A,FA,G)) \text{ if and only if the end}$$

$\int_A H(A,FA,G)$ exists in $[A,\mathcal{B}]$.

Similarly, we obtain:

$$F\backslash G \cong [A,\mathcal{B}](N-, \int_A K(A,FA,G)),$$

and this completes the proof.

Thus the resulting internal-hom functors on $[A,\mathcal{B}]$ have values:

$$\left.\begin{aligned} G/F &= \int_A H(A,FA,G) \\ F\backslash G &= \int_A K(A,FA,G) \end{aligned}\right\} \qquad (5.2)$$

and, by the definitions of H and K in Proposition 5.2, there exist natural isomorphisms:

$$G/N(AB) \cong H(A,B,G)$$

$$N(AB)\backslash G \cong K(A,B,G).$$

The internal-hom functors have an identity object if the identity of $[A,\mathcal{B}]^*$ lies in $[A,\mathcal{B}]$; this identity has the value:

$$I = \int^{AB} I(AB)\cdot EN(AB) \qquad \text{by (3.4)},$$

$$= \int^{AB} (JA \otimes \mathcal{B}(B,I))\cdot(A(A,-)\cdot B)$$

$\cong J\cdot I$ in $[A,\mathcal{B}]^*$, by the representation theorem applied to $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

The tensor product in $[A,\mathcal{B}]^*$ of two functors $F,G \in [A,\mathcal{B}]$ has the value:

$$F \otimes G = \int^{XBYC} (\mathcal{B}(B,FX) \otimes \mathcal{B}(C,GY))\cdot Q(X,B,Y,C) \text{ by (3.3)},$$

$$\cong \int^{XY} Q(X,FX,Y,GY)$$

by the representation theorem applied to $B,C \in \mathcal{B}$, where

$$Q(X,FX,Y,GY) = \int^{AB} (P(XYA) \otimes \mathcal{B}(B,FX\otimes GY))\cdot EN(AB)$$

by definition of the promonoidal category $\mathcal{A} \otimes \mathcal{B}^{op}$,

$$= \int^{AB} (P(XYA) \otimes \mathcal{B}(B,FX\otimes GY))\cdot E(A(A,-)\cdot B)$$

by definition of N,

$$\cong \int^{AB} (P(XYA) \otimes \mathcal{B}(B, FX \otimes GY)) \cdot (A(A,-) \cdot EB)$$

because E preserves tensoring,

$$\cong P(XY-) \cdot (FX \otimes GY)$$

by the representation theorem applied to $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

This establishes the following:

Proposition 5.4. The monoidal structure on $[\mathcal{A},\mathcal{B}]^*$ admits a restriction to $[\mathcal{A},\mathcal{B}]$ if the coend

$$F \otimes G = \int^{XY} P(XY-) \cdot (FX \otimes GY) \qquad (5.3)$$

exists in $[\mathcal{A},\mathcal{B}]$ for all $F,G \in [\mathcal{A},\mathcal{B}]$, and the identity object $J \cdot I$ of $[\mathcal{A},\mathcal{B}]^*$ lies in $[\mathcal{A},\mathcal{B}]$.

The coherence of the monoidal and biclosed structures induced on $[\mathcal{A},\mathcal{B}]$ is a consequence of the coherence of the monoidal biclosed structure on $[\mathcal{A},\mathcal{B}]^*$. In the case where $\mathcal{B} = \mathcal{V}$, the formulas (5.2) and (5.3) reproduce the original convolution structure (1.3) and (1.2) on $[\mathcal{A},\mathcal{V}]$.

Remark. It is straightforward to verify, using (5.3), that if $\mathcal{A}$ and $\mathcal{A}'$ are two small promonoidal categories and if $\mathcal{B}$ is a sufficiently complete and cocomplete monoidal biclosed category then the canonical isomorphism of categories:

$$[\mathcal{A} \otimes \mathcal{A}', \mathcal{B}] \cong [\mathcal{A}, [\mathcal{A}', \mathcal{B}]]$$

admits enrichment to an isomorphism of monoidal biclosed categories, where $\mathcal{A} \otimes \mathcal{A}'$ has the tensor-product promonoidal structure and each functor category has the "convolution" monoidal biclosed structure defined by (5.2) and (5.3).

§6 BICLOSED CATEGORIES OF CONTINUOUS FUNCTORS

If A is a promonoidal category and B is a monoidal biclosed category for which the internal-hom functors defined by (5.2) exist on the functor category $[A,B]$ then we can ask whether a given full subcategory of $[A,B]$ is closed under all exponentiation by these internal-hom functors. The work of Bastiani-Ehresmann [2] on closed categories of sketched structures leads us to examine this question for those full subcategories of $[A,B]$ which consist of the Z-continuous functors from A to B for a given class Z of morphisms in $[A,V]$. Again, the base category is assumed to be V.

For a given category C let C^m denote the class of all morphisms in C. For each $Z \subset C^m$ let C_Z denote the full subcategory of C consisting of the objects orthogonal to Z (following terminology of [12]); that is,

$$C_Z = \{C \in C; C(s,C) \text{ is an isomorphism in } V \text{ for all } s \in Z\}.$$

Let $\bar{Z}$ denote the orthogonal closure of Z in C^m; that is,

$$\bar{Z} = \{s \in C^m; C(s,C) \text{ is an isomorphism in } V \text{ for all } C \in C_Z\}.$$

Then $C_Z = C_U = C_{\bar{Z}}$ for all $Z \subset U \subset \bar{Z}$.

For a given monoidal structure on the category C, a class $Z \subset C^m$ is called monoidal if $C \in C$ and $s \in Z$ imply that $C \otimes s \in Z$ and $s \otimes C \in Z$; that is, if Z is stable under the monoidal action of C on both sides of C^m.

Remark. It was established in [8] §1 that if Z is monoidal with respect to C then the category $C(Z^{-1})$ of V-fractions of C with respect to Z (as constructed in Wolff [21]) has a monoidal structure such that each monoidal functor on C which inverts the elements of Z has a unique monoidal-functor factorisation through the projection $C \to C(Z^{-1})$.

Proposition 6.1. If A is a strongly generating subcategory of a monoidal biclosed category C and $Z \subset C^m$ then the following are

equivalent:

(a) $A \in \mathcal{A}$ and $s \in Z$ imply $A \otimes s \in \bar{Z}$ and $s \otimes A \in \bar{Z}$.

(b) $\mathcal{C}_Z$ is closed under exponentiation in $\mathcal{C}$.

(c) $\bar{Z}$ is monoidal.

Proof. (a) $\Rightarrow$ (b). If $C \in \mathcal{C}_Z$ then $\mathcal{C}(A \otimes s, C) \cong \mathcal{C}(A, C/s)$ and $\mathcal{C}(s \otimes A, C) \cong \mathcal{C}(A, s \backslash C)$ are isomorphisms for all $A \in \mathcal{A}$ and $s \in Z$ by (a). Thus C/s and $s\backslash C$ are isomorphisms for all $s \in Z$ and $C \in \mathcal{C}_Z$ because $\mathcal{A}$ strongly generates $\mathcal{C}$. Hence $\mathcal{C}(s, B \backslash C) \cong \mathcal{C}(B, C/s)$ and $\mathcal{C}(s, C/B) \cong \mathcal{C}(B, s \backslash C)$ are isomorphisms for all $B \in \mathcal{C}$ and $s \in Z$, which implies that $B\backslash C$ and C/B both lie in $\mathcal{C}_Z$ for all $B \in \mathcal{C}$ and $C \in \mathcal{C}_Z$.

(b) $\Rightarrow$ (c). If $B \in \mathcal{C}$ and $s \in \bar{Z}$ then $\mathcal{C}(B \otimes s, C) \cong \mathcal{C}(s, B \backslash C)$ and $\mathcal{C}(s \otimes B, C) \cong \mathcal{C}(s, C/B)$ are isomorphisms for all $C \in \mathcal{C}_Z$ by (b). Thus $B \otimes s$ and $s \otimes B$ are in $\bar{Z}$, as required.

(c) $\Rightarrow$ (a) because $\mathcal{A} \subset \mathcal{C}$.

Let $\mathcal{C}$ be a functor category of the form $[\mathcal{A}, \mathcal{V}]$ and let $Z \subset [\mathcal{A}, \mathcal{V}]^m$. Then we can form (as is done in [13], §8.1) the full subcategory $[\mathcal{A}, \mathcal{B}]_Z \subset [\mathcal{A}, \mathcal{B}]$ consisting of the functors $F \in [\mathcal{A}, \mathcal{B}]$ such that $\mathcal{B}(B, F-) \in [\mathcal{A}, \mathcal{V}]_Z$ for all $B \in \mathcal{B}$. Such a functor F is interpreted as a "Z-continuous" model of $\mathcal{A}$ in $\mathcal{B}$. Clearly $[\mathcal{A}, \mathcal{B}]_Z = [\mathcal{A}, \mathcal{B}]_U = [\mathcal{A}, \mathcal{B}]_{\bar{Z}}$ for all $Z \subset U \subset \bar{Z}$.

We now suppose that $\mathcal{A} = (\mathcal{A}, P, J, \ldots)$ is a small promonoidal category over $\mathcal{V}$ and that $\mathcal{B}$ is a monoidal biclosed category over $\mathcal{V}$ for which the internal-hom functors defined by (5.2) exist on the functor category $[\mathcal{A}, \mathcal{B}]$.

Proposition 6.2. If Z is a monoidal class of morphisms in the convolution functor category $[\mathcal{A}, \mathcal{V}]$ then $[\mathcal{A}, \mathcal{B}]_Z$ is closed under exponentiation in $[\mathcal{A}, \mathcal{B}]$.

Proof. For all $s \in Z$, $B \in \mathcal{B}$, $F \in [\mathcal{A}, \mathcal{B}]$ and $G \in [\mathcal{A}, \mathcal{B}]_Z$, we have:

$$\int_X [s_X, \mathcal{B}(B,(G/F)(X))]$$

$$\cong \int_X [s_X, \mathcal{B}(B, \int_Y H(Y,FY,G)(X))] \qquad \text{by (5.2)},$$

$$\cong \int_Y \int_X [s_X, \mathcal{B}(B,H(Y,FY,G)(X))]$$

by interchanging limits,

$$\cong \int_Y \int_X [s_X, \int_A [P(XYA), \mathcal{B}(B,GA/FY)]] \text{ by (5.1)},$$

$$\cong \int_Y (\int_A [\int^X s_X \otimes P(XYA), \mathcal{B}(B \otimes FY, GA)])$$

by interchanging limits and using the tensor-hom adjunctions of $\mathcal{V}$ and $\mathcal{B}$,

$$\cong \int_Y (\int_A [(s \otimes A(Y,-))(A), \mathcal{B}(B \otimes FY, GA)]) \quad (*)$$

because

$$(s \otimes A(Y,-))(A) = \int^{XX'} s_X \otimes A(Y,X') \otimes P(XX'A) \text{ by (1.2)},$$

$$\cong \int^X s_X \otimes P(XYA)$$

by the representation theorem applied to $X' \in \mathcal{A}$. The morphism (*) is an end over $Y \in \mathcal{A}$ of isomorphisms because $s \otimes A(Y,-) \in Z$ since Z is monoidal and $G \in [\mathcal{A},\mathcal{B}]_Z$ by hypothesis. Thus G/F and, dually, $F\backslash G$ both lie in $[\mathcal{A},\mathcal{B}]_Z$ for all $F \in [\mathcal{A},\mathcal{B}]$ and $G \in [\mathcal{A},\mathcal{B}]_Z$.

Corollary 6.3. If Z is a class of morphisms in $[\mathcal{A},\mathcal{V}]$ for which $[\mathcal{A},\mathcal{V}]_Z$ is closed under exponentiation in the convolution functor category $[\mathcal{A},\mathcal{V}]$ then $[\mathcal{A},\mathcal{B}]_Z$ is closed under exponentiation in $[\mathcal{A},\mathcal{B}]$.

Proof. If $[\mathcal{A},\mathcal{V}]_Z$ is closed under exponentiation in $[\mathcal{A},\mathcal{V}]$ then $\bar{Z}$ is monoidal by Proposition 6.1. Thus $[\mathcal{A},\mathcal{B}]_Z$, which coincides with $[\mathcal{A},\mathcal{B}]_{\bar{Z}}$, is closed under exponentiation in $[\mathcal{A},\mathcal{B}]$ by Proposition 6.2.

Thus, if the Z-continuous models in $\mathcal{V}$ of a promonoidal category $\mathcal{A}$ form an exponentially biclosed subcategory of the convolution $[\mathcal{A},\mathcal{V}]$ then so do the Z-continuous models of $\mathcal{A}$ in any suitably complete monoidal biclosed category $\mathcal{B}$ which is based on $\mathcal{V}$.

This result contains an alternative approach to constructions in [2], Chapter III. For example (cf. [2] §12), let $\mathcal{V} = \mathcal{S}$ and let $\mathbb{\Delta}$ be the simplicial category. Then the category $\mathcal{Cat}$ of small categories and functors is fully embedded in $[\mathbb{\Delta}^{op},\mathcal{S}]$ as a cartesian closed reflective subcategory. This implies that $\mathcal{Cat}$ is defined by a (cartesian) monoidal class of morphisms in $[\mathbb{\Delta}^{op},\mathcal{S}]$; namely, the class of <u>all</u> morphisms inverted by the reflection. It now follows from Corollary 6.3 that the category $[\mathbb{\Delta}^{op},\mathcal{B}]_Z$ of all category objects in $\mathcal{B}$ is closed under exponentiation in $[\mathbb{\Delta}^{op},\mathcal{B}]$ whenever $[\mathbb{\Delta}^{op},\mathcal{B}]$ admits the internal-hom functors (5.2). Moreover, if the monoidal biclosed structure considered in §5 exists on $[\mathbb{\Delta}^{op},\mathcal{B}]$ and the embedding

$$[\mathbb{\Delta}^{op},\mathcal{B}]_Z \subset [\mathbb{\Delta}^{op},\mathcal{B}]$$

has a left adjoint then this adjunction admits monoidal enrichment by the reflection theorem of §1. Note that, by Proposition 6.1, Z may be replaced by <u>any</u> class of morphisms in $[\mathbb{\Delta}^{op},\mathcal{S}]$ which defines $\mathcal{Cat}$ as its class of orthogonal objects.

<u>Remark</u>. The general question of the existence of a left-adjoint functor to an inclusion of the form $[\mathcal{A},\mathcal{B}]_Z \subset [\mathcal{A},\mathcal{B}]$ has been studied in some detail by Freyd-Kelly [12] and by Gabriel-Ulmer [13] §8. Results bearing on special cases of this problem have appeared in [1], [9], [10], and [22].

§7 COMPLETION OF FUNCTOR CATEGORIES

This section is supplementary to §5 and the same $\mathcal{V}$ and $\mathcal{W}$ will be used. We will verify that the completion $[\mathcal{A},\mathcal{B}]^*$ of a functor category $[\mathcal{A},\mathcal{B}]$ is equivalent to the functor category $[\mathcal{A},\mathcal{B}^*]$ when the Yoneda functor

$$N\colon \mathcal{A}^{op} \otimes \mathcal{B} \to [\mathcal{A},\mathcal{B}];\ N(AB) = \mathcal{A}(A,-)\cdot B,$$

exists.

First recall from §2 that $[\mathcal{A},\mathcal{B}]^*$ is characterised as the

largest full reflective subcategory of $[[A,\mathcal{B}]^{op},\mathcal{W}]$ containing the representable functors as a strongly cogenerating subcategory and that $\mathcal{B}^*$ is the largest full reflective subcategory of $[\mathcal{B}^{op},\mathcal{W}]$ containing the representable functors as a strongly cogenerating subcategory.

Proposition 7.1. The full embedding

$$[1,E]: [A,\mathcal{B}] \to [A,\mathcal{B}^*]$$

is dense and strongly cogenerating.

Proof. Because E is a full embedding, so is $[1,E]$. The functor $[1,E]$ is strongly cogenerating because the class $\{EB;\ B \in \mathcal{B}\}$ strongly cogenerates $\mathcal{B}^*$, hence the class $\{A(-,A),EB];\ A \in A,\ B \in \mathcal{B}\}$ of $\mathcal{V}$-cotensor powers in $[A,\mathcal{B}^*]$ strongly cogenerates $[A,\mathcal{B}^*]$. Moreover, because $\mathcal{B}^*$ is $\mathcal{W}$-cocomplete, so is $[A,\mathcal{B}^*]$. This yields the adjunction:

$$\begin{array}{ccc} [[A,\mathcal{B}]^{op},\mathcal{W}] & \underset{R}{\leftarrow} & \\ & \rightarrow & \overline{[1,E]} \\ Y \uparrow & & \\ [A,\mathcal{B}] & \xrightarrow[[1,E]]{} & [A,\mathcal{B}^*] \end{array}$$

where $\overline{[1,E]}$ is the left Kan extension of $[1,E]$ along the Yoneda embedding Y, and R is its right adjoint. The counit of this adjunction has the canonical components:

$$\int^F [A,\mathcal{B}^*](EF,G)\cdot EFA \to GA.$$

It follows from the representation theorem (on taking $G = EF$ and evaluating at the identity morphism of EF) that this transformation coincides with the following composite of isomorphisms:

$$\int^F [A,\mathcal{B}^*](EF,G)\cdot EFA \cong (\int^F [A,\mathcal{B}^*](EF,G))\cdot(\int^B \mathcal{B}(B,FA)\cdot EB)$$

by the representation theorem,

$$\cong \int^B (\int^F [A,\mathcal{B}^*](EF,G) \otimes \mathcal{B}(B,FA)))\cdot EB$$

on interchanging colimits,

$$\cong \int^B (\int^F ([A,\mathcal{B}^*](EF,G) \otimes [A,\mathcal{B}]\ (N(AB),F)))\cdot EB$$

by definition of N,

$$\cong \int^B ([A,\mathcal{B}^*](E(N(AB)),G)\cdot EB)$$

by the representation theorem,

$$\cong \int^B ([A,\mathcal{B}^*](A(A,-)\cdot EB,G)\cdot EB$$

because E preserves tensoring,

$$\cong \int^B \mathcal{B}^*(EB,GA)\cdot EB$$

by the representation theorem,

$$\cong GA \text{ because E is dense.}$$

Thus $\overline{[1,E]} \dashv R$ is a full reflective embedding hence the embedding $[1,E]$ is dense.

Proposition 7.2. $[A,\mathcal{B}^*]$ is a full reflective subcategory of $[A,\mathcal{B}]^*$.

Proof. This follows from Proposition 7.1 and the fact that $[A,\mathcal{B}]^*$ contains every dense and strongly cogenerated completion of $[A,\mathcal{B}]$ as a full reflective subcategory.

Proposition 7.3. $[A,\mathcal{B}^*]$ is equivalent to $[A,\mathcal{B}]^*$.

Proof. Because N is dense (Lemma 5.1), both categories are fully embedded in $[A\otimes\mathcal{B}^{op},\mathcal{W}]$ by the restriction functor

$$[N,1] : [[A,\mathcal{B}]^{op},\mathcal{W}] \to [A\otimes\mathcal{B}^{op},\mathcal{W}].$$

Consider the following diagram:

$$\begin{array}{ccccccc}
[A,[\mathcal{B}^{op},\mathcal{W}]] & \cong & [A\otimes\mathcal{B}^{op},\mathcal{W}] & \underset{[N,1]}{\overset{\bar{N}}{\rightleftarrows}} & [[A,\mathcal{B}]^{op},\mathcal{W}] & & \\
 & & Y\uparrow & & Y\uparrow & R\nwarrow\ \overline{[1,E]}\searrow & \\
 & & A^{op}\otimes\mathcal{B} & \xrightarrow[N]{} & [A,\mathcal{B}] & \xrightarrow[[1,E]]{} & [A,\mathcal{B}^*],
\end{array}$$

where $\bar{N} \dashv [N,1]$ and $[N,1]R$ is a full embedding whose value at $G \in [A,\mathcal{B}^*]$ is given by:

$$R(G)N(AB) = [A,\mathcal{B}^*]([1,E]N(AB),G)$$

by the definition of R,

$$= \int_X \mathcal{B}^*(A(A,X)\cdot EB, GX)$$

by the definition of N,

$$\cong \mathcal{B}^*(EB, GA)$$

by the representation theorem applied to $X \in A$.

Thus $[N,1]R$ is isomorphic to the embedding

$$[A,\mathcal{B}^*] \to [A,[\mathcal{B}^{op},\mathcal{W}]]$$

induced by the inclusion $\mathcal{B}^* \subset [\mathcal{B}^{op},\mathcal{W}]$. Recall from §2 that this inclusion is $[\mathcal{B}^{op},\mathcal{W}]_\Sigma \subset [\mathcal{B}^{op},\mathcal{W}]$ where

$$\Sigma = \{\beta \in [\mathcal{B}^{op},\mathcal{W}]^m;\ \int_B [\beta_B, \mathcal{B}(B,C)] \text{ is an isomorphism for all } C \in \mathcal{B}\}.$$

Also $[A,\mathcal{B}]^* = [A \otimes \mathcal{B}^{op},\mathcal{W}]_Z$ where

$$Z = \{\alpha \in [A\otimes\mathcal{B}^{op},\mathcal{W}]^m;\ \int_{XB} [\alpha_{XB}, \mathcal{B}(B,FX)] \text{ is an isomorphism for all } F \in [A,\mathcal{B}]\}.$$

Thus is remains to verify that HA is orthogonal to Σ for all $A \in A$ if H is orthogonal to Z. But $\int_B [\beta_B, H(AB)]$ is an isomorphism for all $A \in A$ if and only if $\int_{XB} [A(A,X)\otimes\beta_B, H(XB)]$ is an isomorphism for all $A \in A$ by the representation theorem applied to $X \in A$. Thus $\beta \in \Sigma$ implies $A(A,-)\otimes\beta \in Z$ for all $A \in A$. This completes the proof.

Corollary 7.4. If A has a $\mathcal{W}$-promonoidal structure and $\mathcal{B}$ is monoidal biclosed relative to $\mathcal{V}$ then $[A,\mathcal{B}]^*$ is monoidal biclosed.

Proof. If $\mathcal{B}$ is monoidal biclosed then so is $\mathcal{B}^*$ by the application of Proposition 3.1 with $N = 1: \mathcal{B} \to \mathcal{B}$. Thus $[A,\mathcal{B}^*]$ has the convolution monoidal biclosed structure (of §5). The result now follows from the equivalence of $[A,\mathcal{B}]^*$ to $[A,\mathcal{B}^*]$.

Remark. It follows from the above construction that the monoidal biclosed structure produced on $[A,\mathcal{B}]^*$ by the equivalence of $[A,\mathcal{B}]^*$ to $[A,\mathcal{B}^*]$ makes the inclusion $[A,\mathcal{B}]^* \subset [A\otimes\mathcal{B}^{op},\mathcal{W}]$ into a monoidal reflection. Thus the representability hypothesis of Proposition 5.2 is

unnecessary in order that $[A,B]^*$ alone should be monoidal biclosed.

§8. DENSENESS PRESENTATIONS

For a given dense functor $N: A^{op} \to C$, with A promonoidal, the monoidal (biclosed) structure:

$$C \otimes D = \int^{AB} (C(NA,C) \otimes C(NB,D)) \cdot Q(AB) \tag{3.3}$$

obtained in §3 is derived from the coend presentation:

$$C \cong \int^{A} C(NA,C) \cdot NA.$$

In applications, such as in §4 on monoidal monads, an alternative presentation of denseness may be more useful and results in a different colimit expansion of the tensor product (3.3). For this purpose we shall formulate the general concept of a denseness presentation and describe some basic examples.

Given a functor $N: A \to C$, an N-presentation of C will consist of:

(a) an index functor $J: K \to A$

(b) a coefficient functor $K: K^{op} \otimes |C| \to V$

(c) a structure coend:

$$\xi_C: \int^{k} K(k,C) \cdot NJk \cong C$$

for each $C \in |C|$, such that the induced morphism:

$$\int^{k} K(k,C) \otimes A(A,Jk) \to C(NA,C) \tag{8.1}$$

is an isomorphism for all $A \in A$ and $C \in C$. Here $|C|$ denotes the discrete V-category on C.

Remark. If A is small with respect to V and N is a full embedding then (8.1) is an isomorphism if and only if the representation functor:

$$C \to [A^{op}, V], C \mapsto C(N-,C),$$

preserves the structure coend for each $C \in C$.

Proposition 8.1 If C has an N-presentation then N is dense and the structure coend induces an isomorphism:

$$\int_A [C(NA,C), GA] \cong \int_k [K(k,C), GJk] \tag{8.2}$$

for each $C \in \mathcal{C}$ and $G\colon A^{op} \to \mathcal{V}$.

Proof. For each functor $G\colon A^{op} \to \mathcal{V}$ and object $C \in \mathcal{C}$ we have:

$$\int_A [\mathcal{C}(NA,C),GA] \cong \int_A [\int^K K(k,C) \otimes A(A,Jk),GA]$$

by hypothesis on (8.1),

$$\cong \int_k [K(k,C), \int_A [A(A,Jk),GA]]$$

on interchanging limits,

$$\cong \int_k [K(k,C),GJk]$$

by the representation theorem.

This establishes the isomorphism (8.2). On taking $G = \mathcal{C}(N-,D)$ in this isomorphism, we obtain:

$$\int_A [\mathcal{C}(NA,C),\mathcal{C}(NA,D)] \cong \int_k [K(k,C), \mathcal{C}(NJk,D)]$$

$$\cong \mathcal{C}(\int^K K(k,C) \cdot NJk, D)$$

because $\mathcal{C}(-,D)$ preserves limits,

$$\cong \mathcal{C}(C,D)$$ by hypothesis on ξ,

thus N is dense.

Remarks. (a) Neither (8.1) nor (8.2) is required to be natural in $C \in \mathcal{C}$. However, a degree of naturality may occur in practice.

(b) For N to be dense (without Condition (8.2)) it suffices that the morphism (8.1) should be an epimorphism in $\mathcal{V}$ for all $A \in \mathcal{A}$ and $C \in \mathcal{C}$.

Example (a); If $N\colon \mathcal{A} \to \mathcal{C}$ is dense then the coend presentation:

$$\int^A \mathcal{C}(NA,C) \cdot NA \cong C$$

with the "evaluation" structure is natural in $C \in \mathcal{C}$.

Example (b); If $N\colon \mathcal{A} \to \mathcal{C}$ is dense then $\mathcal{C}$ has a comma-category presentation:

$$\int^K \mathcal{C}(Qk,C) \cdot NPk \cong C$$

which is natural in $C \in \mathcal{C}$. Here P and Q denote the projections:

$$A \overset{P}{\leftarrow} N/C \overset{Q}{\rightarrow} C \ .$$

The isomorphisms (8.1) and (8.2) follow from the isomorphism:

$$\int_A \mathcal{B}(FNA,GA) \cong \int_k \mathcal{B}(FQk,GPk)$$

which holds for all functors $F: C^{op} \to \mathcal{B}$ and $G: A^{op} \to \mathcal{B}$.

Example (c); Suppose $T = (T,\mu,\eta)$ is a monad on C with Kleisli category A and algebra category $C(T)$. Let $N: A \to C(T)$ be the comparison functor. Let $\mathcal{D}$ be the free $\mathcal{V}$-category on the category:

$$\{2 \overset{m}{\underset{s}{\rightrightarrows}} 1\},$$

and let $K = \mathcal{D} \otimes C(T)$ with index functor $J: K \to C(T)$ defined by:

$$J(m,(C,\xi)) = \mu_C: T^2C \to TC$$

$$J(s,(C,\xi)) = T\xi: T^2C \to TC$$

for each T-algebra (C,ξ). The coefficient functor $K: K^{op} \otimes C(T) \to \mathcal{V}$ is then defined by:

$$K(d,(C,\xi),(D,\xi)) = I \otimes C(T)(C,D) \cong C(T)(C,D).$$

Then we have:

$$\int^{dD} K((d,D),C) \cdot NJ(d,D) \cong \int^{dD} C(T)(C,D) \cdot NJ(d,D)$$

$$\cong \int^{d} NJ(d,C)$$

by the representation theorem applied to $D \in C(T)$,

$$= \operatorname{colim} NJ(d,C)$$

$$\cong C$$

because

$$T^2C \overset{\mu}{\underset{T\xi}{\rightrightarrows}} TC \overset{\xi}{\rightarrow} C$$

is a coequaliser diagram in $C(T)$. This isomorphism is natural in $C \in C(T)$. The coequaliser is preserved by each functor $C(T)(NA,-): C(T) \to \mathcal{V}$ hence (8.1) is an isomorphism.

On applying the underlying-set functor $V: \mathcal{V} \to S$, Proposition 8.1 becomes:

Proposition 8.2. (the representation theorem for monads). The comparison functor $N: A \to C(T)$ is dense and, for each algebra $(C,\xi) \in C(T)$, the natural transformations from $C(T)(N-,C)$ to a prealgebra $G: A^{op} \to V$ correspond to the elements in the equaliser of:

$$VGC \underset{VGT\xi}{\overset{VG\mu}{\rightrightarrows}} VGTC,$$

where μ and $T\xi$ are regarded as morphisms in A.

In conclusion we note two examples of presentations which are not fully natural in $C \in C$. But first we note the following fact. Suppose the data (J,K,ξ) are given such that (8.1) is an isomorphism but ξ is not known to be an isomorphism.

Proposition 8.3. If (J,K,ξ) are such that (8.1) is an isomorphism then ξ is an isomorphism if and only if N strongly generates C and

$$C(NA, \int^k K(k,C) \cdot NJk) \cong \int^k K(k,C) \otimes A(A,Jk)$$

for all $A \in A$ and $C \in C$.

Example (d); Suppose $V = S$ and C is a category with canonical E-M factorisations for a proper E-M system on C (in the sense of [12]). Suppose that A is a full subcategory of C which is closed under E-images. Let $N: A \subset C$ denote the inclusion and let $K = A \cap M$ with

$$K: K^{op} \times M \to S$$

defined by $K(A,C) = M(A,C)$. Then

$$\int^A M(A,C) \times C(B,A) \cong C(B,C) \tag{8.3}$$

naturally in $C \in M$ for each $B \in C$. Thus, if each object $C \in C$ is the "union" of its subobjects in K in the sense that

$$\int^A M(A,C) \cdot A \cong C, \tag{8.4}$$

then $N: A \subset C$ is dense and

$$\int_{A \in A} [C(A,C), GA] \cong \int_{A \in M} [M(A,C), GA]$$

for each functor $G: A^{op} \to S$. It follows from Proposition 8.3 that the morphism (8.4) is an isomorphism if A strongly generates C and each functor $C(A,-): C \to S$ preserves the coend (8.4). For the case $C = S$, any category $A \subset S$ which is closed under surjective images in S satisfies these requirements.

Example (e); discrete presentations. We suppose that V is sufficiently cocomplete. It is always the case that if $N: A \to C$ is a functor and, for each $C \in |C|$, there exists a free V-category K_C and a functor

$$P = P_C: K_C \to A,$$

together with a colimit cone

$$\xi_C: \operatorname*{colim}_k NP(k) \cong C,$$

such that:

$$\operatorname*{colim}_k A(A,Pk) \cong C(NA,C)$$

for all $A \in A$, then (P,K,ξ) provides a presentation of C. This is achieved by putting:

$$K = \sum_{|C|} K_C \text{ and } J = (P_C): K \to A$$

and defining $K: K^{op} \otimes |C| \to V$ by:

$$\begin{cases} K(k,C) = I \text{ if } k \in K_C, \\ K(k,C) = O \text{ if } k \notin K_C. \end{cases}$$

For example, if $V = S$, then the "discrete" comma-category presentation of a dense functor is given by $K_C = N/C$ for each $C \in C$. Furthermore, given any functor $N: A \to C$, the condition

$$\operatorname*{colim}_k A(A,Pk) \cong C(NA,C)$$

is satisfied for all $A \in A$ and $C \in C$ by the representation theorem. Thus, by Proposition 8.3, N is dense if and only if N is strongly generating and

$$C(NA, \operatorname*{colim}_k NPk) \cong \operatorname*{colim}_k A(A,Pk)$$

for all $A \in A$ and $C \in C$.

In particular, let C be an S-based category and let α be a regular cardinal number. Following the terminology of Gabriel-Ulmer [13], an object $A \in C$ is called α-presentable if $C(A,-)\colon C \to S$ preserves α-filtered colimits. Let C_α denote the full subcategory of C determined by the α-presentable objects. Then, by the preceding observation, a composite inclusion $A \subset C_\alpha \subset C$ is dense if A strongly generates C and A is α-filtered. Thus the observation reproduces a result of Gabriel-Ulmer [13] Theorem 7.4; in this connection, recall that a cocomplete category C is said to be locally α-presentable if it contains a small strongly generating set of α-presentable objects.

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