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Note on monads with rank.

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Author's abstract.

If T is a monad on a category $\underline{C}$ and $N = \underline{A} \subset \underline{C}$ is a dense (= left adequate) subcategory of $\underline{C}$ then we can ask when the class of free algebras on the image of $\underline{A}$ is dense in the Eilenberg-Moore category of T. This turns out to be closely related to the concept of a monad with rank. Several sufficient conditions on T are stated and compared in this note. All categorical algebra is relative to a fixed suitably complete and cocomplete symmetric monoidal closed category.

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Introduction.

In a preceding article [1] §3 the author gave a condition on a monad T for a certain class of free algebras to be dense (= adequate) in the Eilenberg-Moore category of T. It was also shown that this condition was related to the concept of a monad with rank.

The purpose of this note is to give a similar condition which does not seem in general to be equivalent to the preceding notion of rank. Nevertheless, there are close relationships between the two conditions especially in the "classical" theory of monads with a rank.

Throughout this article it is supposed, unless otherwise stated, that all categorical algebra is relative to a given complete and cocomplete symmetric monoidal closed base category $\underline{V}=(\underline{V}, \otimes, I, \ldots)$. The basic terminology and notation is taken from [5], [2], and [1].

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§1. A "rank" condition.

First we recall the representation theorem for monads (see [1] Proposition 2.4). Let $\underline{C}$ be a category with a monad $T = (T, \mu, \eta)$ on it. Let $\underline{C}_T$ denote the Kleisli category of T and let $\underline{C}^T$ denote the Eilenberg-Moore category of T.

1.1 Proposition. (representation theorem for monads). The comparison functor $K : \underline{C}_T \longrightarrow \underline{C}^T$ is dense and, for each algebra $(C, \zeta) \in \underline{C}^T$, the natural transformations from $\underline{C}^T(K-, C)$ to a prealgebra $G : \underline{C}_T^{op} \longrightarrow \underline{V}$ correspond to the elements in the equaliser of

$$VGC \underset{VGT\zeta}{\overset{VG\mu}{\rightrightarrows}} VGTC$$

where μ and $T\zeta$ are regarded as morphisms in $\underline{C}_T$.

We now suppose that $N : \underline{A} \longrightarrow \underline{C}$ is a dense (= adequate) functor and let $\bar{\underline{A}}$ denote the full image of $FN : \underline{A} \longrightarrow \underline{C}_T$ and $\bar{N} : \bar{\underline{A}} \longrightarrow \underline{C}_T$ denote the induced functor.

1.2 Proposition. The functor $K\bar{N} : \bar{\underline{A}} \longrightarrow \underline{C}^T$ is dense if :

(a) $\underline{C}(TC, D) \longrightarrow \int_A [\underline{C}(NA, C), \underline{C}(TNA, D)]$ is an isomorphism for all $C \in \underline{C}$ and $D \in \underline{C}^T$ and

(b) $\underline{C}(T^2C, D) \longrightarrow \int_A [\underline{C}(NA, C), \underline{C}(T^2NA, D)]$ is a monomorphism for all $C \in \underline{C}$ and $D \in \underline{C}^T$.

Proof. The notation U will sometimes be omitted. Suppose $\alpha : \underline{C}^{T}(FNA, C) \longrightarrow \underline{C}^{T}(FNA, D)$ is a transformation which is natural in $FNA \in \underline{\bar{A}}$. An extension $\bar{\alpha}$ is defined by commutativity of :

$$\begin{array}{ccc} \underline{C}^{T}(FB, C) & \xrightarrow{\bar{\alpha}} & \underline{C}^{T}(FB, D) \\ \wr\downarrow & & \downarrow\wr \\ \int_{A} [\underline{C}(NA, B), \underline{C}^{T}(FNA, C)] & \xrightarrow[\int [1, \alpha]]{} & \int_{A} [\underline{C}(NA, B), \underline{C}^{T}(FNA, D)] \end{array} .$$

First we note that $\bar{\alpha} | \underline{\bar{A}} = \alpha$: the diagram

$$\begin{array}{ccc} \underline{C}^{T}(TNA', C) & \xrightarrow{\alpha} & \underline{C}^{T}(TNA', D) \\ \wr\downarrow & & \downarrow\wr \\ \int_{A} [\underline{C}(NA, NA'), \underline{C}^{T}(TNA, C)] & \xrightarrow[\int [1, \alpha]]{} & \int_{A} [\underline{C}(NA, NA'), \underline{C}^{T}(TNA, D)] \end{array}$$

transforms to :

$$\begin{array}{ccc} \underline{C}(NA, NA') \otimes \underline{C}^{T}(TNA', C) & \xrightarrow{1 \otimes \alpha} & \underline{C}(NA, NA') \otimes \underline{C}^{T}(TNA', D) \\ \wr\downarrow & & \downarrow\wr \\ \underline{C}^{T}(TNA, C) & \xrightarrow[\alpha]{} & \underline{C}^{T}(TNA, D) \end{array}$$

which commutes by naturality of α. Secondly, if β is a natural transformation from $\underline{C}^{T}(FB, C)$ to $\underline{C}^{T}(FB, D)$ then $\beta = \bar{\beta}$. This follows from the diagram :

$$\begin{array}{ccc} C^T(TB,\ C) & \xrightarrow{\quad\beta\quad} & \underline{C}^T(TB,\ D) \\ \wr\downarrow & & \downarrow\wr \\ \int_A [\underline{C}(NA,\ B),\ \underline{C}^T(TNA,\ C)] & \xrightarrow[\int [1,\beta]]{} & \int_A [\underline{C}(NA,\ B),\ \underline{C}^T(TNA,\ D)] \end{array}$$

which commutes by naturality of β . By 1.1 Proposition it is required to show that $\bar{\alpha}$ corresponds to the element $\bar{\alpha}(\zeta) \in VGC = \underline{C}_o^T(FC,\ D)$ in the equaliser of :

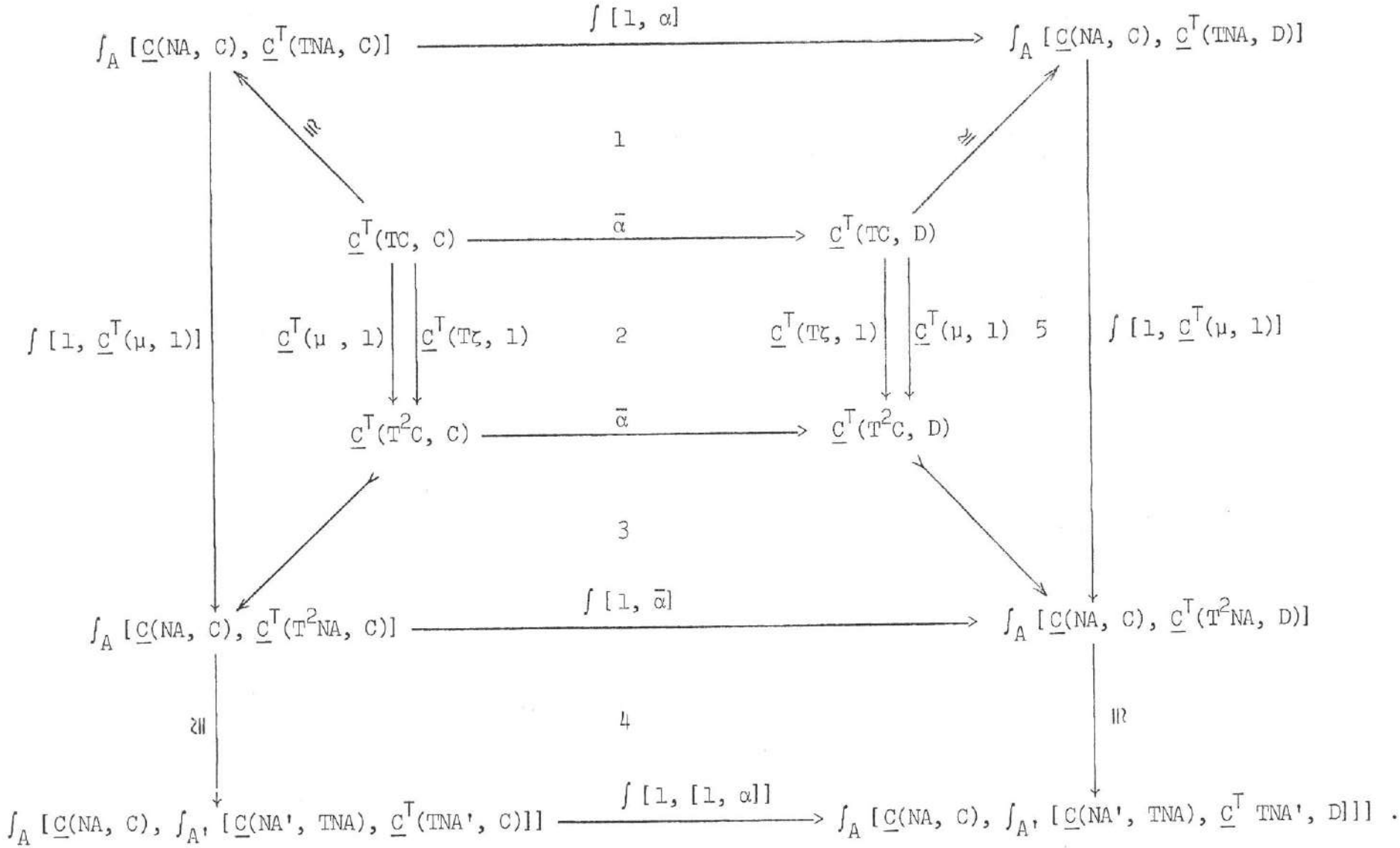
$$VGC \underset{VGT\zeta}{\overset{VG\mu}{\rightrightarrows}} VGTC \ .$$

Firstly, because $\bar{\alpha} = \{\bar{\alpha}_{FB}\}$ is natural in $B \in \underline{C}$, we see that the family $\bar{\alpha}_{FB}$ is derived from $\bar{\alpha}_{FC}(\zeta) : I \longrightarrow \underline{C}_o^T(FC,\ D)$ by the (ordinary) representation theorem :

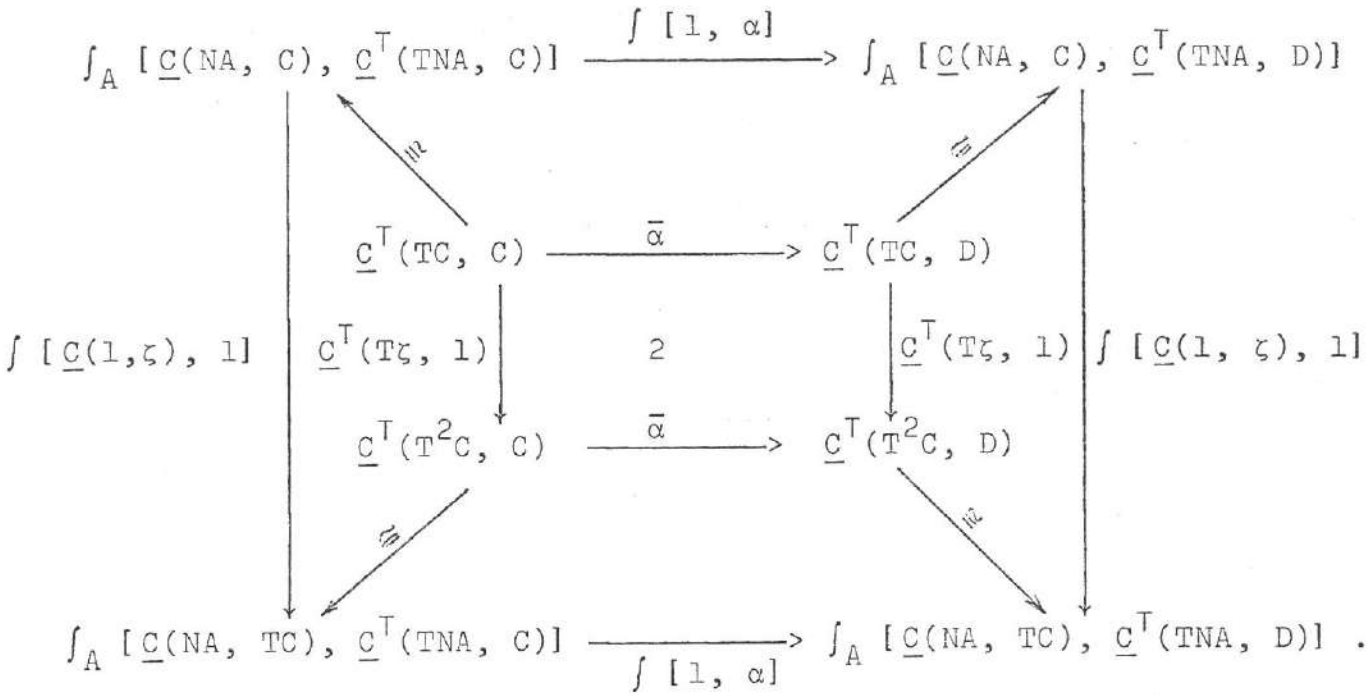
$$\begin{array}{ccc} \underline{C}^T(FB,\ C) & \xrightarrow{\bar{\alpha}_{FB}} & \underline{C}^T(FB,\ D) \\ \wr\downarrow & \nearrow & \\ \underline{C}(B,\ UC) & & \end{array} \ .$$

Thus it remains to verify that $\bar{\alpha}(\zeta)$ is in the equaliser of $(VG\mu,\ VGT\zeta)$. Consider 1.3 Diagram; subdiagrams 1 and 4 commute by definition of $\bar{\alpha}$ so it remains to show that subdiagrams 2, 3, and 5, and the exterior commute. Diagram 2 becomes 1.4 Diagram which clearly commutes. Diagram 3 becomes 1.5 Diagram thus it suffies to show that subdiagram 3' commutes. This follows by applying the representation theorem to $D \in \underline{C}^T$ because both legs are natural in D ; this diagram then becomes 1.6 Diagram.

1.3 Diagram.


$\int_A [\underline{C}(NA, C), \underline{C}^T(TNA, C)]$
$\int [1, \alpha]$
$\int_A [\underline{C}(NA, C), \underline{C}^T(TNA, D)]$
1
$\underline{C}^T(TC, C)$
$\bar{\alpha}$
$\underline{C}^T(TC, D)$
$\int [1, \underline{C}^T(\mu, 1)]$
$\underline{C}^T(\mu, 1)$
$\underline{C}^T(T\zeta, 1)$
2
$\underline{C}^T(T\zeta, 1)$
$\underline{C}^T(\mu, 1)$
5
$\int [1, \underline{C}^T(\mu, 1)]$
$\underline{C}^T(T^2C, C)$
$\bar{\alpha}$
$\underline{C}^T(T^2C, D)$
3
$\int_A [\underline{C}(NA, C), \underline{C}^T(T^2NA, C)]$
$\int [1, \bar{\alpha}]$
$\int_A [\underline{C}(NA, C), \underline{C}^T(T^2NA, D)]$
4
$\int_A [\underline{C}(NA, C), \int_{A'} [\underline{C}(NA', TNA), \underline{C}^T(TNA', C)]]$
$\int [1, [1, \alpha]]$
$\int_A [\underline{C}(NA, C), \int_{A'} [\underline{C}(NA', TNA), \underline{C}^T TNA', D]]$.

1.4 Diagram.



1.5 Diagram.

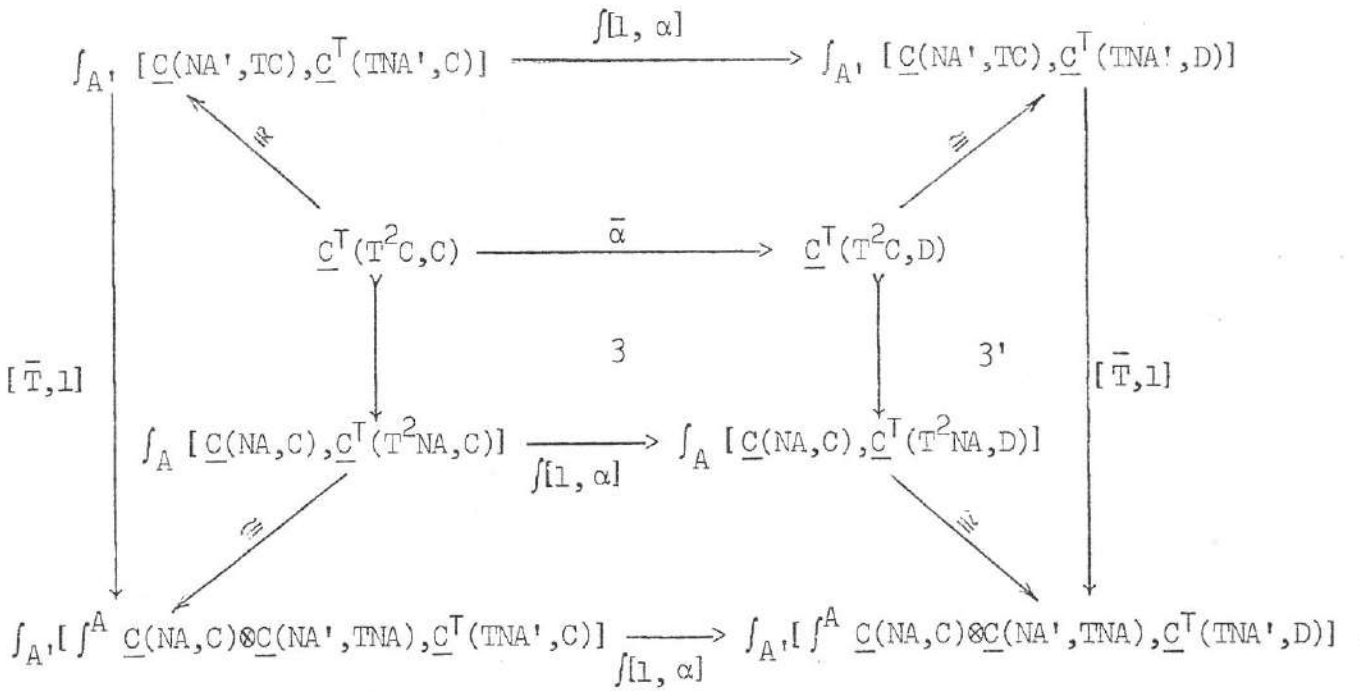


Diagram 1.6.

$$\begin{array}{ccc}
\underline{C}(NA, C) \otimes \underline{C}(NA', TNA) & \xrightarrow{T^2 \otimes 1} & \underline{C}^T(T^2NA, T^2C) \otimes \underline{C}(NA', TNA) \\
\downarrow{\scriptstyle T \otimes 1} & & \downarrow{\scriptstyle 1 \otimes T} \\
\underline{C}(TNA, TC) \otimes \underline{C}(NA', TNA) & & \underline{C}^T(T^2NA, T^2C) \otimes \underline{C}^T(TNA', T^2NA) \\
\downarrow{\scriptstyle \text{compn.}} & & \downarrow{\scriptstyle \text{compn.}} \\
\underline{C}(NA', TC) & \xrightarrow{\quad T \quad} & \underline{C}^T(TNA', T^2C) \ .
\end{array}$$

Again, this diagram commutes by the representation theorem applied to $C \in \underline{C}$. Next consider diagram 5 which transforms to :

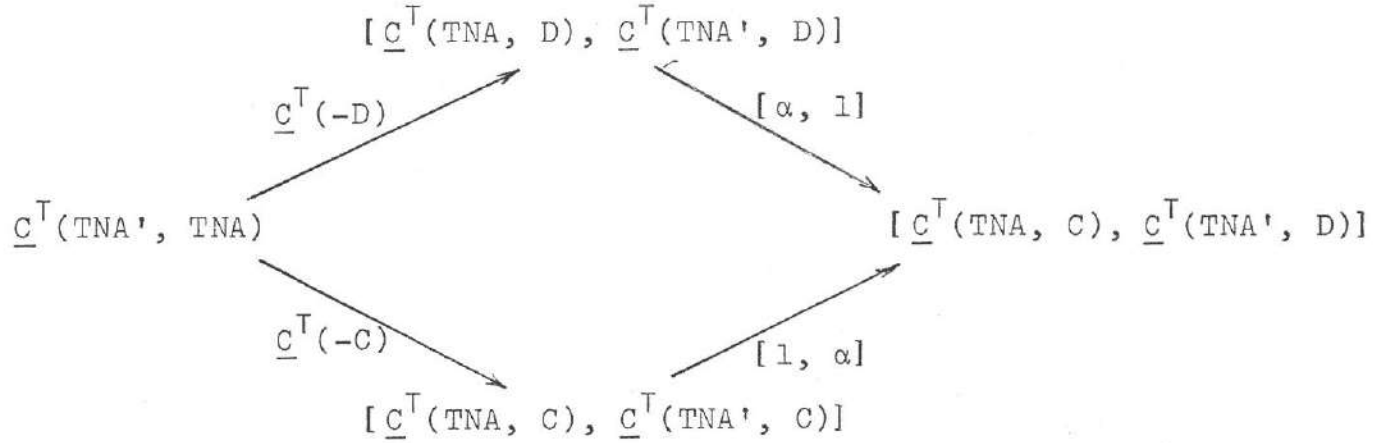
$$\begin{array}{ccc}
\underline{C}(NA, C) & \xrightarrow{\quad \underline{C}^T(T-, D) \quad} & \int_A [\underline{C}(TC, D), \underline{C}(TNA, D)] \\
\downarrow{\scriptstyle \underline{C}^T(T^2-, D)} & & \downarrow{\scriptstyle \int [\underline{C}^T(\mu, 1), 1]} \\
\int_A [\underline{C}^T(T^2C, D), \underline{C}^T(T^2NA, D)] & \xrightarrow[\int [1, \underline{C}^T(\mu, 1)]]{} & \int_A [\underline{C}^T(T^2C, D), \underline{C}^T(TNA, D)]
\end{array}$$

This diagram commutes by naturality of $\mu : T^2 \longrightarrow T$. It remains to check that the diagram

$$\begin{array}{ccc}
* & \xrightarrow{\quad \alpha \quad} & * \\
\downarrow{\scriptstyle \underline{C}^T(\mu, 1)} & & \downarrow{\scriptstyle \underline{C}^T(\mu, 1)} \\
* & \xrightarrow[[1, \alpha]]{} & *
\end{array}$$

commutes.

This diagram transforms to :



composed with :

$$\underline{C}(NA', TNA) \xrightarrow{T} \underline{C}^T(TNA', T^2NA) \xrightarrow{\underline{C}^T(1, \mu)} \underline{C}^T(TNA', TNA) ,$$

thus it commutes by naturality of α . //

§2. Comparison of "rank" conditions.

Again we suppose that $T = (T, \mu, \eta)$ is a monad on $\underline{C}$ and that $N : \underline{A} \longrightarrow \underline{C}$ is dense. Further we suppose, for simplicity, that $\underline{C}$ is cocomplete ; it is left to the reader to write out, if needed, the corresponding general version in terms of ends in $\underline{V}$.

From [1] §3 we obtain a "rank" condition :

RC1. The canonical transformation

$$\int^{A'} \underline{C}(NA', C) \otimes \underline{C}(NA, TNA') \longrightarrow \underline{C}(NA, TC)$$

is an epimorphism for all $A \in \underline{A}$ and $C \in \underline{C}^T$.

From 1.2 Proposition we obtain a "rank" condition :

RC2. (a) $\int^{A} \underline{C}(NA, C)\cdot TNA \longrightarrow TC$

is an isomorphism for all $C \in \underline{C}$,

(b) $\int^{A} \underline{C}(NA, C)\cdot T^2NA \longrightarrow T^2C$

is an epimorphism for all $C \in \underline{C}$.

From the commuting diagram :

$$\begin{array}{ccc} \int^{AA'} \underline{C}(NA', C) \otimes \underline{C}(NA, TNA')\cdot TNA & \longrightarrow & \int^{A} \underline{C}(NA, TC)\cdot TNA \\ \downarrow & & \downarrow \\ \int^{A'} \underline{C}(NA', C)\cdot T^2NA & \longrightarrow & T^2C \end{array}$$

we see that RC2 (b) is satisfied if RC1 holds for all $C \in \underline{C}$

and $\int^A \underline{C}(NA, TC)\cdot TNA \longrightarrow T^2C$ is an epimorphism ; in particular if RC2 (a) holds.

From the diagram :

$$\begin{array}{ccc} \int^{A'} \underline{C}(NA', C) \otimes \underline{C}(NA, TNA') & & \\ \Big\downarrow \kappa & \searrow & \\ \underline{C}(NA, \int^{A'} \underline{C}(NA', C)\cdot TNA') & \longrightarrow & \underline{C}(NA, TC) \end{array}$$

and the fact that N is dense it follows that $\int^A \underline{C}(NA, C)\cdot TNA \to TC$ is an epimorphism if RC1 holds for all $C \in \underline{C}$. Also, if RC2 (a) holds then RC1 holds for all $C \in \underline{C}$ if and only if the canonical comparison transformation κ is an epimorphism.

To relate this to the usual notion of rank we refer to [1] 2.8 Example. Suppose that for each $C \in \underline{C}$ there exists a free $\underline{V}$-category $\underline{K}_C$ and a functor $P = P_C : \underline{K}_C \longrightarrow \underline{A}$ together with a colimit cone

$$\xi_C : \operatorname*{colim}_k NPk \cong C$$

such that $\operatorname{colim} \underline{A}(A, Pk) \cong \underline{C}(NA, C)$ for each $A \in \underline{A}$. Then we define :

$$\underline{K} = \sum_{|\underline{C}|} \underline{K}_C \qquad \text{and} \qquad J = (P_C) : \underline{K} \longrightarrow \underline{A}$$

and $K : \underline{K}^{op} \otimes |\underline{C}| \longrightarrow \underline{V}$ by :

$$\begin{cases} K(k,C) = I & \text{if } k \in \underline{K}_C \\ K(k,C) = 0 & \text{if } k \notin \underline{K}_C \ . \end{cases}$$

with this notation we have :

2.1 Proposition. If the monad T on $\underline{C}$ satisfies RC2 (a) and the induced transformation

$$\int^k K(k,C) \otimes \underline{C}(NA, TNJk) \longrightarrow \underline{C}(NA, T(\int^k K(kC) \cdot NJk))$$

is an epimorphism for all $A \in \underline{A}$ and $C \in \underline{C}$ then T satisfies RC2 (b) and RC1 .

Proof. By inspection of the following commuting diagrams :

$$\begin{array}{l}
\int^{Ak} (K(k,C) \otimes \underline{C}(NA, TNJk)) \cdot TNA \longrightarrow \int^A \underline{C}(NA, T \int^k K(k,C) \cdot NJk) \cdot TNA \\
\quad\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \wr \\
\int^{Ak} (\underline{C}(NJk, C) \otimes \underline{C}(NA, TNJk)) \cdot TNA \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \wr \\
\quad\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \wr \\
\int^k \underline{C}(NJk, C) \cdot T^2 NJk \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \wr \\
\quad\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \wr \\
\int^A \underline{C}(NA, C) \cdot T^2 NA \longrightarrow T^2 C \quad .
\end{array}$$

$$\begin{array}{l}
\int^k K(k,C) \otimes \underline{C}(NA, TNJk) \longrightarrow \underline{C}(NA, T \int^k K(k,C) \cdot NJk) \\
\quad\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \wr \\
\int^k \underline{C}(NJk, C) \otimes \underline{C}(NA, TNJk) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \wr \\
\quad\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \wr \\
\int^{A'} \underline{C}(NA', C) \otimes \underline{C}(NA, TNA') \longrightarrow \underline{C}(NA, TC) \quad . \qquad //
\end{array}$$

For example, let $\underline{V} = \underline{Ens}$ and let α be a regular cardinal number and let $\underline{C}_\alpha$ denote the class of α-presentable objects in $\underline{C}$ (cf. Gabriel-Ulmer [3] §10) . Then an inclusion of the form $N : \underline{A} \subset \underline{C}_\alpha \subset \underline{C}$ is dense if $\underline{A}$ strongly generates $\underline{C}$ and each comma category $\underline{A}/C$ is α-filtered. Let $P_C : \underline{A}/C \longrightarrow \underline{A}$ denote the canonical projection and suppose that $\underline{C}$ is cocomplete (for simplicity).

2.2 Proposition. Under the above assumptions on $N : \underline{A} \longrightarrow \underline{C}$ the following conditions are equivalent :

(i) The canonical transformation

$$\int^{A'} \underline{C}(NA', C) \otimes \underline{C}(NA, TNA') \longrightarrow \underline{C}(NA, TC)$$

is an isomorphism for all $A \in \underline{A}$ and $C \in \underline{C}$,

(ii) T satisfies RC2,

(iii) $\int^A \underline{C}(NA, C) \cdot TNA \cong TC$ for all $C \in \underline{C}$,

(iv) T preserves $\operatorname{colim}_k NP_C k$ for all $C \in \underline{C}$,

(v) T preserves α-filtered colimits.

Proof. (iii) => (v) because $NA \in C_\alpha$ all $A \in \underline{A}$. (v) => (iv) clearly because $\underline{A}/C$ is α-filtered for each $C \in \underline{C}$. (iv) => (iii) because :

$$\int^A \underline{C}(NA, C) \cdot TNA \cong \int^A \underline{C}(NA, \operatorname{colim}_k NP_C k) \cdot TNA$$

$$\cong \operatorname{colim}_k \int^A \underline{C}(NA, NP_C k) \cdot TNA$$

because $NA \in \underline{C}_\alpha$,

$\cong \operatorname{colim}_k TNP_C k$

by the representation theorem,

$\cong T(\operatorname{colim}_k NP_C k)$

by (iv),

$\cong TC$.

Also (ii) => (iii) trivially and (iii) => (ii) by 2.1 Proposition. Finally (i) <=> (iv) by the following :

$$\int^{A'} \underline{C}(NA', C) \otimes \underline{C}(NA, TNA') \cong \int^{A} \underline{C}(NA', \operatorname{colim}_k NP_C k) \otimes \underline{C}(NA, TNA')$$

$$\cong \operatorname{colim}_k \int^{A} \underline{C}(NA, NP_C k) \otimes \underline{C}(NA, TNA')$$

because $NA' \in \underline{C}_\alpha$,

$$\cong \operatorname{colim}_k \underline{C}(NA, TNP_C k)$$

by the representation theorem,

$$\cong \underline{C}(NA, \operatorname{colim}_k TNP_C k)$$

because $NA \in \underline{C}_\alpha$,

$$\longrightarrow \underline{C}(NA, T(\operatorname{colim}_k NP_C k))$$

$$\cong \underline{C}(NA, TC) . \quad //$$

This shows that RC1 is a rather "weak" condition.

2.3 Proposition. RC1 holds if and only if

$$\int^{A'} \underline{C}(NA', TC) \otimes \underline{C}(NA, TNA') \longrightarrow \underline{C}(NA, T^2C)$$

is an epimorphism for all $A \in \underline{A}$ and $C \in \underline{C}$.

Proof. This follows from the preservation of the usual split coequaliser $T^2C \rightrightarrows TC \longrightarrow C$ by each functor $\underline{C}(NA, -)$, $A \in \underline{A}$. //

Interpreted in terms of the Kleisli category RC1 becomes :

$$\int^{A'} \underline{C}_T(FNA', FC) \otimes \underline{C}_T(FNA, FNA') \longrightarrow \underline{C}_T(FNA, FUFC)$$

is an epimorphism for all $A \in \underline{A}$ and $C \in \underline{C}^T$.

A case of special interest which generalises the usual concept of rank (cf. Linton [4] §6) is that where $A_o \in \underline{A}$ is a suitable "base" object and $NA \cong \underline{C}(NA_o, NA) \cdot NA_o$. If, in the presentation described for 2.2 Proposition, each functor $[\underline{C}(NA_o, NA), -] : \underline{Ens} \longrightarrow \underline{Ens}$ preserves epimorphisms then T satisfies RC1 if

$$\int^{A'} \underline{C}(NA', C) \otimes \underline{C}(NA_o, TNA') \longrightarrow \underline{C}(NA_o, TC)$$

is an epimorphism for all $C \in \underline{C}^T$. This follows from the calculation :

$$\int^{A'} \underline{C}(NA', C) \otimes \underline{C}(NA, TNA') \cong \int^{A'} \underline{C}(NA', \operatorname{colim} NP_C k) \otimes \underline{C}(NA, TNA')$$

$$\cong \operatorname{colim} \underline{C}(NA, TNP_C k)$$

$$\cong \underline{C}(NA, \operatorname{colim} TNP_C k)$$

$$\cong [\underline{C}(NA_o, NA), \underline{C}(NA_o, \operatorname{colim} TNP_C k)]$$

$$\cong [\underline{C}(NA_o, NA), \int^{A'} \underline{C}(NA', C) \otimes \underline{C}(NA_o, TNA')]$$

$$\longrightarrow [\underline{C}(NA_o, NA), \underline{C}(NA_o, TC)] \ .$$

Thus, intuitively, we require that each "C-ary operation" factor appropriately through some "NA-ary operation" . With suitable modifications NA_o can clearly be replaced by a dense subcategory of $\underline{C}$ which lies in $\underline{A}$.

2.4 Remark. It is a straightforward exercise to verify that much of what is done in this article using the standard coend presentation may be repeated using an arbitrary density presentation (J, K, ξ) (in the sense of [1]) of the given functor $N : \underline{A} \longrightarrow \underline{C}$. The condition RC1 then becomes :

$$\int^k K(k, C) \otimes \underline{C}(NA, NJk) \longrightarrow \underline{C}(NA, TC)$$

is an epimorphism for all $A \in \underline{A}$ and $C \in \underline{C}^T$, while condition RC2 becomes :

(a) $\int^k K(k, C) \cdot TNJk \cong TC$ for all $C \in \underline{C}$, and

(b) $\int^k K(k, C) \cdot T^2NJk \longrightarrow T^2C$ is an epimorphism for all $C \in \underline{C}$.

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