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Enriched Tannaka reconstruction

Brian J. Day

Department of Mathematics, Macquarie University, N.S.W. 2109, Australia

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Abstract

An abstract approach is made to recent theory on "Tannaka" recovery of coalgebras, bialgebras, and Hopf algebras. It is hoped that the methods used shed some light upon the more quantitative aspects of the mathematics involved.

This paper provides a generalization of the fact that each coalgebra is recoverable from the category of finite dimensional comodules and its underlying functor into vector spaces. We work in the $\mathscr{V}$-enriched context with $\mathscr{V}$ a locally finitely presentable symmetric closed monoidal category for which the finitely presentable objects form a monoidal subcategory (as in [5]). The possibility of such a result may come as a surprise in view of the specialized results used in treatments such as [3, Proposition 6.5] and [7]. Our approach provides a synthesis of Tannaka reconstruction and Gabriel–Ulmer duality.

Let $\mathscr{U}_f$ denote a finitely complete, finitely cocomplete, symmetric compact closed [4] (also called "rigid" [6] or "autonomous" [3]) $\mathscr{V}$-category. Define the $\mathscr{V}$-category

$$\mathscr{U} = \mathrm{Lex}_{\mathscr{V}}(\mathscr{U}_f^{op}, \mathscr{V})$$

which becomes a symmetric monoidal closed $\mathscr{V}$-category via the convolution structure. Evaluation at the unit for tensor product in $\mathscr{U}_f$ determines an equivalence of $\mathscr{U}$-categories

$$\mathrm{Lex}_{\mathscr{U}}(\mathscr{U}_f^{op}, \mathscr{U}) \xrightarrow{\sim} \mathscr{U}.$$

Consider a finitely complete, finitely cocomplete $\mathscr{U}$-category $\mathscr{C}$ and a $\mathscr{U}$-functor $U : \mathscr{C} \to \mathscr{U}_f$ preserving finite $\mathscr{U}$-limits and finite $\mathscr{U}$-colimits and having the effect on homs

$$\mathscr{C}(-, C) \longrightarrow \mathscr{U}(U-, UC)$$

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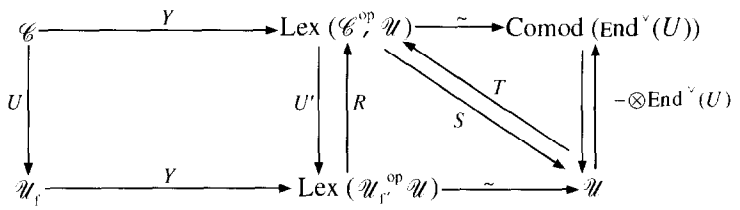


Fig. 1.

a regular monomorphism in the $\mathscr{U}$-functor category $[\mathscr{C}^{op}, \mathscr{U}]$. This leads us to construct and examine the diagram (Fig. 1) which we shall explain below.

The $\mathscr{U}$-functor R is restriction along $U^{op}: \mathscr{C}^{op} \to \mathscr{U}_f^{op}$ while U' is its left adjoint given in terms of the $\mathscr{U}$-coend and $\mathscr{U}$-tensor by the formula

$$U'(F) = \int^C FC \otimes \mathscr{U}_f(-, UC).$$

The $\mathscr{U}$-functors S, T are defined to be the composites of U', R with the bottom right equivalence. Therefore

$$S(F) = \int^C FC \otimes UC.$$

Note that

$$\begin{aligned} S(T(X)) &= S(\mathscr{U}(U-, X)) \\ &\cong \int^C \mathscr{U}(UC, X) \otimes UC \\ &\cong \int^C UC^* \otimes UC \otimes X \\ &\cong \mathrm{End}^{\vee}(U) \otimes X \end{aligned}$$

so that tensoring with the coalgebra

$$\mathrm{End}^{\vee}(U) \cong \int^C UC^* \otimes UC$$

is isomorphic to the comonad generated by the adjunction $S \dashv T$. The top right equivalence in the above diagram now follows from the Beck monadicity theorem and the next result.

Proposition 1. *The $\mathscr{U}$-functor $S: \mathrm{Lex}(\mathscr{C}^{op}, \mathscr{U}) \to \mathscr{U}$ preserves finite $\mathscr{U}$-limits and reflects isomorphisms.*

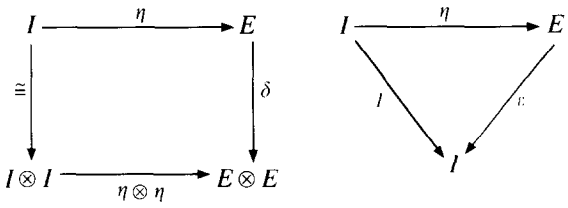


Fig. 2.

Proof. Since S is the left Kan extension of a left $\mathscr{U}$-exact functor into $\mathscr{U}$ along Y, it is also left $\mathscr{U}$-exact [5]. The unit for the adjunction $S \dashv T$ has component at F an arrow

$$F \to \mathscr{U}\left(U -, \int^C FC \otimes UC\right).$$

Since F is left exact, it is a filtered colimit $F \cong \operatorname{colim}_\phi \mathscr{C}(-, C_\phi)$ of representables. Since the values of U are rigid, the above component is induced on colimits by

$$\mathscr{C}(-, C_\phi) \to \mathscr{U}\left(U-, \int^C \mathscr{C}(C, C_\phi) \otimes UC\right) \cong \mathscr{U}(U-, UC_\phi).$$

Since regular monomorphisms are preserved by filtered colimits, we see that the unit is a regular monomorphism. So S reflects isomorphisms. □

It follows that $\mathrm{End}^\vee(U)$ is pointed and flat and that $\mathscr{C}$ strongly $\mathscr{U}$-generates $\mathrm{Comod}(\mathrm{End}^\vee(U))$. Our goal is to show that this characterizes those coalgebras which are recoverable as $\mathrm{End}^\vee(U)$.

Let E be a pointed flat coalgebra in $\mathscr{U}$. That is, suppose that the $\mathscr{U}$-functor $- \otimes E : \mathscr{U} \to \mathscr{U}$ is left exact and there is an arrow $\eta : I \to E$ such that Fig. 2 commutes.

Let $\mathrm{Comod}(E)$ be the $\mathscr{U}$-category of right E-comodules in $\mathscr{U}$. Then let $\mathscr{C}$ be the pullback given in Fig. 3.

By pointedness, $I \in \mathscr{C}$. Clearly $\mathscr{C}$ is finitely $\mathscr{U}$-complete and finitely $\mathscr{U}$-cocomplete, while the effect of U on homs is a regular monomorphism since the following is an equalizer diagram in $\mathscr{U}$:

$$\mathscr{C}(C, D) \to \mathscr{U}(UC, UD) \rightrightarrows \mathscr{U}(UC, UD \otimes E).$$

Proposition 2. *For any pointed flat coalgebra E in $\mathscr{U}$, let $U : \mathscr{C} \to \mathscr{U}_f$ be as in the pullback of Fig. 3, and suppose $\mathscr{C} \to \mathrm{Comod}(E)$ strongly $\mathscr{U}$-generates. Then there is an isomorphism of coalgebras $E \cong \mathrm{End}^\vee(U)$.*

Proof. Since each UC is an E-comodule, we have coactions $UC \to UC \otimes E$ inducing a coalgebra morphism

$$\mathrm{End}^\vee(U) = \int^C UC^* \otimes UC \to E.$$

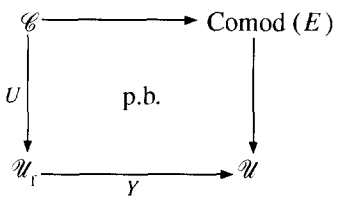


Fig. 3.

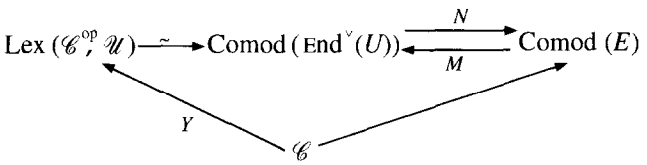


Fig. 4.

This morphism induces a $\mathscr{U}$-functor

$$N : \mathrm{Comod}(\mathrm{End}^{\vee}(U)) \to \mathrm{Comod}(E)$$

which commutes with the underlying functors into $\mathscr{U}$. As N reflects isomorphisms, is $\mathscr{U}$-left exact and has a right adjoint M, it is comonadic. So we have Fig. 4.

Since Comod(E) is now a geometric $\mathscr{U}$-category (that is, satisfies the conditions of [2, Theorem 1(iii)]), it follows from [2] that it is a localization of Lex($\mathscr{C}^{\mathrm{op}}$, $\mathscr{U}$). So M is fully faithful (and N is left $\mathscr{U}$-exact). So the counit $NM \to 1$ is an isomorphism. Since N reflects isomorphisms, the unit $1 \to MN$ is also an isomorphism. So N is an equivalence of $\mathscr{U}$-categories $\mathrm{Comod}(\mathrm{End}^{\vee}(U)) \to \mathrm{Comod}(E)$. Since N also preserves $I \in \mathscr{U}$, it follows that we have $E \cong \mathrm{End}^{\vee}(U)$ as required. □

If $\mathscr{C}$ is now assumed to have a monoidal structure such that $U : \mathscr{C} \to \mathscr{U}_{\mathrm{f}}$ is a strong monoidal functor, then $\mathrm{End}^{\vee}(U)$ has a bialgebra structure with the pointing $\eta : I \to \mathrm{End}^{\vee}(U)$ as the identity. This comes from the monoidal-functor structure induced on $- \otimes \mathrm{End}^{\vee}(U)$. Note that the tensor product on $\mathscr{C}$ preserves finite $\mathscr{U}$-colimits on both sides, so that $\mathrm{Lex}_{\mathscr{U}}(\mathscr{C}^{\mathrm{op}}, \mathscr{U})$ (which is equivalent to $\mathrm{Comod}(\mathrm{End}^{\vee}(U))$) is monoidal biclosed under the resulting convolution data [1].

Conversely, given a flat bialgebra A in $\mathscr{U}$ which, as a coalgebra with the identity pointing, satisfies the hypothesis in Proposition 2 that $\mathscr{C}$ is generating, then $U : \mathscr{C} \to \mathscr{U}_{\mathrm{f}}$ is strongly monoidal, since Comod(A) is strongly monoidal over $\mathscr{U}$; then $A \simeq \mathrm{End}^{\vee}(U)$ as bialgebras.

Next, let $\mathscr{C}$ be a rigid closed category (not merely monoidal), and let $H = \mathrm{End}^{\vee}(U)$. Define the antipode $v : H \to H$ by the commutative diagram given in Fig. 5.

Then $H = \int^{C} UC^* \otimes UC$ is a Hopf algebra. To verify the Hopf-algebra axiom, first fill in the definitions of comultiplication, multiplication, counit, and unit, in a diagram

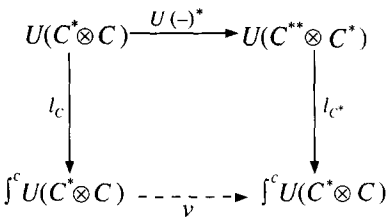


Fig. 5.

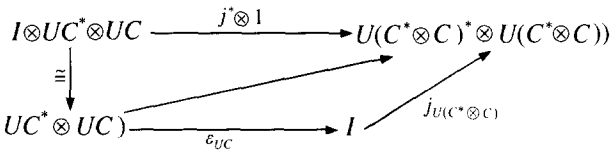


Fig. 6.

with the above antipode. Then the axiom reduces to the requirement of commutativity Fig. 6, which commutes on transposition of UC^*.

In the other direction, if a flat Hopf algebra H is given, with antipode v, with pointing η, and with $\mathscr{C} \to \mathrm{Comod}(H)$ generating, then the monoidal biclosed structure on $\mathrm{Comod}(H)$ can be restricted to $\mathscr{C}$ which becomes a left-rigid category over $\mathscr{V}_f$. Note that the H-coactions

$$\alpha_C : UC \to UC \otimes H$$

give rise to maps

$$UC^* \otimes UC \to H$$

which provide the left-dual H-coactions

$$\beta_C : UC^* \xrightarrow{\alpha^*} UC^* \otimes H \xrightarrow{1 \otimes v} UC^* \otimes H$$

for left-rigidity. The transformations $UC^* \otimes UC \to H$ then exhibit H as the coend $\mathrm{End}^{\vee}(U)$.

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