The Trigonometric Background to Georg Cantor's Theory of Sets

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Communicated by I. B. COHEN & C. B. BOYER

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Introduction

GEORG CANTOR (1845–1918) has won a wide reputation as the originator of set theory. His name is linked in particular with the theory of transfinite numbers and with what has come to be known as the continuum hypothesis.[1] But in contrast to the acclaim and discussion generated by these later works, his earliest achievements, with the possible exception of his theory of irrational numbers, have

[1] CANTOR'S papers on the theory of sets and transfinite numbers have been most widely popularized by the writings of P. E. B. JOURDAIN, and in particular through his translation, GEORG CANTOR: *Contributions to the Founding of the Theory of Transfinite Numbers*, trans: JOURDAIN (Chicago, 1915; Dover reprint: New York, 1955). Despite JOURDAIN'S occasionally torturous rendering of the German into English, his introduction manages to describe the context and development of CANTOR'S mathematical work, though JOURDAIN says little about the papers on trigonometric series. A more detailed account of CANTOR'S mathematical *corpus* is given by JOURDAIN in a series of articles: "The Development of the Theory of Transfinite Numbers," *Archiv der Mathematik und Physik* (Grunert's Archiv), **10** (1906) 254–281; **14** (1909) 289–311; **16** (1910) 21–43; **22** (1913) 1–21. For details on the significance and interpretation of CANTOR'S continuum hypothesis consult KURT GÖDEL: "What is Cantor's Continuum Problem?" *American Mathematical Monthly*, **54**[2] (1947) 515–525. GÖDEL'S paper is reprinted in a revised and expanded version in P. BENACERRAF & H. PUTNAM, eds.: *Philosophy of Mathematics, Selected Readings* (New Jersey, 1964).

remained relatively unknown.[2] And yet these early endeavors clearly bear the stamp of his later mathematical personality. By careful attention to the logical advances made in a series of five articles issued between 1870 and 1872, the restless development and eventual emergence of a rudimentary theory of sets can be traced. In fact, CANTOR'S research devoted to the theory of trigonometric series, inaugurated shortly after his arrival in Halle as a Privatdozent in 1869, reveals the ways in which he was able to proceed to a most general form of the uniqueness theorem for trigonometric representations. In the course of successive generalizations he was led to explore the topology of the real line and to formulate some elementary results in the theory of point-sets. The development of CANTOR'S thinking, culminating in his remarkable article of 1872, is the focus of this paper.

CANTOR'S work on the properties of the representation of arbitrarily given functions by means of trigonometric series assumes and deserves, in its own right, a place in the history of mathematics.[3] Not only does it reveal CANTOR making significant contributions on the frontiers of the mathematical knowledge of his own time, but it also affords a view of how his own kind of mathematical creativity seems to have developed. By concentrating on this early, sustained period of his mathematical productivity, we can hope to discover much about the way CANTOR approached mathematical problems and the ways in which he both formulated and attacked a given mathematical dilemma. More importantly, by tracing the successive development of his theorem devoted to the uniqueness of trigonometric

[2] CANTOR'S papers were collected and edited by ERNST ZERMELO and are conveniently available in a single volume, GEORG CANTOR: *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, ed.: ZERMELO (Berlin, 1932; Olms reprint: Hildesheim, 1966). For a general biography and discussion of CANTOR'S mathematical work, consult A. FRAENKEL: "Georg Cantor," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **39** (1930) 189–266. This biography also appeared separately: *Georg Cantor* (Leipzig, 1930), and is reprinted in an abridged version in ZERMELO'S edition of CANTOR'S works under the title "Das Leben Georg Cantors," *op. cit.* 452–483. The most recent attempt to give both a biography and technical exposition of CANTOR'S mathematics is to be found in HERBERT MESCHKOWSKI: *Probleme des Unendlichen, Werk und Leben Georg Cantors* (Braunschweig, 1967).

[3] It is not the intention of this paper to trace the history of trigonometric series, and I shall draw upon that history only insofar as it helps to illuminate the development of CANTOR'S own thinking. For a more detailed discussion of the broader history, refer to A. SACHSE: *Versuch einer Geschichte der Darstellung willkürlicher Functionen einer Variabeln durch trigonometrische Reihen* (Inaugural-Dissertation, Göttingen, 1879); translated as "Essai historique sur la représentation d'une fonction arbitraire d'une seule variable par une série trigonométrique." *Bulletin des Sciences Mathématiques et Astronomiques*, **15** (1880) 43–64, 83–112. See also the reply to SACHSE by P. DU BOIS-REYMOND: *Zur Geschichte der trigonometrischen Reihen, eine Entgegnung* (Tübinger, 1880). JEAN CAVAILLÈS devotes some space to a discussion of trigonometric series in his "Remarques sur la formation de la théorie abstraite des ensembles," *Philosophie mathématique* (Paris, 1962), as does P. E. B. JOURDAIN: "Georg Cantor's work on trigonometrical series and his theory of irrational numbers," *Archiv*, **16** (1910) 21–43. Most recently, attention to the trigonometric background of the set theory has been given by JEROME MANHEIM: *The Genesis of Point-Set Topology* (New York, 1964), and by А. Б. Паплаускас: *Тригонометрические Ряды от Эйлера до Лебега* (Москва, 1966). See also the article by PAPLAUSKAS: "L'Influence de la théorie des séries trigonométriques sur le développement du calcul intégrale," *Archives internationales d'histoire des sciences*, **84–85** (1968) 249–260.

representations, we can expect to gain some feeling for the way in which a new scientific theory emerged: specifically the origins of CANTOR'S *Mengenlehre.*

In the following discussion the proofs have been sketched with enough detail so that the reader can follow the logic of CANTOR'S thinking and the context of my arguments without having to refer to the original papers themselves. I have not meant to spend more time than seemed necessary recounting technical vicissitudes, but frequently there was no other way to illustrate the salient and operative features of his thinking than by giving careful attention to the mathematical details of his proofs. The level of the mathematical argumentation that CANTOR employs is not of extreme difficulty, and presupposes little more than a basic familiarity on the part of the reader with the theory of functions of a real variable.[4]

The First Paper, March, 1870

The most remarkable aspect of CANTOR'S earliest paper concerning the theory of functions, titled "Über einen die trigonometrischen Reihen betreffenden Lehrsatz," is that it has little to do with trigonometric series.[5] Though this theorem completed for publication in March of 1870 is clearly meant to advance the understanding of convergence properties of the representation of an arbitrarily given function by means of infinite trigonometric series, he never exploits the fundamental properties of the trigonometric functions that make the nature of the representation problem so surprising and significant.

[4] The best and most comprehensive surveys of the purely mathematical aspects of the theory of trigonometric series are to be found in A. ZYGMUND: *Trigonometric Series* (Cambridge, 1959, in two volumes), and NINA BARY: *A Treatise on Trigonometric Series*, trans: MULLINS, (New York, 1964, in two volumes).

[5] GEORG CANTOR: "Über einen die trigonometrischen Reihen betreffenden Lehrsatz," *Journal für die reine und angewandte Mathematik* (Crelle's Journal), **72** (1870) 130–138. This paper appears as the first in the second section, "Abhandlungen zur Funktionentheorie," in the edition of CANTOR'S works edited by E. ZERMELO, *op. cit.* 71–79. For the sake of brevity I shall adopt the following convention, and will hereafter refer to the above paper of CANTOR'S as *II-1* (*March*, 1870). This follows the numbering of CANTOR'S papers given by ZERMELO, and unless otherwise noted citations are from his edition. (Occasionally ZERMELO introduces slight changes from the published texts, usually to correct obvious errors. Cases in which editing has led to important omissions or significant departures from CANTOR'S original papers are cited in appropriate footnotes.)

The theorem which CANTOR proves in the March paper is generally known as the CANTOR-LEBESGUE theorem, for although CANTOR gives the proof for trigonometric series which converge over the interval (a, b), LEBESGUE generalized the result for any interval or set of positive measure in the LEBESGUE sense. See HENRI LEBESGUE: *Leçons sur les séries trigonométriques* (Paris, 1906) 110–111.

We should perhaps note that the converse of CANTOR'S theorem is false. The problem of whether a trigonometric series converges if the coefficients vanish, for sets of positive measure, was discussed by P. FATOU: "Séries trigonométriques et séries de Taylor," *Acta Mathematica*, **30** (1906) 335–400. LUSIN, however, was the first to exhibit the counterexample: a series which diverges almost everywhere and yet has coefficients which approach zero in the limit. Н. Н. Лузин: "Об одном случае ряды Taylor'a," *Математический сборник*, 28 (1911). STEINHAUS later gave an even more comprehensive example, a trigonometric series which diverges at every point though its coefficients vanish. See H. STEINHAUS: "Sur une série trigonométrique divergente," *Comptes Rendus de la Société scientifique de Varsovie*, **5** (1912) 223–227.

CANTOR's proof, in this sense, is disappointing. Rather than offering a demonstration that reveals the nature of trigonometric representations, and eventually the conditions for uniqueness, his proof relies instead upon carefully constructed subsequences of sequences defined over the real numbers. Only in his later papers, when CANTOR comes to consider the number of exceptional points for any given representation, does the nature of the trigonometric functions begin to assume a significant part in the logical structure of his proofs. What is impressive, however, in this early proof is the careful construction of sequences, subsequences, their interrelationships and properties of convergence. For the moment I want to focus our attention upon these aspects of CANTOR's thinking, and to show how the conceptualization of the proof itself reveals elements that later assume greater significance, particularly in his important paper of 1872.

CANTOR sets out to prove the following theorem:

"Wenn zwei unendliche Größenreihen: $a_1, a_2, \ldots, a_n, \ldots$ und $b_1, b_2, \ldots, b_n, \ldots$ so beschaffen sind, daß die Grenze von

$$a_n \sin n x + b_n \cos n x$$

für jeden Wert von x, der in einem gegebenen Intervalle $(a < x < b)$ des reellen Größengebietes liegt, mit wachsendem n gleich Null ist, so konvergiert sowohl a_n wie b_n mit wachsendem n gegen die Grenze Null." [6]

CANTOR needed this result for his proof of the uniqueness theorem: given a function $f(x)$, there is only one trigonometric series representing $f(x)$ and converging for all values of x. The uniqueness theorem, dated *April* of 1870, depended directly upon the *March* result. This earlier theorem however was in itself an important one, and marks the point at which CANTOR probably began to extend and generalize the earlier trigonometric work of HEINE.[7] Assuming that a given trigonometric series represented zero on the interval $(-\pi, +\pi)$ almost everywhere, and, furthermore, that the series converged to zero *uniformly*, HEINE had shown that the coefficients a_n and b_n must likewise vanish. Even earlier RIEMANN[8]

[6] CANTOR: *II-1 (March, 1870)* 71.

[7] See in particular E. HEINE: "Über trigonometrische Reihen," *Journal für die reine und angewandte Mathematik*, (Crelle's Journal), **71** (1870) 353–365. Of special interest is HEINE's remark on page 355:

"Durch Herrn *Cantor* in Halle, den ich mit meinen Untersuchungen bekannt machte, wurde ich veranlaßt, sie auch auf den Unstetigkeitspunkten nicht mehr gefordert wird."

This indicates that by sometime early in 1870 CANTOR had already formulated many of his ideas concerning trigonometric series, in particular the uniqueness result for finite sets of exceptional points, which did not actually appear until January of the following year.

[8] See BERNHARD RIEMANN: "Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe," *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, **13** (1867). This paper was originally written as RIEMANN's *Habilitationsschrift* at Göttingen in 1854, and was published posthumously through the efforts of DEDEKIND. It also appears in B. RIEMANN: *Gesammelte Mathematische Werke und Wissenschaftlicher Nachlaß*, eds.: WEBER & DEDEKIND (Leipzig, 1876; Dover reprint, from the second edition of 1892, including the supplement prepared by M. NOETHER & W. WIRTINGER in 1902: New York, 1953) 213–253.

had managed to prove the theorem assuming only that the coefficients of the trigonometric series be given in the integral form:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt.$$

CANTOR, however, wanted to eliminate as many of these restrictions as possible. In fact, he found that he could give a proof which was free from either of the above assumptions.

Beyond the desire itself to generalize the earlier RIEMANN-HEINE results, there was certainly a further motivation for CANTOR'S proof. Since his development of the uniqueness theorem for trigonometric representations was intimately linked to the conclusion of the March, 1870 theorem, the generality of the latter theorem was clearly essential. Otherwise, whatever limitations were imposed on the earlier result would either have to carry over in the case of the uniqueness theorem, or make the earlier result inapplicable. For a number of reasons, then, CANTOR weakened the assumptions of his theorem as much as possible.

The proof of the March theorem begins with an intricate existence theorem involving a carefully constructed series of positive whole numbers.

"Hat man eine unendliche Reihe ganzer positiver Zahlen

(R) $$u, v, w, x, \ldots$$

von der Beschaffenheit, daß

$$v > 4u, \quad w > 8v, \quad x > 16w, \quad \ldots,$$

so gibt es eine Zahlengröße Ω, welche die Eigenschaft hat, daß das Produkt $n\Omega$, wenn man für n eine der Zahlen (R) setzt, die Form hat

$$n\Omega = 2z_n + 1 \pm \theta_n,$$

wo z_n eine vom Index n abhängende positive ganze Zahl und θ_n eine zu n gehörige positive Größe ist, welche unendlich klein wird, wenn man n in der Zahlenreihe (R) ins Unendliche fortschreiten läßt." [9]

From (R) CANTOR forms the series (S) of odd whole numbers:

(S) $$2g+1,\ 2h+1,\ 2i+1,\ \ldots,$$

where $2g+1$ is determined by the condition that $\left|2g+1-\frac{v}{u}\right| \leqq 1$. Similarly $\left|2h+1-(2g+1)\frac{w}{v}\right| \leqq 1$, $\left|2i+1-(2h+1)\frac{x}{w}\right| \leqq 1$, and so on.

If there are ever two values of $g, h, i, \ldots$ that might satisfy the given inequalities, CANTOR takes the smaller. From (R) and (S) he now builds an infinite series of rational numbers (N):

(N) $$\frac{1}{u}, \quad \frac{2g+1}{v}, \quad \frac{2h+1}{w}, \quad \ldots.$$

[9] CANTOR: *II-1 (March, 1871)* 72.

CANTOR shows that these approach a limit which is different from zero, and which he designates by Ω. The demonstration depends upon the careful way in which he has defined (S) from (R). Taking successive differences between the terms of (N):

$$\left|\frac{1}{u}-\frac{2g+1}{v}\right|<\frac{1}{v}, \quad \left|\frac{2g+1}{v}-\frac{2h+1}{w}\right|<\frac{1}{w}, \quad \ldots,$$

CANTOR considers the nature of

$$\frac{1}{v}, \quad \frac{1}{w}, \quad \frac{1}{x}, \quad \ldots,$$

concluding that

$$\left|\frac{1}{u}-\Omega\right|<\frac{1}{v}+\frac{1}{w}+\cdots.$$

Continuing his argument in a similar manner, CANTOR finally shows that the differences:

$$u\Omega-2z_u-1, \; v\Omega-2z_v-1, \; w\Omega-2z_w-1, \; \ldots,$$

denoted

$$\pm\theta_u, \pm\theta_v, \pm\theta_w, \ldots,$$

diminish more rapidly than $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$.

This appeal to convergence relations between two sequences, one dominating the other, allows CANTOR to assert the conclusion of his lemma.

I have given this brief résumé of CANTOR's proof of the lemma in the first section of the March, 1870 paper because it demonstrates with such clarity the nature of his approach to the representation problem. The most important element in terms of the later proof of CANTOR's uniqueness theorem lies here: in the ability to construct Ω within the limitations of the lemma. The nature of this Ω depends, not on any trigonometric properties, but solely upon the ingenious way CANTOR has constructed the infinite series (N) from the elements of (R) and (S), infinite series drawn completely from the system of non-negative natural numbers.

In section two CANTOR manages to modify his original lemma, and shows that, "Die Zahlengröße Ω kann durch eine Modifikation des für sie festgestellten Entstehungsgesetzes so bestimmt werden, daß sie in ein gegebenes Intervall des reellen Größengebietes zu liegen kommt." [10] By dividing the interval $[0, 2]$ into 2ν equal intervals, CANTOR shows how to define Ω so that it will lie in an arbitrarily given μ^{th} interval. This depends on restricting the construction of the series (S) in such a way that the fraction $\frac{2f+1}{u}$ will fall within the following extremes:

$$\frac{3\mu-2}{3\nu}<\frac{2f+1}{u}<\frac{3\mu-1}{3\nu}.$$

Again CANTOR's interest is strictly limited to the construction and behavior of infinite sequences. As before, the importance of carefully defining the construction of Ω to fit a predetermined set of assumptions, ones that will enable him to establish the proof of his theorem, remains uppermost in his mind.

[10] *Ibid.* 74.

Before proceeding to the details of the demonstration of the theorem, CANTOR needed one additional lemma, which he sets out in section three. Considering an infinite series (G), he assumes that there exists an infinite subsequence (G′) of (G) containing elements all of which are less than an arbitrarily given number $\delta > 0$. It then follows that (G) must converge to zero. As a corollary CANTOR offers the following:

"Ist eine unendliche Größenreihe (G) gegeben und kann man aus jeder aus (G) gehobenen Größenreihe (G′) eine neue Größenreihe (G″)

$$\varrho_u, \varrho_v, \varrho_w, \cdots$$

ausheben, in welcher die Glieder mit wachsendem Index unendlich klein werden, so ist

$$\lim \varrho_n = 0."\ ^{11}$$

CANTOR now combines the above results to prove the theorem towards which the previous lemmas have led. In order to establish the fact that $\lim a_n = 0$ and $\lim b_n = 0$ if one assumes that $\lim (a_n \sin n x + b_n \cos n x) = 0$, CANTOR appeals to a subsequence of terms which form a series analogous to (G″). The construction of this series follows closely the outlines of the lemma given in section one of the paper.

The crux of CANTOR's argument, at least initially, involves an important trigonometric transformation:

$$a_n \sin n x + b_n \cos n x$$

is considered in its alternate polar form:

$$\varrho_n \cos (\phi_n - n x),$$

where $\varrho_n = \sqrt{a_n^2 + b_n^2}$ and $\sin \phi_n = \frac{a_n}{\varrho_n}$, $\cos \phi_n = \frac{b_n}{\varrho_n}$.

As CANTOR notes: "es ist bei dieser Bezeichnung nur zu zeigen, daß $\lim \varrho_n = 0$, um alsdann unmittelbar schließen zu können, daß $\lim a_n = 0$, $\lim b_n = 0$." [12]

CANTOR first uses (G) to denote the series

(G) $$\varrho_1, \varrho_2, \varrho_3, \cdots, \varrho_n, \cdots.$$

(G′) then denotes any infinite subsequence taken from (G):

(G′) $$\varrho_\alpha, \varrho_\beta, \varrho_\gamma, \cdots.$$

He now sets out to show that one can form from the elements of (G′) another infinite subsequence:

(G″) $$\varrho_u, \varrho_v, \varrho_w, \cdots,$$

the terms of which, with increasing index, become arbitrarily small. Before showing this, however, CANTOR turns to examine more carefully the nature of the infinite series:

(ϕ) $$\phi_\alpha, \phi_\beta, \phi_\gamma, \cdots.$$

[11] *Ibid.* 76.
[12] *Ibid.*

He narrows his focus to the interval $\Phi < \phi_{\{\alpha\}} < \Phi + \frac{\pi}{4}$, and picks from (ϕ) an infinite subsequence, all of whose terms lie in this Φ interval. The resulting sequence he denotes as follows:

$$(\phi') \qquad \phi_{\alpha'}, \phi_{\beta'}, \phi_{\gamma'}, \ldots .$$

From the index sequence $\alpha', \beta', \gamma', \ldots$ CANTOR now extracts an infinite index subsequence which corresponds to the conditions placed upon the sequence (R) in the lemma of section one:

$$(R) \qquad u, v, w, \ldots .$$

Moreover, Ω, as in section two, is defined to lie in the open interval $\left(\frac{a}{\pi}, \ldots, \frac{b}{\pi}\right)$, or in $\left(\frac{2a}{\pi}, \ldots, \frac{2b}{\pi}\right)$, depending upon which of two cases we follow below. Having emphasized the fact that every element of the subsequence extracted from (ϕ') by the index sequence (R):

$$(F) \qquad \phi_u, \phi_v, \phi_w, \ldots,$$

lies in the interval $\left(\Phi, \Phi + \frac{\pi}{4}\right)$, CANTOR now distinguishes two cases:

Case I: The interval $\left(\Phi, \Phi + \frac{\pi}{4}\right)$ contains neither $\frac{\pi}{2}$ nor $\frac{3\pi}{2}$.

Case II: The interval $\left(\Phi, \Phi + \frac{\pi}{4}\right)$ contains either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Case I is the easier to follow, and since Case II, with slight modifications, parallels the logic of CANTOR'S argument used for Case I, we shall discuss only the latter. For Case I it is easy for CANTOR to establish his claim that there exists a non-zero number ε such that the absolute value of $\cos\phi$ is greater than ε for all ϕ in $\left(\Phi, \Phi + \frac{\pi}{4}\right)$, since only at $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ would $\cos\phi = 0$.

He now determines, as in section two of the paper, a number Ω constituted such that:

1) Ω lies between $\frac{a}{\pi}$ and $\frac{b}{\pi}$.

2) $\Omega n - (2z_n + 1) = \pm\theta_n$, where θ_n becomes arbitrarily small if one takes for n the increasingly large numbers of the series (R).

This done, CANTOR has only to examine the series:

$$(P) \qquad \varrho_u \cos(\phi_u - u x),\ \varrho_v \cos(\phi_v - v x),\ \varrho_w \cos(\phi_w - w x), \ldots .$$

By showing that the cosines involved in the series (P) are always larger in absolute value than any predetermined $\frac{\varepsilon}{2}$ as the indices become increasingly large, he is able to conclude that:

"Die Glieder selbst der Reihe (P) werden der Voraussetzung gemäß, welche in Theorem liegt, für jeden Wert von x in den Grenzen a und b, mithin auch für $x = \pi\Omega$, mit wachsendem index unendlich klein; daraus folgt, daß die Glieder der Reihe (G'') mit wachsendem Index unendlich klein werden." [13]

[13] *Ibid.* 78.

The corollary to section three, the convergent subsequence theorem, allows CANTOR to conclude that $\lim \varrho_n = 0$, and thus to prove the main theorem.

The most interesting feature of CANTOR's argument is the ease with which he cajoles his infinite sequences, subsequences and indices into setting up the structure of the theorem. Then, by carefully selecting and defining the elements of his proof, and in particular the series (N) and the limit Ω, constructed so that $n\Omega = 2z_n + 1 \pm \theta_n$, he shows how to reach $\lim \varrho_n = 0$, and consequently $\lim a_n = 0$, $\lim b_n = 0$. It is only at this point that the specifically trigonometric properties of the series come into play. The theorem begins with the assumption that $\lim (a_n \sin n x + b_n \cos n x) = 0$ for all $x \in [a, b]$. By exploiting the transformation from $a_n \sin n x + b_n \cos n x$ to $\varrho_n \cos (\phi_n - n x)$, and taking $x = \pi\Omega$, CANTOR shows that $\cos (\phi_n - n x)$ is non-zero for all ϕ_n. From the assumptions of the theorem it must then follow that $\lim \varrho_n = 0$. However, the proof of the theorem depends, essentially, on the definition and properties of infinite sequences, and only incidentally upon the trigonometric nature of the functions involved.

The Second Paper, April, 1870: The Uniqueness Theorem

CANTOR'S first theorem of 1870 was really a prelude to the paper dated *April* and published in the same volume of Crelle's *Journal für die reine und angewandte Mathematik*.[14] In this second paper CANTOR was able to establish the uniqueness of the trigonometric representation of a given function. It was an impressive achievement. Working within the tradition of RIEMANN'S analysis of the representation problem, CANTOR successfully applied a number of techniques that led almost directly to the uniqueness theorem itself. For this proof CANTOR was compelled to exploit more fully than he had in the March paper the nature of the trigonometric representation. In the process he was led to consider in greater detail not only properties of convergence, but aspects of periodicity and orthogonality as well.

Uniqueness theorems for special classes of functions under certain assumptions had already been given by others. HEINE in particular, who was an important influence upon CANTOR in this period, had outlined early in 1870 the following theorem:

> "Eine im allgemeinen stetige, nicht notwendig endliche Funktion $f(x)$ läßt sich höchstens auf eine Art in eine trigonometrische Reihe von der Form
>
> $$f(x) = \tfrac{1}{2} a_0 + \sum (a_n \sin n x + b_n \cos n x) \qquad (1)$$
>
> entwickeln, wenn die Reihe der Bedingung unterworfen ist, im allgemeinen in gleichem Grade zu convergieren. Die Reihe stellt die Funktion im allgemeinen von $-\pi$ bis π dar." [15]

[14] GEORG CANTOR: "Beweis, daß eine für jeden reellen Wert von x durch eine trigonometrische Reihe gegebene Funktion $f(x)$ sich nur auf eine einzige Weise in dieser Form darstellen läßt," *Journal für die reine und angewandte Mathematik* (Crelle's Journal), **72** (1870) 139–142. Henceforth: *II-2 (April, 1870)*.

[15] E. HEINE: *op. cit.* 355. It should also be noted that for HEINE the expression "im allgemeinen stetige" assumes a technical significance. It is used to express the property of almost-everywhere continuity.

Requiring almost-everywhere continuity and uniform convergence, HEINE's theorem invited direct generalizations. The challenge he presented in the opening paragraphs of his paper emphasized that even the great minds, including DIRICHLET, LIPSCHITZ and RIEMANN, had been unable to solve the uniqueness problem. They had only managed to confirm that in a number of specific cases, and under certain restrictions, a given function could be represented by a trigonometric series. But as HEINE stressed, they had failed to show "auf wie vielen Arten die Entwickelung geschehen könne." [16] With this in mind, CANTOR set out to show that there was only one way in which such a development could be given, assuming that one existed. He was also determined to do away with as many assumptions as he could, giving the most general theorem possible. As CANTOR soon discovered, the problem was a much deeper one than anyone seems theretofore to have expected, and required, for the most general solution, entirely new concepts.

WEIERSTRASS provided CANTOR with an important tool that eventually assumed a prominent function in CANTOR'S proof of the uniqueness theorem as it was originally given, namely WEIERSTRASS' concept of uniform convergence. CANTOR himself noted in the April paper that the uniqueness problem could not be solved, as was generally assumed, by multiplying each term of the representation series (1) by $\cos n(x-t)\,dt$, followed by a term-by-term integration from $-\pi$ to π. This process assumed not only the integrability of $f(x)$, but it further had to assume uniform convergence of the series in order to permit the integration itself. CANTOR cast the whole matter in terms of a given series and the uniform convergence of its remainder term R_n to zero:

"Also muß, wenn man

$$f(x) = A_0 + A_1 + \cdots + A_n + R_n \tag{2}$$

setzt, bei gegebener Größe ε, eine ganze Zahl m vorhanden sein, so beschaffen, daß für $n \geqq m$, R_n seinem absoluten Betrage nach kleiner ist als ε für alle Werte von x, welche in Betracht kommen." [17]

Denoting the value m by which uniform convergence is established as a discontinuous function of x and ε, that is $m(x, \varepsilon)$, CANTOR summarized the difficulty of the traditional approach:

"So weiß man nicht, ob die Funktion $m(x, \varepsilon)$ bei gegebenem ε für alle Werte von x unterhalb einer endlichen Grenze liegt; es ist sogar leicht einzusehen, daß, wenn $f(x)$ für $x = x_1$ eine Unstetigkeit hat, die Funktion $m(x, \varepsilon)$ Werte annehmen muß, welche jede angebbare Grenze übersteigen, wenn, bei festgehaltenem ε, x dem Werte x_1 unendlich nahe rückt." [18]

Thus it was clear to him that he could not expect to probe the uniqueness of functional representations by convergent trigonometric series with the approach outlined above. Instead, turning first to RIEMANN'S *Habilitationsschrift*, he showed how a proof suggested by SCHWARZ could be used to demonstrate the uniform

[16] *Ibid.* 353.
[17] CANTOR: *II-2 (April, 1870)* 80.
[18] *Ibid.* 80–81.

convergence of the remainder terms of a special series, which took CANTOR easily to his conclusion. The April paper of 1870 shows how such a number $m(x, \varepsilon)$ can be found.

Having assumed that at least one representation of the function $f(x)$ exists, the demonstration begins by supposing that there are two. Subtracting these two supposed representations, CANTOR writes:

$$0 = C_0 + C_1 + C_2 + \cdots + C_n + \cdots, \tag{3}$$

where $C_0 = \frac{1}{2} d_0$ and $C_n = c_n \sin n x + d_n \cos n x$, and where the coefficients c_n, d_n, with increasing index, become arbitrarily small. Thus the problem, in this first step, reduces to showing that the coefficients on all indices must coincide. Following RIEMANN,[19] CANTOR constructs the following function:

$$F(x) = C_0 \frac{x \cdot x}{2} - C_1 - \cdots - \frac{C_n}{n \cdot n} - \cdots. \tag{4}$$

The RIEMANN function $F(x)$ is not only continuous over the entire interval, but, as RIEMANN had shown, the second derivative (CANTOR calls it "der zweite Differenzenquotient"[20])

$$\frac{F(x+\alpha) + F(x-\alpha) - 2F(x)}{\alpha^2} \tag{5}$$

approaches zero as α also diminishes without limit.

We shall defer for the moment discussion of the importance and implications of the SCHWARZ result that $F(x)$ is in fact a linear function. For the present it is sufficient to stress that, knowing he could describe $F(x) = c x + c'$ as a linear function, CANTOR was thus able to conclude, from (4), that:

$$C_0 \frac{x \cdot x}{2} - c x - c' = C_1 + \frac{C_2}{2^2} + \cdots + \frac{C_n}{n^2} + \cdots. \tag{6}$$

Rewriting this result as

$$C_0 \frac{x^2}{2} - c x - c' = \sum (a_n \sin n x + b_n \cos n x)/n^2, \tag{7}$$

it is clear that $\sum (a_n \sin n x + b_n \cos n x)/n^2$ has a period of 2π. This can only happen in equation (7) if $C_0 = 0$ and $c = 0$. CANTOR can immediately reduce (7) to the following:

$$-c' = C_1 + \frac{C_2}{2^2} + \cdots + \frac{C_n}{n^2} + R_n, \tag{8}$$

where we now have an equation for which, given any $\varepsilon > 0$, CANTOR is able to give a whole number m such that for all values of $n \geqq m$, the absolute value of R_n is less than ε. This holds, as CANTOR italicizes, "*für alle Werte von x.*"[21] Since the convergence of R_n is therefore uniform, he now employs the conclusions from WEIERSTRASS' work on uniform convergence and is thus able to multiply each

[19] RIEMANN: *op. cit.* 245.

[20] CANTOR: *II-2 (April, 1870)* 81. Though neither SCHWARZ nor CANTOR gives the SCHWARZ second derivative any special designation, I will follow currently standard practice and use the notation $D^2F(x)$ to represent this function.

[21] *Ibid.* 83.

term in (3) by $\cos n(x-t)\,dx$. Because of the orthogonality relationships of the trigonometric functions, by performing the term-by-term integration from $-\pi$ to π, CANTOR therefore has that:

$$c_n \sin n x + d_n \cos n x = 0. \tag{9}$$

Now it is a simple matter of appealing to the March, 1870 theorem, and concluding that $c_n = 0$, $d_n = 0$. "Es ergibt sich also das Resultat, daß eine für jeden einzelnen reellen Wert von x konvergente Darstellung der Null durch eine trigonometrische Reihe (3) nicht anders möglich ist, als wenn die Koeffizienten d_0, c_n, d_n sämtlich gleich Null sind."[22] CANTOR's uniqueness theorem follows immediately.

I have chosen to follow rather carefully these details of CANTOR's argument because they provide some important contrasts with his proof one month earlier on the convergence properties of trigonometric series. While the March paper seemed to delight in the intricate construction of number series, CANTOR could avoid such complications in establishing the uniqueness theorem. The April paper is much more satisfying aesthetically and capitalizes on the combination of results, primarily from RIEMANN and SCHWARZ, to achieve the theorem. Not only does the second paper seem less tortured in its argumentation, it also depends, as has been suggested above, more heavily upon the properties of trigonometric representation. The periodic character of the sine and cosine functions permit the reduction from (7) to (8) and yield a form from which CANTOR can more readily deduce the uniform convergence of the series. Once this is established, he is free to exploit the orthogonality of the trigonometric functions. It is this property, upon term-by-term integration, that allows CANTOR to press his conclusion that $c_n \sin nt + d_n \cos nt = 0$. Going back to the result from the previous month, the uniqueness theorem follows nicely. Clearly, in the April paper of 1870 CANTOR had managed a more assiduous application of the essential properties of the trigonometric functions themselves.

Finally, CANTOR's use of SCHWARZ's theorem that under certain conditions a given function $F(x)$ is linear bears emphasizing, since it is this point in particular about which the uniqueness theorem revolves.[23] Given a continuous function of a real variable $F(x)$ for all x in the closed interval $[a, b]$, the SCHWARZ condition requires that the second derivative function of $F(x)$:

$$\frac{F(x+\alpha) - 2F(x) + F(x-\alpha)}{\alpha^2},$$

vanish for all x in the open interval (a, b), for arbitrarily small values of α. The significance of this condition, however, is never emphasized by CANTOR, but the usefulness of its application merits some comment. It is clear that SCHWARZ was generalizing the normal second derivative in an important way. Whenever the

[22] *Ibid.*

[23] See HERMANN A. SCHWARZ: *Gesammelte mathematische Abhandlungen* (Berlin, 1890), volume 2, 341–343. Volume one contains SCHWARZ's work between 1865–1887 on surfaces of minimal area. Volume two contains all other mathematical papers published after 1863. The paper here in question concerning the SCHWARZ second derivative was never published separately, and appears only in SCHWARZ's collected works.

usual second derivative $F''(x)$ exists, it is easy to show that the SCHWARZ second derivative $D^2F(x)$ also exists and that, in fact: $D^2F(x) = F''(x)$. Assuming only that $F(x)$ is continuous, however, does not always insure that $F''(x)$ will exist. $D^2F(x)$, however, can exist even if $F''(x)$ does not. Moreover, the analogy between $D^2F(x)$ and $F''(x)$ is strongly revealed in SCHWARZ's theorem: if $D^2F(x) = 0$ in the limit on (a, b) for $F(x)$ continuous on $[a, b]$, then $F(x)$ must be linear. CANTOR follows SCHWARZ, with certain minor exceptions, in considering first the function:

$$\phi(x) = i\left\{F_1(x) - F_1(a) - \frac{x-a}{b-a}\left(F_1(b) - F_1(a)\right)\right\} - \frac{\varkappa_2}{2}(x-a)(b-x), \quad (10)$$

where $\varkappa$ is any real number. The value i is taken as ± 1. From the assumed properties of the function $F_1(x)$ it is clear that $\phi(x)$ is continuous in (a, b). It is also easy to see that $\phi(a) = \phi(b) = 0$, and that the SCHWARZ second derivative $D^2\phi(x) = \varkappa^2$. A neat argument by contradiction, following largely an approach due to WEIERSTRASS,[24] shows that $\phi(x) \leqq 0$ for all values of the argument x in (a, b). From (10) it must then follow that

$$i\left\{F_1(x) - F_1(a) - \frac{x-a}{b-a}\left(F_1(b) - F_1(a)\right)\right\} < \frac{\varkappa^2}{2}(x-a)(b-x). \quad (11)$$

Letting $\varkappa^2$ become arbitrarily small, we now have that

$$F_1(x) = F_1(a) + \frac{x-a}{b-a}\left(F_1(b) - F_1(a)\right). \quad (12)$$

This equation clearly exhibits $F_1(x)$ as a linear function.

CANTOR's presentation and use of the SCHWARZ result is revealing. Though he never manages to make clear the essential characteristics of the SCHWARZ second derivative, and therefore fails to exhibit the deeper reasons for its usefulness, he does manage to put this seemingly unrelated piece of mathematical work to good use. By developing his theorem in such a way that the SCHWARZ conclusion is crucial to the uniqueness proof itself, he again displays his ability to draw upon a wide variety of techniques and results, while specifically tailoring them to the demands of his objectives. In this, CANTOR echoes unmistakably his earlier paper, where the string of lemmas used to prove the final theorem were concerned with a careful construction of functions and sequences fulfilling certain very special conditions. In all of this the expected results dictated, to a large extent, the framework of the proofs themselves. The essential nature of the trigonometric functions appears almost secondarily, while CANTOR's focus seems to be primarily concerned with the arithmetic properties of certain sequences,

[24] CANTOR himself credits WEIERSTRASS with having given the theorem upon which the argument in the April paper is based. See CANTOR: *II-2 (April, 1870)* 82. Essentially WEIERSTRASS had shown that given a continuous function f of a real variable on a closed interval $[a, b]$, then for at least one value in $[a, b]$ f must assume a maximal value. That the proof of this theorem was frequently given, as CANTOR suggests, in lectures by WEIERSTRASS is clearly evidenced in the latter's discussion of the theory of maxima and minima for functions of one or more real variables; C. WEIERSTRASS: *Theorie der Variationsrechnung* (lectures delivered in the summer semester, 1879, preserved in a typescript copy, Harvard University, from an original set of hand-written notes in the library of The Johns Hopkins University).

or the fact that a given function is linear over intervals defined by exceptional points, and identically the same over all such intervals.

Perhaps it was CANTOR'S ability to see the uniqueness problem less in terms of the trigonometric series themselves and more as a general problem in the theory of functions, that allowed him to make the broad connections between various results in analysis, often seemingly unrelated to the special properties of the trigonometric relations. By dexterously manipulating the trigonometric elements of his theorems, he could hope to make them fit the results others had already obtained. CANTOR'S advances in the early 1870's, however, underscore an important phase in his mathematical development. He had clearly mastered the techniques necessary for creating the requisite frameworks within which he could mold, to his own specifications, the elements of a given theorem.

The Third Paper: Cantor's "Notiz" and the Uniqueness Theorem for Finite Exceptional Sets

By January of 1871 CANTOR had obtained several significant refinements of his theorem for the uniqueness of trigonometric representations. The paper dated January 7 was offered only as a "Notiz"[25] to the April 1870 paper, but the implications of the newly achieved results were to have important consequences.

The "Notiz" is divided into three parts, beginning with a simplification of the uniqueness theorem verbally suggested, as CANTOR notes, by KRONECKER.[26] With u as any arbitrarily given number, CANTOR begins with a trigonometric representation of zero:

$$0 = C_0 + C_1 + \cdots + C_n + \cdots, \tag{13}$$

where $C_n = a_n \sin n x + b_n \cos n x$. He then substitutes for x, $x = x + u$, $x = u - x$, and adds the resulting two series to reach:

$$0 = e_0 + e_1 \cos x + e_2 \cos 2x + \cdots + e_n \cos n x + \cdots, \tag{14}$$

where $e_n = c_n \sin n u + d_n \cos n u$.

This may have been KRONECKER'S suggestion, but it seems unlikely since RIEMANN had used this same method of double substitution in his *Habilitationsschrift*,[27] which CANTOR had cited in both of his previous papers and with which CANTOR was thus undoubtedly familiar. KRONECKER had probably alerted CANTOR'S attention to the fact that, if one makes the substitution $x = u$ in (13), the coefficients of (14) become arbitrarily small since they correspond to the terms C_n. This enabled CANTOR to conclude that

$$e_n = c_n \sin n u + d_n \cos n u = 0. \tag{15}$$

Since u was taken as arbitrary, the fact that c_n and d_n vanish is now an immediate consequence. This result in particular was an important one for CANTOR, and assumes greater significance in connection with his second remark in the

[25] GEORG CANTOR: "Notiz zu dem Aufsatz: Beweis, daß eine für jeden reellen Wert von x durch eine trigonometrische Reihe gegebene Function $f(x)$ sich nur auf eine einzige Weise in dieser Form darstellen läßt," *Journal für die reine und angewandte Mathematik* (Crelle's Journal), **73** (1871) 294–296. Henceforth: *II-3 (January, 1871)*.

[26] *Ibid.* 84.

[27] RIEMANN: *op. cit.* 255.

"Notiz" concerning the exceptional points of a given representation.[28] Early in the April paper of 1870 devoted to the uniqueness of trigonometric representations, CANTOR had stressed the conclusion of his first paper on trigonometric series and had used that conclusion to establish, from

$$c_n \sin n t + d_n \cos n t = 0, \tag{16}$$

the fact that the coefficients c_n and d_n must also vanish.[29] Employed at the end of the April proof, the March result was prominently used as a part of the apparatus needed to establish the theorem. But the earlier result had assumed that $\lim (a_n \sin n x + b_n \cos n x) = 0$ for *all* x in $a < x < b$. Thus the theorem could not have been used in discussing the problem of representation at exceptional points, and so CANTOR needed to make the uniqueness theorem independent of his earlier conclusion. This independence, though indebted to KRONECKER, was clearly a necessary prelude to the extension of the uniqueness theorem for finite exceptional sets.

The major result, undoubtedly, of the 1871 "Notiz" is its modification of the theorem concerning exceptional points of the trigonometric representation of zero. Originally CANTOR had set out the theorem in the following form:

> "Hat man *eine für jeden Wert* von x gültige, d.h. konvergente Darstellung des Wertes Null durch eine trigonometrische Reihe (13); so sind die Koeffizienten c_n, d_n dieser Darstellung gleich Null."[30]

It is through his modification of the initial assumptions, however, that he was able to transform his early uniqueness theorem into an embryonic mathematical idea, an approach which, from this point forward, grew in CANTOR'S mind, eventually yielding the CANTORIAN theory of irrationals, the transfinite numbers, and set theory itself. CANTOR weakens his assumptions as follows: "Es lassen sich nun hierbei die Voraussetzungen in dem Sinne modifizieren, daß man für gewisse Werte von x entweder die Darstellung der Null durch (13) oder die Konvergenz der Reihe aufgibt."[31]

To do this, CANTOR assumes the existence of an infinite series of values $\ldots x_{-1}, x_0, x_1, \ldots$, increasing with index such that for these x_n, either convergence of (13), or the representation of zero by (13), is given up. The important restriction, that these x_n occur in finite intervals in *only finite numbers*, is stressed by CANTOR. It is this restriction that CANTOR began to explore in future papers, showing how, under certain conditions, the sets of exceptional points in finite intervals could actually contain an infinite number of values x_n.

Having set out this modified framework for the uniqueness theorem, CANTOR now returned to the April 1870 paper and began to assemble his arguments for the new proof. He remarks that the RIEMANN function $F(x)$, defined earlier, assumes the form of a linear function $k_\nu x + l_\nu$ in each interval $(x_\nu \ldots x_{\nu+1})$. To

[28] In what follows, the abbreviated terminology 'exceptional sets' is used to designate those sets of values x of the domain of a given function f for which either the representation of f by a trigonometric series, or the convergence by the series itself to f, is given up.

[29] See CANTOR: *II-2 (April, 1870)* 81.

[30] CANTOR: *II-3 (January, 1871)* 85.

[31] *Ibid.*

apply the conclusions of the April paper to the 1871 "Notiz", CANTOR only has to show the identity of the linear functions over all the intervals. He takes each successive pair of intervals, and hence the neighboring linear functions $k_v x + l_v$, $k_{v+1} x + l_{v+1}$, employing a method HEINE had already exploited in his paper "Über trigonometrische Reihen."[32] The continuity of $F(x)$, plus a result from RIEMANN'S *Habilitationsschrift*[33] for the value x_{v+1} of x, give

$$F(x_{v+1}) = k_v x_{v+1} + l_v \tag{17}$$

and

$$\lim \frac{x_{v+1}(k_{v+1} - k_v) + l_{v+1} - l_v + \alpha(k_{v+1} - k_v)}{\alpha} = 0 \quad \text{for} \quad \lim \alpha = 0. \tag{18}$$

As CANTOR shows, this is only possible if $k_v = k_{v+1}$, $l_v = l_{v+1}$, and thus the identity of $F(x)$ over the $(x_v \ldots x_{v+1})$ intervals is established for all v. Now CANTOR applies his theorem of April to establish the uniqueness result and the fact that the trigonometric coefficients must vanish.

It would be tempting to suggest that this paper marks a transition in CANTOR'S thinking. In a certain sense it does. It was here, with the 1871 "Notiz," that he began the important work on exceptional sets. But though he made the extension, it was without any recourse to mathematical elements of his own creation. CANTOR was still very much at work within the established traditions of mid-nineteenth century mathematics, tinkering with theoretical tools borrowed from RIEMANN, HEINE and SCHWARZ. As in the two papers of 1870, the conclusions CANTOR reached and the theorems he was able to establish were significant contributions to the mathematical literature of this time, but they were achieved primarily through previously available results. No essentially new elements appear in any of these early proofs. By April of 1872, however, he had created some important new techniques, innovations which grew directly out of this theorem from the 1871 "Notiz." The "Notiz," in fact, was only the beginning: "Diese Erweiterung des Satzes ist keineswegs die letzte; es ist mir gelungen, eine ebenfalls auf strengem Verfahren beruhende, um vieles weitergehende Ausdehnung desselben zu finden, welche ich bei Gelegenheit mitteilen werde."[34] The opportunity was not long in coming. By November of 1871 CANTOR had finished the second extension of his uniqueness theorem, and by early 1872 the paper had appeared in the *Mathematische Annalen.*

CANTOR, clearly, was well aware of the importance of his early "Notiz" and realized that its significance lay in the modifications he was able to make on the assumptions of the original uniqueness theorem. But if he could prove uniqueness for an infinite set of exceptional points, finite in finite intervals, was there also some way to advance the proof for *infinite* sets of exceptional points in finite intervals? By the end of 1871 CANTOR was able to give at least some preliminary answers.

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[32] HEINE: *op. cit.* 359.

[33] See RIEMANN: *op. cit.* 248. CANTOR refers specifically to RIEMANN'S second theorem: "$\frac{F(x+2\alpha)+F(x-2\alpha)-2F(x)}{2\alpha}$ wird stets mit α unendlich klein."

[34] CANTOR: *II-3 (January, 1871)* 85.

Before he had finished preparing this further extension of the uniqueness theorem, however, CANTOR returned briefly to his proof of March, 1870. KRONECKER'S suggestion, as well as HEINE'S result used in the "Notiz," had given CANTOR ideas that he could immediately exploit to simplify his earliest work on trigonometric series. Immediately upon publication of his "Notiz," CANTOR'S refurbished proof of the 1870 result appeared in April of 1871, simply titled "Über trigonometrische Reihen." Even the title suggested his indebtedness to HEINE.

The Fourth Paper, April, 1871

Historians of science have long been aware of the value in comparing the various editions of scientific works. Whether it be NEWTON'S *Principia* or DARWIN'S *Origin*, transformations in both theory and interpretation become clear if first the significant changes in text can be traced. For the historian of mathematics textual changes are no less helpful in revealing the ways in which the understanding of a given mathematical problem is deepened. Frequently such advances lead to simplifications in proof, to the weakening of assumptions in theorems themselves, and, in the most important instances, to the discovery and establishment of new interrelations between the various branches of mathematics, or to the creation of wholly new branches.

The value, in this context, of CANTOR'S second proof of his March, 1870 theorem,[35] lies not so much in the simplifications as in the manner by which he manages to achieve simplification. In this second version of the theorem, CANTOR is able to reach his result without having to rely on trigonometric transformations, case-by-case examinations during the final stages of the proof, or a host of other mathematical complications. Thus his proof, that the coefficients of trigonometric series vanish under certain assumptions, was refurbished within the year and reappeared, as CANTOR himself put it, with the hope that "vielleicht in bezug auf Übersichtlichkeit und Einfachheit nichts zu wünschen übrig lassen wird."[36] In the course of this new proof, CANTOR makes clear the lessons he had learned from KRONECKER and HEINE, lessons he had already applied in the "Notiz," and now put to good use in the simplification of his earlier theorem.

The first lemma of the new proof stands as follows, and though seemingly very much like the original 1870 form of the lemma, the changes become significant:

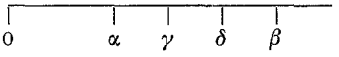
> "Ist $x_1, x_2, \ldots, x_n, \ldots$ eine gegebene unendliche Reihe ganzer positiver Zahlen, welche nur an die Bedingungen gebunden ist, daß
>
> $$x_2 \geqq k\, x_1;\ x_3 \geqq k^2\, x_2;\ \ldots;\ x_n \geqq k^{n-1}\, x_{n-1};\ \ldots$$
>
> wo $k > 1$ ist, so gibt es Zahlengrößen Ω, welche eine solche Beziehung zur Zahlenreihe $x_1, x_2, \ldots$ haben, daß das Produkt $x_n \Omega$ sich von einer ungeraden ganzen Zahl $2y_n + 1$ um eine Größe θ_n unterscheidet, welche unendlich klein wird, wenn man n ins Unendliche wachsen läßt, und zwar kann die

[35] GEORG CANTOR: "Über trigonometrische Reihen," *Mathematische Annalen*, **4** (1871) 139–143. Henceforth: *II-4 (April, 1871)*.

[36] *Ibid.* 87.

Zahlengröße Ω innerhalb eines willkürlich vorgegebenen Intervalles $(\alpha \dots \beta)$ gefunden werden."[37]

Though he had felt no such need for pictorial clarification in the 1870 version of the theorem, CANTOR begins the proof with a diagram. In denoting the size of the interval $(\alpha \dots \beta)$ by i, he proceeds to subdivide it into three equal parts, $(\alpha \dots \gamma) = (\gamma \dots \delta) = (\delta \dots \beta) = \frac{1}{3}$.



The significance of this geometric appeal is not so important here as it is in the fifth paper, the 1872 proof of the uniqueness theorem on certain exceptional sets. Nevertheless, this is an early clue to the way in which CANTOR must have visualized his problems, and then translated his spatial intuitions into mathematical terms.[38]

Assuming that i is positive, CANTOR begins by taking x_ν, the first x_n which is larger than both $\frac{3}{(k-1)i}$ and $\frac{6}{i}$, and then constructing the series of uneven numbers $2y_n+1$ such that $\frac{2y_n+1}{x_n} \in (\gamma \dots \delta)$. This, essentially, is the first major modification of the earlier proof, and manages to accomplish several things. The statement of the lemma in 1871 joins two lemmas from the earlier proof. Instead of first showing the existence of an Ω with certain given properties, and then showing that Ω can be constructed so that it falls within any arbitrarily given interval, CANTOR now fulfils both conditions at once. Since $x_\nu > \frac{i}{6}$, he can show that $\frac{2y_n+1}{x_n}$ must lie in the interval $(\gamma \dots \delta)$. Later the fact that x_ν is also greater than $\frac{3}{(k-1)i}$ will insure that Ω also falls within the arbitrarily given interval $(\alpha \dots \beta)$.

Having chosen $\frac{2y_n+1}{x_\nu}$ so that it lies in $(\gamma \dots \delta)$, CANTOR now generates the series of uneven numbers $2y_{\nu+1}+1, 2y_{\nu+2}+1, \dots$ such that:

$$\text{(A)} \qquad \begin{aligned} &\left|2y_{\nu+1}+1-(2y_{\nu+1}+1)\frac{x_{\nu+1}}{x_\nu}\right| \leqq 1 \\ &\qquad\vdots \\ &\left|2y_{n+1}+1-(2y_{n+1}+1)\frac{x_{n+1}}{x_n}\right| \leqq 1 . \end{aligned}$$

The numbers x_n and y_n now determine an infinite series of rational numbers, much like the series (N) in the 1870 proof. CANTOR denotes this new series by $(\mathscr{A})$:

$$(\mathscr{A}) \qquad \frac{2y_1+1}{x_1}, \quad \frac{2y_2+1}{x_2}, \quad \dots .$$

[37] *Ibid.*

[38] This reliance upon his intuition concerning the geometric character of space, here of the line, is revealed more precisely in the 1872 paper, and was later to cause CANTOR much distress in his inability to prove the impossibility of mapping Euclidean 2-space onto the real line. The problem, of course, was that the mapping theorem is counter-intuitive.

The series ($\mathscr{A}$) approaches a definite limit, which he again denotes by Ω, for arbitrarily increasing values of n. Now, by appealing to the system of conditions (A), CANTOR, as he did in the earlier paper, concludes that:

$$\left|\frac{2y_n+1}{x_n}-\Omega\right| \leqq \frac{1}{x_{n+1}}+\frac{1}{x_{n+2}}+\cdots.$$

Thus

$$|\Omega x_n - 2y_n - 1| \leqq \frac{x_n}{x_{n+1}}+\frac{x_n}{x_{n+2}}+\cdots.$$

Since

$$\frac{x_n}{x_{n+1}} \leqq \frac{1}{k^n}; \quad \frac{x_{n+1}}{x_{n+2}} \leqq \frac{1}{k^{n+1}}; \quad \cdots,$$

CANTOR now has that

$$|\Omega x_n - 2y_n - 1| \leqq \frac{1}{k^n}+\frac{1}{k^{n+1}}+\frac{1}{k^{n+2}}+\cdots < \frac{1}{k^n}+\frac{1}{k^{2n}}+\cdots.$$

Thus at last

$$|\Omega x_n - 2y_n - 1| < \frac{1}{k^n - 1}, \tag{B}$$

and since $k>1$, it follows immediately that $\theta_n = x_n\Omega - 2y_n - 1$ vanishes as n increases without limit.

Since $x_v > \frac{3}{(k-1)\,i}$, it follows that Ω lies in the given interval $(\alpha \ldots \beta)$, for by setting $n=v$ in (B) we have:

$$\left|\Omega - \frac{2y_v+1}{x_v}\right| < \frac{1}{x_v(k^v-1)} < \frac{1}{x_v(k-1)} < \frac{i}{3}.$$

Since $\frac{2y_v+1}{x_v} \in (\gamma \ldots \delta)$, it is not difficult to see that Ω must lie in $(\alpha \ldots \beta)$.

The contrasts between this new approach to the proof of the first lemma and CANTOR'S original development given in 1870 are striking. No longer does he have to worry about squeezing Ω between values like $\frac{11}{16u} < \Omega < \frac{21}{16u}$ or of later having to redefine the elements of the uneven number series (S) so that $\frac{2f+1}{u}$ would lie between $\frac{3\mu-2}{3v}$ and $\frac{3\mu-1}{3v}$ in order to place Ω in a predetermined interval. Undoubtedly the simplifications wrought in the later proof were suggested by these early, cumbersome manipulations of conditions to insure that Ω would fall in a given interval under the assumptions of the lemma itself. CANTOR was always impressed by simple, concise, seemingly self-evident kinds of proof, and it was probably at this level that his genius excelled most.

Lemma II of the 1871 paper offers nothing new in terms of conceptualization or methodology. It is essentially the same proof given earlier as the lemma in section three of the 1870 paper, but now, instead of relegating the useful result to a corollary, CANTOR sets it out at the very beginning. As before, he shows that given an infinite sequence $\{c_n\}$, and any infinite subsequence $\{c_{v_n}\}$, if it is possible to extract from $\{c_{v_n}\}$ another infinite subsequence $\{c_{\mu_{v_n}}\}$ such that $\lim c_{\mu_{v_n}} = 0$, then it must follow that $\lim c_n = 0$ as n increases without limit.

Before proceeding to the theorem itself, however, CANTOR adds a new lemma III: "Wenn für jeden Wert von x zwischen 0 und $i/2$ (wo i eine beliebige positive Größe ist) $\lim(c_n \sin n x) = 0$ ist, so ist $\lim c_n = 0$."[39]

The proof rests on a careful choice of the index sequence and a trigonometric substitution. Selecting

$$\{c_{v_{\mu_n}}\} \tag{19}$$

so that for $k > 1$ we always have $v_{\mu_n} \geqq k^{n-1} v_{\mu_n}$, CANTOR returns to the opening lemma to find a number Ω in $(0 \ldots i/\pi)$ such that $\Omega v_{\mu_n} - (2y+1) = \theta_n$ becomes arbitrarily small as n increases without limit. It is now clear that $\Omega' = \Omega \cdot \frac{\pi}{2}$ lies in the interval $(0 \ldots i/2)$, and further that

$$\Omega' v_{\mu_n} - \frac{\pi}{2}(2y_n + 1) = \theta'_n, \tag{20}$$

which also vanishes as n increases without limit. Since the theorem assumes $\lim c_n \sin n x = 0$ for all $x \in (0 \ldots i/2)$, it follows that for the case $x = \Omega'$ we have $\lim c_n \sin n\Omega' = 0$, and thus also $\lim c_{v_{\mu_n}} \sin v_{\mu_n}\Omega' = 0$ for $n = \infty$. But (20) also allows CANTOR to substitute for $\sin v_{\mu_n}\Omega'$ the value $\pm \cos \theta'_n$, and thus:

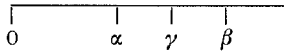
$$\lim(c_{v_{\mu_n}} \cos \theta'_n) = 0 \quad \text{for} \quad n = \infty. \tag{21}$$

But since $\lim \theta'_n = 0$, it follows immediately that $\lim c_{v_{\mu_n}} = 0$, and from lemma II CANTOR has that $\lim c_n = 0$ for $n = \infty$.

In the course of this development CANTOR has paralleled his earlier proof, arguing that for the sequence P: $\{\varrho_u \cos(\phi_u - u x)\}$ he can give an $\varepsilon > 0$ such that $|\cos \phi| > \varepsilon$ for all values of ϕ in the intervals with which he is working. Nevertheless, this fundamental idea has undergone some important modifications. There is still the matter of carefully choosing the index sequence (19) and the Ω in particular intervals. But no longer does he have to complicate the proof by considering first the sequence (G'') of $\varrho_u, \varrho_v, \varrho_w, \ldots$ determined by the indices from $\phi_{\alpha'}, \phi_{\beta'}, \phi_{\gamma'}, \ldots$, and by then examining the behavior of $\varrho_n \cos(\phi_n - n x)$ for different cases (specifically whether or not uneven multiples of $\pi/2$ will fall in the given interval). While earlier CANTOR had sought to avoid the problem of $\cos \phi = 0$, now he is working towards a solution where $\lim \cos \phi_n = 1$. Thus the result in 1871 emerges with much less effort; the conclusion proceeds much more smoothly and generally.

[39] *Ibid.* 89. HEINE had proven this same result in a somewhat different form. Considering the sine series $\{b_n \sin n x\}$, he assumed that it uniformly converged almost-everywhere to zero on $(0, \pi)$, the exceptional points being those at which either the representation failed or where the convergence was not uniform. In considering the function $F(x) = \sum b_n \frac{\sin n x}{n^2}$ over intervals $(x_{\sigma-1}, x_\sigma)$, where the x_σ represent exceptional points, HEINE showed that $F(x)$ was both linear and identically the same over all the $x_{\sigma-1}$-intervals. It was at this point, where he concluded that $F(x) = c_\sigma + x k_\sigma$, that HEINE used the method which CANTOR adopts in his "Notiz" to establish the identity of the linear representation of $F(x)$ over the various $x_{\sigma-1}$-intervals. CANTOR had already shown how the CANTOR-LEBESGUE result could be obtained for ordinary convergence of trigonometric series in his first paper of 1870. The fact that he is drawing upon HEINE's paper of the same year is evident, though instead of requiring, as HEINE has, uniform convergence in order to follow a proof relying on term-by-term integrations, CANTOR still appeals to his own method of carefully examining sub-sequences and fiddling with Ω's and θ's. For further details on HEINE's proof consult E. HEINE: *op. cit.* 357–359.

The final argument, which establishes the theorem itself, comes as a *tour-de-force*, particularly when compared with the earlier proof. Again CANTOR reveals the geometric character of his argument by setting out the following diagram, with γ mid-way between $(\alpha \dots \beta)$:



The length of $(\alpha \dots \beta)$ is denoted, as before, by i. Now CANTOR sets

$$a_n \cos n\gamma - b_n \sin n\gamma = c_n,$$
$$a_n \sin n\gamma + b_n \cos n\gamma = d_n.$$

Thus

$$a_n = c_n \cos n\gamma + d_n \sin n\gamma,$$
$$b_n = -c_n \sin n\gamma + d_n \cos n\gamma,$$

from which he will show that $\lim c_n = 0$, $\lim d_n = 0$, and hence $\lim a_n = 0$, $\lim b_n = 0$. But by the theorem's assumption, for $x = \gamma$,

$$\lim (a_n \sin n\gamma + b_n \cos n\gamma) = \lim d_n = 0,$$

and so he has only to establish that $\lim c_n = 0$.

For all positive $x < i/2$ CANTOR concludes that

$$\lim \big(a_n \sin n(\gamma + x) + b_n \cos n(\gamma + x)\big) = 0,$$
$$\lim \big(a_n \sin n(\gamma - x) + b_n \cos n(\gamma - x)\big) = 0.$$

This follows from the assumptions of the theorem itself, since $(\gamma \pm x) \in (\alpha \dots \beta)$ for $x < i/2$. By subtracting these two equations, and dividing by 2, CANTOR has

$$\lim (c_n \sin n x) = 0 \quad \text{for} \quad n = \infty.$$

Judiciously applying his lemma III, CANTOR has the proof.

This last result demonstrates that the usefulness of the approach taken in his modification of the April 1870 proof of the uniqueness theorem had not been lost on CANTOR. While earlier he had used the simultaneous substitutions of $x = u \pm x$ to produce a cosine series, he now uses that same technique to generate a sine series where $\lim (c_n \sin n x) = 0$. Thus, by introducing the new sine-series lemma, and by perspicuously using the technique KRONECKER had previously brought to his attention, CANTOR has achieved a more unified and satisfying proof of his earlier theorem.

But there was clearly another factor which by April of 1871 had come to impress itself upon CANTOR's mind. The relationship, heretofore implied, between the real numbers and the points on the real-line had run as an undercurrent throughout his early papers, interchangeably, and without much concern for whether points or numbers were at issue. Thinking first of the sequence $\{x_n\}$ as a number-sequence such that $x_n \geqq k^{n-1} x_{n-1}$, CANTOR then considers the interval $(\alpha \dots \beta)$ neatly pictured geometrically, whose magnitude is the distance-number i. However, he also refers to x_v as a number larger than $6/i$. And again, he speaks of the number $\frac{2y_n+1}{x_n}$ as lying in the interval $(\gamma \dots \delta)$. This may at first have seemed only a nominalistic rather than a mathematical problem, but as CANTOR began to ponder the difficulties of the irrational numbers, their construction was clearly a mathematical concern, and the approach he had adopted included, if for convenience's sake alone, the double use of elements of series as numbers and as

points. By November of 1871 he had finally come to terms with this problem. He had managed, in conjunction with his construction of the irrational numbers, to describe essentially the nature of derived sets and to present a remarkable extension of his uniqueness theorem.

The Fifth Paper: Cantor's Theory of Irrational Numbers

Having finished in 1871 the first extension of his uniqueness theorem for finite exceptional sets, CANTOR was well aware that he was on the threshold of something new. Unable to know or to evaluate the true significance of his achievement, he was nevertheless able to add that "Diese Erweiterung des Satzes ist keineswegs die letzte."[40] Hindsight alone allows us to appreciate the understatement of that comment. By 1872, in less than a year, he had managed to work out the details of a remarkable proof[41]: the uniqueness theorem for trigonometric representation can be shown even if convergence or representation is given up for an *infinite* number of points in the interval $(0 \ldots 2\pi)$. The result must have been startling, just as FOURIER's original memoir had been surprising in its development of the idea of trigonometric representations of arbitrary, even discontinuous functions.[42] Mathematicians such as HEINE, for example, had become accustomed to dealing with convergence properties for functions that exhibited discontinuities for *finite* numbers of points. HEINE, in fact, whom CANTOR acknowledged as having been an important instigator for his own work, had devoted a paper early in 1870 to various theorems on trigonometric series, emphasizing that a proof is possible "im allgemein, wenn die Ausnahme sich auf eine *endliche Anzahl von Punkten* beziehen soll."[43] CANTOR, however, had now managed to show how one could deal with *infinite* sets of exceptional points on finite intervals. This extension, as he noted in the 1872 paper, was to have farreaching ramifications.[44]

The logical path CANTOR must have taken from the 1871 result in order to reach the conclusion found in his paper of 1872, is one that can be reconstructed with reasonable assurance. The earlier paper had considered an infinite series $\ldots x_{-1}, x_0, x_1, \ldots$ of values for which either convergence or the representation is given up. CANTOR had only stipulated that the elements of the series of exceptional points x_n occur, in finite intervals, in finite numbers. The crux of the proof, then, was to show that the function $F(x)$ is linear on each interval $(x_\nu \ldots x_{\nu+1})$, and that in fact it is identically the same over all such intervals. The question must then have been whether a set of exceptional points could be constructed in such a way as to allow a similarly constituted proof to proceed for the case of infinite exceptional sets. This would involve defining infinite sets with some property which, upon closer examination, could be broken down into intervals that would allow CANTOR to show the linearity of $F(x)$ over all such intervals.

[40] GEORG CANTOR: *II-3 (January, 1871)* 85.

[41] GEORG CANTOR: "Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen," *Mathematische Annalen*, **5** (1872) 123–132. Henceforth: *II-5(1872)*.

[42] See the discussion of FOURIER's trigonometric work by I. GRATTAN-GUINNESS: "Joseph Fourier and the Revolution in Mathematical Physics," *Journal of the Institute of Mathematics and its Applications*, **5** (1969) 230–253.

[43] E. HEINE: *op. cit.* 354.

[44] CANTOR stressed in particular his belief that the new results would provide new concepts which could profitably be applied in analysis. For further details see CANTOR: *II-5 (1872)* 96.

The uniqueness conclusion would then easily follow as before. CANTOR actually managed the construction of such infinite sets in the 1872 paper, and called them "derived sets of the v^{th} kind."

Wanting to give as clear and concise a presentation of his proof as possible, CANTOR stressed that

> "Zu dem Ende bin ich aber genötigt, wenn auch zum größten Teile nur andeutungsweise, Erörterungen voraufzuschicken, welche dazu dienen mögen, Verhältnisse in ein Licht zu stellen, die stets auftreten, sobald Zahlengrößen in endlicher oder unendlicher Anzahl gegeben sind; dabei werde ich zu gewissen Definitionen hingeleitet, welche hier nur zum Behufe einer möglichst gedrängten Darstellung des beabsichtigten Satzes, dessen Beweis in § 3 gegeben wird, aufgestellt werden."[45]

The first two sections, laid as the preliminary groundwork for the proof of the theorem itself, are really the heart of the paper. Devoted to his theory of irrational numbers and the geometric representation of the real numbers in general, this early example of CANTOR's thinking about sets is important for multiple reasons.

The problem of irrational numbers in mathematics dates from the time of the Pythagoreans.[46] By the late nineteenth century the development of analysis had led mathematicians to the realization that understanding the nature of the irrationals was of fundamental importance. Generally WEIERSTRASS, CANTOR and DEDEKIND are credited with having made the basic advances in building more rigorous theories of the system of real numbers.[47]

[45] *Ibid.* 92.

[46] CANTOR suggests EUCLID as the best reference for the problem of irrationals and their identification with the real line: "Zum Vergleiche mit § 1 and § 2 sei das 10. Buch der *Elemente des Euklides* erwähnt, welches für den darin behandelten Gegenstand maßgebend bleibt," *Ibid.* 96.

[47] No account given by WEIERSTRASS himself of the theory of irrational numbers survives. However, several of his students, preserving their notes of his lectures, published them in various forms. See for example ERNST KOSSAK: *Die Elemente der Arithmetik* (Berlin, 1872); S. PINCHERLE: "Saggio di una introduzione alla Teoria delle funzioni analitiche secondo i principii del Prof. C. Weierstrass," *Giornale di Matematiche*, **18** (1880) 178–254, 317–357. In 1887 OTTO BIERMANN, following WEIERSTRASS, KOSSAK and especially PINCHERLE, published his *Theorie der analytischen Functionen* (Leipzig, 1887). Though BIERMANN claims to be developing an essentially WEIERSTRASSIAN theory of the irrationals, he devotes an entire section to "abgeleitete Punktmengen," and declares his indebtedness to CANTOR for the concept of the fundamental series (see G. CANTOR: *III-4*, 186). BIERMANN, in fact, is clear evidence that not only were CANTOR's ideas being circulated and discussed, they were also being put to good use. For R. DEDEKIND's work on the irrationals see his famous *Stetigkeit und irrationale Zahlen* (Braunschweig, 1892, reprinted with *Was sind und was sollen die Zahlen?* by Vieweg: Braunschweig, 1965). An English edition, translated by W. W. BEMAN, is available under the title: *Essays on the Theory of Numbers, Continuity of Irrational Numbers, The Nature and Meaning of Numbers* (Chicago, 1901).

The history of the irrational numbers has been dealt with extensively. Of particular interest is the translation by J. MOLK of A. PRINGSHEIM's article for the *Encyklopädie der mathematischen Wissenschaften*: "Nombres irrationnels et notion de limite," *Encyclopédie des sciences mathématiques*, **1** (3) (Paris, 1904) 133–208. See also JOURDAIN's article in the *Archiv*, **16** (1910) 21–43, and the section devoted to irrational numbers in JEAN CAVAILLÈS: *op. cit.* 35–44.

Before setting out his own theory, CANTOR was clearly aware of the shortcomings of previous attempts to devise a theory of irrational numbers.[48] Indeed, in a later paper[49], CANTOR explicitly stressed the logical objections to previous attempts to define the irrationals in terms of infinite approximative sequences:

> "Es würde hierin ein *logischer Fehler* liegen, weil vielmehr die Definition der Summe $\sum a_v$ erst durch Gleichsetzung mit der notwendig vorher schon definierten *fertigen* Zahl b gewonnen wird. Ich glaube, daß dieser erst von Herrn Weierstrass vermiedene logische Fehler in früheren Zeiten fast allgemein begangen und aus dem Grunde nicht bemerkt worden ist, weil er zu den seltenen Fällen gehört, in welchen wirkliche Fehler keinen bedeutenderen Schaden im Kalkül anrichten können."[50]

Anxious to put the theory of irrational numbers on a strong arithmetic base, CANTOR set himself the task of developing a satisfactory theory of the irrationals which in no way presupposed their existence. His 1872 paper begins from the domain of rational numbers, denoted (A), and then carefully specifies the way in which the word "number" is being used. CANTOR writes: "Wenn ich von einer Zahlengröße im weiteren Sinne rede (i.e. other than as a rational number), so geschieht es zunächst in dem Falle, daß eine durch ein Gesetz gegebene unendliche Reihe von rationalen Zahlen

$$a_1, a_2, a_3, \ldots, a_n, \ldots \tag{22}$$

vorliegt."[51]

Thus CANTOR is dealing primarily with infinite sequences of rational numbers. These sequences are further subject to the condition that, for whatever m, if n is taken large enough, $|a_{m+n} - a_n| < \varepsilon$ for however small an $\varepsilon > 0$. He also stresses, for the present, that ε must itself be a rational number. If a series satisfies this condition, he expresses it as follows: "Die Reihe (22) hat eine bestimmte Grenze b"[52]. This has a very special meaning for CANTOR and must be taken as a convention to express, not that the series a_n actually has the limit b, or that the number b is presupposed as the limit, but merely that with each such sequence $\{a_n\}$

[48] Consult in particular BERTRAND RUSSELL'S discussion of irrational numbers in *The Principles of Mathematics* (Cambridge, 1903) 276–286.

[49] GEORG CANTOR: "Über unendliche lineare Punktmannigfaltigkeiten," *Mathematische Annalen*, **15** (1879) 1–7; **17** (1880) 355–358; **20** (1882) 113–121; **21** (1883) 51–58, 545–586; **23** (1884) 453–488. ZERMELO reprints the entire series of articles: G. CANTOR: *III-4*, 139–244. A substantial part of this series also appeared separately (see ZERMELO, pages 165–244) as the *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (Leipzig, 1883).

[50] G. CANTOR: *III-4*, 185.

[51] G. CANTOR: *II-5 (1872)* 92–93.

[52] *Ibid.* 93. For conciseness I shall call such series, as CANTOR later does himself, "fundamental series." See G. CANTOR: *III-4*, 186. CANTOR also noted the connection between his theory of irrationals and the fundamental series in "Bemerkung mit Bezug auf den Aufsatz: Zur Weierstrass-Cantorschen Theorie der Irrationalzahlen," *Mathematische Annalen*, **33** (1889) 476. See also ZERMELO'S edition, G. CANTOR: *II-9 (1889)* 114. It should also be noted that in the first published mention of CANTOR'S work on the irrationals HEINE called the fundamental series "Elementarreihen." Refer to the paper by E. HEINE: "Die Elemente der Functionenlehre," *Journal für die reine und angewandte Mathematik* (Crelle's Journal), **74** (1872) 173.

a definite symbol b is associated with it.[53] He then defines the order relations between the fundamental sequences as follows: if a_n is associated in the limit with b, a'_n with b', then, if $a_n - a'_n < \varepsilon$, this may also be expressed as $b = b'$; if $a_n - a'_n > \varepsilon$, $b > b'$; if $a_n - a'_n > -\varepsilon$, $b < b'$, for n arbitrarily large and any rational number $\varepsilon > 0$. Thus the sequence (22), associated with the limit b, can have only one of three relations with a rational number a (where a can be defined by the constant sequence $\{a\}$, which is clearly a fundamental sequence and for which the value a is actually the value reached by the series in the limit). CANTOR notes, by comparing the sequences, that either $b = a$, $b < a$ or $b > a$, and concludes:

> "Aus diesen und den gleich folgenden Definitionen ergibt sich als *Folge*, daß, wenn b die Grenze der Reihe (22) ist (in the special sense noted above), alsdann $b - a_n$ mit wachsendem n unendlich klein wird, womit *nebenbei* die Bezeichnung "Grenze der Reihe (22)" für b eine gewisse Rechtfertigung findet." [54]

CANTOR now denotes the collection of all numbers b associated with infinite sequences $\{a_n\}$ by B. CANTOR shows that the arithmetic operations hold in B by examining the associated sequences $\{a_n\}$. The relations $b \pm b' = b''$, $b \cdot b' = b''$, $b/b' = b''$, are expressed as follows:

$$\lim (a \pm a'_n - a''_n) = 0,$$
$$\lim \left((a_n \cdot a'_n) - a''_n\right) = 0,$$
$$\lim \left((a_n / a'_n) - a''_n\right) = 0.$$

What CANTOR has shown, essentially, is that the space of fundamental sequences is closed under the arithmetic operations. It should also be noted that by virtue of the fact that any constant sequence $\{a\}$ is a fundamental sequence, then a must be an element of B. Thus elements of A may enter into the above relationships, as well as elements from B.

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[53] This association is very important, because it allows CANTOR to avoid the accusations RUSSELL makes: "There is nothing to show that a rational can be subtracted from a real number, and hence the supposed proof is fallacious." If CANTOR suggests such a subtraction, it is only symbolically, and not meant to be taken in any literal sense. CANTOR makes all of this clear in his response to ILLIGENS, who had also objected to CANTOR's theory of irrationals. See CANTOR: *II-9 (1889)* 114. For BERTRAND RUSSELL's critique of CANTOR, consult RUSSELL: *op. cit.* 283–286.

[54] G. CANTOR: *II-5 (1872)* 93. Here CANTOR may have been relying too heavily on geometric intuition. Interpreting $b - a_n$ in the language of infinite series helps to legitimate his claim. The argument also appears, however, quite circular, since b is defined, or merely associated, with $\{a_n\}$. CANTOR's remark in the limit reduces to the tautology $(a_n - a_n) = 0$. What CANTOR probably intended to say, if I read him correctly, was that since all sequences in the class of sequences associated with b differ in the limit from a_n by an arbitrarily small amount, there is a clear justification for thinking of b as the limit of $\{a_n\}$. CANTOR would have avoided RUSSELL's criticism in particular had he not relaxed his terminology and returned either to more familiar notations or intuitions.

Now, in an analogous fashion, CANTOR builds the domain C from the domain B, considering infinite sequences:

$$b_1, b_2, b_3, \ldots, b_n, \ldots, \tag{23}$$

and defining the fundamental sequences as before. With each sequence $\{b_n\}$ there is associated a number c. For all sequences generated from B, these numbers constitute the domain C. Now CANTOR stresses the fundamental and crucial difference between the three domains A, B and C. While for every $a \in A$ there is a corresponding $b \in B$, the converse is false. However, not only can we identify each $b \in B$ with an element $c \in C$, but every $c' \in C$ also has an image $b' \in B$. Although in this isomorphic sense the elements of the domains B and C are seen to correspond reciprocally, CANTOR underscores their very important conceptual difference. This difference is most easily explained by the fact that even in the domain B the "identity" of two elements $b = b'$ is not really an identity between the elements themselves, but expresses instead, as CANTOR says, a certain relation between the *series* to which the limits b, b' are related.[55]

This raises as well a philosophical problem concerning the nature and existence of the domain B generated from A, and the subsequent domains generated serially from B. CANTOR takes these numbers in themselves as meaningless. On this point he is explicit and emphatic, arguing that he has built a theory "in welcher die Zahlengröße, zunächst an sich im allgemeinen gegenstandslos, nur als Bestandteil von Sätzen erscheint, welchen Gegenständlichkeit zukommt, des Satzes z. B., daß die entsprechende Reihe die Zahlengröße zur Grenze hat."[56] Hence the objectivity of the numbers of domain B, and of the domains successively generated from B, receive whatever objectivity they may have from the sequences to which they are related. Clearly their objectivity is of a very different kind from that enjoyed by the rational numbers of domain A. Consequently an element $b \in B$ should be thought of as a number only for the sake of convenience. It really represents a fundamental sequence. Since each such limit $b \in B$ can be associated with an infinite number of rational sequences of elements from A, B may be thought of as having been determined by sets of equivalence classes of rational sequences. This gives the elements of B an objectivity in terms of their corresponding equivalence classes, which, apart from an illegitimate identification with the objectivity of number in the sense of the integers or rationals, is much closer to the kind of objectivity CANTOR had in mind.[57]

With these caveats in mind, CANTOR proceeds to define higher-order domains from C. Proceeding through λ such constructions, he reaches the domain L,

[55] This further emphasizes CANTOR's view that the elements of the set B should not be taken as actual limits. This, coupled with CANTOR's earlier expression for $b \pm b' = b''$ as $\lim(a_n \pm a'_n - a''_n) = 0$, further confirms my interpretation of the meaning CANTOR had intended for $b - a_n$. See footnote 54 and G. CANTOR: *II-5 (1872)* 94.

[56] *Ibid.* 95.

[57] CANTOR must have felt insecure leaving the numbers b as merely associations with infinite sequences $\{a_n\}$. He might have avoided both criticism and some conceptional hedging had there been a more developed theory of iso- and homomorphisms available. Lacking these, and about to stress the identifiability of his new numbers with the points of the line, he perhaps overplayed their ontological status as numbers in the sense of limits. It was then too easy a step, and a wrong step, to begin thinking of the elements of B as limits in the actual sense.

which is closed under the arithmetic operations and upon which order relationships are defined as above for the domain B. Just as B and C are in a special sense reciprocally identifiable, so too is the domain L reciprocally coincident with all the generated domains $B, C, \ldots, K$. A of course is still excepted, and though A can be described as embedded in L, L is much richer in elements than the merely rational set A. Given an element l in the domain L, reached by λ successive generalizations from the domain A, CANTOR writes: "*Sie ist als Zahlengröße, Wert oder Grenze* λ^{ter} *Art gegeben,* woraus ersichtlich ist, daß ich mich der Worte *Zahlengröße, Wert* und *Grenze* im allgemeinen in gleicher Bedeutung bediene."[58] The equivalence of CANTOR's vocabulary is here justifiable, again by the special way in which he has taken the elements of the λ^{th} domain L as numbers. He stresses that these λ^{th} domains are valuable conceptually: "der Zahlenbegriff, soweit er hier entwickelt ist, den Keim zu einer in sich notwendigen und absolut unendlichen Erweiterung in sich trägt."[59] He reserves, however, the opportunity for discussing the absolute extensions for a later occasion.[60] Clearly he already had in mind the possibility of extending his definitions and operations for the benefit and advancement of analysis itself, though he was well aware that this would require a more careful and detailed approach than that given in the 1872 paper. It was beyond the scope of his extension of the uniqueness theorem itself. Nevertheless, as CANTOR himself had best described it, the germ of a new and significant development for mathematics was here laid out, embryonically, for the first time.

With the foregoing construction of the real number system at hand, CANTOR next turned to the task of identifying numbers with points on the real line. For the rational numbers this was not a difficult problem. CANTOR knew that given a point, if it had a rational relation to the unit abscissa, then it could be expressed by an element from the domain A. Otherwise, it could be approached by a sequence of rational points

$$a_1, a_2, a_3, \ldots, a_n, \ldots, \tag{22}$$

each of which corresponds to an element in A. Furthermore, CANTOR could take $\{a_n\}$ as a fundamental sequence which approached the given point arbitrarily closely. He expresses this condition as follows: "*Die Entfernung des zu bestimmenden Punktes von dem Punkte o ist gleich b, wo b* die der Reihe (22) entsprechende Zahlengröße ist."[61] Since every element from A is uniquely embedded in the domain B, the uniqueness of this representation of the points of the line and the domain B follows easily. What, of course, CANTOR was unable to show is the converse, that for every element of the domain B there corresponds a unique point of the real line. Thus he had to invoke the well-known axiom, "daß auch umgekehrt zu jeder Zahlengröße ein bestimmter Punkt der Geraden gehört, dessen Koordinate gleich ist jener Zahlengröße, und zwar in dem Sinne

[58] G. CANTOR: *II-5 (1872)* 95.

[59] *Ibid.*

[60] See G. CANTOR: *III-4.* See also the section below titled: "Sets of the Second Species," pp. 213–215.

[61] G. CANTOR: *II-5 (1872)* 96.

gleich, wie solches in diesem § erklärt wird".[62] This axiom served two functions for CANTOR. Not only did it give a sense of symmetry and completeness to the relation between points of the line and real numbers, a relation which would have been one-sided otherwise, with no guarantee that for all elements of the domain B there corresponded unique elements of the line itself, but also, and perhaps equally important, the identification of points with the derived domains $B, C, \ldots$, gave the elements of these domains a further objectivity. CANTOR stressed, however, that the numbers in these various domains remain entirely independent of this geometric identification, and the isomorphism serves, really, as an aid in thinking about the numbers themselves.[63] CANTOR thus summarized his description of the identification of points with the elements of the various derived sets as follows: "Dem Obigen gemäß betrachte ich einen Punkt der Geraden als bestimmt, wenn seine Entfernung von o mit dem gehörigen Zeichen versehen, als Zahlengröße, Wert oder Grenze λ^{ter} Art gegeben ist."[64]

The Fifth Paper: Derived Sets

Having completed his discussion of the real numbers, CANTOR now translates his results from the previous section of his paper into the language of point-sets, cautioning that, though the geometric character of a point-set makes it somewhat easier to visualize, whenever the word "point" occurs, the associated number-value by which the point is given should always be kept in mind. He notes that a given finite or infinite collection of numbers will be called a *Wertmenge*; correspondingly, a finite or infinite collection of points on the line a *Punktmenge*.

> "Unter einem 'Grenzpunkt einer Punktmenge P' verstehe ich einen Punkt der Geraden von solcher Lage, daß in jeder Umgebung desselben *unendlich* viele Punkte aus P sich befinden, wobei es vorkommen kann, daß er außerdem selbst zu der Menge gehört. Unter 'Umgebung eines Punktes' sei aber

[62] *Ibid.* 97. DEDEKIND notes in the preface to his *Stetigkeit und irrationale Zahlen* that his own work was sent to press upon having read HEINE's development of irrationals (which drew heavily on CANTOR's work, as HEINE acknowledged), and immediately after he had seen CANTOR's own paper of 1872. DEDEKIND, in describing CANTOR's exposition of the irrationals, was particularly struck by the CANTORIAN axiom "Wie ich bei raschem Durchlesen finde, so stimmt das Axiom in § 2 derselben, abgesehen von der äußeren Form der Einkleidung, vollständig mit Dem überein, was ich unten in § 3 als das Wesen der Stetigkeit bezeichne," *op. cit.* 3. But the axiom by which DEDEKIND attributes continuity to the line is given as follows: "Zerfallen alle Puncte der Geraden in zwei Classen von der Art, dass jeder Punct der ersten Classe links von jedem Puncte der zweiten Classe liegt, so existiert ein und nur ein Punct, welcher diese Eintheilung aller Puncte in zwei Classen, diese Zerschneidung der Geraden in zwei Stücke hervorbringt," *op. cit.* 11. Though this is interpreted as equivalent to CANTOR's assumption that to every real number there corresponds a unique point of the EUCLIDEAN line, the identification is by no means made explicit in DEDEKIND's form of the axiom. This, of course, is one of CANTOR's great contributions and insights, clearly and precisely presented. The assumption that such a correspondence is possible led CANTOR to the axiom that essentially underlies the complete understanding of the nature of continuity in any continuous domain.

[63] DEDEKIND makes this same point concerning the value of geometric guides, though stressing their lack of rigor, in the preface to his *Stetigkeit und irrationale Zahlen*. *op. cit.* 1.

[64] G. CANTOR: *II-5 (1872)* 97.

hier ein jedes Intervall verstanden, welches den Punkt *in seinem Innern* hat. Darnach ist es leicht zu beweisen, daß eine aus einer unendlichen Anzahl von Punkten bestehende Punktmenge stets zum wenigsten *einen* Grenzpunkt hat." [65]

CANTOR's conclusion is the Bolzano-Weierstrass theorem that any infinite bounded point-set must have at least one limit point. With this result, CANTOR can now describe the immediate relationship between a given point-set P and any point of the real line, namely, that every point of the line is either a limit point of the set P, or it is not. With any given point-set P there is always given, conceptually, its set of limit points, which CANTOR denotes as P' and calls: "die *erste abgeleitete Punktmenge* von P." [66] And now, just as CANTOR was able to generate from B an entire system of λ-domains, he does the same with P'. If P' is an infinite point-set, then it gives rise to a second derived point-set P'', and so on, until "Man findet durch v solcher Übergänge den Begriff der v^{ten} abgeleiteten Punktmenge $P^{(v)}$ von P." [67]

What CANTOR has achieved is a translation of ideas from his fundamental sequences $\{a_n\}$, which had served to define the domain B, into the language of point-sets, where the limit points correspond to the elements of B and are defined by an infinite set of points (corresponding to the fundamental sequence) in any arbitrarily small neighborhood of that point. The case which now becomes interesting for CANTOR's uniqueness theorem is the one in which, after v repetitions, the derived point-set P^v consists only of a finite number of points, thus making the extension to further derived sets impossible,[68] that is, P^{v+1} does not exist.

[65] *Ibid.* 98. It should be stressed that the concept of limit-point and the associated idea of the first derived set P' of limit-points of a set P are the fundamental, basic elements of CANTOR's development of the set theory. Not only do they play an obvious part in proving CANTOR's uniqueness theorem as published in 1872, they also contained the seeds of important consequences. One such development included the transfinite numbers, as discussed in greater detail below, pages 213–215. The concept of the first derived set of P, however, was also the means by which CANTOR was later to categorize sets as being, for example, closed, perfect or dense. For a more detailed discussion of these and other, more sophisticated ways in which CANTOR was able to characterize sets in terms of P and P' alone, see the paper by I. GRATTAN-GUINNESS: "An Unpublished Paper by GEORG CANTOR: Principien einer Theorie der Ordnungstypen, erste Mittheilung" *Acta Mathematica*, **124** (1970) 70–71. These various types of sets, in turn, permitted a more exacting analysis of general sets, and led to a host of decomposition theorems, including the well-known CANTOR-BENDIXSON theorem. For CANTOR's own discussion of this theorem, see G. CANTOR: *III-4*, in particular §16, 215–225, and the shorter paper by CANTOR: "Sur divers théorèmes de la théorie des ensembles de points situés dans un espace continu a n dimensions," published as a letter to the editor, *Acta Mathematica*, **2** (1883) 409–414, and reproduced in ZERMELO's edition, pages 247–251. It was through such means as the decomposition theorems that CANTOR had hoped to find a solution to the problem of the power of the continuum. The skeleton of just such a proof is outlined by CANTOR in a letter to MITTAG-LEFFLER dated August 26, 1884. See MESCHKOWSKI, *op. cit.* 242.

[66] G. Cantor: *II-5 (1872)* 98.

[67] *Ibid.*

[68] CANTOR, however, had not forgotten the importance of the case where $P^{(\infty)} \neq 0$. The significance of such point-sets of the second species, as CANTOR was later to call them, is discussed in greater detail below, pages 213–215.

When this occurs, CANTOR gives the original set P the following designation: "in diesem Falle wollen wir die ursprüngliche Punktmenge P *von der* ν^{ten} *Art* nennen, woraus folgt, daß alsdann P', P'', ... von der $(\nu-1)^{\text{ten}}$, $(\nu-2)^{\text{ten}}$, ... Art sind."[69]

CANTOR now shows that such point-sets exist by appealing to a point whose abscissa is determined by a number of the ν^{th} kind. Retracing his steps from P^{ν} to $P^{\nu-1}$, $P^{\nu-2}$, ..., he finally reaches a set Q in the domain A which produces an infinite set of rational numbers. Translated back into the language of point-sets, the point-set Q is now a set of the ν^{th} kind, and $Q^{\nu+1}$ does not exist.[70]

What CANTOR achieved through this development of derived sets was a simplification of the proof which extends the uniqueness theorem to infinite exceptional sets on finite intervals. The actual value of this theorem, however, rests in the fact that it led CANTOR to consider the topology of the real line and to describe its nature as precisely as possible in non-geometric terms. Like DEDEKIND, he must have felt that while the geometric intuition of the nature of continuity was a valuable guide, it was by no means a rigorous one. Therefore, following WEIERSTRASS, CANTOR turned to a strictly arithmetic formulation, proceeding on logical principles and working towards a conclusion which, though capable of being translated into geometric terms, was nevertheless totally independent of any spatial intuition. Having begun with the rationals, he avoided the problem of defining the irrationals by limits of infinite sequences of rational numbers, and thereby presupposing the existence of the limits themselves. Instead, he associated with each fundamental sequence a symbol $b \in B$, and demonstrated further that there are more elements in B than merely the rational numbers. Showing that B is closed under the arithmetic operations, and that order can be defined on the elements of B, CANTOR proceeded to define the series of domains derived from B. Ultimately the objectivity of the elements of these derived domains rests with the infinite fundamental sequences of rational numbers associated with them.

CANTOR then managed to establish the relations between point-sets and the value-sets themselves, thus formulating in a precise way the notion of derived point-sets of the ν^{th} kind. Essentially, these sets returned CANTOR to the more familiar context of point-set intervals on the real line, while mathematically he had rigorously established the fundamental nature and characteristics of the real numbers themselves. And though always keeping in mind the idea of the infinite fundamental sequence as the defining property of the real numbers, he felt he had won a more satisfying objectivity for his numbers through their geometric representation, independent though they were of this visualization. The proof of the uniqueness theorem itself was to make good use of all this preparation.

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[69] G. CANTOR: *II-5 (1872)* 98.

[70] Meanwhile a shift in terminology has occurred. Earlier a value of the λ^{th} kind meant that the value was contained in the derived set L obtained after λ extensions from the set of rationals A. Now, in describing a *point-set* P of the λ^{th} kind, CANTOR means that $P^{(\lambda+1)}$ does not exist.

The Fifth Paper: The Uniqueness Theorem for Infinite Exceptional Sets

The final section of CANTOR's 1872 paper begins with a statement of the extended uniqueness theorem for exceptional sets:

"Wenn eine Gleichung besteht von der Form

$$0 = C_0 + C_1 + \cdots + C_n + \cdots, \tag{24}$$

wo $C_0 = \frac{1}{2} d_0$; $C_n = c_n \sin n x + d_n \cos n x$, für alle Werte von x mit Ausnahme derjenigen, welche den Punkten einer im Intervalle $(0 \ldots 2\pi)$ gegebenen Punktmenge P der v^{ten} Art entsprechen, wobei v eine beliebig große ganze Zahl bedeutet, so ist $d_0 = 0$, $c_n = d_n = 0$." [71]

The proof itself proceeds in a fashion quite similar to the methods used in the 1870 uniqueness theorem and the first extension of that theorem to finite exceptional sets in 1871. Considering the RIEMANN function,

$$F(x) = C_0 \frac{x\,x}{2} - C_1 - \frac{C_2}{4} - \cdots - \frac{C_n}{n \cdot n} - \cdots, \tag{25}$$

CANTOR shows that given the nature of the set P of the v^{th} kind, there must be an interval $(\alpha \ldots \beta)$ in which no points of P lie. On such an interval $(\alpha \ldots \beta)$

$$\lim (c_n \sin n x + d_n \cos n x) = 0 \quad \text{for all values of } x \text{ in } (\alpha \ldots \beta).$$

Thus, by appealing to the 1871 paper on trigonometric series, he immediately has that $\lim c_n = 0$ and $\lim d_n = 0$. Now, by introducing certain results from RIEMANN'S *Habilitationsschrift*,[72] CANTOR claims the following properties for $F(x)$:

1) $F(x)$ is continuous in the neighborhood of any value x.
2) $\lim \frac{F(x+\alpha)+F(x-\alpha)-2F(x)}{\alpha \cdot \alpha} = 0$ if $\lim \alpha = 0$ for all values of x, with the exception of those values which correspond to points of the set P.
3) $\lim \frac{F(x+\alpha)+F(x-\alpha)-2F(x)}{\alpha} = 0$ if $\lim \alpha = 0$ for all values of x without exception.[73]

He now claims that $F(x)$ is a linear function, identically the same over the entire interval $(0 \ldots 2\pi)$. With this result he will be able to prove the theorem itself, following the same procedure he had already applied successfully to the two earlier versions of the theorem. For sub-inervals $(p \ldots q)$ which contain only a finite number of points from P, the linearity of $F(x)$ follows directly from the earlier "Notiz," where the uniqueness theorem had already been established for a finite number of exceptional values of x. The difficulty now, of course, was to advance beyond the finite number of exceptions to the infinite case. Before doing this, however, CANTOR was careful to emphasize the above result. He sets it apart as follows:

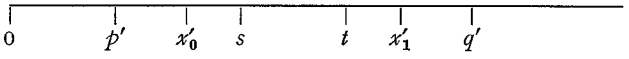
[71] G. CANTOR: *II-5 (1872)* 99.

[72] B. RIEMANN: *op. cit.*, in particular section 8 of his paper, pages 245–251.

[73] Condition 2) involves the SCHWARZ second derivative, discussed above, pages 192 and 193. Condition 3) however involves the idea generally referred to as the "smoothness" of the function $F(x)$, which implies that if $F(x)$ possesses a derivative $F'(x)$ at x, then both its left derivative $F'(x^-)$ and right derivative $F'(x^+)$ exist and are equal.

(A) "Ist $(p \ldots q)$ irgend ein Intervall, in welchem nur eine endliche Anzahl von Punkten der Menge P liegt, so ist $F(x)$ in diesem Intervalle linear." [74]

Now CANTOR advances to the consideration of intervals $(p' \ldots q')$ which contain only a finite number of points $x'_0, x'_1, \ldots, x'_v$, of the *first derived set* P' of P. Here he introduces a geometric diagram, and sets out to show that not only is $F(x)$ linear over each sub-interval $(x'_v \ldots x'_{v+1})$, but that it is, in fact, identical over the entire interval $(p' \ldots q')$.



CANTOR considers, in particular, the sub-interval $(x_0 \ldots x_1)$. Since, in general, $(x_0 \ldots x_1)$ will contain an infinite number of points from the originally given point-set P, the "Notiz" result (A) cannot be applied. However, each sub-interval $(s \ldots t)$ falling entirely within $(x'_0 \ldots x'_1)$ can have only a finite number of points from P, otherwise there would be a limit point in $(s \ldots t)$. But it has been assumed that there is no limit point of P between x_0 and x_1. Now applying the result (A), the function $F(x)$ must be linear over $(s \ldots t)$. Since the endpoints s and t can be taken arbitrarily near the endpoints x'_0 and x'_1, it follows immediately that the continuous function $F(x)$ is also linear over $(x'_0 \ldots x'_1)$. CANTOR applys this conclusion over each sub-interval $(x'_v \ldots x'_{v+1})$ of $(p' \ldots q')$, and deduces as before the following:

(A') "Ist $(p' \ldots q')$ irgend ein Intervall, in welchem nur eine endliche Anzahl von Punkten der Menge P' liegt, so ist $F(x)$ in diesem Intervalle linear." [75]

The proof now proceeds analogously, considering sub-intervals $(p^k \ldots q^k)$ which contain only finitely many points of the k^{th} derived point-set P^k of P. $F(x)$ is linear over all such intervals. Since P is of the v^{th} kind, however, CANTOR can now press his conclusion. After a finite number of steps he reaches the intervals which contain points of P^v. There can only be a finite number of such points, and no derived sets can exist beyond P^v. Thus he concludes that $F(x)$ is linear in any arbitrarily given interval $(a \ldots b)$. $F(x)$ can therefore be given the form $F(x) = c\,x + c'$, and the proof now follows exactly as it had in both of the previous versions of the uniqueness theorem which CANTOR had given.

The step from the 1871 conclusion for a finite set of exceptional points to the infinite case was easy once CANTOR had managed to solve the difficult problem of explaining the linearity of $F(x)$ over such intervals. Given $(p \ldots q)$ containing finitely many points from P, he already had the result. Given an interval $(p' \ldots q')$ with infinitely many points of P, he eventually realized he could single out certain sub-intervals of $(p' \ldots q')$ which would contain only finitely many points of P, except at certain points in the neighborhoods of which infinite numbers of the exceptional points occured. These of course were the limit points as CANTOR defined them in section two of the 1872 paper. It was only a matter of advancing analogously a finite number of times before he had not only the proof he needed, but the concept of derived sets as well.

[74] G. CANTOR: *II-5 (1872)* 100.
[75] *Ibid.* 101.

Now the problem was to find a convenient way of expressing the peculiar and important properties of the derived sets of the v^{th} kind. This could not be done geometrically or descriptively in any precise way, and so CANTOR was forced to think of his theorem, for the sake of rigor and exactness, in purely arithmetic terms. Once he had cast the problem in this manner it was impossible to avoid the difficulty of identifying the points of the line with the real numbers themselves. Since this identification was already available for the mapping between the domain of rational numbers and the real line, it was natural for CANTOR to start there, and to advance from the rationals to the reals, and to derived sets of higher order.[76]

Once confronted with the question of whether the uniqueness theorem could be shown for infinite exceptional sets, CANTOR became involved with the topological properties of linear point-sets, which required, eventually, his reformulation of the theory of irrational numbers and the fundamental concept of derived sets. All of this was to lead, in a natural and direct way, to the first stage of the *Mengenlehre* itself, the *Punktmannigfaltigkeitslehre.*

Sets of the Second Species

In order to prove his uniqueness theorem over infinite exceptional sets, CANTOR was restricted to the consideration of point-sets of the first species, those for which $P^{(v)}=0$ for some $n<\infty$. But it is clear, even from his own remarks, that by 1872 he had proceeded well beyond the simple sequence of derived sets of the first kind, $P^{(1)}, P^{(2)}, \ldots, P^{(v)}$, to consider the *entire sequence of derived sets of the second species* as well, including

$$P^{(\infty)},\ P^{(n\infty^{\infty})},\ P^{(\infty^{\infty+1})},\ P^{(\infty^{\infty+n})},\ P^{(\infty^{n\infty})},\ P^{(\infty^{\infty^{n}})},\ P^{(\infty^{\infty^{\infty}})},\ \ldots. \qquad (26)$$

CANTOR is explicit about his earliest formulation of this sequence and stresses, as a footnote to his paper of 1880 in the *Punktmannigfaltigkeiten* series (a footnote which ZERMELO in his edition of CANTOR'S *Gesammelte Abhandlungen* failed to include), that

> "Zu derselben bin ich vor nun zehn Jahren gelangt, bei Gelegenheit einer eigenthümlichen Darstellung des Zahlbegriffs (Math. Ann. 5) habe ich entfernt darauf hingewiesen." [77]

[76] DEDEKIND, from his remark in the preface to *Stetigkeit und irrationale Zahlen,* seems to have been unable to appreciate the significance of CANTOR'S development of the domains of higher order: "Welchen Nutzen aber die, wenn auch nur begriffliche Unterscheidung von reellen Zahlgrößen noch höherer Art gewähren wird, vermag ich gerade nach meiner Auffassung des in sich vollkommenen reellen Zahlgebietes noch nicht zu erkennen," *op. cit.* 3.

[77] GEORG CANTOR: "Über unendliche lineare Punktmannigfaltigkeiten," *Mathematische Annalen,* **17** (1880) 358. I. GRATTAN-GUINNESS emphasizes this same point and underscores that it was in this context of CANTOR'S early work that the transfinite numbers arose without ever referring to the questions of cardinality and ordinality of point-sets themselves. See the article by I. GRATTAN-GUINNESS, previously cited: *Acta Mathematica,* **124** (1970) 69. I am particularly grateful for the opportunity to have discussed various aspects of the present paper with Dr. GRATTAN-GUINNESS during the conference on "Problemgeschichte der Mathematik" held at the Mathematisches Forschungsinstitut, Oberwolfach, West Germany, in October of 1970.

In fact, CANTOR's reference as it appeared in the 1872 paper was not so oblique as to make his meaning unclear: "... der Zahlenbegriff, soweit er hier entwickelt ist, den Keim zu einer in sich notwendigen und absolut unendlichen Erweiterung in sich trägt." [78]

What CANTOR, looking back in 1880 to his paper of 1872, had described as a "dialektische Begriffserzeugung," [79] was the conceptual generation of derived sets of the *second species* which had grown out of his general idea of derivation in a *necessary* and *absolute* way. Thus the derived set of second species, $P^{(\infty)}$ where $P^{(\infty)} \neq 0$, led to a derived set of $P^{(\infty)}$ itself, namely $P^{(\infty)'}$, which CANTOR denoted $P^{(\infty+1)}$. This was the important and crucial step. He had already stressed the fact, in discussing his construction of the irrational numbers, that the set B derived from the rationals A forms essentially the real numbers. From B proceeds the unending sequence of further derived sets $B, C, D, \ldots, L, \ldots$, a development which DEDEKIND, from his own perspective, was unable to appreciate.[80] CANTOR himself had noted that further derivations beyond B yield no new elements in the sense that each domain derived from B is isomorphic to B. Thus, as DEDEKIND remarked, these "higher order real numbers" contributed nothing that the first derived set B from A did not already contain. But CANTOR in 1872 was interested in more than just an analysis of the real number system and a rigorous construction of the irrationals. In order to prove the uniqueness theorem over infinite exceptional sets, the derived sets of the *first species* were essential. While DEDEKIND would have left the sets of higher order derived from B to collapse, by virtue of their isomorphisms, to the set B itself, CANTOR stressed important distinctions as outlined in an earlier section of this paper.[81] At first these differences were significant because they allowed him to establish the linearity of the function $F(x)$ over such derived sets of the *first* species. But it was also the important differences between the sets derived from B, considered dialectically in the case of *point sets of the second species*, that provided CANTOR'S germinal idea: the sequence (26).

Eventually CANTOR was led to consider this sequence in a form abstracted from the point sets of the second species. In his *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* [82] he generates, in an entirely formalistic way, the entire sequence of transfinite numbers. In contrast to the denumerable first number class (I) of integers $1, 2, \ldots, \nu, \ldots$, CANTOR describes the transfinite numbers as comprising the second number class (II), and denotes them as follows:

$$\omega, \omega+1, \ldots, \nu_0\omega^{\mu}+\nu_1\omega^{\mu-1}+\cdots+\nu_{\mu-1}\omega+\nu_{\mu}, \ldots, \omega^{\omega}, \ldots, \alpha, \ldots .$$

Once these transfinite numbers were defined, CANTOR was then able to proceed with the development of the transfinite arithmetic itself, which also received its first published mention in the *Grundlagen* of 1883.

Thus the entire concept of derived sets, as CANTOR developed it for his 1872 paper, was a sword with many edges. Not only did it serve the purpose, then

[78] G. CANTOR: *II- 5(1872)* 95.
[79] G. CANTOR: *III-4*, 148.
[80] Consult DEDEKIND's opinion as quoted above, footnote 76.
[81] Refer in particular to pages 208–210.
[82] G. CANTOR: *III-4*, 197.

uppermost in his mind, of solving the uniqueness problem itself, but it went much further. It provided the point-sets of the first species over which he could show the linearity of $F(x)$ and thereby the uniqueness theorem itself. It was instrumental in his rigorous formulation of the irrational numbers. It led to the first example in CANTOR'S published work of infinite point-sets, and these with specially determined properties. And above all, it led "necessarily and absolutely" to even further extensions. In the language of point-sets this meant the entire sequence of point-sets of the second species, and eventually the transfinite numbers and their corresponding arithmetic.

Conclusion

GEORG CANTOR, having begun his researches into the complexities of trigonometric series with a short paper in 1870, had finally reached, in less than two years, the rudimentary outline of a new mathematical theory. Meager though the suggestion was, CANTOR already had in mind further extensions of his new ideas, and must have begun immediately to explore the consequences of his results of 1871. Having shown that his uniqueness theorem could be proven even on certain infinite exceptional point-sets, CANTOR knew that the nature of these sets and the derived sets themselves posed problems closely related to the structural properties of the real line. His development of the concepts of limit point and of derived sets clearly reveals the direct association between the two in CANTOR'S own mind. Since the nature of CANTOR'S proofs of the uniqueness theorem depended heavily on the nature of point-relationships on the line, and only to a lesser extent on the trigonometric functions themselves, it would have been natural for CANTOR to turn his attention to the former. In fact, his next published paper in analysis contained important results concerning the non-denumerability of the continuum, as well as the problem of algebraic numbers and the relationship they bore to the real numbers in particular.[83] Although both the rational numbers and a large class of irrational numbers are contained in the class of algebraic numbers, CANTOR was well aware of LIOUVILLE'S proof that there are, nevertheless, an infinite number of transcendental numbers. CANTOR'S 1874 proof not only demonstrated that the algebraic numbers are denumerable, but that the real numbers are not. Taken together, as CANTOR notes: "so ist damit ein neuer Beweis des zuerst von *Liouville* bewiesenen Satzes gegeben, daß es in jedem vorgegebenen Intervalle $(\alpha \ldots \beta)$ unendlich viele *transzendente*, d.h. nicht algebraische reelle Zahlen gibt." [84] It was then a natural step to begin to ponder the relations between various sets and classes of numbers, and to explore in more detail the nature of correspondences between them.

The development of CANTOR'S ideas over the two-year period 1870–1872 offers interesting evidence of the way in which his mathematical mind formulated questions and then set out to answer them. Knowing what the important problems were constituted but one half of the problem. CANTOR was fortunate in that he was guided at this crucial point in his career by EDUARD HEINE, whose own

[83] GEORG CANTOR: "Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen," *Journal für die reine und angewandte Mathematik* (Crelle's Journal), **77** (1874) 258–262. See also G. CANTOR: *III-1 (1874)* 115–118.

[84] *Ibid.* 116.

ideas provided the basis and framework for CANTOR'S early work in analysis: specifically his concentration upon the uniqueness of trigonometric representations.

The earliest paper on trigonometric series was hardly exciting. The result had been achieved in various forms by others, though CANTOR did manage to give his proof for the most general case. But the proof was long, complicated, and suggests that CANTOR was just beginning to come to terms with his subject. Later, when he returned to give an alternate proof, his improved mastery of the field is clearly revealed by a more straightforward and elegant proof, cleverly incorporating a number of previously known results.

The evolution of the uniqueness theorem itself displays even more directly this mark of CANTOR'S early work: the indispensable facility for knowing how to tailor a theorem to the mathematical tools at his disposal. As CANTOR developed each new extension of the uniqueness theorem, he more-and-more successfully marshalled the previous work of RIEMANN, HEINE and SCHWARZ to his own ends, modifying lemmas or the details of proof to obtain, in a clear and natural way, the uniqueness theorem in considerable generality. But it was only in the final stage of his work on that theorem that CANTOR displayed the most important feature of his mathematical ability: the capacity for creating new forms and concepts when the limitations of existing approaches became evident, and neither suggested nor seemed to permit further progress. At this point CANTOR established himself as an important member of his own generation of mathematicians, capable of forging new concepts and tools which could be used to obtain radically new kinds of solution to known problems.

Thus growing out of the specific problems posed by the nature of trigonometric representations, set theory nevertheless soon gained an autonomy of its own. But just as FOURIER had shocked LAGRANGE and the mathematicians of his own day with his results on the trigonometric representation problem, CANTOR'S work was to provide equally distressing dilemmas. Yet in his early papers on the nature and uniqueness of such representations, there was no hint of the paradoxes to come. Instead, CANTOR had managed to find impressive solutions to problems well-known to mathematicians of his own time. As CANTOR turned away from the strictly trigonometric context of his early work and began to focus on set theory itself, a new era in mathematics was about to emerge.

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(Received November 11, 1970)