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PEIRCE'S PLACE IN MATHEMATICS

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SUMMARIES

This article compares treatments of the infinite, of continuity and definitions of real numbers produced by the German mathematician Georg Cantor and Richard Dedekind in the late 19th century with similar interests developed at virtually the same time by the American mathematician/philosopher C. S. Peirce. Peirce was led, not by the internal concerns of mathematics which had motivated Cantor and Dedekind, but by research he undertook in logic, to investigate orders of infinite sets (multitudes, in his terminology), and to introduce the related concept of infinitesimals. His arguments in support of the mathematical and logical validity of infinitesimals (which were rejected by such eminent mathematicians as Cantor, Peano, and Russell at the turn of the century) are considered. Attention is also given to the connections between Peirce's mathematics, his philosophy, and especially his interest in continuity as it was related to his Pragmatism.

Cet article compare, d'une part, les façons d'aborder l'infinie et la continuité ainsi que les définitions des nombres réels élaborées par les mathématiciens allemands Georg Cantor et Richard Dedekind vers la fin du XIX^{e} siècle, à, d'autre part, l'intérêt de même nature développé quasiment à la même époque par le mathématicien-philosophe américain C. S. Peirce. Peirce fut amené à étudier les ordres d'ensembles infinis (multitudes dans sa terminologie) et à introduire le concept d'infinitésimal non pas, comme dans le cas de Cantor et Dedekind, par des préoccupations purement mathématiques, mais plutôt à la

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suite des ses recherches en logique. Nous nous penchons sur son argumentation visant à montrer le bien-fondé mathématique et logique des infinitésimaux (que rejetaient au tournant du siècle d'aussi éminents mathématiciens que Cantor, Peano et Russell). De plus nous nous attardons aux relations entre les mathématiques de Peirce et sa philosophie, plus particulièrement à son intérêt pour la notion de continuité, dans la perspective de son pragmatisme.

In diesem Beitrag werden die Behandlung des Unendlichen, der Stetigkeit und die Definition reeller Zahlen der deutschen Mathematiker Georg Cantor und Richard Dedekind im ausgehenden 19. Jahrhundert verglichen mit ähnlichen Bestrebungen, die der amerikanische Mathematiker-Philosoph C. S. Peirce zu genau der gleichen Zeit verfolgte. Während Cantor und Dedekind zu ihren Untersuchungen durch innermathematische Probleme geführt wurden, gaben für Peirce Forschungen auf dem Gebiet der Logik den Anlaß, Vielheiten von unendlichen Mengen zu untersuchen und den damit zusammenhängenden Begriff infinitesimaler Größen einzuführen. Es werden die Argumente betrachtet, die er zugunsten der Verwendung des Begriffs infinitesimaler Größen in mathematischer und logischer Hinsicht vorbrachte (und die um die Jahrhundertwende von so bekannten Mathematikern wie Cantor, Peano und Russell zurückgewiesen wurden). Ferner wird auf die Zusammenhänge hingewiesen, die zwischen der Mathematik, der Philosophie und dem besonderen Interesse von Peirce an der Stetigkeit (in Verbindung mit seinem Pragmatismus) bestehen.

For the past thirty years Carolyn Eisele has been responsible, almost single-handedly, for bringing the many-faceted mathematical talents of Charles S. Peirce to light. She has accomplished this, in part through her edition of Peirce's *New Elements of Mathematics* [Peirce 1976], as well as through her writings and lectures on Peirce, the first of which was published in *Scripta Mathematica* in 1951. In recognition of her preeminence as a Peirce scholar, she has been awarded an honorary doctorate from Texas Tech University, where she is also a member of the Institute for Studies in Pragmaticism, and has even been commissioned as an honorary member of the crew of the National Oceanic and Atmospheric Administration's surveying ship *Peirce*.

The recent publication of Professor Eisele's collected papers is one more indication of the influence of her work. The book, *Studies in the Scientific and Mathematical Philosophy of Charles S. Peirce* [Eisele 1979], includes thirty of her most important essays and papers, and is a necessary and valuable companion to *The New Elements of Mathematics*. Moreover, these papers are a testimony not just to the work of Peirce, but to Carolyn Eisele herself. As her *Studies* volume makes clear, there has been growing interest among scholars over the last few decades in Peirce as a mathematician. This in large measure is due to her sustained and persistent efforts.

It is particularly fitting that the Hunter College colloquium honoring her works should have taken place in New York City, for Professor Eisele's initial contact with Peirce, which was entirely fortuitous, happened as she was reading manuscripts from the Plimpton Collection at Columbia University. In her first paper on Peirce, "The *Liber Abaci*," she describes what happened: "As I continued to handle the volume [MS 189] a sheaf of notes on faded blue-grey paper fell out from under the back cover" [Eisele 1951, 15]. When she eventually realized that the pages were not student notes, but a letter by Peirce to Plimpton, she knew that she had made an important discovery. In the letter, Peirce revealed himslef to be not only a well-informed antiquarian, but a perspicacious historian of mathematics as well.

A grant from the American Philosophical Society soon made it possible for Professor Eisele to begin archival research using collections of Peirce papers in Cambridge, Massachusetts; New York; and Washington, D.C. It was her hope, as she put it rather modestly in retrospect, "to collect materials pertinent to a critical and comprehensive study of the activities of Charles S. Peirce as a historian of science" [Eisele 1954, 35].

It soon became apparent, however, that in order to do justice to Peirce's work in the history of science, a much deeper and more wide-ranging study of his work in philosophy, science, and mathematics would be necessary. Over the years, the results of this work began to accumulate. Articles soon appeared on a wide variety of subjects ranging from Peirce's work in probability and logic to map projections, binary arithmetic, economics, semiotics, scientific method, and the four-color problem. Several articles included analysis of Peirce's work in the history of science, especially his interest in such figures as Duns Scotus, Galileo, Kepler, and Darwin. Studies of Peirce at the Smithsonian Institution, his correspondence with Simon Newcomb, his importance as a "nineteenth-century man of science," and his significance as a philosophical critic of figures like Poincaré and Mach all found their way into lectures, or into print.

As her research delved further and further into the writings and mind of Charles Peirce, Professor Eisele began to realize

that the key to understanding the connections between his diverse undertakings was his mathematics. Among the earliest papers to hint at this was "Binary Arithmetic," in which Professor Eisele showed that "Peirce was cognizant of the mathematical basis of all his thought" [Eisele 1966, 201]. Above all, his work in geometry, his analysis of continuity, infinitesimals, and the infinite, as well as his development of the logic of relatives, were especially significant.

By 1971, at the Fourth International Congress for Logic, Methodology, and Philosophy of Science (Bucharest), Professor Eisele was ready to make a prominent issue of the fact that mathematics was indispensable for any proper understanding of Peirce's thought: "the philosophy of his later years was grounded in mathematical foundations that he had investigated and had personally helped to establish, mostly for logical purposes" [Eisele 1971, 238]. By 1975, the need to make scholars aware of the significance of Peirce's mathematics had become an energetic crusade: "any study of Peirce's thoroughly integrated thought must start with an acquaintance with his productivity in pure and applied mathematics" [Eisele 1975, 245]. Professor Eisele's recent achievement, in fact, making good her claims for the pervasive influence of mathematics in every aspect of Peirce's work, is her edition of the *New Elements of Mathematics.*

Carolyn Eisele has labored unstintingly to bring the message of Peirce's mathematics not just to a wider forum of historians of science, but to philosophers and intellectual historians as well. As she once put it, quite unabashedly in her opening remarks at the Vth International Congress for Logic, Methodology, and Philosophy of Science: "I have come to London [Ontario, Canada] on the usual mission of propagandizing for an appreciation and celebration of the endlessly fruitful thought of Charles S. Peirce" [Eisele 1975, 245]. In what follows, I shall take my turn in propagandizing for Peirce, and in particular, shall consider an especially important aspect of his work as it relates to the history of mathematics at the turn of the last century.

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In 1881 Charles S. Peirce published a paper, "On the Logic of Number," in the *American Journal of Mathematics.* Later, he was always proud to emphasize that he had characterized the difference between finite and infinite sets well before Dedekind had done so in 1888. Peirce asserted that Dedekind's *Was sind und was sollen die Zahlen?* was doubtless influenced by his own paper, because Peirce had sent Dedekind a copy of his article of 1881 [1]. But the most interesting feature of Peirce's

entire approach to mathematics was not the way in which it resembled the research then being conducted in Europe, but in the ways it differed from the approaches taken by Richard Dedekind or his equally famous colleague, Georg Cantor, to the problems of continuity and infinity.

Cantor, for example, had been motivated to study the continuum of real numbers as a result of his early study of the representation theorem for trigonometric series [Dauben 1979, 37-40]. Dedekind's characterization of the continuum and his introduction of the "Dedekind cut" to define the real numbers were inspired by his desire to provide rigorous foundations in teaching calculus [Dedekind 1872, 1-3]. Specifically, in trying to teach the basic elements of the differential calculus, particularly theorems involving limits, Dedekind realized that geometric intuition, though a guide, was not rigorously satisfactory. And so he produced a purely arithmetic study of continuity and the irrational numbers.

Peirce, on the contrary, took an entirely different approach. Unlike Cantor and Dedekind, whose interests in foundations, irrational numbers, and continuity proceeded from problems arising *within* the subject--Cantor from his investigation of trigonometric series, and Dedekind from his teaching--Peirce was inspired by his studies in logic. In this difference lies the key to understanding why Peirce differed so markedly from Cantor and Dedekind in his approach to the problems of continuity and the infinite.

Peirce's first publication describing the difference between finite and infinite classes appeared in 1881, while he was lecturing on logic at Johns Hopkins. His paper began with a definition (though insufficient) of continuity:

> *A continuous system is one in which every quantity greater than another is also greater than some intermediate quantity greater than that other.* [Peirce 1881, 159; 1931, 3.256]

But since the rationals would be continuous under this definition, Peirce's description is inadequate, although it does represent a necessary condition. Peirce, however, was only beginning his study of continuity at the time; the most interesting feature of his paper appeared toward the end, where he offered a distinction between finite and infinite collections. He announced that a set was finite if no one-to-one correspondence could be found between the set and any proper subset. Peirce's favorite example was a syllogism which appeared in numerous equivalent forms throughout his mathematical and logical writings:

Every Texan kills a Texan
No Texan is killed by more than one Texan
Hence every Texan is killed by a Texan [2]

The syllogism is true only if the set of Texans is finite. The form of the syllogism, Peirce noted, was due to De Morgan, who called it the *syllogism of transposed quantity* [De Morgan 1860]. Thus Peirce's interest in the difference between finite and infinite classes did not stem directly from *within* mathematics as it had for Cantor and Dedekind. Instead, Peirce's interest in the infinite was inspired by his studies in logic, and the consequences one might draw from the syllogism of transposed quantity.

In keeping with his interests, including metaphysical concerns related to the continuity of space and time, and in his desire to further his study of quantity, both finite and infinite, as far as he could, Peirce decided that a logical definition of continuity was needed, and in 1897, when he published "The Logic of Relatives" in *The Monist*, he wrote that

> *a perfectly satisfactory logical account of the conception of continuity is required. This involves the definition of a certain kind of infinity, and in order to make that quite clear, it is requisite to begin by developing the logical doctrine of infinite multitude. This doctrine still remains, after the works of Cantor, Dedekind, and others, in an inchoate condition. For example, such a question remains unanswered as the following: is it, or is it not, logically possible for two collections to be so multitudinous that neither can be put into a one-to-one correspondence with a part or the whole of the other? To resolve this problem demands, not a mere applicaiton of logic, but a further development of the conception of logical possibility.* [Peirce 1897, 3.526]

But what did Peirce mean by the need to define a certain kind of infinity before the concept of continuity could be accounted for logically? What was "inchoate" about the work of Cantor and Dedekind? Why was the comparability of cardinals, in Peirce's view, impossible to establish without developing further the concept of logical possibility? What, in fact, did Peirce mean by logical possibility? The answers to all these questions hinge on Peirce's view of the infinite, and upon a very important discovery, one he apparently made independently of Georg Cantor, and one for which Peirce was always, and justifiably so, very proud.

Peirce proved (though exactly when he did so for the first time is unclear) that the power of the set of all subsets of a given set is always greater than the power of the original set itself. In other words, for any exponent N, $2^N > N$, a result, as Peirce put it, of "prime importance" [3]. Beginning from the smallest infinite set, the set of all integers, he concluded that it was always possible to produce increasingly larger sets of greater and greater power. Peirce designated the collection of all integers "denumerable." The set of all real numbers comprised what he called the "first abnumeral" or the "primi-postnumeral" multitude. (Note that Peirce has tacitly assumed as true the validity of what has come to be known as Cantor's Continuum Hypothesis.)

The set of all subsets of the real numbers produced the "second abnumeral," or the "secondo-postnumeral" multitude, and so on. Since one could always form from such sets the set of all subsets, Peirce noted that there could be no maximal multitude. But Peirce also commented that as for the second abnumeral, mathematics could offer no example of such a multitude. This comment reveals that Peirce was apparently unfamiliar with a paper Cantor had published in 1891, in which he showed by his famous method of diagonalization that the set of all continuous functions on [0,1]--the unit interval--was indeed of a power, or cardinality, greater than the set of all real numbers [Dauben 1979, 165-168]. Peirce, however, maintained that in fact mathematics had no occasion to consider multitudes as great as secondo-postnumeral multitudes--a comment that is all the more puzzling, as we shall see, in light of his construction of infinitesimals and his assertion that continua were greater in power than *any* postnumeral multitude.

In a letter to his friend E. H. Moore, Peirce commented upon the significance of his discovery that there was no maximum multitude:

> *Here we have a hint about continuity.... The continuum is a General. It is a General of relation. Every General is a continuum vaguely defined.* [4]

By a "General" Peirce seems to have meant that which was neither discrete nor definite, "General" as opposed to particular or to something completely specified. But what did Peirce see in all this to help solve the mystery of continuity?

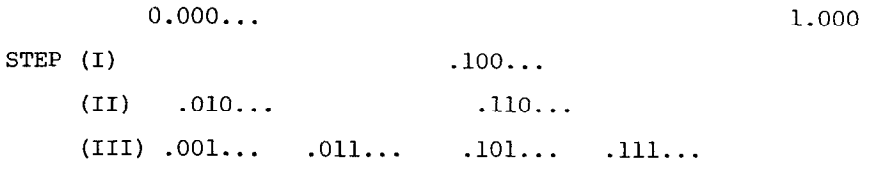
Peirce argued that if a continuum did not contain all the points that it possibly could, then there would be gaps or discontinuities present [Peirce 1976, 3, 880-881]. Thus it was a problem of the utmost importance to determine what the maximum possible multitude *was* in order to determine the power of continua. But Peirce had already shown that there could be *no* such

greatest possible multitude, that the process of forming power sets was unending, and that consequently it was a process that remained potential, indefinite. Similarly, if the continuum were to contain the maximum possible multitude of points, it had to be correspondingly potential, indefinite.

Since Peirce regarded the essence of multitudes to be their definiteness, thus making it possible to determine their powers or cardinalities, it was reasonable for him to assert that since the collection of all abnumerals was unending, potentially infinite, thus entirely indefinite, it could not properly be called a set. Or, in Peirce's terminology, it could not be called a multitude. Likewise, the elements constituting a continuum could not be regarded as comprising a set or multitude of objects. Thus the complete determination of continua was impossible, for Peirce regarded the concept as intrinsically potential, essentially general [Peirce 1976, 3, 62, 981].

Consequently, Peirce was led to reject Cantor's view that the geometric continuum was somehow made up of a multitude of points. Peirce realized that there were two features of such continua that had to be considered: one involved quantity, the other involved order. In 1895 Cantor had published his analysis to what he called the simply-ordered type θ of the continuum of real numbers [Cantor 1895]. Peirce, however, disagreed on two counts. The collection of points comprising any continuum must be infinitely larger than any Cantor contemplated because, Peirce claimed, the real numbers R as defined by Cantor were grossly insufficient to account for the geometric continuum [Peirce 1976, 3, 122]. This was so, he argued, because of his discovery that the set of abnumerals was unending. Since the line must contain all points possible, and since the set R corresponded to the multitude represented by $2^{\aleph_0}$, it could not possibly, as a completed multitude, account for the nature of continuity. As for the question of the order of elements constituting a continuum, Peirce suggested that something like the proper idea was approximated if between every pair of rational numbers one inserted a sequence of irrational numbers. Between any two irrationals of this sequence one could pack yet another such sequence, and so on, without end. Thus Peirce asserted that it was always possible to pack sets of points of higher and higher multitude--or cardinality--between any two points of a continuum, however close. The continuum would eventually be "cemented together," and not by virtue of discrete points [Peirce 1976, 3, 98].

To illustrate his idea of continuity, Peirce outlined a procedure which he called interpolation on the unit interval, and which involved decimal representations given with only the digits 0 and 1. At step (I), there was only one interpolation; at step (II) there were 2; at step (III) there were four; at step (IV) there were 8; and at step (N) there would be 2^{N-1}. Schematically:



Here Peirce found, as he put it, a "premonition of continuity" [Peirce 1976, 3, 87-88]. In carrying out this procedure, a nondenumerably infinite number of interpolations would be made. Somehow this process was to help Peirce explain how the continuum was able, in his words, to "stick together" [Peirce 1976, 3, 88]. No finite collection could ever "stick together" in any order, nor could any denumerably infinite collection if considered in any well-ordered form. The difficulty in describing the continuity of the real line, Peirce believed, reduced to the fact that numbers per se could never account for continuity [Peirce 1931, 3.568; 1976, 3, 93-94]. Numbers expressed nothing but the order, he believed, of discrete objects. Nothing discrete could possibly be multitudinous enough to account for the continuum.

For example, Peirce noted that in supposing the countable collection of the set of rational points to be completely present on the line, one was in effect supposing also the collection of irrational points in the sense that the irrationals could be considered as interpolated between the rationals. Hence the denumerable set of rationals implied the existence of a nondenumerable collection of points, namely, the first abnumeral multitude of irrational numbers. In exactly the same way, said Peirce, the system of irrational points on a line led to a secundo-postnumeral collection of points interpolated between the irrationals. This secundo-postnumeral collection involved, for Peirce, his infinitesimals, and it is possible to understand the role they played more clearly by returning for a moment to his diagram of decimal interpolations.

In a letter to M. F. C. S. Schiller of 1906, Peirce explained that by a Leibnizian infinitesimal of the first order he meant an assumed quantity smaller than any finite positive quantity [Peirce 1976, 3, 989]. It was the *first* quantity after the sequence .1, .01, .001, .0001, .00001, ... Peirce believed it was impossible to prove that there was *no* such quantity. In fact, he believed his infinitesimals were given an imprimatur of sorts by nature, since he took their existence to be necessary for physics. This is clear from letters, among others, that he sent to C. J. Keyser in 1908, and to Josiah Royce in 1902, [Peirce 1976, 3, 898 and 957, respectively].

Peirce's most important reason for insisting that his infinitesimals were acceptable was their self-consistency. Logically, being noncontradictory, there was no reason not to admit them

into mathematics. Although Peirce did not undertake a careful arithmetic investigation of the properties of his infinitesimals, nor did he undertake any investigation of non-Archimedean systems in general, he was a protoproponent of nonstandard analysis in believing that there was perhaps more to understanding the nature of continuity than the rationals and irrationals together could manage to explain.

To make Peirce's point as clear as possible, it may be helpful, in closing, to sketch his reasons for arguing that the rationals and irrationals together were insufficient to account for the nature of continuity. While Cantor and Dedekind regarded the irrational numbers as completing the rationals, and thus conferring completeness on the real numbers, Peirce saw the relation between rationals and irrationals somewhat differently. He concluded that there was a kind of "nextness" in the reals which actually constituted a breach of continuity [Peirce 1976, 3, 121-122, 125].

Suppose, he argued, that we do have a clear idea of a sequence of real numbers. If we have a clear idea of their order, it can be assumed that any set of objects sufficiently multitudinous can be similarly ordered. Assume each of these objects to be replaced with a sequence of points, similar in order to the real numbers on the open interval (0, 1). But there is no reason to stop here, and Peirce went on to replace each point of such series by yet other series, and so on, without end. In his own words:

> *The result is, that we have altogether eliminated points. We have a series of series of series,* ad infinitum. *Every part, however closely designated, is still a series and divisible into further series. There are no points in such a line. There is no exact boundary between any parts.* [Peirce 1976, 3, 126]

It would be easier to interpret the significance Peirce attached to his infinitesimals for mathematics had he ever commented explicitly upon Cantor's and Dedekind's axioms of continuity, which hypothesized the equivalence of the standard Archimedean arithmetic continuum and the geometric continuum. But all we have is Peirce's allegation that the real numbers were insufficient to account for the continuity of space and time. As for analysis, however, it never had need to consider multitudes greater in magnitude than that of the first abnumeral, which meant that the set of real numbers was enough for the interests of analysis and presumably for all of mathematics [Peirce 1976, 3, 85].

Ultimately Peirce's infinitesimals remained vague rather than rigorously defined mathematical entities. He never suggested how they might be useful in analysis. But then Peirce's

interests had never been inclined toward analysis. From the very beginning he had been inspired by the purely logical implications of the syllogism of transposed quantity, and the logic of relations. Thus, unlike Cantor, he was not concerned to develop the arithmetic properties of his ideas, since their existence as numbers was not for him of great consequence. He was interested in illuminating a deep philosophical problem of long standing, namely, that of the continuum, and he felt that conceptually he had found an approach to the subject that was the most satisfying of all.

Finally, how can we now draw all of these ideas together in order to interpret Peirce's claim that "the continuum is a General," meaning that it could not be defined as a set in Cantor's sense of a collection of distinct elements, and Peirce's statement that "infinity is nothing but a peculiar twist given to generality" [Peirce to William James, June 8, 1903, in Peirce 1931, 8.268]? Peirce took generality to involve the potential infinite, in that anything not general, i.e., specific, was completed in some sense or other. He also took infinity to be a potentiality, something never completed, and thus it too reposed upon generality. Reasoning about such matters, he noted with perhaps too little emphasis, was always exceedingly puzzling. Almost all metaphysicians, and even mathematicians, he added, had fallen into pitfalls concerning the infinite where the "ground was spongy" [Peirce 1976, 3, 79]. Trouble inevitably stemmed from existential quantifiers like "all and every," but if one were always certain to determine *how* objects in question were to be selected, he believed that erroneous reasoning could easily be avoided. The trouble with the collection of all abnumerable multitudes was simply that it could never be considered as completed. It was self-generating, without end. Such supermultitudinous collections were so great that they were no longer discrete, and not being complete, no definite determination of their magnitude could be given. They were potential, and as such, general. In the same way the line was general, indeterminate, since between any of its points, Peirce imagined that more could be packed representing supermultitudinous collections. To quote Peirce directly:

> *Such supermultitudinous collections stick together by logical necessity. Its constituent individuals are no longer distinct, or independent. They are not subjects but phases expressive of the properties of the continuum.* [Peirce 1976, 3, 95]

Peirce offered a graphic illustration. Suppose a collection of blades were to cut the line. So long as they did not comprise a supermultitudinous collection, the line would be cut up

into bits, each of which was still a line. Peirce therefore urged mathematicians to discard all analytical theories about lines, and recommended that they begin from his view, a synthetic point of view [Peirce 1976, 3, 96]. By a simple mental experiment, he believed he had shown that the line refused to be cut up into points. But even if mathematicians refused to accept his arguments, Peirce insisted that only one thing mattered: his idea of continuity and of infinitesimals involved no contradiction. In closing, he warned:

> *I am careful not to call supermultitudinous collections multitudes. Multitudes imply an independence of the individuals of one another which is not found in supermultitudinous sets. Here the elements are cemented together, they become indistinct.* [Peirce 1976, 3, 87-89]

Here Peirce has reached the potentiality, the generality that he had earlier said was essential if one were to understand properly the structure of continuity. For Peirce, the essence of continuity depended upon cardinality (i.e., the supermultitudinousness of the elements of the line), and ordinality (or, as he put it, "cohesion"--"the intrinsic arrangement which is inseparable from the particular grade of multitude in which those phenomena of cohesion are found" [Peirce 1976, 3, 98-99]).

It is now possible to see what Peirce meant when he wrote to Paul Carus, editor of *The Monist*, to say that at last he had seen where Cantor had gone wrong [in a letter of August 17, 1899: Peirce 1976, 3, 792]. Continuity could not come from any collection of points because points were discrete, determined, and if anything, points represented *discontinuities* when removed from the line. In summarizing his views for Judge Francis Russell in 1909, Peirce interpreted the essence of continuity in terms of the potential and completely general nature of the ideas involved. "As to a straight line not having any definition proper, it is demonstrable that it cannot be, properly speaking, defined" [Peirce 1976, 3, 981].

If we may take "properly speaking" to mean "mathematically speaking," then there was not much Peirce could hope to offer in such a view to his mathematical colleagues. But it is also true that he did not feel mathematics needed to go so far in the analysis of the continuum as he had gone. Peirce was interested in pushing the logical consequences of his ideas to their ultimate conclusions for the sake of philosophy, but mathematics, he seemed to say, could stop at something less.

For Peirce, such ideas justified themselves as a matter of instinct, of common sense [Peirce 1976, 3, 56]. He had always held that his intuitive understanding of the continuum, or the continuity of space and time, was the ultimate guide in his analysis of continuity. Nothing could have been further from the aims of Weierstrassian analysis, which sought to reject all such intuitions and aimed to base mathematics upon more certain foundations of arithmetic. Weierstrass had the example of everywhere continuous, nowhere differentiable functions to show the inadequacy of intuition. But Peirce was not convinced, and once even commented that Weierstrassian mathematics, in showing a distrust of intuition, betrayed an ignorance of fundamental principles of logic [Peirce 1976, 3, 968]. Peirce followed his intuitions as far as they would carry him, and it may have been this feature of his thinking, as much as his interest in philosophical and metaphysical arguments, that prevented his being more readily accepted by his contemporaries.

Peirce once said that it was the business of science and mathematics to guess. For mathematics this reduced to the fabrication of hypotheses to be tested for logical self-consistency [Peirce 1976, 3, 893]. If no contradictions could be deduced, then the hypothesis, or the mathematical theory in question, stood as acceptable. This was the basis upon which Peirce argued most persuasively for the respectability of his ideas concerning infinitesimals and continuity. Kepler, to Peirce's mind, was the greatest guesser that science had ever seen. But in terms of American mathematics at the turn of the century, Peirce may easily have been an equally impressive guesser, producing a mathematical hypothesis of infinitesimals.

Kepler lacked sufficient mathematical techniques and a theory of gravitation to establish a convincing explanation of his laws, deficiencies Isaac Newton would later remedy. In much the same way, Peirce lacked sufficient techniques to produce a rigorous theory of non-Archimedean systems. But his hypotheses were eventually vindicated, in the 20th century, by mathematicians like Schmieden, Laugwitz, Robinson, and Luxemburg [5]. Perhaps until the middle of this century, only a mathematician as interested in philosophy as was Peirce, and as isolated from the prevailing assumptions of established mathematics of the 19th century, could have pursued the problem of continuity in the way he did. Out of touch with most European mathematicians, Peirce considered infinitesimals with an unprejudiced eye, not only to affirm their existence, but to argue as well that the arithmetic continuum of real numbers was only a very incomplete picture of the actual richness of any continuum. Working in obscurity, often in penury and isolation, it was nevertheless this American mathematician who saw, if only a glimmer, what later generations might be more willing to accept. In light of current research in nonstandard analysis,

it is now possible to consider, more rigorously than did Peirce, alternatives to the traditional 19th-century view of the standard Archimedean continuum.

NOTES

1. Peirce explained this in a letter to P. E. B. Jourdain, December 5, 1908, in Peirce [1976, 3, 883], and in the article "Multitude," in appendixes to Volume 3 [Peirce 1976, 3, 1117]. Refer as well to a letter Peirce sent to the Editor of *Science* on March 16, 1900, "Infinitesimals," in Peirce [1931, 3.564].

2. Peirce gave the Texan version of the syllogism in a paper of 1881, in Peirce [1931, 3.288]. In a letter to Georg Cantor of December 23, 1900, he used Hottentots: Peirce [1976, 3, 772].

3. [Peirce 1976, 3, 51.] See as well Peirce's letters to George Cantor (December 23, 1900), and to F. W. Frankland (May 8, 1906), in Peirce [1976, 3, 777 and 785, respectively]. Peirce also discusses transfinite exponentiation and power sets in what appears to have been a draft for a lecture on "Multitude and Number," probably from 1897. Turn in particular to the section on "The Primipostnumeral," in Peirce [1931, 4.200-4.212].

4. Peirce in a letter to E. H. Moore, March 20, 1902, in Peirce [1976, 3, 925]. For a discussion of "Generals," consult "Notes on Symbolic Logic and Mathematics," in Peirce [1931, 3.642].

5. Schmieden and Laugwitz 1958; Luxemburg 1962; Robinson 1966; Luxemburg and Robinson 1972.

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