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OPTIMAL SOLUTION OF A DYNAMIC LEONTIEF MODEL WITH SUBSTITUTION[1]

By George B. Dantzig

A fairly general dynamic Leontief System with discrete time periods is considered in which (a) alternative substitute type activities are allowed, (b) a bill of goods is given over time, and (c) the unknown quantities of activities satisfying the system are to be determined so as to minimize a specified linear objective function. The model has two features (a) the "general Leontief aspect" which assumes that there is one and only one output for each activity, and (b) a "dynamic (or one-way flow) aspect" which assumes that no inputs for an activity occur later in time than the time of its output.

In the optimal solution certain activities appear in positive amounts and others in zero quantities. Once they are known, the quantitative solution is obtained readily by direct solution of the equations. Hence, the basic problem is to select the activities which appear in positive quantity in the optimal solution. The main results are (a) the "Samuelson Property" due to the Leontief aspect; *i.e.*, the selection of these activities can be made independent of the bill of goods; (b) the local optimization property due to the inclusion also of the dynamic aspect; *i.e.*, when the activities are ordered over time according to the time of production of their principal item, the selection of activities for the first k time periods, $k = 1, 2, \cdots$ can be made without consideration of the nature of activities producing in time periods later than k, moreover, if the coefficients in the linear objective function are interpreted as "costs" to perform a unit quantity of an activity, then the optimum selection for successive time periods can be obtained inductively as a sequence of local optimizations using an implicit cost for each time period by subtracting from original costs those implicit costs associated with earlier time periods.

The two properties, together, lead to a special computational procedure which is more efficient than the ordinary procedure of the generalized simplex method and which, of course, is not applicable to a general linear programming problem. Numerically, the number of multiplications, required for the optimal solution by the conventional simplex method, is reduced by a factor p^2 where p is the number of discrete time periods considered.

The Leontief Equilibrium model assumes that each activity (*e.g.*, industry) has one principal output. The unit quantity of the jth activity is defined as one producing a unit quantity of some item (commodity) i_j. It will be assumed here (and later in its dynamic aspect) that there are *no other items* produced by activity j. Accordingly, adopting the usual convention that inputs are of negative sign and outputs are of positive sign, the jth activity can be represented by a column vector

$$P_j = (a_{1j}, a_{2j}, \cdots, a_{mj})'$$

with the properties

$$a_{ij} = 1, \qquad i = i_j$$

$$a_{ij} \leq 0, \qquad i \neq i_j.$$

[1] Presented before the Eugene, Oregon, meeting of the Econometric Society, June, 1952 (abstract in Econometrica, January, 1953).

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The original Leontief Equilibrium model assumes the non-existence of joint production; *i.e.*, that there is exactly one item i produced by an activity. When there are two or more activities that produce item i, these activities are called *substitute* activities, and the model is referred to as *Leontief Equilibrium Model with Substitution*. For a given bill of goods represented as a column vector $Q = b_1, b_2, \cdots, b_m$, the basic problem is to determine quantities of activities $x = (x_1, x_2, \cdots, x_n)$, nonnegative such that

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1, 2, \cdots, m), \tag{1}$$

or in vector notation

$$\sum_{j=1} x_j P_j = Q, \qquad x_j \geq 0. \tag{2}$$

In order not to complicate the mathematical discussion, it will be assumed here that the given b_i are *positive* (*i.e.*, $b_i \neq 0$):

$$b_i > 0, \qquad (i = 1, 2, \cdots, m).$$

We now specialize the type of generalized Leontief matrix $A = [a_{ij}]$ by assuming that it can be partitioned into "block triangular" form

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1p} \\ 0 & \alpha_{22} & \cdots & \alpha_{2p} \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ 0 & \cdots & \cdots & \alpha_{pp} \end{bmatrix} \tag{3}$$

where α_{tr} is a submatrix of A with m_t rows and n_τ columns such that for $t \neq \tau$ all elements of α_{tr} are negative or zero, and such that diagonal submatrices α_{tt} contain exactly one positive element in each column.

The class of models satisfying (3) has application to a broad class of non-dynamic models. An example[2] of a *static input-output system* follows: The three blocks (here $p = 3$) can stand for the industrial groups clustered around the manufacturing of "raw sugar," "refined sugar," and "candy," and the essence of the structure is *one-way feeding*. The block triangular structure means that first stage products (raw sugars) are required to produce second stage products (refined sugars) and that second stage products are required to produce final stage products (candies). Hence the blocks below the diagonal α_{21}, α_{31}, α_{32} vanish. The same type of structure can be obtained in a *dynamic* situation by associating first a time index with an item i, which is the time period t_i when

[2] This example was given by one of the referees of this paper.

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the item is either required or produced and letting

$$
\begin{aligned}
&t_i = 1 \quad \text{for} \quad i = 1, 2, \cdots, m_1, \\
&t_i = 2 \quad \text{for} \quad i = m_1 + 1, \cdots, m_1 + m_2, \\
&t_i = 3 \quad \text{for} \quad i = m_1 + m_2 + 1, \cdots, m_1 + m_2 + m_3, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \text{ etc.}
\end{aligned} \tag{4}
$$

Secondly, associating with each activity j a time period, τ_j, which is made the same as the time of its single item of production (*i.e.*, $\tau_j = t_{i_j}$); moreover, the subscripts of the activities are arranged so that

$$
\begin{aligned}
&\tau_1 = 1 \quad \text{for} \quad j = 1, 2, \cdots, n_1, \\
&\tau_2 = 2 \quad \text{for} \quad j = n_1 + 1, \cdots, n_1 + n_2, \\
&\tau_3 = 3 \quad \text{for} \quad j = n_1 + n_2 + 1, \cdots, n_1 + n_2 + n_3, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ etc.}
\end{aligned} \tag{4'}
$$

Finally, the dynamic model is specialized so that no inputs to an activity occur after the period of the activity. It is *assumed*, therefore, that

$$a_{ij} = 0 \quad \text{if} \quad t_i > \tau_j . \tag{5}$$

Much of our later discussion will depend on an implicit assumption that we can find at least one set of values x_j that satisfy the general system (1) or the specialized system which is characterized by the additional restriction (5). We shall say a few words as to when we can be sure we can solve the system. It is well known that if activity levels x_j can be found for at least *one* bill of goods Q with all positive components (or if a solution exists for a system based on the *transpose* of matrix a_{ij}), then solutions exist for *any* bill of goods Q with non-negative components.[3] For our purpose here it is assumed for the given values of $b_i > 0$ that at least one solution can be obtained for (1). Because each $b_i > 0$, *it is necessary that one or more of the substitute activities that produce item* i *appear with positive activity level in the solution.* If more than m activities appear in the solution in positive quantity, it is well known, from the theory of linear programming, that it is possible to produce a second solution in which no more than m activities are required to have positive activity levels (the remainder are zero). In the Leontief model, however, it is clear that this cannot be done with fewer

[3] A classical case of a solution existing for the original matrix occurs when the a_{ij} are derived from an empirical situation. A classical case of a solution occurring for the transpose matrix is when input and output coefficients are measured in dollars and each activity is operating at a profit. In this case, the sum of the coefficients in any column is positive; *i.e.*,

$$\sum_{i=1}^{m} a_{ij} > 0, \qquad j = 1, 2, \cdots, n.$$

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than m activities; indeed, the new solution must have exactly one of each type of substitute activity at a positive level. In a model which has *no* substitute activities, the solution can be shown to be unique; *i.e.*, the matrix a_{ij} is square and *nonsingular*. In a model with substitute activities the square submatrix of the coefficients, *i.e.*, involving just m–activities for the same reason, must be *nonsingular*.

In a model with substitution, however, the solution to (1) or (2) need not be unique. Because of non-uniqueness the essential problem considered then is to choose the values x_j satisfying (1) or (2) such that some *linear objective function* Z in the variables x_j is *minimized*:

$$\sum_{j=1}^{m} c_j x_j = Z \ (= \text{Min}). \tag{6}$$

For example, c_j may be the amount of labor required per unit of activity, in which case it is desired to find activity levels x_j which produce the desired bill of goods and which, at the same time, minimize Z, the total labor required.

It is intuitively obvious, then, for a structure of "one-way feeding" just described, that the reasonable thing to do is to optimize the primary group of industries *first*, *i.e.*, the selection and activation of those activities in the first block column in such a way as to minimize the labor requirement. *After* this is done, the secondary group of industries, *i.e.*, the second block column, can be optimized. This second optimization will not, in turn, affect the structure of the primary group, which is already well organized, because of the one-way-feeding pattern. Similarly, the tertiary group can be optimized *after* the first two groups are optimized. It is our purpose to establish this rigorously.

A *general* linear programming problem is concerned with minimization of a linear function of the nonnegative variables $X = (x_1, x_2, \cdots, x_n)'$ such as (6) subject to a system of linear restraints $AX = b$ where $A = [a_{ij}]$ is a general $m \times n$ matrix and $b = (b_1, b_2, \cdots, b_m)'$ a general m-component column vector. A set of m variables $(x_{j_1}, x_{j_2}, \cdots, x_{j_m})$ is called a *basic set* if the matrix (called a *basis*) of coefficients of these variables is nonsingular. The values of these variables obtained by setting all remaining variables equal to zero is called a *basic solution*. Moreover, if an *optimizing* solution exists, *there is an optimizing solution which is a basic solution* [2]. The *simplex method* for obtaining an optimizing solution is a finite iterative technique which starts with an assumed basic solution and proceeds to a second basic solution that has $m - 1$ variables in common with the first and such that the value of Z for the second basic solution is less than the value of Z for the first. Because of the special nature of the Leontief model, it is clear that the change in the corresponding basis consists in replacing a column of the basis by a column of coefficients associated with a *substitute* activity [1].

The simplex criterion for determining whether any substitute activity will do better than a given activity in a basic solution can be determined by first finding a "price" vector $\pi = (\pi_1, \pi_2, \cdots, \pi_m)$ such that

$$\sum \pi_i a_{ij} = c_j \qquad \text{or} \qquad \pi P_j = c_j \tag{7}$$

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for those activity vectors P_j in the basis. If now for any other j

$$\pi P_j > c_j , \tag{8}$$

the introduction of the substitute activity, j, into the basis will decrease the value of Z. If, however, for all j it is true that $\pi P_j \leq c_j$, then the solution is optimal.

When the matrix $[P_{j_1}, P_{j_2}, \cdots, P_{j_m}]$ of the basis is partitioned into rows and columns by time periods, then, since $a_{ij} = 0$ for all $t_i > \tau_j$, it has the partitioned form[4]

$$B = \begin{bmatrix} A_{11} & A_{12} \cdots A_{1p} \\ & A_{22} \cdots A_{2p} \\ & & A_{pp} \end{bmatrix} \tag{9}$$

where A_{tr} is the submatrix of a_{ij} in which $t_i = t$ and $\tau_j = \tau$. Since in a basic solution exactly one activity which produces an item can appear, it follows that $A_{11}, A_{22}, \cdots$ are *square matrices*. Moreover, since the entire basis (9) is nonsingular, the diagonal submatrices must be nonsingular also.

From this it follows that the first m_1 components of the price vector π depend only on the activities in the basis associated with the first time period. Indeed

$$(\pi_1, \pi_2, \cdots, \pi_{m_1})A_{11} = (c_{j_1}, c_{j_2}, \cdots, c_{j_{m_1}}), \tag{10}$$

or

$$\pi^{(1)}A_{11} = c^{(1)} \tag{10'}$$

where $\pi^{(1)}$ and $c^{(1)}$ denote the corresponding row vectors in (10). Moreover, since any vector P_j not in the basis can also be partitioned in time periods and all components i of P_j such that $t_i > t_j$ have the property $a_{ij} = 0$. It follows that the criterion (8) for determining whether any substitute activity associated with the first time period will do better, depends only on activity vectors P_j in the basis in the first time period. The same statement, of course, can be made for the first k time periods.

Suppose now that some basis is given initially. The first step then is to consider whether any substitute activity for the first time period will decrease the value of the objective function. To determine whether an activity associated with the first time period should substitute for one already in the basis, it is only necessary to determine $\pi^{(1)}$ the first m_1 components of the price vector π from (10) since $\pi P_j = \pi^{(1)}P_j^{(1)}$ for $j = 1, 2, \cdots, n_1$, where $P_j^{(1)}$ represents the first m_1 components of P_j and one notes the remaining components for $j \leq n_1$ are zero. If so, this process is iterated with each new basis until, after a finite number of iterations, a basis is obtained for which $\pi P_j = \pi^{(1)}P_j^{(1)} \leq c_j$ for $j \leq n_1$; *i.e.*, it is not possible to bring any new activity vector P_j into the basis to replace any substitute activity vector already in the basis and decrease the value of Z. It is important to note, however, that the entire analysis of what changes in the

[4] Called "block triangular" in [3].

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columns of the basis are desirable is made locally *using information derived from the first block only, i.e.,* α_{11} in (3); moreover, the shifts in basis are made independently of the bill of goods or of the corresponding basic solutions associated with the basis.

Turning our attention now to the second or later time periods, it will be noted that any replacements of a column in the basis by an activity vector associated with the second period or later will only eliminate a vector belonging to the same period; it will leave A_{11} in (9) untouched and thus can have no effect on the first m_1 components of π. Thus, it is clear that once the set of first period activity vectors in the basis has been selected such that $\pi^{(1)}P_j^{(1)} \leq c_j$ for $j \leq n_1$, no other shifts in the basis will ever affect this set or the values of $\pi^{(1)}$.

The best choice of activity vectors for the second period can also be made locally by using information derived from the second block α_{22} only if an adjustment is made first in the values of c_j corresponding to activities in the second block. Indeed the next m_2 components of π are given by

$$(\pi_{m_1}, \pi_{m_1+1}, \cdots, \pi_{m_1+m_2})A_{22} = (c'_{j_{m_1}}, c'_j, \cdots, c'_{j_{m_1+m_2}}) \tag{11}$$

where c'_j is given by

$$c'_j = c_j - \sum_{i=1}^{m_1} a_{ij}\pi_i \qquad (j = n_1, n_1+1, \cdots, n_1+n_2). \tag{12}$$

Denoting by $\pi^{(2)} = (\pi_{m_1}, \pi_{m_1+1}, \cdots, \pi_{m_1+m_2})$ and by $P_j^{(2)}$ that partition of P_j belonging to the second time period (block). The simplex criterion of which vector P_j to replace a column in the basis associated with the second block is given by

$$\pi^2 P_j^2 > c'_j \qquad (j = n_1, \cdots, n_1+n_2); \tag{13}$$

i.e., it depends on the information contained in α_{22} only and the modified c_j values c'_j. Hence a locally optimum basis A_{22} can be obtained from α_{22} by repeating the algorithm until $\pi^2 P_j^2 \leq c'_j$ for all second block j. The columns corresponding to this A_{22} form the best choice for B.

For the third time period it is clear that the values of $(\pi_1, \cdots, \pi_{m_1+m_2})$ are used in (12) to compute c'_j where the summation in (12) is modified so that it extends over the range 1 to $m_1 + m_2$. Similarly, $\pi^{(3)} = (\pi_{m_1+m_2}, \cdots, \pi_{m_1+m_2+m_3})$ is computed using A_{33} in (11) in place of A_{22} and condition (13) is replaced by $\pi^{(3)}P_j^{(3)} \geq c'_j$, etc. The advantage of this procedure is that the determination of those columns in the optimum basis associated with the tth period can be made in terms of the information contained in the tth period block α_{tt} and the corresponding modified c'_j values which depend on the components of the price vector for earlier periods previously determined optimally. Thus, obtaining the optimum over all time periods is reduceable to p local optimum solutions with the additional effort of obtaining the adjusted cost values c'_j.

The π_i can be interpreted as prices of items in time periods earlier than the period t of the activity P_j. In which case, c_j is total costs per unit of activity j,

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and c_j' is the implicit cost of the inputs and outputs for period t alone. Thus the optimal choice of activities for some current period t depends only on the input-output coefficients and the implicit costs of these activities for the period under consideration.

It will be noted that the determination of the optimal basis has been made independent of the bill of goods b_i.[5] The last step is the computation of the values of the basic variables $V = [X_{j_1}, X_{j_2}, \cdots, X_{j_m}]$. Referring to (9) let us assume that this is the B of the optimal basis, then V is given by

$$BV = Q \qquad Q = (b_1, \cdots, b_m)'.$$

Because of the form of B, the sub-matrix A_{pp} would be used first to determine the last m_p components of V and by successive substitutions of these values into the proceeding equations the values of X_{j_i} will be determined starting with last period, next to last period, etc.

In this model, if we interpret the difference costs $c_j' - \pi^{(t)}P^{(t)}$ as short-run profits, for some current period t, then it is clear that immediate profitability determines which activities will be used among those with immediate output. These in turn will effect the *choice* of activities to be initiated whose output is more distant. Once the class of activities is selected, the future bill of goods will determine the *quantities* of future activities first, and these in turn will effect the quantities of activities with outputs in earlier time periods.

This model is essentially of a flow type without capital structure or initial inventory limitations. In working with more complex models these results have served as a useful guide; see [3].

I am indebted to Dr. John Fei for a number of contributions in the way of economic interpretation that have been interwoven into the present version of this paper. The concluding remarks which follow, and which are by him, may interest the reader.

"From the viewpoint of the development of research in linear programming (after the problem of 'degeneracy' is solved, *i.e.*, after *every* typical problem can be solved by the simplex method), the natural mathematical development is to analyze the implications for computational efficiency of more restrictive assumptions imposed on a 'general' system. In this development, the only meaningful thing to do is to concentrate on the analysis of those restrictive types of matrices which have a wide range of applications, *from the economic point of view.*

"Since the 'triangular block Leontief matrix' is certainly a matrix of general economic significance, *e.g.*, the dynamic problem and the static structural prob-

[5] We review the reasons for this: (a) the original basis or subsequent basis is feasible whatever be the bill of goods if it is feasible for at least one bill of goods with $b_i > 0$ for all i; (b) the determination of which activity to introduce into the basis (for improvement in the value of the objective form) is always independent of the constant terms, even in a general linear programming problem; (c) the activity to be dropped from the basis in a Leontief system is always the substitute activity. Hence, none of the steps depend on the constant terms.

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lem as illustrated here, the analysis of a computationally efficient method of handling such matrices is not only interesting in its own right, but also serves to indicate one of the main trends of research in activity analysis."

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