Cyclic polycategories and the contour/splicing adjunction*

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In [6] the contour/splicing adjunction $C \dashv W \colon \mathsf{MultiCat} \to \mathsf{Cat}$ was introduced in a categorical study of context-free grammars to analyze the Chomsky–Schützenberger representation theorem. Subsequently, it has been adapted to promonoidal and produoidal categories with application to process algebra [3].

There are two observations that can be made about this adjunction which motivate the work in this talk. The first is that the operations of the spliced arrow multicategory have a natural cyclic structure, making $W(\mathsf{C})$ a cyclic multicategory in the sense of [1] (see also [2]). The second is that the counit of the adjunction is *almost* an equivalence, as it is faithful and surjective on objects but narrowly fails to be full as some 'contours' are missing.

This latter observation motivates the extension of this adjunction to the setting of (planar) polycategories, whilst the former suggests focusing our attention on a special class of polycategories. At first it seems that the most appropriate setting might be *cyclic polycategories with duals*, which are reminiscent of Hyland's $*$-polycategories [4]. These are polycategories where inputs and outputs of polymaps may be provided in any order up to cyclic rotation, and where every object A is equipped with a dual object A^*.

However, even duals of objects are not needed if in each Hom-set we label each object with an i or o, intuitively denoting whether the object should be treated as an input or an output, in an "entries only" formulation of cyclic polycategories. This may be thought of as shifting the dualising functor $(-)^*$ to the level of judgements and suggests links with proof theory [7, 8]. These cyclic-polymorphisms may be visualised as oriented planar trees with edges coloured by the associated objects. We compose by gluing operations along a pair of external edges of the same colour and of opposite orientations (see Figure 1).

The contour/splicing adjunction may then be described on objects as follows. Given a category C, the objects of $W(\mathsf{C})$ are pairs $A{-}B$ of objects A and B in C and morphisms in $\mathrm{Hom}(A_1{-}B_1 \colon x_1, \ldots, A_n{-}B_n \colon x_n)$ with $x_1, \ldots, x_n \in \{\mathsf{i}, \mathsf{o}\}$, $A_{n+1} = A_0$, $B_{n+1} = B_0$ given by tuples $(f_1, \ldots, f_n)$ where $f_j \colon B_j \to A_{j+1}$ if $x_j = \mathsf{i} = x_{j+1}$, $f_j \colon B_j \to B_{j+1}$ if $x_j = \mathsf{i}$ and $x_{j+1} = \mathsf{o}$, $f_j \colon A_j \to A_{j+1}$ if $x_j = \mathsf{o}$ and $x_{j+1} = \mathsf{i}$ and $f_j \colon A_j \to B_j$ of $x_j = \mathsf{o} = x_{j+1}$ (see Figure 2 for an example). Composition is handled in an appropriate way.

For the reverse direction, the contour category $C(\mathsf{P})$ has objects R^u and R^d for each object R in the cyclic polycategory P, which may be thought of as two possible orientations of each object. Morphisms then correspond to the contours of the polymorphism as in Figure 3 subject to some reasonable relations.

We find that the splicing functor embeds categories into cyclic polycategories in the sense that the counit is an equivalence. Moreover these categories have strong connections with cyclic multicategories and $*$-autonomous categories. These structures may be viewed fruitfully in a combinatorial light where we find connections with representations of unrooted planar trees and more general planar maps by permutations [5], but this will require further exploration.

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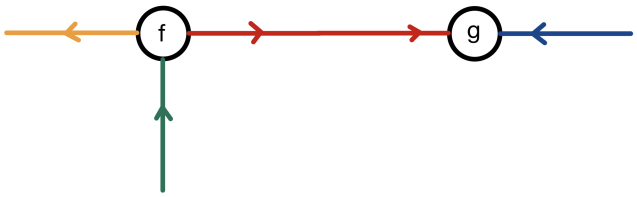


Figure 1: Here $f \in \mathrm{Hom}(R\colon \mathsf{o}, G\colon \mathsf{i}, Y\colon \mathsf{o})$ and $g \in \mathrm{Hom}(R\colon \mathsf{i}, B\colon \mathsf{i})$ compose along R to yield $g \circ f \in \mathrm{Hom}(G\colon \mathsf{i}, Y\colon \mathsf{o}, B\colon \mathsf{i})$.

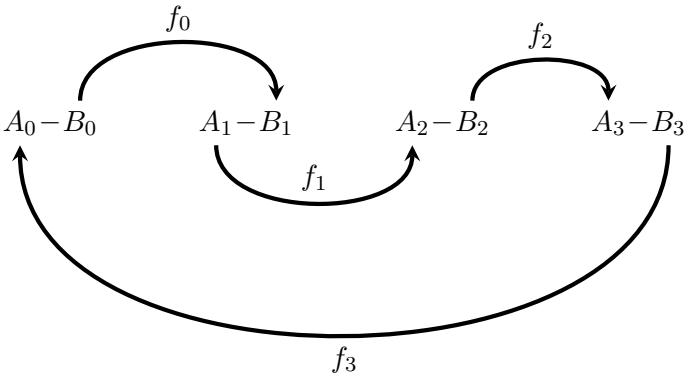


Figure 2: A morphism of $W(\mathsf{C})$ contained in $\mathrm{Hom}(A_0{-}B_0\colon i, A_1{-}B_1\colon o, A_2{-}B_2\colon i, A_3{-}B_3\colon i)$

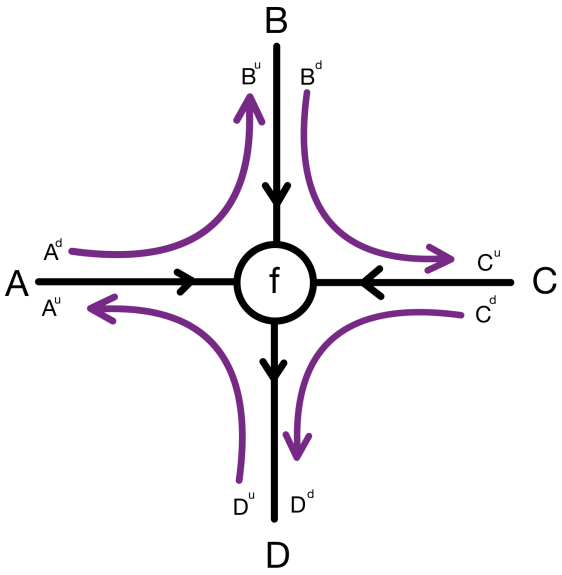


Figure 3: Four contours of the polymorphism f.

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