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UNITIES AND NEGATION: ON THE REPRESENTATION OF FINITE LATTICES

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1. Introduction

To any binary relation $\varrho \subseteq J \times M$ between two sets J and M we associate a certain preorder $I(\varrho)$ on the direct sum (formally-disjoint union) $J \oplus M$, called the *unity* (of opposites) of the relation ϱ. This unity is the strongest preorder on $J \oplus M$ whose restriction to the product $J \times M$ is equal to ϱ (Fig. 1).



Fig. 1. A binary relation, and its unity.

The idea in constructing the unity of a relation is accurately to reflect as internal structure those contradictions (relations) which were originally seen only as external, existing only as links from J to M. The concept of unity emphasizes the aspect of mutual-definedness present in any relation.

We view this particular study, which illustrates the manner in which external relations give rise to internal structure, both of the parties to a relation and of their combination in a unity of opposites, as being one aspect of a more general program, a program whose aim is the formalization of dialectical logic.

This paper is concerned with the role of unities in the duality and representation theory of finite lattices. One preliminary comment is in order, to explain the

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connection between unities and lattices. In our opinion, the decisive aspect of combinatorial lattice theory is the conflict between elements-beneath with elements-above, an interplay heretofore worked out by means of Galois connections [6, Section 3]. Where there are enough irreducibles, as say in finite lattices, the binary relation '$\leq$' between the set J of join-irreducibles and the set M of meet-irreducibles carries the entire lattice structure. For example, finite-dimensional geometric lattices are given by the incidence relation of points with hyperplanes. (Such *geometric* incidence relations are characterized by the exchange property: for any pair A, B of distinct hyperplanes, and for any point $p \notin A \cap B$, there is a hyperplane C containing both the point p and the intersection $A \cap B$ [7].)

The main outlines of the duality and representation theory presented below are as follows. In any finite lattice L, the induced preorder $I(L)$ on the direct sum $J \oplus M$ on its sets J and M of irreducible elements is a unity, from which the lattice may be reconstructed by Dedekind–MacNeille completion [2]. Another unity I° on $J \oplus M$ is *dual* to the lattice L, in the sense that the lattice is representable as

$$L \cong \mathrm{Map}(I^\circ(L), 2),$$

once maps of unities, and the unity 2, are appropriately defined. The unity $I^\circ(L)$ is the intersection of certain preorders on $J \oplus M$ associated with individual elements of L, or equivalently with maximal chains in L. $I^\circ(L)$ turns out to be the *negation* of the unity $I(L)$, in the sense that I and I° have underlying binary relations which are complementary in the subset $J \times M$. Dedekind–MacNeille completion of the unity I° leads to a lattice L^*, also given as the dual of the unity I:

$$L^* = \mathrm{Map}(I(L), 2).$$

Finally, the attempt to extend maps of unities to maps of lattices, in order to realize the pairing L, L^* as a duality in some appropriate category of lattices, leads to a study of *determinacy* of maps at elements of the MacNeille completion.

This research began as an effort to extend to arbitrary lattices the Priestley representation [10] of distributive lattices (see Section 7 below), and thus has its origins also in the Stone representation of distributive lattices [13] and the Raney–Balachandran–Bruns theory [12, 1, 4] of completely distributive lattices. We had the good fortune to discuss the Priestley representation at length with Don Brunker [3] at the University of Waterloo. The main ideas in this paper were worked out in marathon with Mike Parmenter at the Memorial University of Newfoundland. Herb Gaskell, also at Memorial, led us to the then unpublished work of Alisdair Urquhart, whose superb paper [14] represents arbitrary lattices as lattices of doubly-closed stable sets in L-spaces, these being certain doubly-disconnected compact spaces. Other papers in this general direction, with relevance to finite lattices, include works by Wille [15] and Markowsky [9]. More recent developments, including the definition of a *map* between unities, a drastic simplification of several proofs originally encumbered by excessive detail, and an understanding of the role of coherent spaces in representation theory, follow helpful discussions with Bill Lawvere [8].

2. Terminology

We shall use the abbreviated form *ordered set* to mean partially ordered set. A *preorder* is any reflexive and transitive relation on a set. We use the term *increasing* to refer to a subset $A \subseteq X$ of an ordered set X with the property $x \in A$ and $x \leq y$ imply $y \in A$. *Decreasing* sets are increasing in the opposite order, that is in the ordered set X^{opp}.

In a finite lattice L, an element a is *join-irreducible* if and only if

$$\text{for all subsets } B \subseteq L,\ a = \vee B \text{ implies } a \in B;$$

this occurs exactly when the element a covers a unique lattice element. With respect to the opposite order, join-irreducibles become *meet-irreducible*. Note that in this usage, the least element 0 in a lattice is never join-irreducible, because it is representable as the join of the empty set.

Throughout the paper we shall denote two sets by the letters J and M, to remind the reader of the main example wherein J and M are the join- and meet-irreducibles, respectively, in a finite lattice. Elements of J will be indicated by lower case letters $a, b, \ldots, x$, those of M by *primed* lower case letters $a', b', \ldots, x'$. The prime is *not* intended to indicate an operator: a and a' are simply different symbols.

The cover relation "y covers or is equal to x" is written $y \downarrow x$.

3. Unities

By the distributivity of direct product across direct sum (formally-disjoint union) in sets, for any sets J and M we may write:

$$(J \oplus M) \times (J \oplus M) = (J \times J) \oplus (J \times M) \oplus (M \times J) \oplus (M \times M).$$

Thus any preorder on $J \oplus M$ partitions naturally into a preorder on J, a relation from J to M, a relation from M to J, and a preorder on M. The *unity* $I(\varrho)$ *of a binary relation* $\varrho \subseteq J \times M$ is defined as the largest preorder on $J \oplus M$ whose restriction to $J \times M$ is equal to ϱ. So, given a relation $\varrho \subseteq J \times M$, we must construct the preorders on J and M, and the relation from M back to J, which complete the unity $I(\varrho)$. Using the notations $a \leq a'$ and $a \varrho a'$ interchangeably, we establish necessary conditions for inclusion of an ordered pair in each of the four parts of the unity:

In $J \times J$, an ordered pair $a \leq b$ can be included in the preorder on J only if the required transitivity of the expression $a \leq b \leq b'$, for some $b \leq b'$ in ϱ, does not force a relation $a \leq b'$ not in ϱ.

In $J \times M$, a pair $a \leq a'$ must be included if and only if $a \varrho a'$. In $M \times J$, a pair $a' \leq a$ can be included in the relation from M back to J only if no pairs $b \leq a'$ and $a \leq b'$ in ϱ force, by transitivity of the expression $b \leq a' \leq a \leq b$, a pair $b \leq b'$ not in ϱ.

In $M \times M$, as in the first case, we can include a pair $a' \leq b'$ only if the required

transitivity of the relation $a \leq a' \leq b'$ with $a \leq a'$ in ϱ does not force a relation $a \leq b'$ not in ϱ.

Surprisingly, if a pair is not explicitly *excluded* by one of the above conditions, *it can be included,* and the resulting preorder on $J \oplus M$ is clearly maximal among those with a given restriction to $J \times M$. This is the substance of the following theorem.

Theorem 1. *The unity $I(\varrho)$ of a relation $\varrho \subseteq J \times M$ consists of the following ordered pairs:*

(1) *$a \leq b$ in $J \times J$ iff for all $b' \in M$, $b \varrho b'$ implies $a \varrho b'$.*

(2) *$a \leq a'$ in $J \times M$ iff $a \varrho a'$.*

(3) *$a' \leq a$ in $M \times J$ iff for all $b \in J$ and $b' \in M$, $b \varrho a'$ and $a \varrho b'$ imply $b \varrho b'$.*

(4) *$a' \leq b'$ in $M \times M$ iff for all $a \in J$, $a \varrho a'$ implies $a \varrho b'$.*

Proof. By the previous discussion, it suffices to prove that this relation '$\leq$' is a preorder. It is clearly reflexive, so we check transitivity. In what follows, elements $a, b, \ldots$ are always in the set J, while $a', b', \ldots$ are always in the set M.

For any elements a, b, c, and c', from $a \leq b \leq c$ and $c \leq c'$ we may conclude from line (1) that $b \leq c'$, then $a \leq c'$, so $a \leq c$. Similarly, $a' \leq b' \leq c'$ implies $a' < c'$.

For all elements a, b, b', from $a \leq b \leq b'$ we conclude from line (1) that $a \leq b'$. From $a \leq a' \leq b'$ and line (4) we conclude $a \leq b'$.

For all elements a, b, a', b', from $a \leq a' \leq b$ and $b \leq b'$ we conclude from line (3) that $a \leq b'$. Thus, by line (1), $a \leq b$. Similarly, from $a' \leq b \leq b'$ and $a \leq a'$ we conclude that $a \leq b'$, so $a' \leq b'$.

Finally, for all elements a, b, c, a', c', from $a' \leq a \leq b$, $c \leq a'$ and $b \leq c'$ we conclude from line (1) that $c \leq a' \leq a \leq c'$, and from line (3) that $c \leq c'$. Thus, again by line (3), $a' \leq b$. The proof is similar, that $a \leq a' \leq b'$ implies $a \leq b'$. □

As a consequence of Theorem 1, we may characterize these preorders on a direct sum $J \oplus M$ which are representable as unities of relations from J to M. In any such preorder, its restrictions to $J \times J$, $M \times J$, and $M \times M$ are determined by its restriction to $J \times M$, as stated in lines (1), (3) and (4). That is, for all $a, b \in J$, $a', b' \in M$:

(5) $a \leq b$ iff for all $c' \in M$, $b \leq c'$ implies $a \leq c'$,

(6) $a' \leq a$ iff for all $c \in J$ and $c' \in M$, $c \leq a'$ and $a \leq c'$ imply $c \leq c'$,

(7) $a' \leq b'$ iff for all $c \in J$, $c \leq a'$ implies $c \leq b'$.

These relations can be combined into the following single simple statement of separation by both J and M, which we use to define the concept of unity without explicit reference (as before) to any underlying relation. Thus a *unity* of sets J and M is any preorder $\leq$ on the direct sum $I = J \oplus M$ such that

(8) for any elements $x, y \in I$, if $x \not\leq y$, then there exist elements $a \in J$ and $a' \in M$ such that both $a \leq x$, $a \not\leq y$ and $x \not\leq a'$, $y \leq a'$.

Theorem 2. *Any unity of sets J and M is representable as the unity of a relation.*

Proof. Let I be a unity of sets J and M. The relation in question is the restriction of the preorder '$\leq$' on I to the product $J \times M$. It suffices to establish lines (5) through (7). Lines (5) and (7) are conclusions to be drawn directly from transitivity of the preorder $\leq$, upon separation by M (line (5)) and J (line (7)). Line (6) is as follows: Say $a \in J$ and $a' \in M$. For all $c \in J$, $c' \in M$, if $c \leq a'$ and $a \leq c'$, then $a' \leq a$ implies $c \leq a' \leq a \leq c'$, which implies $c \leq c'$, by transitivity. If $a' \leq a$, separation (line (8)) guarantees the existence of an element $c \in J$ such that $c \leq a'$, $c \not\leq a$. Then, using separation by M on the pair $c \not\leq a$, we find an element $c' \in M$ such that $c \not\leq c'$, $a \leqq c'$. Since $c \leqq a'$ and $c \not\leq c'$, it follows that $a' \not\leq c'$. Thus $c \leq a'$, $a \leq c'$, but $c \not\leq c'$, which completes the other half of the proof of line (6). □

The most immediate example of a unity is that obtained by the joint inclusion of join-irreducible and meet-irreducible elements into a finite lattice L. We denote this unity $I(L)$. Fig. 2 shows two examples, $I = I(L)$, $I^\circ = I(L^*)$. The notations I° and L^* will be explained later.

Theorem 3. *The preorder $I(L)$ induced by inclusion of irreducibles into a finite lattice L is a unity.*

Proof. Since every lattice element is expressible both as a join of join-irreducibles and as a meet of meet-irreducibles, any statement of the form $x \not\leq y$ for elements x and y in a finite lattice guarantees the existence of a join-irreducible c and a meet-irreducible c' such that $c \leq x$, $c \not\leq y$ and $x \not\leq c'$, $y \leq c'$. This holds in particular when x and y are taken to be irreducible elements of either type. Thus line (8) holds, and the preorder induced by joint inclusion of $J \oplus M$ in L is a unity. □

4. Negation

We introduce the concept of negation of a unity in such a way that the negation of the unity $I(\varrho)$ of any relation ϱ will be the unity of the complementary relation 'not ϱ' (the set of pairs (a, a') not in the relation ϱ). Thus we simply define the *negation* I° of a unity I on $J \oplus M$ to be that preorder $\leq^\circ$ on $J \oplus M$ which satisfies, for all $a, b \in J$, $a', b' \in M$,

(9) $a \leq^\circ b$ iff $b \leq a$,

(10) $a \leq^\circ a'$ iff $a \not\leq a'$,

(11) $a' \leq^\circ a$ iff for all $c \in J$ and $c' \in M$, $c \not\leq c'$ or $c \leq a'$, or $a \leq c'$,

(12) $a' \leq^\circ b'$ iff $b' \leq a'$.

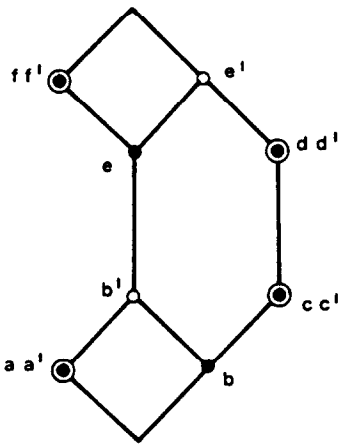
In this context, line (11) may be more simply written as

(13) $a' \leq^\circ a$ iff for all $c \in J$, $a \leq c$ or $c \leq a'$,

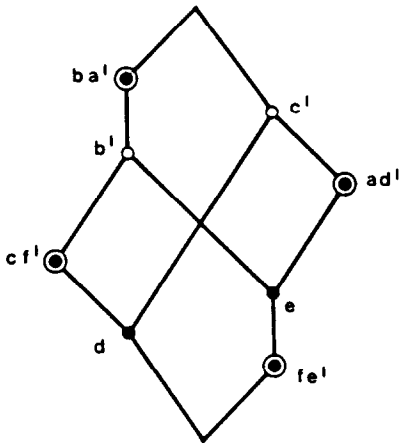
or as

(14) $a' \leq^\circ a$ iff for all $c' \in M$, $a \leq c'$ or $c' \leq a'$.

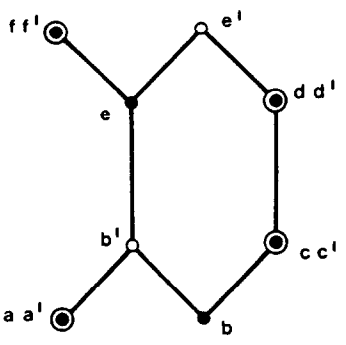
(The reader may wish to check that this definition guarantees I° is the unity of the relation $\not\leq$ from J to M.)



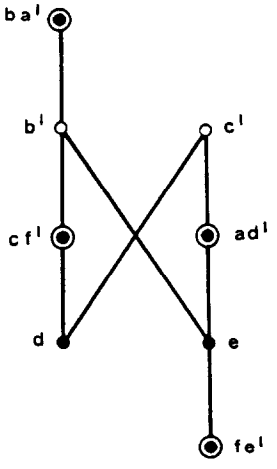
L



L*



I



I^{o}

Fig. 2.

Theorem 4. *$I^\circ(\varrho) = I(\text{'not } \varrho\text{'})$, and $I^{\circ\circ} = I$.*

Proof. It follows from the fact that $I^\circ = I(\not\leq)$ that $I^\circ(\varrho) = I(\not\varrho)$. Then

$$I^{\circ\circ} = I^{\circ\circ}(\leq) = I^\circ(\not\leq) = I(\leq) = I. \qquad \square$$

Of particular interest is the negation $I^\circ(L)$ of the unity of irreducibles in a finite lattice. As we mentioned in the introduction, $I^\circ(L)$ is a dual object for the lattice L. Surprisingly, it also arises directly from a relation between irreducibles and covering pairs in L. This we investigate as follows.

In a finite lattice L, every element is expressible both as a join of join-irreducibles and as a meet of meet-irreducibles. Thus for every covering pair $y \downarrow x$ there exist join irreducibles $a, \ldots$ such that $x \not\leq a'$, $a \leq y$ and meet-irreducibles $a', \ldots$ such that $x \leq a'$, $y \not\leq a'$. We *label* each covering pair with its corresponding irreducibles, as in Fig. 3.

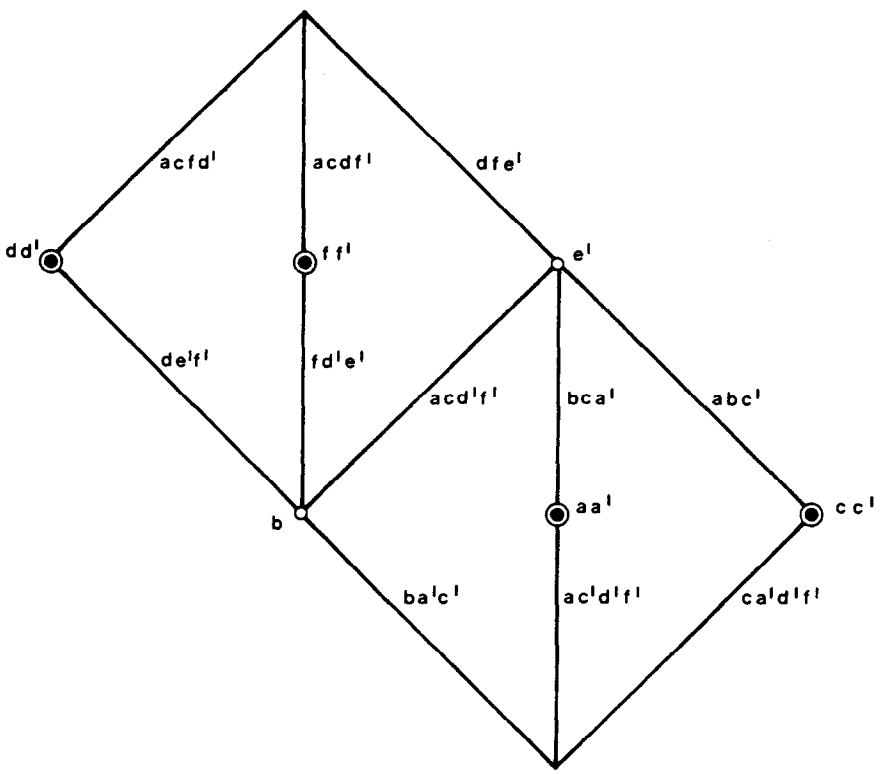


Fig. 3.

Fixing our attention on any one maximal chain C in L, we note that the labels, in their order along the chain (each irreducible occuring exactly once as a label), determine a preorder on the set $J \oplus M$. The *opposite* of this preorder we call $\leq^C$. By

forming the intersection of these preorders $\leq^C$ for all maximal chains C, we obtain a preorder on $J \oplus M$ which we shall show is equal to the negation $I^\circ(L)$ of the unity of irreducibles in L. (The preorder thus obtained for the lattice in Fig. 3 is the unity I° in Fig. 4.)

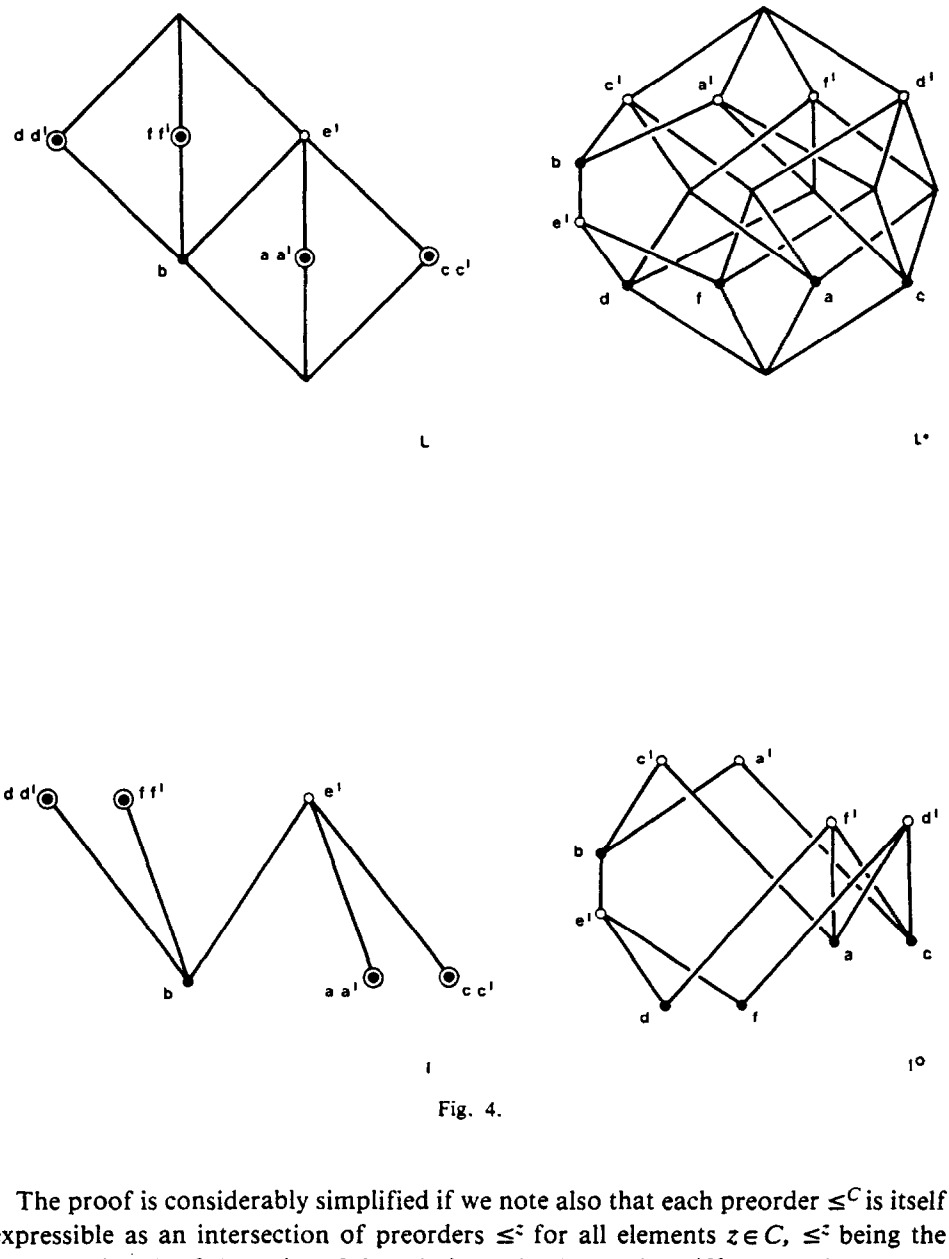


Fig. 4.

The proof is considerably simplified if we note also that each preorder $\leq^C$ is itself expressible as an intersection of preorders $\leq^z$ for all elements $z \in C$, $\leq^z$ being the negation $I^\circ(\text{'}z\text{'})$ of the unity of the relation 'z', where a 'z' a' iff $a \leq z \leq a'$.

Theorem 5.

$$\bigcap_{\text{max. chains } C \subseteq L} \leq^C = \leq^\circ = \bigcap_{z \in L} \leq^z.$$

Proof. Since $\leq^z$ is the preorder of the unity $I^\circ(\text{'}z\text{'})$, we know that for all $a \in J$, $a' \in M$,

$$a \leq^z a' \quad \text{iff} \quad a \not\leq z \text{ or } z \not\leq a'.$$

This relation (the complement of 'z') and its unity are shown in the schematic diagram, Fig. 5. The restrictions of this unity to the products $J \times J, \ldots, M \times M$ are given by the formulas: for all $a, b \in J$, $a', b' \in M$,

$$a \leq^z b \quad \text{iff} \quad a \not\leq z \text{ or } b \leq z, \tag{15}$$

$$a \leq^z a' \quad \text{iff} \quad a \not\leq z \text{ or } z \not\leq a', \tag{16}$$

$$a' \leq^z a \quad \text{iff} \quad z \leq a' \text{ or } a \leq z, \tag{17}$$

$$a' \leq b' \quad \text{iff} \quad z \leq a' \text{ or } z \not\leq b'. \tag{18}$$

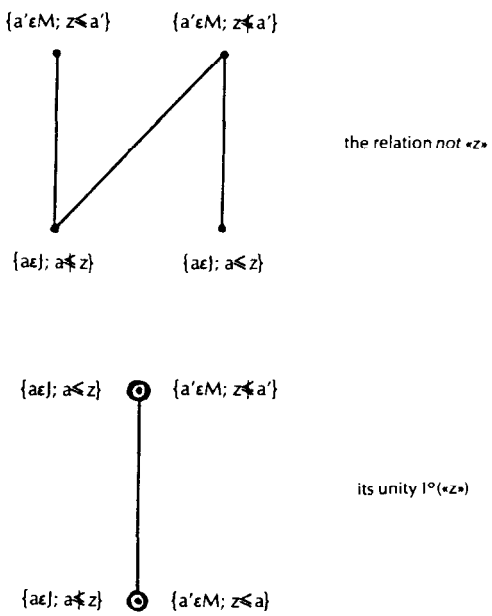


Fig. 5.

For any elements $x, y \in J \oplus M$, $x \leq^C y$ iff the covering pair labelled x is *above* or equal to the covering pair labelled y in the chain C. The four cases to be considered are illustrated in Fig. 6. We find respectively:

$$a \leq^C b \quad \text{iff} \quad \forall z \in C,\ a \leq z \Rightarrow b \leq z,$$

$$a \leq^C a' \quad \text{iff} \quad \forall z \in C,\ a \not\leq z \text{ or } z \not\leq a',$$

$$a' \leq^C a \quad \text{iff} \quad \forall z \in C,\ a \leq z \text{ or } z \leq a',$$

$$a' \leq^C b' \quad \text{iff} \quad \forall z \in C,\ z \leq b' \Rightarrow z \leq a'.$$

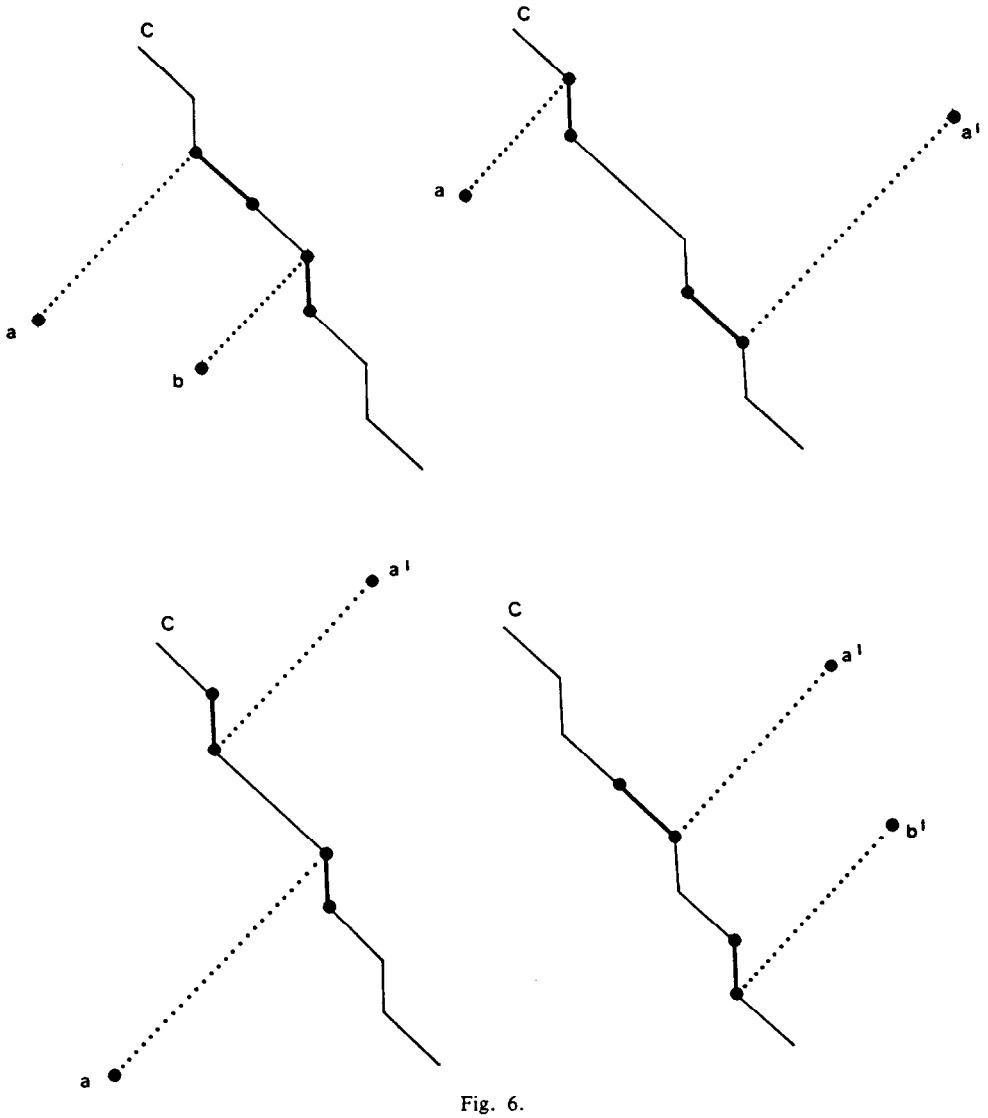


Fig. 6.

If we remove the quantifications '$\forall z \in C$' from all those statements, we obtain exactly lines (15) through (18), which characterize the preorder $\leq^z$. Thus the preorder $\leq^C$ is the intersection of all preorders $\leq^z$ for $z \in C$.

Finally we show that the preorder $\leq^\circ$ of the unity $I^\circ(L)$ is the intersection of the preorders $\leq^z$ for all elements $z \in L$. We use elementary properties of the lattice L, and lines (9), (10), (13) and (12) respectively:

(19) $((\forall z \in L)\ a \not\leq z$ or $b \leq z)$ iff $b \leq a$ iff $a \leq^\circ b$,

(20) $((\forall z \in L)\ a \not\leq z$ or $z \not\leq a')$ iff $a \not\leq a'$ iff $a \leq^\circ a'$,

(21) $((\forall z \in L)\ a \leq z$ or $z \leq a')$ iff $((\forall b \in J)\ a \leq b$ or $b \leq a')$ iff $a' \leq^\circ a$,

(22) $((\forall z \in L)\ z \leq a'$ or $z \not\leq b'$ iff $b' \leq a'$ iff $a' \leq^\circ b'$. □

Caution. Theorem 5 may suggest that unities on $J \oplus M$ form a lattice in the order given by inclusion as preorders on a single set. This is not true. Taking $J = \{a, b\}$, $M = \{a', b'\}$, the unities of the singleton relations $\{(a, b')\}$ and $\{(b, b')\}$ have as intersection a preorder consisting of a' as a lower bound for three incomparable elements: a, b, b'. This is not a unity. When ordered according to inclusion of their underlying relations (restrictions to $J \times M$), unities form a complete Boolean algebra. But this is not the order involved in the above theorem.

In this section we have seen how the preorders $\leq^z$ of the unities $I^\circ(\text{'}z\text{'})$ for all elements $z \in L$ may be intersected to yield the preorder $\leq^\circ$ of the unity $I^\circ(L)$. Below, in Section 6, we shall reverse this process. We find that certain maps from the unity $I^\circ(L)$ to a fixed unity '2' correspond exactly to the elements of L, and when appropriately ordered form a lattice isomorphic to L. In fact, the quotients of those maps, which are preorders dominating the preorder $\leq^\circ$, are exactly the preorders $\leq^z$ for $z \in L$.

5. Completion by nodes

The relationship between the unity of irreducibles in a finite lattice L and the lattice L itself is the analogue of the relationship between an ordered set and its Dedekind–MacNeille completion. The relation $\leq$ on $J \times M$ sets up a Galois connection [6] between decreasing subsets of J and increasing subsets of M, the quotient of which is the required completion. In what follows, we shall use the term 'node' to replace the term 'cut', which would be less accurate in this context.

A *node* $(A : B)$ in a unity I on $J \oplus M$ is a pair of subsets $A \subseteq J$, $B \subseteq M$ such that

$$\forall a \in A,\ \forall b' \in B, \qquad a \leq b', \tag{23}$$

$$\forall a \in J \setminus A,\ \exists b' \in B, \qquad a \not\leq b', \tag{24}$$

$$\forall b' \in M \setminus B,\ \exists a \in A, \qquad a \not\leq b'. \tag{25}$$

More simply, a node is a maximally totally-related pair of subsets (see Fig. 7 below).

Given an arbitrary subset $C \subseteq J$, let $B \subseteq M$ be the set of M-upper-bounds for C, and let $A \subseteq J$ be the set of J-lower-bounds for B. Then $(A : B)$ is a node in the unity I. Underlying this construction is a familiar Galois connection, which we shall specify in a moment. As a consequence, the set of nodes $(A : B)$ in a unity I, ordered by

$$(A_1 : B_1) \leq (A_2 : B_2) \quad \text{iff} \quad A_1 \subseteq A_2,$$

or equivalently

$$\text{iff} \quad B_2 \subseteq B_1,$$

is a complete lattice $\mathrm{Node}(I)$. For the purposes of the following theorem, we replace the distributive lattices $\mathrm{Decr}\, X$ and $\mathrm{Incr}\, X$ of decreasing and increasing subsets of an ordered set X by the lattice

$$\mathrm{Ord}(X, 2) \simeq (\mathrm{Decr}\, X)^{\mathrm{opp}} \simeq \mathrm{Incr}\, X$$

of order-preserving functions from X to 2. Each such lattice is a complete category whose maps are the ordered pairs in X. Between such categories, order-preserving maps are functors, and Galois connections are adjoint situations.

Theorem 6. *For any unity I, the maps*

$$\mathrm{Ord}(J, 2) \underset{\alpha}{\overset{\beta}{\leftrightarrows}} \mathrm{Ord}(M, 2)$$

defined for all $s \in \mathrm{Ord}(J, 2)$, $t \in \mathrm{Ord}(M, 2)$, $x \in J$, $x' \in M$, by

$$t^{\beta}(x) = 0 \quad \textit{iff} \quad t(x') = 1 \Rightarrow x \leq x',$$

$$s^{\alpha}(x') = 1 \quad \textit{iff} \quad s(x) = 0 \Rightarrow x \leq x'$$

is an adjoint situation with β left adjoint to α:

$$t^{\beta} \leq s \quad \textit{iff} \quad t \leq s^{\alpha}.$$

The composites are idempotent, with $\beta\alpha$ decreasing on $\mathrm{Ord}(J, 2)$ *and $\alpha\beta$ increasing on* $\mathrm{Ord}(M, 2)$. *The images of these idempotent operators are both isomorphic to a complete lattice Q, where*

$$Q^{\mathrm{opp}} = \mathrm{Node}(I).$$

Proof. The adjointness is given by

$$t^{\beta} \leq s \quad \text{iff} \quad \forall x \in J,\ x' \in M,\ s(x) = 0 \Rightarrow (t(x') = 1 \Rightarrow x \leq x')$$

$$\text{iff} \quad \forall x \in J,\ x' \in M,\ t(x') = 1 \Rightarrow (s(x) = 0 \Rightarrow x \leq x')$$

$$\text{iff} \quad t \leq s^{\alpha}.$$

In any such adjointness, the composites have the stated properties, and their images are isomorphic complete lattices. For each 'image' pair s, t with $s = t^{\beta}$, $s^{\alpha} = t$, let $A = s^{-1}(0)$, $B = t^{-1}(1)$. Since $t^{\beta} \leq s$, we know that for any $x \in A$ and $x' \in B$, $x \leq x'$. Since $s^{\alpha} = t$, B is the set of M-upper-bounds for A, so any element $x' \notin B$ is not an upper bound for A, and there is an element $x \in A$ with $x \not\leq x'$. Similarly, for each $x \notin A$, there is an element $x' \in B$ with $x \not\leq x'$. Thus $(A : B)$ is a node.

Conversely, if $(A : B)$ is any node in I, we construct the corresponding order-preserving maps s, t by setting $A = s^{-1}(0)$, $B = t^{-1}(1)$. The order on the lattice

Node(I) is the opposite of that on Ord($J,2$), and is therefore opposite to that of the image of the map $\beta\alpha$ in Ord($J,2$). □

For any unity I, the lattice Node(I) is the *completion* of the unity. In Figs. 2 and 4, the lattice L is the completion of the unity I, and the lattice L^* is the completion of the unity I°.

For any element $x \in J$, there is a *J-node*, N_x,

$$(\{y \in J; y \leq x\}, \{x' \in M; x \leq x'\}).$$

Similarly, there is an *M-node* N_x, for every element $x' \in M$. Together we refer to *J*-nodes and *M*-nodes as *principal nodes*. These principal nodes give the embedding of the unity I in the completion Node(I).

We now confirm that every finite lattice is the completion of its unity of irreducibles.

Theorem 7. *If I is the unity of irreducibles, under inclusion, in a finite lattice L, then $L \simeq$ Node(I).*

Proof. For any node $(A:B)$ in I, $\vee A = \wedge B$ is an element $z \in L$, and

$$A = \{x \in J; x \leq z\}, \qquad B = \{x' \in M; z \leq x'\}.$$

Conversely, for any element $z \in L$, the above expression defines a pair of sets which form a node $(A:B)$ in I, with $A = z = B$. □

For any lattice L, we can construct also the *completion of the negation* $I^\circ(L)$ of its unity $I(L)$ of irreducibles. This lattice we call L^*, examples of which we have already drawn in Figs. 2 and 4. We may mark the principal nodes in L^*, but there is no reason to believe that these will all be irreducibles in L^*. See for example Fig. 8 below, where L is distributive but not Boolean. Thus, $I(L^*)$ is not necessarily equal to $I^\circ(L)$, for a lattice L. For what is true, see Theorem 8 below: the dual of a unity is the completion of its negation.

6. Splits of a unity

To arrive at the duality theory sketched in the introduction, wherein any finite lattice L is represented as the lattice of maps from the unity $I^\circ(L)$ to a fixed unity 2, we must define 'maps' between unities. We make some suggestions along these lines in Section 8 below. For present purposes it will suffice to describe all possible ways in which a unity may be separated into two parts (the inverse images of 0 and 1) under the action of a 'map to 2'. We call such partitions 'splits' of a unity, and define them as follows.

Given two subsets S, T in an ordered set X, the set T is *final* in S iff

$$\forall s \in S \ \exists t \in S \cap T \text{ such that } s \leq t.$$

The set T is *initial* in S iff

$$\forall s \in S \ \exists t \in S \cap T \text{ such that } t \leq s.$$

A *split* of a unity I on $J+M$ is an order-preserving function $h: I \to 2$ such that

$$M \text{ is final in } h^{-1}(0), \qquad J \text{ is initial in } h^{-1}(1). \tag{26}$$

Some examples of splits are in order.

Any $(0, 1, \wedge, \vee)$-homomorphism f of a lattice to the 2-element lattice $2 = \{0, 1\}$ has the property that $f^{-1}(0)$ is a prime ideal, while $f^{-1}(1)$ is the complementary prime filter. In a finite lattice, any prime ideal has as its least element a join-irreducible, and any prime filter has as its greatest element a meet-irreducible. Thus the restriction of such a lattice homomorphism to the unity $I(L)$ of irreducibles is a split.

At the other extreme, consider a unity formed by 'doubling' a partially ordered set. Here the partially ordered sets J and M are isomorphic, and are identified in the unity. In this case, a split is simply any order-preserving function from the unity to 2. This is the case which arises in the representation of distributive lattices, as we shall see in the next section.

Fig. 7 shows the representation of one lattice element by a split of $I^\circ(L)$.

Let Split(I) be the set of splits of I, ordered pointwise as functions from 1 to 2. In the following theorem, we show among other things that Split(I) is a complete lattice.

Theorem 8. *For any unity I with negation I°*

$$\text{Split}(I) \simeq \text{Node}(I^\circ).$$

Proof. If $h: I \to 2$ is a split, let

$$A = J \cap h^{-1}(1), \qquad B = M \cap h^{-1}(0). \tag{27}$$

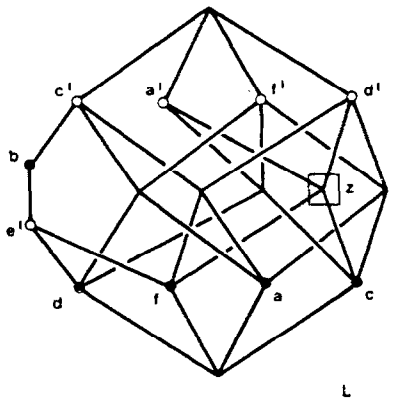
Since h is order-preserving, for every $x \in A$ and $x' \in B$, $x \not\leq x'$, so $x \leq^\circ x'$. If $y \in J \setminus A$, $h(y) = 0$, and since M is final in $h^{-1}(0)$, there is an element $x' \in M$, with $y \leq x'$, $h(x') = 0$. But then $x' \in B$, and $y \not\leq^\circ x'$, as required. Similarly, for any $y' \in M \setminus B$, there is an element $x \in A$ with $x \not\leq^\circ y'$. Consequently, $(A : B)$ is a node in I°.

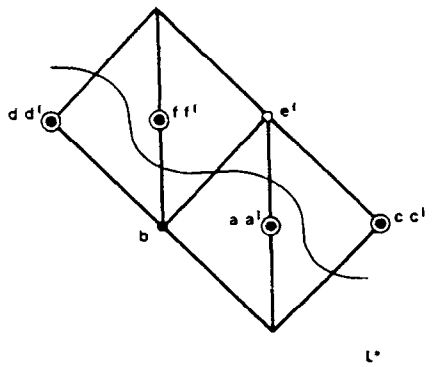
A reverse argument shows that for any node $(A : B)$ in I, if A, B are considered as subsets of I, then A is initial in $A \cup M \setminus B$, while B is final in $B \cup J \setminus A$. Line (27) thus defines a split $h: I \to 2$.

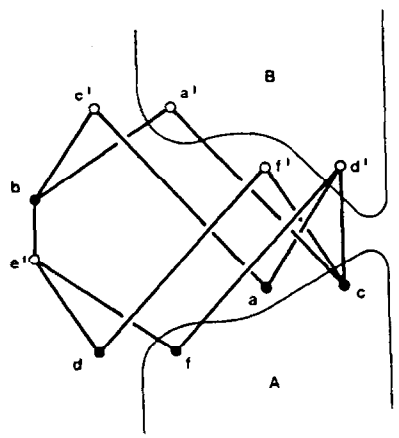
Given two splits h_i $(i = 1, 2)$ of I, let $(A_i : B_i)$ be the corresponding nodes of I°. Then $h_1 \leq h_2$ iff $A_1 \subseteq A_2$, iff $B_2 \subseteq B_1$, so

$$\text{Split}(I) \simeq \text{Node}(I)$$

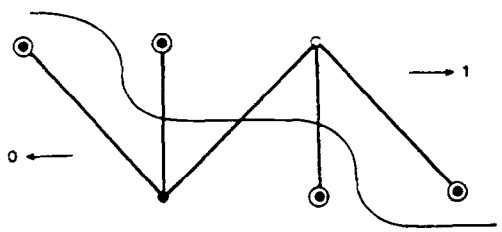
as required. □







The node (A:B) of I(L) at the element z



The split (A:B) of I°(L) representing the element z∈L

Fig. 7.

Theorems 7 and 8 now yield directly the main *representation theorem* for finite lattices.

Theorem 9. *For any finite lattice L,*

$$L \simeq \text{split}(I^\circ(L))$$

where I° is the negation of the unity I of irreducibles in L.

Proof. $L \simeq \mathrm{Node}(I) \simeq \mathrm{Split}(I^\circ)$. □

Splitting a finite lattice at an irreducible produces a 'principal split', as follows. Let I be any unity. For any fixed element $a \in J$,

$$h^a(z) = 1 \quad \text{iff} \quad a \leq z$$

defines an *upper-principal* map from $J \oplus M$ to 2. Similarly, for any fixed $a' \in M$,

$$h_{a'}(z) = 0 \quad \text{iff} \quad z \leq a'$$

defines a *lower-principal* map.

Theorem 10. *Upper-principal and lower-principal maps are splits.*

Proof. Say $a \in J$ and h^a is the associated upper-principal map. For any $x \in J$ with $h^a(x) = 0$, we have $a \not\leq x$, so there is an element $x' \in M$ with $x \leq x'$, $a \not\leq x'$, by line (5) in the definition of unity. Then $x \leq x'$ and $h^a(x') = 0$, so M is final in $(h^a)^{-1}(0)$.

For any $x' \in M$ with $h(x') = 1$ we have $a \leq x'$, where $a \in J$ and $h^a(a) = 1$. Thus J is initial in $h^{-1}(1)$, and h^a is a split. Similarly, $h_{a'}$ is a split. □

7. Distributive lattices

Every finite distributive lattice L is isomorphic to the lattice $\mathrm{Ord}(X, 2)$ of order-preserving functions from a set X to the 2-element chain $2 = \{0, 1\}$. Within the lattice L there are two natural choices for this ordered set: $X \simeq J^{\mathrm{opp}} \simeq M^{\mathrm{opp}}$. (In the discrete topology, this ordered set X is the Priestly space [10] of the lattice.)

Identification of J^{opp} with M^{opp} under this isomorphism is precisely what occurs in the construction of the unity $I^\circ(L)$ when L is distributive. An example is shown in Fig. 8. Note that for any covering pair $y \downarrow x$ in a finite distributive lattice L, there is a *unique* join irreducible element a such that $a \leq y$, $a \not\leq x$, and a *unique* meet irreducible element a' such that $x \leq a'$, $y \not\leq a'$.

Fig. 9 shows the labelling for the lattice in Fig. 8.

The identification of such pairs a, a' in the unity $I^\circ(L)$ occurs because the filter generated by a and the ideal generated by a' are *complementary subsets* of the lattice. Since $a \not\leq a'$, we have $a \leq^\circ a'$ by line (10), and since *every* element in L is either $\geq a$ or $\leq a'$, we have $a' \leq^\circ a$ by line (11).

On such a unity, where every element is both a J-element and an M-element, the defining properties (26) of a split are trivially satisfied, and any order-preserving map from $I^\circ(L)$ to 2 represents an element of L, as required in the usual representation theory.

The lattice L^*, which is representable as either $\mathrm{Node}(I^\circ(L))$ or $\mathrm{Split}(I(L))$, is simply the Dedekind–MacNeille completion of the Priestley space.

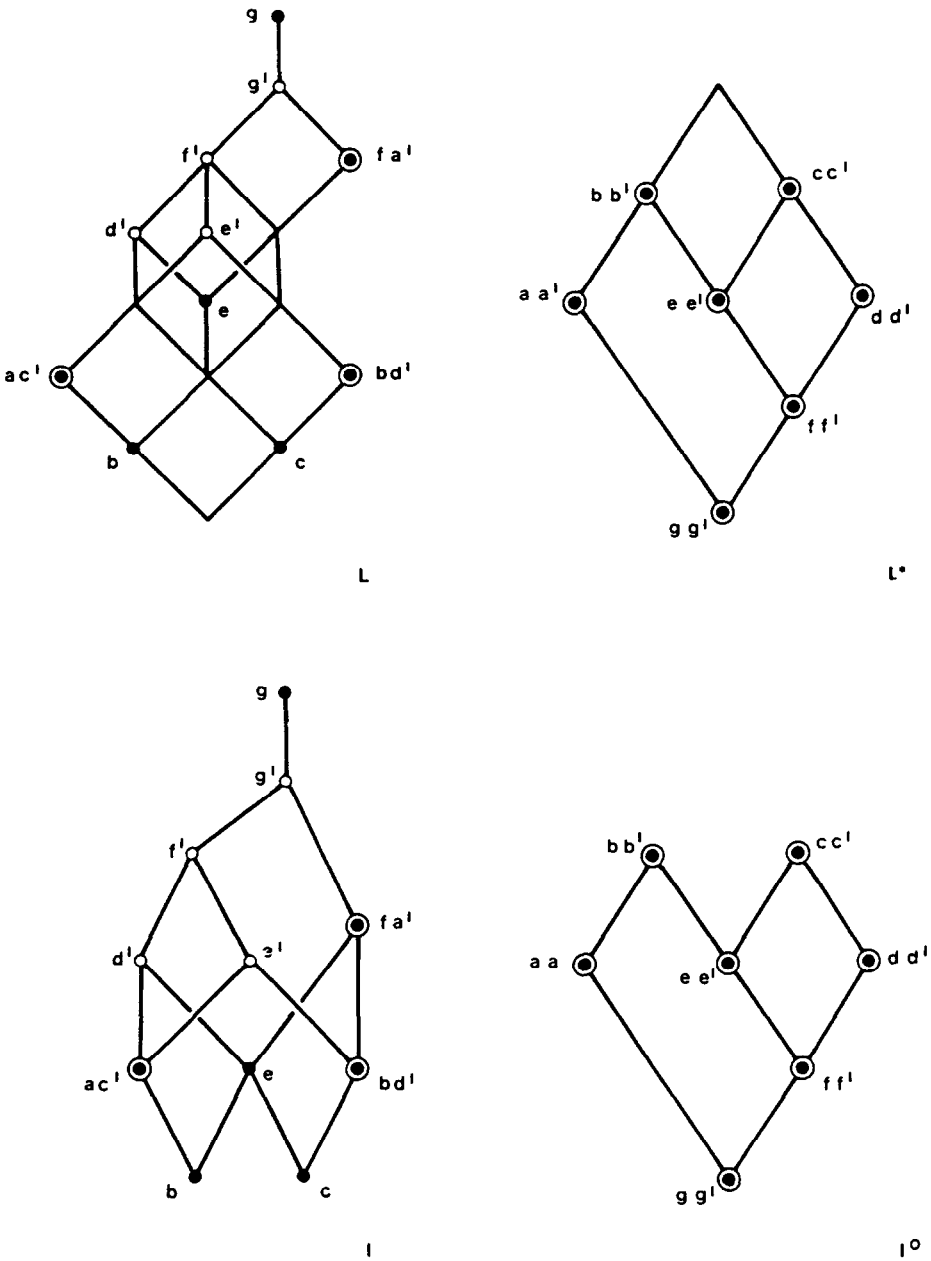


Fig. 8.

8. Problems and examples

In Sections 1–7 above, we have presented the fundamental facts concerning representation and duality of finite lattices. A number of problems should now be settled.

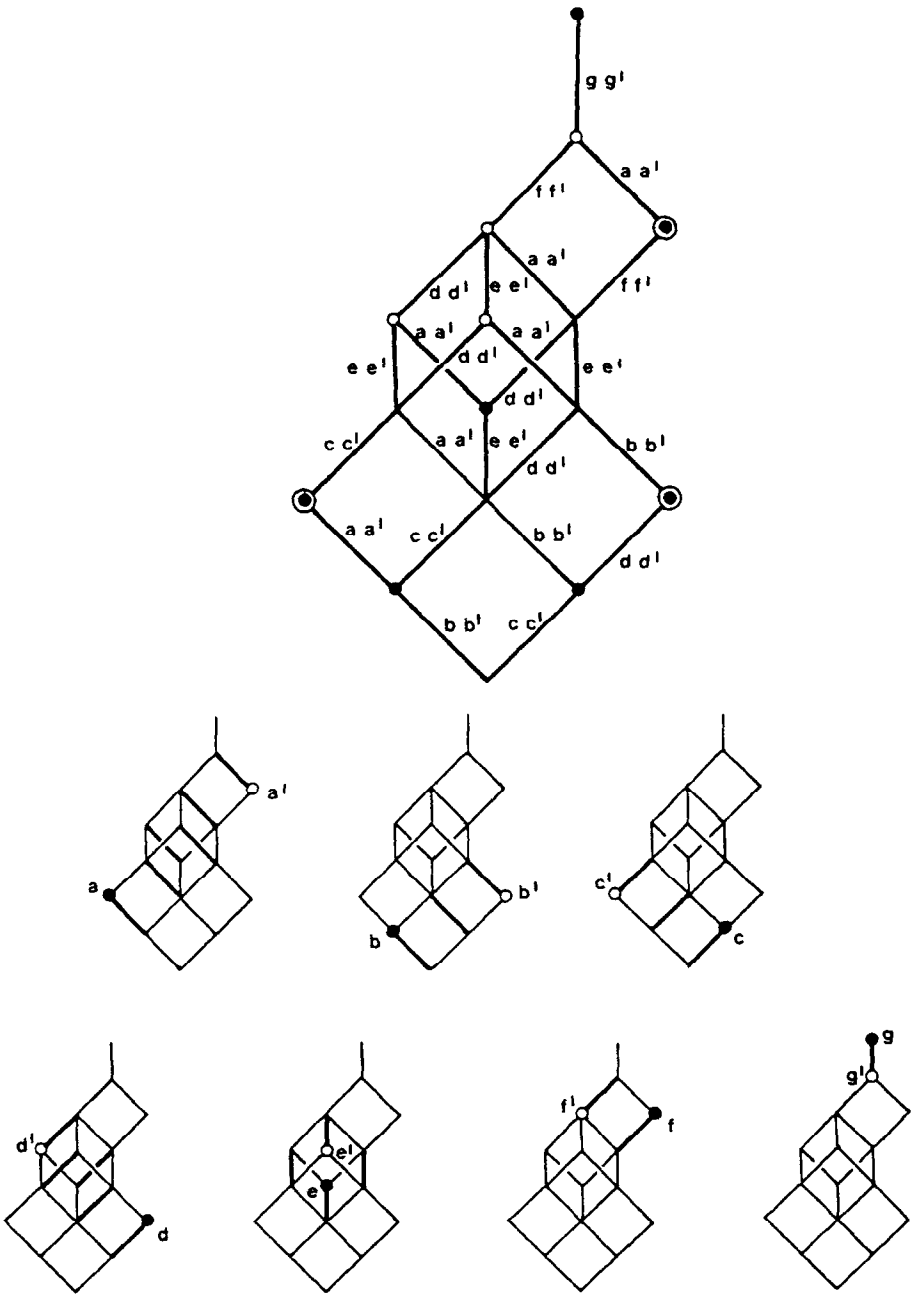


Fig. 9.

Problem 1. Extend the representation theory $L \simeq \mathrm{Split}(I^{\circ}(L))$ to arbitrary lattices.

Here we can replace join- and meet-irreducibles by those filters and ideals which

occur in *maximal filter-ideal pairs* (A, B) where A and B are disjoint, but any larger filter $A_1 \supset A$ meets B, and any larger ideal $B_1 \supset B$ meets A. The set of such pairs (A, B), in a topology determined by the lattice elements, is Urquhart's L-space. It remains to define the correct topology on $I^{\circ}(L)$ and to determine which splits of $I^{\circ}(L)$ correspond to his 'doubly-closed stable sets'.

Since Urquhart was forced to impose a number of tedious conditions to single out L-spaces, it will probably be better to develop this theory along the lines of the representation of the category of distributive lattices as opposite (and dual) to the category of coherent spaces and coherent morphisms [8].

Problem 2. Define maps between unities, in such a way that $L \simeq I(L)$ becomes a covariant functor for the category Lat of lattices and $(0, 1, \wedge, \vee)$-homomorphisms into (not onto) the category Unity of unities and maps.

Whatever definition is proposed, any composite of maps should be a map, and any composite of a map $I_1 \to I_2$ with a split of I_2 should be a split of I_1. Also, the ordered set $2 = \{0, 1\}$ used to define splits should be a unity on some pair of sets.

We have recently made some progress in that direction, and offer the conjecture that the following, to the point formulated only in terms of *relations* rather than *unities*, is the correct idea.

We define a *map* from a unity I_1 on $J_1 \oplus M_1$ to a unity I_2 on $J_2 \oplus M_2$ to be any pair S, T of unities

$$S \text{ on } J_2 \oplus J_1, \qquad T \text{ on } M_1 \oplus M_2,$$

having underlying relations

$$\sigma \subseteq J_2 \times J_1, \qquad \tau \subseteq M_1 \times M_2$$

satisfying, for all $a' \in M_1$, $b' \in M_2$,

(28) $a' \tau b'$ iff $\forall a \in J_1$ and $b \in J_2$, $b \sigma a \leq a'$ implies $b \leq b'$ and for all $a \in J_1$, $b \in J_2$

(29) $b \sigma a$ iff $\forall a' \in M_1$ and $b' \in M_2$, $a \leq a' \tau b'$ implies $b \leq b'$.

These conditions (28) and (29) are designed to guarantee that maps $f: L_1 \to L_2$ between lattices, defined by

$$f(x) = \vee\{b \in J_2; b \sigma a \leq x \text{ for some } a \in J_1\} \tag{30}$$

$$= \wedge\{b' \in M_2; x \leq a' \tau b \text{ for some } a' \in M_1\} \tag{31}$$

will be join-preserving and meet-preserving respectively.

We have checked that if 2 is chosen to be the unity shown in Fig. 10, then maps (σ, τ) from a unity I to 2 correspond exactly to splits $f: I \to 2$ of the unity I. The correspondence is, for all $a \in J$ and $a' \in M$,

$$r \sigma a \quad \text{iff} \quad f(a) = 1,$$

$$a' \tau u' \quad \text{iff} \quad f(a') = 0.$$

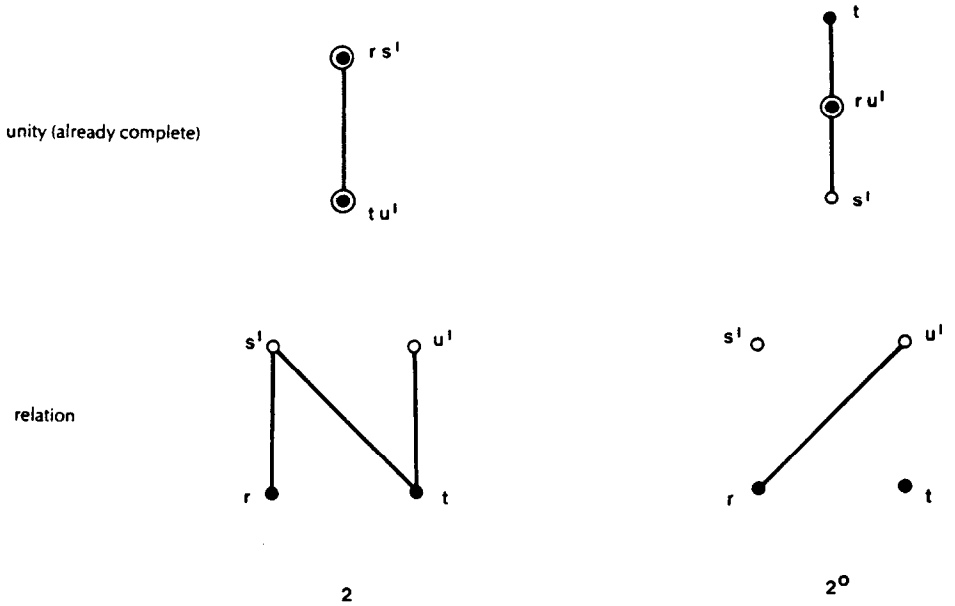


Fig. 10.

We expect that all maps between unities of lattices came from lattice homomorphisms, under the functor $L \to I(L)$. If so, we might also expect that the duality

$$L \simeq \text{Unity}(I^{\circ}(L), 2)$$

would lift (by completion) to a duality

$$L \simeq \text{Lat}(L^{*}, \text{Two})$$

for some lattice 'Two'. We know this is false, because there are simple lattices L whose negations L^{*} are nontrivial (see Fig. 11 below, the nonmodular 5-element lattice). The solution to this apparent paradox lies in the fact that the unity 2 is *not* the unity of a lattice.

Problem 3. Show that the operators $^{\circ}$ and * are simply two restrictions of a single functor, say 'negation', defined on a category which includes both Lat and Unity.

It is conceivable that this functor is also a duality over some fixed object. The objects in this larger category will be certain pairs of inclusions maps $J \to C \leftarrow M$ of two ordered sets into a third, not necessarily jointly onto, as in the case of a unity, but certainly such that every element of C is represented as a node of the underlying unity for J to M.

The following examples will suggest further problems concerning the representation of products and coproducts of lattices, and the representation of free lattices in various varieties.

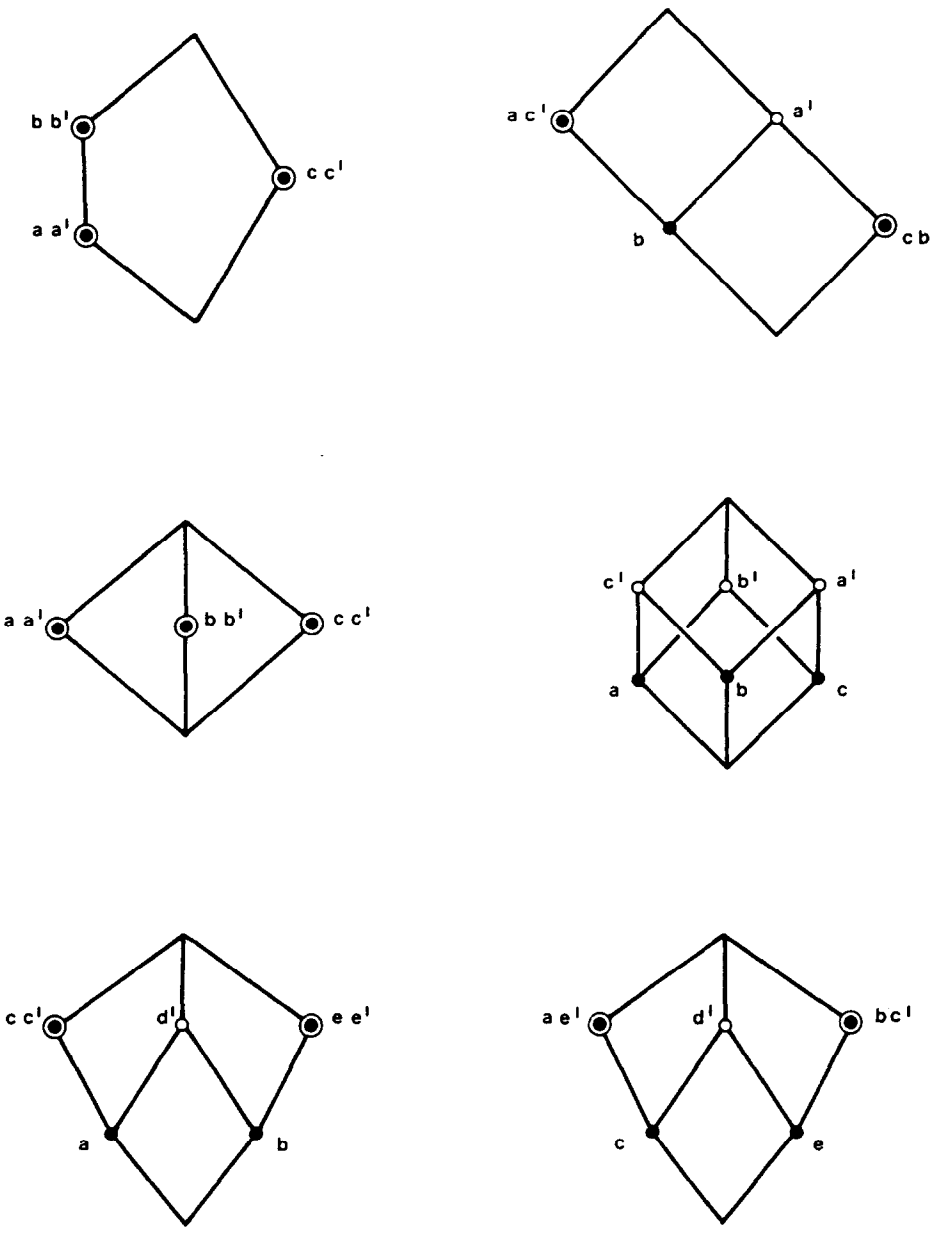


Fig. 11.

The pairs of lattices in Fig. 11 are completions of unities which are negations of one another. Observe that the negation of a 'Boolean' unity is a 'doubled' unordered set whose completion is a lattice with rank 2. In the final pair of drawings in Fig. 11, the lattices L and L^* are isomorphic, but the rearrangement of irreducibles to pass from L to L^* is nontrivial.

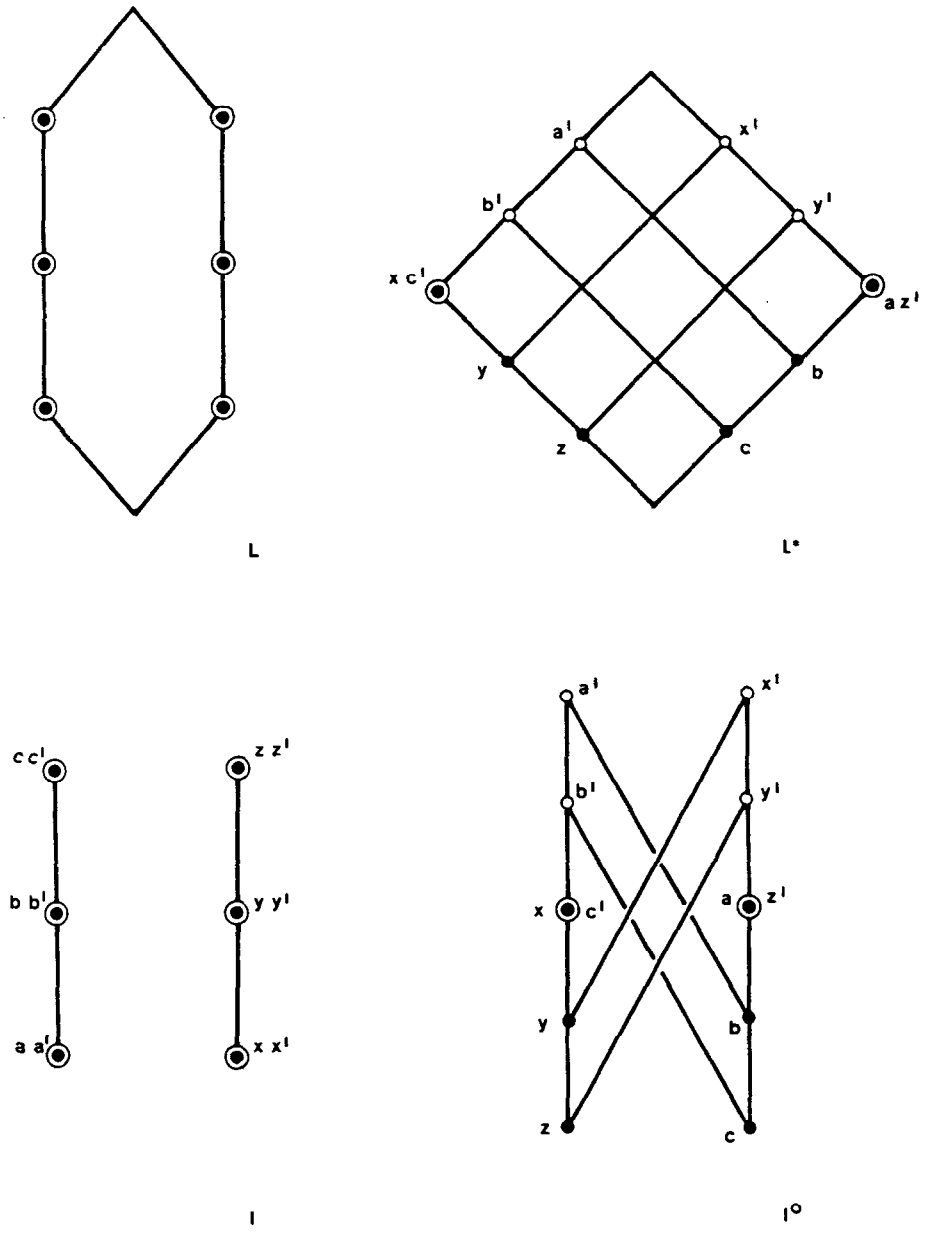
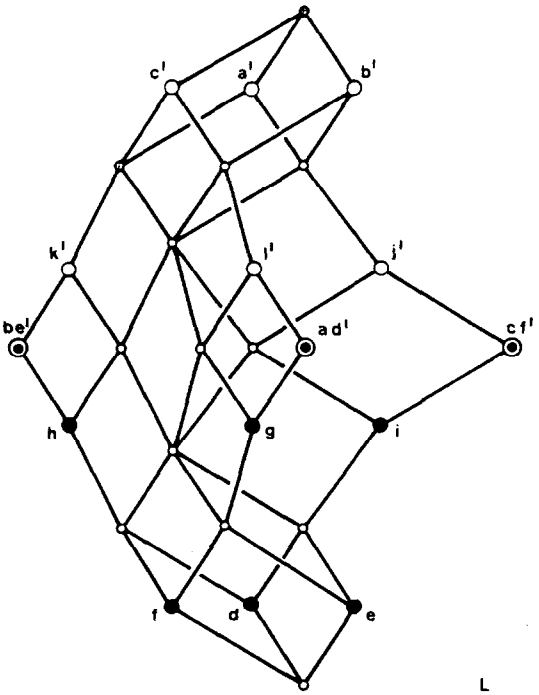


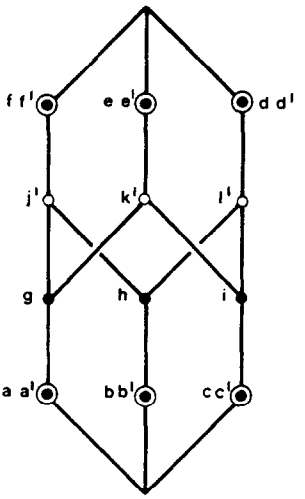
Fig. 12.

In Fig. 12, the unity I is a sum of chains, while the lattice L^* is a product of chains.

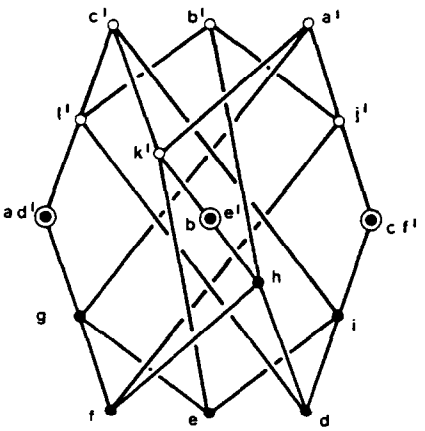
Fig. 13 gives the negation of the unity of irreducibles in the free modular lattice on three unordered generators. Note how simple the negation is, compared to the free


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a d'
c f'
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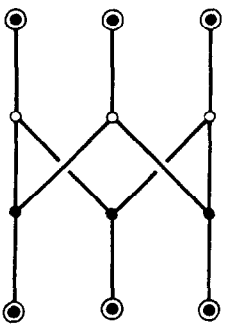
L


f f'
e e'
d d'
j'
k'
l'
g
h
i
a a'
b b'
c c'

L*


c'
b'
a'
l'
j'
k'
a d'
b
e'
c f'
h
g
i
f
e
d

I



I°

Fig. 13.

modular lattice. This suggests that it may be preferable to approach various word problems for lattices by regarding the words as descriptions of splits of the negation.

9. Determinacy

For any finite lattice L, we have the representation

$$L^* \simeq \mathrm{Split}(I).$$

Perhaps it is possible to extend splits from I to all the completion L, thereby obtaining a family of mappings from L to 2, and a representation in the form

$$L^* \simeq \mathrm{Map}(L, 2).$$

This is 'almost' possible, as follows.

Every split $h: I \to 2$ of a unity I yields a node $(C:D)$ of I° by

$$C = J \cap h^{-1}(1), \qquad D = M \cap h^{-1}(0).$$

For the purpose at hand, it is convenient to specify splits in this way also, that is: to speak of the split $(C:D)$ of I, which maps $C \cup (M \setminus D)$ to 1, $(J \setminus C) \cup D$ to 0.

Given a split $(C:D)$ and a node $(A:B)$ of a unity I, either $A \cap C = \emptyset$ or $B \cap D = \emptyset$. The reason is that $x \leq x'$ holds for *all* $x \in A$, $x' \in B$, and for *no* $x \in C$, $x' \in D$. If the subsets A and C have an element $x \in J$ in common, then the unity $(C:D)$ maps x to 1, and $(A:B) \geq N_x$ with respect to the principal node N_x in the completion Node(I), so any order-preserving function from L to 2 which extends the split $(C:D)$ must map the node $(A:B)$ to 1. By the same reasoning, if the subsets B and D have an element $x' \in M$ in common, then the node $(A:B)$ is mapped to 0 by any order-preserving extension of the split.

However, and this is the interesting part, *both* intersections $A \cap C$ and $B \cap D$ may be empty. In this event, there is no natural choice for the value of the split $(C:D)$ at the node $(A:B)$. The split is *undetermined* at that node.

The unities in Fig. 14 exhibit this possibility. The node $(ab, c'd')$ of I, as a split of I°, meets neither of the nodes $(cd, a'b'e')$, nor $(cde, a'b')$ of I°.

Given any unity I, we say a node $(A:B)$ *meets* a split $(C:D)$ of I iff either A meets C or B meets D. This relation 'meets' yields a Galois connection between subsets of Node(I) and subsets of Split(I). The image of any set of nodes is the set of splits which meet all those nodes; the image of a set of splits is the set of nodes which meet all those splits. Two important problems arise.

Problem 4. Characterize those sets of nodes which are closed in this Galois connection.

Problem 5. Characterize those sets of splits which are the image of some sublattice of the completion Node(I).

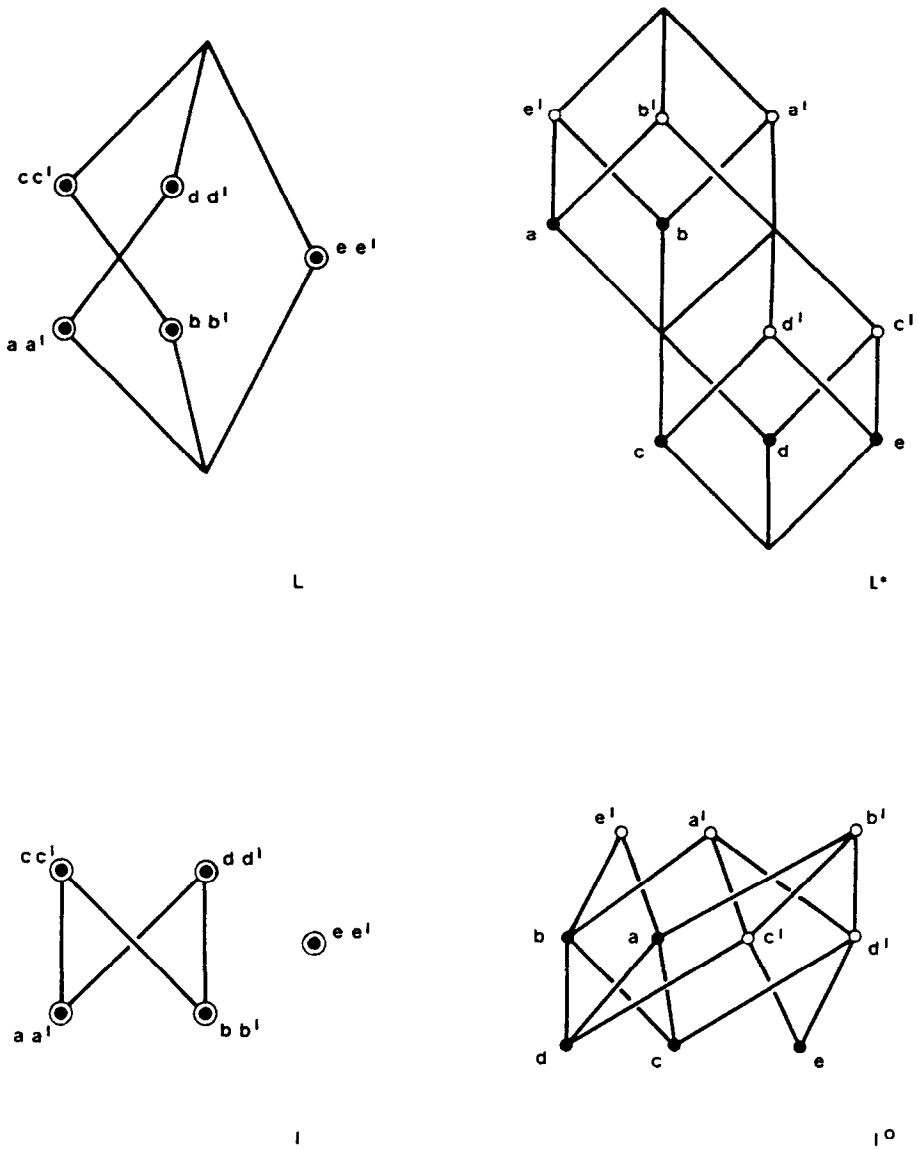


Fig. 14.

We give one result in this direction.

Theorem 11. *Every split meets every principal node.*

Proof. Let $h = (C : D)$ be a split, and $N_y = (A : B)$ be a J-node. Then $y \in A$. If C does not meet A, then $y \notin C$, so $h(y) = 0$. Since h is a split, there is an element $y' \in M$ with $y \leq y'$ and $h(y') = 0$. Thus B meets D in the element y'. □

10. Extending splits to finite lattices

Let I be the unity of irreducibles of a finite lattice L. Certain splits of I meet every node of I, and thus define mappings from L to 2. We conclude this paper by characterizing those mappings.

First, define operators $S\uparrow$, $H\downarrow$ on subsets of L by:

$S\uparrow(A)$ is the set of joins of subsets of A,

$H\downarrow(A)$ is the largest subset of A which is decreasing in L. $S\downarrow$ and $H\uparrow$ are the same operators, relative to the opposite order, so $S\downarrow(A)$ is a meet-sub-semilattice, and $H\uparrow(A)$ is an increasing set. The composites $H\downarrow S\uparrow$ and $H\uparrow S\downarrow$, as operators acting on subsets, are closure operators. (They are inclusion-preserving, increasing, and idempotent.)

Using these operators, we define a *map* from a lattice L to 2 as any order-preserving function $h: L\to 2$ such that if $E = h^{-1}(0)$, $F = h^{-1}(1)$, then

$$H\downarrow S\uparrow(E) = E, \qquad H\uparrow S\downarrow(F) = F.$$

That is: elements not a join of elements mapped to 0 are initial among elements mapped to 1, and elements not a meet of elements mapped to 1 are final among elements mapped to 0.

(The reader may prefer to view $H\downarrow S\uparrow$ and $H\uparrow S\downarrow$ as operators on order-preserving functions from L to 2. In that case $H\uparrow S\downarrow$ is a closure operator, $H\downarrow S\uparrow$ is a coclosure operator (i.e. decreasing), and a map is a function which is simultaneously closed and coclosed.)

Theorem 12. *The restriction of any map from L to 2, for a finite lattice L, is a split of the unity I of irreducibles, a split which meets every node of I. Conversely, given a split of I which meets every node of L, it extends uniquely to a map of L to 2.*

Proof. Let h be a map from L to 2. Any element which is mapped to 1 has beneath it a join-irreducible element also mapped to I. The reason is that $H\downarrow S\uparrow(E) = E$, i.e.: every element s not in E, being also not in $H\downarrow S\uparrow(E)$, has an element t beneath it *not* in $S\uparrow(E)$. Thus t is not a join of join-irreducibles which are mapped to 0. Since every element is a join of join-irreducibles, some join-irreducible x with $x\le t\le s$ must be mapped to 1. Of course the converse is true: $x\in J$, $x\le s$ and $h(x) = 1$ imply $h(s) = 1$. A similar argument shows that an element is mapped to 0 iff it lies beneath a meet-irreducible element which is also mapped to 0. Consequently, the restriction of h to I is a split.

We now show that every map $L\to 2$ is the extension of its restriction to the unity I of irreducibles. Given an element $z\in L$, let $(A:B)$ be its representation as a node of I. By the preceding paragraph, we know $h(z) = 1$ iff there is a join-irreducible $x\le z$ such that $h(x) = 1$. That is, $h(z) = 1$ iff in the split $(E:F)$ obtained by restricting h to I, A meets E. Thus the map h is equal to the extension of its restriction to I, and $(E:F)$ meets every node of I.

Finally we prove that the extension of any split which meets all nodes of I is a map. For any node $(A:B)$ which is mapped to 1 in the extension of $(E:F)$ to L, there is an element $x \in A \cap E$. The element x is join-irreducible in L, so the node N_x is join-irreducible in the completion. Since x is join-irreducible and $h(x)=1$, x is not a join of elements mapped to 0. Similarly, any node $(A:B)$ which is mapped to 0 lies beneath an M-node which is also mapped to 0, and which is therefore not a meet of nodes mapped to 1. □

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