Categorification as a Heuristic Device

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I was drawn into the philosophy of mathematics by the writings of Lakatos, Brunschvicg and Lautman. When I first started to read the mainstream English-language philosophy of mathematics literature, I was immediately struck by its almost complete lack of interest in what I considered to be the treasure house of mathematics. The philosophers I read seemed to think that the best way to get a handle on mathematics was to find some or other formal calculus which could be said to represent it in its totality. Set theory or second-order logic appeared to be just the thing for the job. To me this was like badly packing some pieces of jewellery in a huge cardboard box, then speaking indiscriminately of the whole of the space enclosed within the box as though it were the precious contents. It seemed to me that were we to pose the right questions, we would be required to look at the finer filigree. Or to put it the other way around, if we find we are not required to look at the details of mathematical thinking, we are not posing the right questions.

Michael Polanyi wrote in *Personal Knowledge*

> Between the practice of hackneyed exercises on the one hand and the heuristic visions of the lonely discoverer on the other, lies the major domain of established mathematics on which the mathematician consciously dwells by losing himself in the contemplation of its greatness. A true understanding of science and mathematics includes the capacity for a contemplative experience of them, and the teaching of these sciences must aim at imparting this capacity to the pupil. (Polanyi 1958: 195-6)

I have always been fascinated by the interface between these “heuristic visions” and “established mathematics”. Visions tell stories of how mathematics has come to look as it does and of how it ought to develop in the

future. But, how does a novel vision get taken up by the rest of the community? In what sense can this taking up process be called rational? Those who dwell on the static appraisal of the truth of mathematics within an overarching framework are inclined to see these questions as lying wholly within the context of discovery, and so consider them outside the bounds of philosophy. I reject this charge completely. I am very much interested in justification, but not primarily at the level of single propositions where so much work has already been done. Rather, I want to explore how powerful visions can spawn thriving research programmes, in which fruitful concepts and theories can be developed. There can be no absolute standpoint from which judgements about these good qualities can be made. They take place in 'real time', and so one must enter into the thickets of the history of mathematical practice.

Those who agree with me can choose to study mathematical reasoning at a very local level, local in historical terms and in terms of the patch of mathematics studied. The risk here, however, is in losing oneself in the minutiae of specific fragments of reasoning. On the other hand, the temptation to over-generalise from a detailed case must be resisted. We need detailed historical studies of allusive examples of reasoning which represent the established and emerging styles of thinking of an epoch, along with accounts of the way they transmogrify.

I also recognised that it was very important that we do not neglect the relatively recent past. Otherwise, the impression that mathematics has stabilised and can be treated now in timeless fashion would not go properly challenged. In recent decades, with the proliferation of mathematics, the issue that has come to the fore has been how best to organise the field. What I sought, then, was a reasonably large-scale contemporary effort to organise the contents of mathematics. One that makes genuine contact with, and goes along the grain of, valued findings. One that gives a retrospective account of the development of episodes from the past, and that suggests lines for future expansion. Those who have read my book, *Towards a Philosophy of Real Mathematics*, will know that I think I've found the answer in

'higher-dimensional algebra'.[1]

Now, I certainly do not wish to claim that higher-dimensional algebra has all the answers to the way mathematics is today, how it has emerged and the path it is travelling along, but I do believe that it possesses sufficiently many noteworthy attributes that a number of philosophers might usefully engage with it. If we could have just half the resources currently allocated to the reconstruction of mathematics in terms of second-order logic, we would be able to cast in sharp relief a host of important features. These features, in my opinion, merit the kind of attention Russell lavished on the logics of Cantor, Frege and Peano.

In chapter 10 of my book, I began to sketch some of these features. More recently, when invited to speak to a group of sixty or so mathematicians in Minneapolis, who had gathered together to compare the dozen or so existing definitions of higher-dimensional categories (see http://www.ima.umn.edu/categories), I outlined two broad approaches as to how mathematics might come to inspire philosophy once more. One of these approaches would attempt to link mathematics via philosophy of science to epistemology, aesthetics and ethics; the other, I dubbed the *Russellian* approach, would use higher-dimensional algebra to run through traditional trading links between mathematical logic and philosophy, reforming metaphysics on the way. Here are suggestions for the Russellian approach:

a) "Every interesting equation is a lie": behind every interesting equation there lies a richer story of isomorphism or equivalence. Much more subtle conceptions of sameness are made available by higher-dimensional algebra.
b) Reappraisal of property and structure: maps between different categories of entity factor to reveal what is property and what structure.
c) New insights into logic: for example, a form of modal logic appears in a natural sequence, coming after propositional logic and predicate logic. Just

[1] For the visionary aspect of higher-dimensional algebra, see in particular p. 249 of Corfield 2003, and also its elaboration in section 3 of Corfield *forthcoming*.

as toposes throw light on constructive logic, their categorifications, i.e., 2-toposes should be important.

d) New notions of complex structures for biology, neurology, computer science, etc.: as yet perhaps the most speculative of the entries here.

e) Part-whole relations, the nature of space: one of the inspirations for higher-dimensional algebra was the revolutionary vision of Alexandre Grothendieck. Conceptions of space are currently being radically transformed. Higher-dimensional algebra is finding increasing use in leading approaches to quantum gravity.

f) Diagrammatics to question the "transparency" of logic: the blurring of the distinction between topology and algebra in the notation of higher-dimensional algebra reveals its iconic and symbolic roles (see the paper by Emily Grosholz, this volume).

g) "The sphere spectrum is the true integers": this is an example of taking the process of categorification to its extreme, starting out with the integers. It illustrates the idea that higher-dimensional algebra can carve out the 'right' concepts. If you start out with an important construction and categorify it 'well', you will end up with something else important.

In the brief space of this paper I can hardly begin to indicate the full significance of any one of these features. Instead, I'll give an overview of some of them. As ever when I write about higher-dimensional algebra, I am enormously indebted to John Baez and James Dolan for their expository work. My borrowings are too numerous to detail. I strongly advise the interested reader to look up their papers and web material at Baez's home page http://math.ucr.edu/home/baez/.

Higher-dimensional algebra

In higher-dimensional algebra, also known as higher-dimensional category theory, you encounter a ladder which you're irresistibly drawn to ascend.[2] Let

[2] For a more leisurely account of this material, read Baez's 'Tale of *n*-categories', starting at

us begin with a finite set. About two elements of this set you can only say that they are the same or that they are different. Thinking about sets a little harder, you are led to consider what connects them, namely, functions (or perhaps relations). Taken together, sets and functions form a category. Now, there are two levels of entity, the objects (sets) and the arrows (functions), satisfying some conditions, existence of identity arrows and associative composition of compatible arrows.

There are plenty of examples of categories. For example, categories of structured sets, such as groups and homomorphisms, but also spatial ones, such as 2-Cob, whose objects are sets of circles and whose arrows are (diffeomorphism) classes of surfaces between them. A pair of trousers represents an arrow from a single circle (the waist) to a pair of circles (the trouser cuffs) in this latter category.

In a category, the arrows between two objects *A* and *B*, Hom(*A*, *B*) forms a set. The only choice for two arrows in Hom(*A*, *B*) is whether they are the same or different. At the level of objects, however, there is a new option. *A* and *B* need not be the same, but there may be arrows between them which compose to the identity arrows on each object. This kind of sameness is often called isomorphism. So-called 'structuralists' are aware of this degree of sameness.

Let us continue up the ladder. Consider categories long enough, and before you know it you're thinking of functors between categories, and natural transformations between functors. Functors are ways of mapping one category to another. If the first category is a small diagram of arrows, a copy of that diagram within the second category would be a functor. Natural transformations are ways of mediating between two images within the target category. Taking all (small) categories, functors, and natural transformations, we have an entity with three levels, which we draw with dots, arrows and double-arrows. This is an example of a 2-category, the next rung of the ladder. In this setting, whether objects are the 'same' is not treated most

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http://math.ucr.edu/home/baez/week73.html .

generally as isomorphism, but as equivalence. Between equivalent objects there is a pair of arrows which do not necessarily compose to give identity arrows, but do give arrows for which there are invertible 2-arrows to these identity arrows.

This is something structuralists have missed. At the level of categories, isomorphism is too strong a notion of sameness. Anything you can do with, say, the category of finite sets, you can do with the equivalent (full) sub-category composed of a representative object for each finite number and functions between them.

Then, one more step up the ladder, 2-categories form a 3-category, with four levels of entity. Another example of a 3-category is the fundamental 3-groupoid of a space. Take the surface of a sphere, such as the world. Objects are points on the globe. 1-arrows between a pair of objects, say the North and South Poles, are paths. 2-arrows between pairs of 1-arrows, say the Greenwich Meridian and the International Data Line, are ways of sweeping from one path to the other. Finally, a 3-arrow between a pair of 2-arrows, say one that proceeds at a uniform rate between the Greenwich Meridian and the International Data Line and the other that tarries a while over New Delhi, is represented by a way of interpolating between these sweepings.

Mathematicians hate to stop a good thing when it's rolling, so are aiming to extend this process infinitely far to omega-categories, by defining them at one fell swoop. The idea for doing so was inspired by Alexandre Grothendieck, who realised that there was a way of treating spaces up to homotopy in algebraic terms. Already at the 2-category level there are many choices of shape to paste together. There are thus many ways of defining an omega-category. At present, twelve definitions have been proposed. It is felt, however, that the choice is in a sense immaterial, in that all ways will turn out to be the 'same' at the level of omega-categories, although each may be best suited to different applications. I was invited to a fortnight long workshop, hosted by the Institute for Mathematics and its Applications in Minneapolis, whose aim was to compare these definitions, and assess their potential applications. Regarding the latter, computer scientists from the French

nuclear industry are using omega-categories to analyse potential deadlocks in multi-processor computations.

Adding in some structure

Let's look at some applications not too far up the ladder. Plenty of categories have extra structure. Sets do not just form a category, but a highly structured kind of one, known as a topos. Toposes are environments for the constructive reasoning. At a less structured level, we can view the category of sets as having two obvious binary operations, disjoint union and cartesian product. If we just take one of these and analyse the way the binary operation coheres with the functions, we extract what is called a monoidal category. These possess a 'multiplication' acting on objects, such as the tensor product of Hilbert spaces or vector spaces. A way to generate a monoidal category is to take any object *A* in a 2-category and consider only the arrows from *A* to *A*, and the 2-arrows between these. Reindex these 1-arrows so that they count as objects, and the 2-arrows as 1-arrows and you have a monoidal category. This process of killing off the lowest level is known as *delooping*. It can be systematised to allow the generation of the following table. Delooping moves you in a south-westerly direction.

Table of k-tuply monoidal n-categories

		n = 0	n = 1	n = 2
	…			
k = 0	sets	categories	2-categories	
k = 1	monoids	monoidal categories	monoidal 2-categories	

k = 2	commutative	braided monoids	braided monoidal categories
k = 3	" "	symmetric	weakly monoidal involutory monoidal 2-categories
k = 4	" "	" "	strongly involutory monoidal 2-categories
k = 5	" "	" "	" "

…

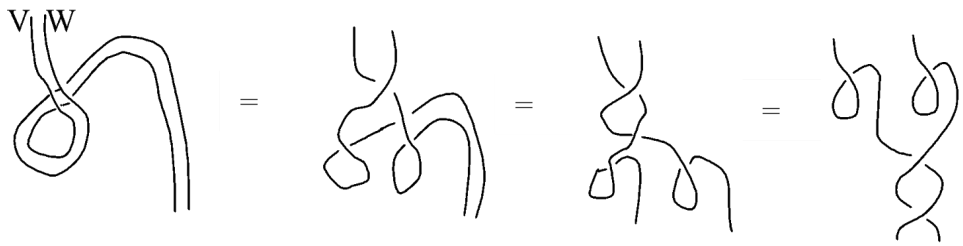
(Baez & Dolan 1999).

This table comes in three flavours. The first is as above. The second requires inverses at each level of arrow. But the most interesting of the three requires a weaker form of inverse known as a dual, a generalised form of the adjoint construction from Hilbert space theory. 2-Cob has these. For instance, the adjoint of a pair of trousers is another pair of trousers oriented in the opposite direction. Gluing these arrows at the trouser cuffs, you see a map from the

waist of one pair to the waist of the other. This is not an identity arrow from the waist to the waist, nor is it equivalent to such an identity.

The mathematics relevant to the (*k*, *n*) position concerns the world of *n*-dimensional things living in an (*n* + *k*)-dimensional world. For example, in position (1, 2) of the table flavoured with duals, we are dealing with lines and circles living in a 3-dimensional world. This is where knots, loops and tangles live. To find ways of distinguishing them, we need something algebraic which belongs to that same position. In the 1980s candidates were found, namely, representations of quantum groups. What is so unusual about this work is that the algebra has to be tailored to the dimension. Where ordinary algebraic topology was happy to use groups to pick up information in all dimensions, so-called *quantum topology* requires specific kinds of algebra for specific dimensions.

The pay-off is two way. The quantum groups help classify knots, while tangles help us calculate with quantum groups. Regarding the latter we are increasingly finding dimensioned notation appearing in textbooks, such as the following 'proof' by Majid (1995: 444-446):



To write out the steps of this calculation in the usual linear form would be horrendous. We can think of the notation used in this process as both an iconic representation of the topological object, the tangle, while at the same time a symbolic representation of the algebra, a mapping between two algebraic entities. This is an excellent example of the phenomenon Emily Grosholz treats in her paper. Higher-dimensional algebra also throws light on the iconic aspects of very simple symbolism, where the letters of algebra and logic are zero-dimensional entities living in some higher-dimensional environment (see pp. 242-251 of Corfield 2003 for further details).

Categorification

Given that mathematicians and physicists want to know about how spheres can tie up in 4-dimensional space, they would be keen to discover a method of moving eastwards on the *n*-categories table. Such an operation is known as *categorification*. Categorified quantum groups would constitute the right algebra for the (2, 2) position. The opposite process, known as *decategorification* is algorithmic. Take your *n*-category, throw away all top level arrows apart from invertible ones, then collapse (*n* – 1)-arrows into equivalence classes. E.g., take the category of finite sets, retain only bijections, then counts as the same any objects (0-arrows) linked by bijections. You should be able to see that what emerges is the set of natural numbers. In the process there is a loss of information. Can we reverse this process? Can any construction be categorified? No, there is no recipe. In fact, there is always more than one answer, although at times a best one is all but forced upon you.

The job of categorification may not be at all easy. At present, people are working hard to categorify the tools of physics: groups, Lie groups, Lie algebras, vector bundles, Hilbert spaces, etc. It is intriguing to find proponents of the two leading theories of quantum gravity, loop quantum gravity and string theory, discussing the role of 2-groups in higher-dimensional gauge theory,[3] but ordinary gauge theory is beyond most people's ken, so let us here run through some simpler examples:

(a) Algebraic topology from mid-19th century to 1930

This is an example of how one may reconstruct episodes from the past. Another categorification of the natural numbers besides the category of finite sets is the category of finite-dimensional vector spaces over a given field. Riemann had characterised the torus as the closed orientable surface which could be made simply connected by cutting it along two lines to produce a

[3] This can be read in the archives of the sci.physics.research newsgroup.

rectangle. But associating '2' to the torus tells us nothing about the maps between two tori. By 1930, mathematicians were now associating the two-dimensional real vector space R^2 to the torus. Corresponding to any map between tori is a certain linear map between vector spaces, i.e., one representable as a 2 by 2 matrix. This association can be used to distinguish between mappings. You can also use these ideas to show that there is no homeomorphism between the torus and the sphere, since there can be no linear mapping between R^2 and R^2 which factors through R^0.

(b) From series to species

This example derives from the work of categorically-minded mathematicians. Consider series in one variable. Let's take those of the form $\Sigma\ a_i\ x^i/i!$, a_i a natural number. To categorify we need to replace the a_i by sets of that cardinality. What you do is define a *species* (first defined by André Joyal) as something which when fed a collection will give you back the set of structures of a certain kind that can put on that collection. For example, the species *singleton* spits out a one-element set if given a collection of one element, but otherwise gives the empty set. It is denoted *X*. The species *set* when given a collection spits out the set of ways of putting a set structure on this collection. There's only ever one way of doing this. All the resulting sets have cardinality 1, so *set* is a categorified version of the series with $a_i = 1$, in other words of the exponential function exp *x*. A species which only does this for even-numbered sets is the categorification of cosh *x*, and for odd-numbered sets sinh *x*. On the other hand, the species of ordered sets or permutations spits out the set of *n*! such structures when fed an *n* element set. This is a categorified version of $1/(1 - x) = 1 + x + x^2 + \ldots$. Finally, the species *cycle* which gives the set of ways of ordering a collection in a circle, where there is no distinguished first element, is a categorification of the series $x + x^2/2 + x^3/3 + \ldots$.

Species compose in different ways:

F + G places an F-structure *or* a G-structure on a collection. E.g., $2X = X + X$

can be thought of as colouring a singleton in two colours, say, either a blue singleton or a red singleton.

$F \cdot G$ places an F-structure on part of the input and a G-structure on the remainder. $\cosh^2 X$, for example, forms ordered pairs of even-numbered sets out of the input.

$F \circ G$ collects the ways of partitioning the input, placing a G-structure on each component of the partition, and placing an F-structure on the collection of components. For example, $\mathrm{Exp}(2X)$ when given a collection looks to form a set of 2-coloured singletons.

With these tools we can now categorify some identities:

$\cosh 2x = \cosh^2 x + \sinh^2 x$ is a decategorification of the fact that the set of ways of 2-colouring an even-numbered collection is naturally isomorphic to the union of the set of ways of partitioning the collection into two even-numbered sets and the set of ways of partitioning the collection into two odd-numbered sets.

The following calculation is a decategorification of the fact that the species of permutations on a collection is isomorphic to the species which gives the ways of writing the elements of the collection in the form of a set of cycles:

$$1 + x + x^2 + \ldots = 1/(1 - x) = \exp(-\ln(1 - x)) = \exp\left(x + x^2/2 + x^3/3 + \ldots\right).$$

We already know that any permutation can be written as a set of disjoint cycles, but the species isomorphism gives us extra information. Just because we have an identity of series does not mean it is a decategorification of an isomorphism of species on collections. For example, $\cosh^2 x = \sinh^2 x + 1$, tells us that for a non-empty collection, there are as many ways of splitting it into two even-numbered sets as there are ways of splitting it into two odd-numbered sets. But we require more structure on our input before we can talk about an isomorphism.

In a paper (Corfield, *forthcoming*) I have written on the subject of 'natural kinds' in mathematics, I have proposed that we draw a distinction between 'law-like' and 'happenstantial'[4] mathematical facts. We could use higher-dimensional algebra to make claims such as:

A happenstantial equation is one which cannot be categorified productively.

The fourth triangular number, 10, is one more than the third square number, 9. Were there a law-like relation occurring here, something like the (n + 1)th triangular number is one greater than the n th square number, you would be very confident that there would be systematic species isomorphisms. On the other hand, one might observe that the number of ways of 2-colouring a 6 element set is 64, while the number of ways of partitioning a 6 element set into two sets each with an even number of elements is 32, and there are also 32 ways to partition it into two sets with an odd number of elements. 64 = 32 + 32. Is this just happenstantial? No, there is a species isomorphism lurking behind the scenes.

A further illustration of this phenomenon of categorification and lawlikeness begins with the observation that the third coefficient of the j-function from nineteenth century work on the moduli space of elliptic curves is one greater than the dimension of the smallest non-trivial irreducible representation of the monster finite simple group is (see pp.125-6 of Corfield 2003). Is this happenstantial? No, a regularity was found linking further coefficients with the dimensions of the irreducible representations, which is now understood to be a decategorification of an isomorphism at the level of modules.

Species are having a large impact on the field of combinatorics. Baez and Dolan have taken the next step by categorifying species to *stuff types*, which they use to analyse the combinatorics of Feynman diagrams.

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[4] My thanks to Jeremy Butterfield for suggesting the term.

(c) Categorifying logic

Looking at the *n*-categories table, you may wish for aesthetic reasons to fill in a missing triangle at the top left hand corner. Let's try to extend the top line to the left for the table taken with inverses. Here, instead of 'category' we should write 'groupoid'. As Emily Grosholz has described in her (1985), there are strong correspondences between logic and topology. Let's use topology to find what should be the entry to the left of 'Set'.

When I talked earlier about the ways of constructing a 3-category out of paths, and paths between paths, on the surface of the Earth, I was producing what is known as the fundamental 3-groupoid of the sphere. Let's go down a couple of dimensions. Take a space *X* and form the groupoid whose objects are the points of *X*, and whose arrows are classes of paths between two points, where any two paths which can be deformed to each other within *X* keeping the end points fixed are taken to belong to the same path. On the Earth, the Greenwich Meridian and the Date Line would be counted as the same path. This *fundamental groupoid* is used to measure whether there are any ways of mapping a circle into *X* which cannot be extended to the mapping of a complete disk. Down a dimension, we look for the set of connected components of *X*. This is counting as the same any two points which form the end points of a line in *X*. In other words, it measures whether mappings of the 0-sphere (2 points) can be filled by the 1-disk (a line). To descend a further dimension we need to complete the last of the following sentences:

$\Pi_1(X)$ is a groupoid, which measures ways that circles S^1 can't be filled by D^2.

$\Pi_0(X)$ is a set, which measures ways that S^0 can't be filled by D^1.

$\Pi_{-1}(X)$ is a ?, which measures ways that S^{-1} can't be filled by D^0.

Well D^0 is just a point, and so its boundary S^{-1} is the empty set. $\Pi_{-1}(X)$ measures whether when the empty set maps into *X* it can be filled by a point. In other words it asks of *X*, are you empty? This is a yes/no question. The '?' must be filled with 'truth value'.

With this result in mind, and the idea that modal logic is a categorification

of predicate logic, I worked out with John Baez one evening in Minnesota how it would have to look. There are more reasons as to why it looks right to categorify the way we did, but I hope the following table is suggestive.

Logic | Extension	Propositional	Predicate (typed)	Modal Predicate
Groupoids			S, T, …
Sets		A, B, …	$A_S(u)$, $B_{T,U}(v,w)$
Truth values	P, Q, …	$P_A(x)$, $Q_{A,B}(x, y)$	$P_{A(u)}(x)$, $Q_{A(u),\, B(v,w)}(x, y)$
Axioms	$P \& Q \rightarrow R$	$\forall x \in A. \exists y \in B. Q(x, y)$	$\forall u \in S.$ $\exists x \in A(u). P(x)$
Models	set (lines of truth table)	groupoid	2-groupoid

The 'Axioms' line gives an example of an axiom of a theory expressible in the logic. The version of predicate logic is most naturally constructed as a typed logic. This accords with the thinking of theoretical computer scientists, who view philosophers as crazy to deal with some kind of universal set, relying on predicates to do the work of types. The highest level of syntax in the modal logic, such as S, refers to an accessibility relation between worlds. Notice there may be more than one of these. Researchers in artificial intelligence use such a logic to describe multi-agent systems. A predicate symbol is typed by an accessibility relation and has to be interpreted as a set at each world covered by the relation. I am not certain if this logic has ever seen the light of day before. It's something like a modal typed predicate S5n, but where the *n* equivalence relations jointly cover the set of worlds but need not do so individually.

This table suggests a highly degenerate logic to the left, with no symbols, and two models – true and false. It also suggests a kind of meta-modal logic to the right. If we worked with categories rather than groupoids, it seems as though we would arrive at a parallel S4 kind of modal logic, but details need to be checked. Another possibility is to work with directed homotopy, rather than ordinary homotopy.[5]

Can it be a coincidence that Charles

[5] For this suggestion and many other relevant ideas see Porter 2003.

Peirce's existential graphs grew in complexity from the alpha part to the beta part to the gamma part, which deal respectively with propositional logic, predicate logic and modal logic. Did he presage the first few steps of a ladder of categorifications of logic?

Conclusion

I certainly do not wish to give the impression that higher-dimensional algebra has a monopoly on the large-scale transformations possible in mathematics. Other candidates include *q-deformation*, finding non-commutative versions (e.g., C*-algebras are non-commutative topological spaces), and the many discussed by Vladimir Arnold in his fascinating 'Polymathematics' paper (Arnold 1999). I do, however, think this is an extremely important one.

Some of the issues it raises do seem to touch on what might be called 'foundational' questions, if the term weren't reserved by others. But perhaps it is time to recover the term. Listen to the highly respected Russian mathematician Yuri Manin:

> I will understand 'foundations' neither as the para-philosophical preoccupation with the nature, accessibility, and reliability of mathematical truth, nor as a set of normative prescriptions like those advocated by finitists or formalists. I will use this word in a loose sense as a general term for the historically variable conglomerate of rules and principles used to organize the already existing and always being created anew body of mathematical knowledge of the relevant epoch. At times, it becomes codified in the form of an authoritative mathematical text as exemplified by Euclid's Elements. In another epoch, it is better expressed by the nervous self-questioning about the meaning of infinitesimals or the precise relationship between real numbers and points of the Euclidean line, or else, the nature of algorithms. In all cases, foundations in this wide sense is something which is relevant to a working mathematician, which refers to some basic principles of his/her trade, but which does not constitute the essence of his/her

work. (Manin 2002: 6)

Manin continues by discussing the foundational role of Cantorian sets, before proceeding to explain how this role was taken up first by categories, and now by higher-dimensional categories.

Philosophers, we know so little about the mathematics of the past seventy years. Higher-dimensional algebra presents you with the opportunity to work your way quite rapidly to be able to deal with extremely important ideas. Even if you choose to look at other features of contemporary mathematics, it provides an excellent platform for your research. We need to understand how the 'vision thing' works in mathematics. By drawing philosophical attention to it, we can prompt existing and future mathematicians to be more open about their visions, or 'bottom lines' to use a term of Gian-Carlo Rota, which can only help them and our successors.

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