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The Form and Function of Duality in Modern Mathematics

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The Form and Function of Duality in Modern Mathematics

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Résumé : Des phénomènes compris sous le terme de dualité se produisent tout au long de l'histoire des mathématiques dans toutes ses branches, de la dualité des polyèdres à la dualité de Langlands. En considérant une « épistémologie interne » de la dualité, nous essayons de comprendre les avantages trouvés par les mathématiciens dans l'exploitation de situations duales. Nous abordons ces questions au moyen d'une compréhension inspirée de la théorie des catégories. Suivant Mac Lane et Lawvere-Rosebrugh, nous distinguons entre dualités « axiomatiques » ou « formelles » (ou de type Gergonne) d'une part et « fonctionnelles » ou « concrètes » (ou de type Poncelet) de l'autre. Alors que les premières sont souvent utilisées dans le cadre d'une stratégie « deux théorèmes par une preuve », les secondes permettent souvent d'étudier des « espaces » par des fonctions définies sur eux, ce qui, dans la terminologie de Grothendieck, revient à prouver un théorème en travaillant dans un cadre dualement équivalent où la preuve correspondante est plus facile à obtenir. Nous essayons de montrer par quelques exemples que dans le premier cas, les objets duaux ont tendance à être plus idéaux (épistémologiquement plus éloignés) que ceux d'origine, tandis qu'il n'en va pas nécessairement de même dans le second cas.

Abstract: Phenomena covered by the term duality occur throughout the history of mathematics in all of its branches, from the duality of polyhedra to Langlands duality. By looking to an "internal epistemology" of duality, we try to understand the gains mathematicians have found in exploiting dual situations. We approach these questions by means of a category theoretic understanding. Following Mac Lane and Lawvere-Rosebrugh, we distinguish

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between "axiomatic" or "formal" (or Gergonne-type) dualities on the one hand and "functional" or "concrete" (or Poncelet-type) dualities on the other. While the former are often used in the pursuit of a "two theorems by one proof"-strategy, the latter often allow the investigation of "spaces" by studying functions defined on them, which in Grothendieck's terms amounts to the strategy of proving a theorem by working in a dually equivalent framework where the corresponding proof is easier to find. We try to show by some examples that in the first case, dual objects tend to be more ideal (epistemologically more remote) than original ones, while this is not necessarily so in the second case.

1 Introduction

Phenomena covered by the term *duality* have long fascinated mathematicians, from the duality of polyhedra and the logical duality captured by de Morgan's Laws to projective duality and the duality of Fourier transforms. This fascination has only increased with the passage of time right up to the current intense investigation of Langlands duality. A broader perspective orients us towards general dualities between algebra and geometry, and between syntax and semantics, and teaches us much about the content of mathematics. Yet, it seems that the role of the concept of duality in modern mathematics has been the subject of very few philosophical studies.

One such study by Ernest Nagel concerns projective duality [Nagel 1939]. In Euclidean geometry, two points determine a line, and two non-parallel lines determine a point. By adding points at infinity as the intersection of two parallel lines, we can omit the word "non-parallel" in the last sentence, and thereby achieve duality of points and lines in plane projective geometry. Nagel claimed that the discovery of this duality freed mathematics from the idea that it was dealing with specific elements bearing a set of defining properties.

> The liberation of geometrical terms from their usual but narrow interpretation first required a thoroughgoing denial of the need for absolute simples as the foundation for a demonstrative geometry. Such a liberation was in large measure the consequence of the discovery of the principle of duality and of the manifold extensions and applications which were made of it. [Nagel 1939, 179]

Nagel points to what we may call the "internal ontology" of mathematics, that is, the content of mathematics as seen by the working mathematician at a moment in history. We can also look to an "internal epistemology" of duality, which tries to understand the gains mathematicians have found in exploiting dual situations. In this direction, a philosophical study, related to projective duality, has been envisaged by Michael Detlefsen in his "Ideals of proof"-project run at Nancy and Paris 2007-2011.

> It has frequently been claimed that the use of ideal elements [...] somehow shortens or simplifies proofs and problem-solutions without compromising their reliability or other epistemic virtues.
>
> Sometimes these efficiencies seem striking, as in the case of the so-called "dualities" that are made possible by the introduction of elements at infinity in projective geometry.[1]

Detlefsen, after describing how by interchanging the terms "point" and "line" one basically gets two theorems for one proof, proposes to submit to a critical scrutiny the conviction that the "reliability or other epistemic virtues" aren't compromised by this procedure. However, this part of the project has not been pursued since. From Detlefsen's approach, we can derive a couple of questions to be asked regarding a much wider field of dualities:

- Do we generally find it possible to exchange parts of a given language with others *salva veritate*? And is it equally the purpose to get two theorems by one proof?
- Are there features analogous to, say, points at infinity in projective geometry?[2]

We shall approach these questions by means of a category theoretic understanding. It will become clear in the next section why we have chosen this strategy. The overarching aims of the present paper then are (1) to make progress on the classification of situations involving dualities; (2) to investigate the internal epistemological and ontological significance of such dualities, notably in comparison to classical dualities such as projective geometry or vector space theory. There is an enormous amount of work to be done here, and in this paper we can only hope to make a start.[3]

2 Two kinds of duality

One key problem to address when we confront duality is that there is no definitive agreement about what the term means. The Princeton Companion of Mathematics tells us that

> Duality is an important general theme which has manifestations in almost every area of mathematics [...]. Despite the importance of duality in mathematics, there is no single definition which covers

1. See https://mdetlefsen.nd.edu/research/ideals-of-proof-ip/.

2. This is not necessarily so since unlike in projective duality, ideal elements are not needed for the logical duality captured by de Morgan's Laws.

3. A more extended but still preliminary version of our investigation is available as a preprint; see [Krömer & Corfield 2013]. We are currently organizing a workshop aiming at producing a collective volume covering large parts of the history and philosophy of duality in mathematics.

> all instances of the phenomenon. [Gowers, Barrow-Green *et al.* 2008, III. 19 Duality, 187]

This claim notwithstanding, over the past few decades attempts have been made (especially in the framework of category theory) to give precise mathematical definitions of the concept of duality in general. The key ingredient of category theoretic dualities very often is the notion of dual category, of course. Historically, this very notion has been motivated by a number of dualities similar to the duality of finite dimensional vector spaces, due to the fact that the constructions involved can be seen as contravariant (arrow-reversing) functors. We will have occasion to label this type of duality as "functional" or "concrete" duality. The notion of dual category then was used by Mac Lane and Buchsbaum in a more "axiomatic" or "formal" way in the pursuit of a "two theorems by one proof"-strategy in, e.g., homological algebra, eventually arriving at dual categories epistemologically more remote (in a sense related to Detlefsen's "ideal") than the original categories; see section 5 below.

While these enterprises don't seem to have led very far, we find that in a further development due to Grothendieck, again dualities of a more "functional" or "concrete" type have been achieved by explicitly defining dual equivalences (i.e., functors); compare section 6. In these cases, ideal elements have to be added in the sense that the given category has to be enlarged in order to become dually equivalent to some other (thus *completing* the duality or "analogy" between two theories); on the other hand, the objects in the dually equivalent category (or rather: the "dualizing object", see below) are more "accessible", or "manageable" than the original objects.[4]

These dualities typically do *not* yield two theorems by one proof; rather, we will be able to relate the epistemic gain of many of them to what we consider as basic methodological principles of modern mathematics, namely studying "spaces" by studying functions defined on them, the counterpart of another principle that one can study algebras by devising a space on which they are algebras of functions.

Above, we relied on two distinctions drawn in the literature between kinds of duality. The distinction *axiomatic vs. functional* duality has been drawn by Saunders Mac Lane [Mac Lane 1950] while *formal vs. concrete* duality has been referred to by Lawvere and Rosebrugh [Lawvere & Rosebrugh 2003]. Mac Lane's distinction actually was of historical significance for the development studied here and thus will be presented in its historical place (see section 4).

Lawvere and Rosebrugh define *formal* duality in terms of the reversal of arrows in a category. So an epimorphism becomes a monomorphism, a product becomes a coproduct, etc. As with Mac Lane's axiomatic duality, proofs may come in dual pairs. On the other hand, *concrete* duality arises when an arrow $f : A \to B$ is "exponentiated" by some object V, to $V^f : V^B \to V^A$.

4. For the sake of avoiding terminological confusion, we refrain from describing them as more "concrete".

V might then be called the "dualizing object". Pontrjagin duality is an example of this, choosing V to be the circle group $\mathbb{R}/\mathbb{Z}$. Exponentiation sends a group to its group of characters. For example, the circle group is sent to the groupof the integers.[5]

3 From the dual vector space to category-theoretic dualities

The category-theoretic conception of duality historically emerged from a well-known "classical" duality, namely the notion of the vector space $L(V)$ dual to a finite-dimensional real vector space V. $L(V)$ is the set of all linear mappings $f : V \to \mathbb{R}$, again a finite-dimensional real vector space. Let $\dim V = n$; to subspaces of dimension r of V correspond subspaces of dimension $n-r$ of $L(V)$ (see [Birkhoff & Mac Lane 1965, 185 f.] for details). There is a simple mathematical connection (which is actually also a historical connection) between this construction and projective duality.[6] What is new here is to consider a dual to the entire space V instead of considering just duals of subspaces. We think that this difference points to an important step in the development of the mathematical concept of space: while in the original situation of projective geometry, there is just "the" space, parts of which can be dual to each other, in vector space theory there are various spaces which can be dual to each other. In the case of finite-dimensional vector spaces, $L(V)$ is just isomorphic to V; the situation gets more interesting when passing to infinite-dimensional vector spaces.[7]

Now, this last example leads us to the category-theoretic conception of duality. (The example actually played a role when category theory was first introduced historically: it is discussed in the introduction of [Eilenberg & Mac Lane 1945], but in a different context.) For let $g : V_1 \to V_2$ be a linear mapping between two such spaces, and let $f_2 \in L(V_2)$. Then by the composition

$$V_1 \xrightarrow{g} V_2 \xrightarrow{f_2} \mathbb{R},$$

an element of $L(V_1)$ is defined; thus we can define a linear mapping $L(g) : L(V_2) \to L(V_1)$ by setting

$$[L(g)](f_2) := g \circ f_2.$$

5. For another philosophical use of this distinction, see [Corfield 2010]. See also the distinction of *Eckmann-Hilton vs. strong* duality contained in [Becker & Gottlieb 1999].

6. In a projective space of finite dimension n, subspaces of dimension r are dual to (interchange with) subspaces of dimension $n - r - 1$. Thus, in projective 3-space, points are dual to planes and lines are dual to lines.

7. Even in this case, evidence for the historical continuity with projective geometry can be found. For instance, Hans Hahn in his proof of the Hahn-Banach theorem calls the dual of a real vector space its "polaren Raum" [Hahn 1927, 219].

This defines a functor L from the category of finite-dimensional real vector spaces to itself. This functor is *contravariant* (the direction of the arrow is reversed).

This reversion of the direction of the arrows occurs quite often: in the passage from an abelian topological group to its group of characters with values in some specific group (Pontrjagin duality), or in the relation between homology and cohomology groups, or between direct and inverse limits, to name just a few examples. Early category theory was devised exactly for dealing with such constructions. Thus, they have motivated, historically, the working out of the very notion of category-theoretic duality.

This notion basically is the following: statements of category theory typically concern the composition of arrows (which might be thought of as functions); in the statement dual to a given statement, the arrows are reversed. Now, there are other occurrences of this which were only arrived at through a consequent application of the dualization strategy to the original situation, namely projective and injective objects in abelian categories, a category and its dual category, or some technically even more involved constructions from Grothendieck's mathematics.

Thus, we find that many concepts of modern mathematics fall under this notion of duality. However, our enumeration of examples actually includes very different types of situations. The first three of them are of the "spaces-functions" type (see below), and are concrete dualities in the sense of Lawvere-Rosebrugh, while the next three are formal dualities. In the following sections, we shall follow up the historical development of category-theoretic dualities, and the epistemological properties of the conceptions at the various stages of development.

4 Mac Lane and "functional" vs. "axiomatic" duality

In [Mac Lane 1950], Saunders Mac Lane makes an attempt inspired by category theory to cope with certain incomplete dualities in group theory.[8] He focuses on group-theoretical notions which can be expressed in terms of arrow composition.[9] To make clear what his aim is, he is led to distinguish between two types of duality:

> [...] in the case of vector spaces [...] there is a process assigning to each object a dual object and to each transformation a dual

8. See [Krömer 2007, section 2.4.3]. for a detailed historical account.

9. Mac Lane is aware that the "formulation of duality in terms of homomorphisms does not suffice to subsume all known 'duality' phenomena" [Mac Lane 1950, 494]; he refers to [Hall 1940] for phenomena not subsumed.

> transformation, so that a "functional" duality is present. Similarly, the duality of (plane) projective geometry may be formulated in two ways: *functional*, by assigning to each figure its polar reciprocal with respect to a fixed conic; *axiomatic*, by observing that the axioms for plane projective geometry are invariant under the interchange of "point" with "line".
>
> Even for discrete abelian groups or for discrete (infinite-dimensional) vector spaces, a functional duality does not exist. We aim to provide an axiomatic duality covering such cases. [Mac Lane 1950, 494f]

Let us stress that Mac Lane interprets the role category theory can play in the context of group theory as analogous to the axiomatic way of speaking about projective duality, rather than the functional one.[10] He is quite closely sticking to the idea of replacing terms by others in expressions (namely reverse arrows or rather, reverse the order of the factors in products, interchange the terms monomorphism and epimorphism and so on). Historically speaking, this path to axiomatic duality was prepared by the fact that functional duality (where available as with finite-dimensional vector spaces or locally compact Hausdorff abelian groups) happens to come with a contravariance.

Mac Lane isolated this feature to make it the basic ingredient of his axiomatic approach in cases where a functional duality is lacking. For example, in both the category of groups and the category of abelian groups, many constructions and results may be dualized. However, these categories are not self-dual. Mac Lane's axiomatization of the duality present in the category of abelian groups was later modified by Buchsbaum and Grothendieck, yielding the self-dual notion of abelian category. Now the dual of any result which may be proved from the axioms for an abelian category also holds in such a category.

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10. To distinguish these two approaches to projective duality constitutes a kind of standard history of projective geometry.

> The principle of duality [in the sense of syntactically interchanging terms in propositions] may properly be ascribed to Gergonne [...]. Poncelet protested that it was nothing but his method of reciprocation with respect to a conic (polarity), and Gergonne replied that the conic is irrelevant—duality is intrinsic in the system. Thus Gergonne came nearer to realizing how the principle rests on the symmetrical nature of the axioms of incidence" [Coxeter 1961, 15].

With his last remark, Coxeter is certainly suggestive of the considerable influence Gergonne's approach had on Hilbert's axiomatic geometry. For different interpretations of this history, see [Bioesmat-Martagon 2010].

5 Buchsbaum, Grothendieck, and duality in homological algebra

In categories with algebraic objects, one often studies whether a given functor preserves exact sequences or not. Homological algebra answers this question by considering an exact sequence as a complex and calculating its cohomology. This yields the "derived functors" of the given functor, an approach developed by Cartan and Eilenberg in [Cartan & Eilenberg 1956] (written in 1953) for categories of modules.

Much like Eilenberg and Steenrod in their 1952 axiomatization of homology and cohomology theories [Eilenberg & Steenrod 1952] faced the repetition of dual argumentations (a situation Mac Lane in his 1950-paper tried to cope with), Cartan and Eilenberg, while being perfectly aware of a latent "symmetry" (related to reversion of arrows, p. 53), they couldn't help treating separately right and left derived functors respectively and even distinguishing the different possible variances of the functors. David Buchsbaum, in an appendix written in 1955, developed the functor derivation procedure for general exact[11] categories, and eventually resolved the problem of avoiding dual argumentations by using the concept of dual category; see [Krömer 2007, section 3.1.2.2] for a detailed description of the results obtained by him.

What can be said, from what we have seen so far, as to the epistemological comparison of category-theoretic and classical dualities? Reversion of arrows can be seen as a purely formal exchange of some parts of the language, like in classical dualities. But reversion of arrows applied to a true theorem does not yield systematically a true theorem. Counterexamples occur as with injective and projective objects;[12] in general, this occurs when the categorical environment is not self-dual.[13] One has to distinguish between the dualization procedure for obtaining the dual statement and duality principles which assert the truth of the dual of a true statement. (The duality principle established by Buchsbaum reads: with the category $\mathcal{A}$, also its dual category $\mathcal{A}^*$ is exact.)

For our epistemological purpose, it is worthwile to consider the explanation Buchsbaum gives for the fact that his duality theory was outside the scope of Cartan-Eilenberg (where only categories of modules are considered):[14]

11. His notion of "exact category" is very close to the now standard notion of abelian category.

12. And with direct and inverse limits as well; we will however not discuss this case here.

13. In the case of injective and projective modules, the internal minutes of the Bourbaki meetings relate this to the fact that the category of sets is not self-dual. See *La Tribu 56* concerning the Bourbaki *Rédaction* n° 373. These documents are not among those available online; they can be seen in the "Archives Delsarte" at the Institut Elie Cartan, Université de Lorraine, Nancy.

14. $H(A, B)$ denotes the homology functor construed in their manner.

> In [the] category [of all left Λ-modules $\mathcal{M}_\Lambda$], $H(A,B) = \mathrm{Hom}_\Lambda(A,B)$. However, the dual category $\mathcal{M}_\Lambda^*$ admits no such concrete interpretation. This explains the fact that the duality principle could not be efficiently used, as long as we were restricted to categories concretely defined, in which the objects were sets and the maps were maps of those sets. [Cartan & Eilenberg 1956, 382]

To put it in more general terms: when starting from a category C composed of (structured) sets and functions (or, more technically, from a "concrete" category, that is, a category C with an underlying functor $U : C \to Set$),[15] its dual category C^{op} obtained by arrow reversion is often more "ideal" than C itself in that the arrows so obtained need not be set functions (C^{op} need not be concrete). Therefore, Buchsbaum considered the step to pass to axiomatically given categories (not necessarily concrete in this sense) as the crucial step for making use of a duality principle.

We should compare this "ideality" and the ideal elements of projective geometry carefully. In the case stressed by Buchsbaum, the dual objects as a whole are more "ideal" than the original objects. In projective geometry, on the other hand, it is not just the dual objects in general but only the ideal ones (the objects dual to parallel lines) which are less accessible. When applied as we suggest to do in Buchsbaum's case, the usage of the term "ideal" seems to be not identical to its usage in classical ontological doctrines like realism, etc. Rather, it concerns whether something is representable as set and structure.

We should add, moreover, that there are many situations where there are dual concrete categories. This occurs when their underlying functors, which throw away the extra structure, are representable, that is, are of the form $U(-) = C(c,-)$, for some object c, see [Porst & Tholen 1991]. Here the object c is a free object on one free generator. The duality between the category of finite dimensional vector spaces and its opposite is of this form, the base field playing the role of c.

The history of homological algebra didn't stop with Buchsbaum's achievements; actually, Grothendieck around 1955 became interested in applying the Cartan-Eilenberg derivation procedure to functors defined on categories of sheaves, eventually showing the limitations of Buchsbaum's "two theorems for one proof"-strategy [Grothendieck 1957] (see [Krömer 2007, section 3.3.3.3] for details). In later work by Grothendieck, category-theoretic dualities rather generalize the situation in vector space theory (study an object by studying its dual). There is still an introduction of "ideal elements" in these cases, but they aren't introduced any longer to obtain a *salva veritate*-duality. So what is the epistemic gain one has in mind instead?

15. Note that this technical usage of the word "concrete" quite closely corresponds to what Buchsbaum in a still non-technical manner called "categories concretely defined". It is also related to the Lawvere-Rosebrugh conception of "concrete dualities" in that the category in which the objects of the form V^A live often is concrete. We will elaborate on this point further on.

6 Grothendieck, spaces and functions, and the epistemic gain

We take the following as a basic methodological principle of modern mathematics: In order to learn something about an object which could be called a space, one studies the functions defined on that space and having values in a similar but "simpler" space (concrete duality in the sense of Lawvere-Rosebrugh). The elements of dual vector spaces and character groups are representations of certain other spaces or groups with values in some particularly simple space or group. This is similar with cohomology groups at the level of chain complexes being dualised to cochain complexes.

Similar strategies are central to functional analysis. For example, consider the Banach-Alaoglu theorem and Gelfand's representation theorem for Banach algebras. Gelfand substituted an algebra of functions on a space for an arbitrary Banach algebra A by defining a mapping $A \to C(X)$, where X is $\mathrm{Spec}(A)$, the compact Hausdorff space of all multiplicative linear forms of A (which can also be interpreted as the set of maximal ideals of A equipped with a certain topology).[16] Actually, you have to choose the weak* topology to get the very important property of compactness—and this topology is related from the outset to the concept of dual vector space. Thus, not only is the space X used in the theorem a subset of the space dual to A as a vector space, but it comes equipped with a topology closely related to this dual space.

We find that there are at least three levels on which a given algebra is made more accessible by representing it as a $C(X)$, and that on at least two of these levels, vector space duality plays a central role:

- The first level is that an "arbitrary" algebra is replaced by a space of functions (we know now what the elements of the algebra are).
- The second level is that we study a complicated object (an element of the algebra) by studying its values under linear forms, and these values are simple objects (elements of the base field);
- But the usefulness of the second idea depends largely on the properties of X (its compactness) furnished by the consideration of topologies related to the dual space.

The study of elements of Banach algebras by studying their values under linear forms actually incorporates a very subtle "duality" (in the sense of an exchange

16. The mapping $A \to C(X)$ is actually defined as the composite mapping $A \to \mathrm{Spec}(A) = X \to C(X)$; this is an isomorphism which sends $x \in A$ to a map from $X = \mathrm{Spec}(A)$ to $\mathbb{C}$, which is evaluating $f \in \mathrm{Spec}(A)$ at x. $\mathrm{Spec}(A)$ is the space of characters of A, i.e., the set of its continuous characters, that is continuous nonzero linear homomorphisms into the field of complex numbers, and canonically equipped with a so-called spectral topology. Elements of $\mathrm{Spec}(A)$ are a kind of function on A, so can be evaluated against a member of A.

of parts of the language), namely the idea to change the roles in the expression $f(x)$, i.e., keep x fixed and vary f instead.[17] This observation allows us to elaborate a little on the matter of "simplicity". There would be no point in saying that in $C(X)$, the X is simpler than A, given that it is a space of certain linear forms on A and that $\mathbb{C}$ (the dualizing object, the object in which the functions of $C(X)$ take values) is simpler. The idea rather is: replace one complicated object (an element x of A) by *many* simple objects (the values of x under all these linear forms); *this* is the motivation to keep x fixed and vary f instead. Thus, dual objects should not be thought of as being more accessible than original objects (this being true only of the dualizing object); rather, they *make* accessible the original objects—and *this* is the epistemic gain, of course.

Our history of how duality of space and function allowed for mathematical progress entered a second, overtly category-theoretic stage, when Grothendieck (having begun his mathematical career in the field of functional analysis) started to adapt Gelfand's strategy for use in algebraic geometry, see [Cartier 2001, 397]. Grothendieck's strategy very closely parallels Gelfand's: he substituted functions for algebraic objects (the elements of an arbitrary commutative ring A) by mapping $A \to \Gamma(X, \mathcal{O})$ where $X = \mathrm{Spec}(A)$, the set of maximal ideals of A equipped with the Zariski topology, $\mathcal{O}$ a sheaf defined on that space and Γ the section functor of that sheaf, yielding a set of functions as values.

We think that this line of development (transporting ideas from functional analysis to algebraic geometry by stressing the category-theoretic aspect) played the major role in the development of duality as a central theme in structural mathematics while the Mac Lane-Buchsbaum-Tohoku line of development (the axiomatic approach to duality) was far less important. While Mac Lane and Buchsbaum pursued a two theorems by one proof-strategy, Grothendieck pursued the strategy to prove a theorem by working in a dually equivalent framework where the corresponding proof is easier to get.

To understand this point, note that in all cases discussed so far of passage from an original space (considered as an object of some category) to a space of functions, arrows are reversed. The new objects are dual in the sense that they are objects of a dually equivalent category. Here, the epistemic gain seems to occur on the first of the three levels discussed above: instead of studying "remote", "abstract" objects (like arbitrary Banach algebras or arbitrary commutative rings), we have the result that these categories are dually equivalent to categories of function spaces of certain types; thus we can study these more "accessible" objects instead. But a more detailed study of what Grothendieck actually did might show that there are other levels in his case as well.

If Grothendieck's approach to mathematics can be characterized by an overall strategy or method, such a characterization certainly would involve

17. See [Birkhoff & Mac Lane 1965, 185], for instance. An application of this fundamental idea in the field of Hopf Algebras used as physical models is described in [Corfield 2003, 24].

the theme of analogy between different fields of mathematics. It is clear, for instance, that Grothendieck's algebraic geometry heavily relies on Dedekind's idea of an analogy between number and function, see [Corfield 2003, section 4.3]. In fact, we can find repeatedly that Grothendieck aimed at making analogies complete in the sense of working with a pair of dually equivalent categories; therefore he enlarged one of the two categories involved. Examples are the analogy between algebraic geometry and commutative algebra, made complete by the passage from varieties to affine schemes, or the analogy between Galois theory and the theory of coverings, made complete by the introduction of Grothendieck topologies (see [Gelfand & Manin 1996, 76] for more examples). Then, the passages from one category to the dually equivalent one often are passages from something more "remote" to something more "accessible".

And the enlarging of the category studied originally in order to obtain a dual equivalence with some "tame" category is the introduction of ideal elements in this case. Rather as the addition of ideal elements in the projective case led to a geometry with pleasanter features, namely, self-duality, many constructions of Grothendieck were motivated by the idea that rather than work in a category of nice objects, which often itself doesn't possess nice qualities, it is better to "complete" into a nice category. For example, we embed a category into the category of presheaves on it, which is the free cocompletion. This is an extension of Cayley's theorem, embedding a group G in the category of G-sets, as the group acting on its underlying set. We can recover G from this category of G-sets, and this is part of a very large story of Tannaka duality, whereby one recovers an algebraic entity from a category of "geometric" representations. This extends even to a duality between theories and their categories of models, see [Awodey & Forssell 2013].

Let us relate the rather vague notions of remoteness, accessibility and niceness ascribed to Grothendieck's strategy in what precedes to the category-theoretic notion of concreteness discussed in the preceding section. The following quote illustrates the idea that there's something more manageable on the concrete side of a duality.[18]

> [...] for a given category A, the existence of a duality with some concrete category B might give considerable additional information about A: if e.g., B has limits—often quite obvious constructions in concrete categories—the category A automatically will have colimits which, moreover, can be described explicitly (for A algebraic usually a difficult task) as S-images of limits in B. [Porst & Tholen 1991, 111–112]

18. The S is just one of the adjoints involved in the dual equivalence.

7 Conclusions

In the settings of Mac Lane and Buchsbaum (sections 4 and 5), category theoretic duality was used for enhancing (doubling) the set of proved theorems by linguistic exchange. In these cases, dual objects tended to be more ideal (epistemologically more remote) than original ones. On the other hand, section 6 suggests that when category theoretic duality is employed in the "space-functions" way, the key issue seems to be that the dualizing object is less ideal than the original objects.

As we mentioned, this paper marks only the first steps towards a treatment of the internal epistemological and ontological features of duality in mathematics. First of all, the paper is meant to be largely descriptive; we didn't intend to criticise in Mik Detlefsen's sense the epistemic status of the knowledge gained by the uses of duality described. And the descriptive work is not finished; for instance, a finer analysis in our opinion should focus on the fact that different identification criteria are used in each case.

Moreover, further work should look beyond the issue of having one side more concrete than the other. For example, duality may relate different structures of the same domain to each other, some of which are easier to work with. For example, in the case of Fourier analysis, the convolution of two functions is transformed into a multiplication. A more modern and involved example occurs with mirror symmetry for Calabi-Yau manifolds, where the Kähler and complex structures are exchanged as one passes between mirror manifolds. It turns out that to perform calculations on one of these structures for a particular manifold, it may be easier to work on the other structure on the mirror.

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