The duality of space and function, and category-theoretic dualities

Ralf Krömer & David Corfield

1 Introduction

Phenomena covered by the term *duality* have long fascinated mathematicians, from the duality of polyhedra, the symmetric duality of Pascal's triangle, and the logical duality captured by de Morgan's Laws to projective duality and the duality of Fourier transforms. This fascination has only increased with the passage of time right up to the current intense investigation of Langlands duality. A broader perspective orients us towards general dualities between algebra and geometry, and between syntax and semantics, and teaches us much about the content of mathematics. Yet, it seem that the role of the concept of duality in modern mathematics has been the subject of very few philosophical studies.

Ernest Nagel in [Nag39] claimed that the discovery of duality in projective geometry liberated mathematics from the idea that it was dealing with specific elements bearing a set of defining properties.

> It is a fair if somewhat crude summary of the history of geometry since 1800 to say that it has led from the view that geometry is the apodeictic science of space to the conception that geometry, in so far as it is part of natural science, is a system of "conventions" or "definitions" for ordering and measuring bodies. (p. 143)
>
> The liberation of geometrical terms from their usual but narrow interpretation first required a thoroughgoing denial of the need for absolute simples as the foundation for a demonstrative geometry. Such a liberation was in large measure the consequence of the discovery of the principle of duality and of the manifold extensions and applications which were made of it. (p. 179)

This interesting paper points to what we may call the 'internal ontology' of mathematics, that is, the content of mathematics as seen by the working mathematician at a moment in history. We can also look to an 'internal epistemology' of duality, which tries to understand the gains mathematicians have found in exploiting dual situations. In this direction, a philosophical study, related to projective duality, has been envisaged by Michael Detlefsen in his "Ideals of proof"-project run at Nancy and Paris 2007-2011.

> It has frequently been claimed that the use of ideal elements [...] somehow shortens or simplifies proofs and problem-solutions without compromising their reliability or other epistemic virtues.
>
> Sometimes these efficiencies seem striking, as in the case of the so-called "dualities" that are made possible by the introduction of elements at infinity in projective geometry.[1]

Detlefsen, after describing how by interchanging the terms 'point' and 'line' one basically gets two theorems for one proof, proposes to submit to a critical scrutiny the conviction that the "reliability or other epistemic virtues" aren't compromised by this procedure. However, as Detlefsen told one of us recently, this part of the project has not been pursued since.

We will not do this either at present. Rather, we will derive from Detlefsen's approach a couple of questions to be asked regarding a much wider field of dualities:

- Do we generally find it possible to exchange parts of a given language with others *salva veritate*? And is it equally[2] the purpose to get two theorems by one proof?
- Are there features analogous to, say, points at infinity in projective geometry?

We shall approach these questions by means of a category theoretic understanding. It will become clear in the next section why we have chosen this strategy. The overarching aims of the present paper then are (1) to make progress on the classification of situations involving dualities; (2) to investigate the internal epistemological and ontological significance of such dualities, notably in comparison to classical dualities such as projective geometry, boolean algebra or vector space theory. There is an enormous amount of work to be done here, and in this paper we can only hope to make a start. We will find that the answers to our questions largely depend on the actual historical state of development of category theory

[1] See http://www.univ-nancy2.fr/poincare/idealsofproof/subproj_Ib.html

[2] It is certainly up to further historical study whether this actually was the purpose when projective duality was first introduced.

we're talking about. We will not elaborate on the historical details here, but rather stress the respective epistemological situation—however in the sense of internal epistemology alluded to above, thus perfectly historically: we retrace the history of this epistemology.[3]

2 Two Kinds of Duality

One key problem to address when we confront duality is that there is no definitive agreement about what the term means. The Princeton Companion of Mathematics tells us that "Duality is an important general theme which has manifestations in almost every area of mathematics [...] Despite the importance of duality in mathematics, there is no single definition which covers all instances of the phenomenon." ([GBGL08], III. 19 Duality, p. 187). This claim notwithstanding, over the past few decades attempts have been made (especially in the framework of category theory) to give precise mathematical definitions of the concept of duality in general. The key ingredient of category theoretic dualities very often is the notion of dual category, of course. Historically, this very notion has been motivated by a number of dualities similar to the duality of finite dimensional vector spaces, due to the fact that the constructions involved can be seen as contravariant (arrow-reversing) functors. We will have occasion to label this type of duality as "functional" or "concrete" duality. The notion of dual category then was used by Mac Lane and Buchsbaum in a more "axiomatic" or "formal" way in the pursuit of a "two theorems by one proof"-strategy in, *e.g.*, homological algebra, eventually arriving at dual categories epistemologically more remote (in a sense related to Detlefsen's "ideal") than the original categories; see section below.

While these enterprises don't seem to have led very far, we find that in a further development due to Grothendieck, again dualities of a more "functional" or "concrete" type have been achieved by explicitly defining dual equivalences (i.e., functors); compare section . In these cases, ideal elements have to be added in the sense that the given category has to be enlarged in order to become dually equivalent to some other (thus *completing* the duality or "analogy" between two theories); on the other hand, the objects in the dually equivalent category (or rather: the "dualizing object", see below) are more "accessible", or "manageable" than

[3]Several of the more historical passages of the present paper have already been included in [Krö07], but scattered around in various investigations without stress of the common theme of duality. So we find it convenient to reissue them here under a common heading.

the original objects.[4]

These dualities typically do *not* yield two theorems by one proof; rather, we will be able to relate the epistemic gain of many of them to a basic methodological principle of modern mathematics, namely studying "spaces" by studying functions defined on them, the counterpart of another principle that one can study algebras by devising a space on which they are algebras of functions. One important difference between the two last-mentioned historical situations is that in the Grothendieck situation, equivalence of categories (and not something like having identical object classes) is made the relevant criterion of identification.

In what precedes, we relied on two distinctions drawn in the literature between kinds of duality. The distinction between *axiomatic vs. functional* duality has been drawn by Saunders Mac Lane [ML50] while *formal vs. concrete* duality has been referred to by Lawvere and Rosebrugh [LR03]. During the pursuit of our aim to classify the situations, we will have many occasions to use these two distinctions when discussing particular cases of duality. Mac Lane's distinction actually was of historical significance for the development studied here and thus will be presented in its historical place (see section); let us thus just say some words about formal vs. concrete duality here.[5]

Lawvere and Rosebrugh define *formal* duality in terms of the reversal of arrows in a category. So an epimorphism becomes a monomorphism, a product becomes a coproduct, etc. As with Mac Lane's axiomatic duality, proofs may come in dual pairs. On the other hand, *concrete* duality arises when an arrow $f : A \to B$ is 'exponentiated' by some object V, to $V^f : V^B \to V^A$. V might then be called the "dualizing object". Pontrjagin duality is an example of this, choosing V to be the circle group $\mathbb{R}/\mathbb{Z}$. Exponentiation sends a group to its group of characters. For example, the circle group is sent to the group of the integers.[6]

[4] For the sake of avoiding terminological confusion, we refrain from describing them as more "concrete". The distinction we wish to make is an epistemological one and concerns accessibility, in contrast to "ideality" in Detlefsen's usage.

[5] For another philosophical use of this distinction, see [Cor10].

[6] There are more such distinctions in the literature. Becker and Gottlieb, for instance, restrict their attention to duality in algebraic topology to *Eckmann-Hilton vs. strong* duality [BG99]. *Eckmann-Hilton* duality is named after two mathematicians who noticed that for many constructions in some category of topological spaces it is possible to form their duals, for instance, the loop space and the suspension of a space. But this is merely a heuristic principle, suggestive of new constructions and results. In the same way as the category of abelian groups, sometimes Eckmann-Hilton duality fails in the sense that the dual of a true theorem is not itself a true theorem. The axiomatisation of the self-dual properties in the form of model categories, resembles that of abelian categories. On the other hand, the *strong* dual of an object in a symmetric monoidal category is defined via the possession of certain properties. In the category of vector spaces, a finite vector space has a strong dual. This is actually an instance of Lawvere and Roseburgh's concrete duality with $\mathbb{R}$ as the dualizing object.

3 From "classical" to category-theoretic dualities

In what follows, we focus on both historical and epistemological relations between the category-theoretic conception of duality and "classical" dualities like in projective geometry, boolean algebra, and the vector space dual to a finite-dimensional real vector space. Therefore we first briefly recall these classical cases. The first two provide examples of "Two theorems by one proof"-strategies (with the important difference that in the second case, no ideal elements are needed, as far as we can see). The third case, though being closely related to the first, is of a different kind in that it concerns a study of objects by studying their duals, or, more precisely, the study of a space by studying its functions into a "simpler" space.

Duality in Projective Geometry In Euclidean geometry, two points determine a line, and two non-parallel lines determine a point. By adding points at infinity as the intersection of two parallel lines, we can omit the word "non-parallel" in the last sentence, and thus get duality of points and lines in plane projective geometry. More generally, in a projective space of finite dimension n, subspaces of dimension r are dual to (interchange with) subspaces of dimension $n-r-1$. Thus, in projective 3-space, points are dual to planes and lines are dual to lines.

Duality of projective geometry historically has been expressed in two different ways by Poncelet on the one hand and Gergonne on the other.

> The principle of duality [in the sense of syntactically interchanging terms in propositions] may properly be ascribed to Gergonne [...]. Poncelet protested that it was nothing but his method of reciprocation with respect to a conic (polarity), and Gergonne replied that the conic is irrelevant—duality is intrinsic in the system. Thus Gergonne came nearer to realizing how the principle rests on the symmetrical nature of the axioms of incidence" [Cox61, p.15].

With his last remark, Coxeter is certainly suggestive of the considerable influence Gergonne's approach had on Hilbert's axiomatic geometry. We should however be more careful about the historical facts. In the preface of [BM10], we can read

Another distinction is contained in the following quote (where the word "concrete" is used in a nontechnical sense, of course):

> The formal inversion of arrows furnished in the definition of dual category in concrete examples often produce [sic!] *duality theorems* (when categories coincide or are close to each other), or some relations of the type *geometry vs algebra* (when categories are not alike, e.g., rings and their spectra). [GM96, 76]

> Gergonne défend une version de la dualité assez proche de celle expliquée dans les exposés axiomatisés des manuels actuels. [...] Néanmoins, sa tentative n'est pas totalement achevée puisqu'il n'osera pas affirmer que deux droites coplanaires se coupent toujours [p.5f]

Thus, Gergonne didn't go as far as we go today.[7]

Duality in Boolean Algebra In Boolean Algebra, we have a duality for logical connectives or, alternatively, basic operations on subsets of a given set. In the case of connectives, negation reverses the direction of implication, and, by de Morgan's law, replaces "and" by "or". Similarly for sets, complementation reverses the direction of inclusion, and replaces intersection by union. Consequently, a theorem of boolean algebra stays valid if the signs are interchanged correspondingly. A related example is the duality of open and closed sets in a topological space.

As with the case of projective duality, we have a "Two theorems by one proof"-situation here: we can systematically interchange some terms of the language *salva veritate*. The important epistemological difference between the two cases is that in the second case, no ideal elements are needed.

The dual vector space Let V be a finite-dimensional real vector space. Its dual space $L(V)$ is the set of all linear mappings $f : V \to \mathbb{R}$, again a finite-dimensional real vector space. There is a simple mathematical connection (which is actually also a historical connection) between this construction and projective duality. Let $\dim V = n$; to subspaces of dimension r of V correspond subspaces of dimension $n - r$ of $L(V)$ (see [BML65] p.185f for details). What is new here with respect to dualities yielding two theorems by one proof is to consider a dual to the entire space V instead of considering just duals of subspaces. We think that this difference points to an important step in the development of the mathematical concept of space: while in the original situation of projective geometry, there is just "the" space, parts of which can be dual to each other, in vector space theory there are various spaces which can be dual to each other. In the case of finite-dimensional vector spaces, $L(V)$ is just isomorphic to V; the situation gets more interesting when passing to infinite-dimensional vector spaces.[8]

[7] In [BM10], the reader finds many informations on the history of projective geometry and pointers to additional literature on the subject.

[8] Even in this case, evidence for the historical continuity with projective geometry can be found. For instance, Hans Hahn in his proof of the Hahn-Banach theorem calls the dual of a real vector space its "polaren Raum" [Hah27] p.219.

Now, this last example leads us to the category-theoretic conception of duality. (The example actually played a role when category theory was first introduced historically: it is discussed in the introduction of [EML45], but in a different context.) For let $g : V_1 \to V_2$ be a linear mapping between two such spaces, and let $f_2 \in L(V_2)$. Then by the composition

$$V_1 \stackrel{g}{\to} V_2 \stackrel{f_2}{\to} \mathbb{R},$$

an element of $L(V_1)$ is defined; thus we can define a linear mapping $L(g) : L(V_2) \to L(V_1)$ by setting

$$[L(g)](f_2) := g \circ f_2.$$

This defines a functor L from the category of finite-dimensional real vector spaces to itself. This functor is *contravariant* (the direction of the arrow is reversed).

Reversion of arrows The reversion of the direction of the arrows, as present in the last example, occurs quite often: along with the passage to the dual of a vector space, it takes place in the passage from an abelian topological group to its group of characters with values in some specific group (Pontrjagin duality), or in the relation between homology and cohomology groups, or between direct and inverse limits. Early category theory was devised exactly for dealing with such constructions. Thus, they have motivated, historically, the working out of the very notion of category-theoretic duality.

This notion basically is the following: Statements of Category theory typically concern the composition of arrows (which might be thought of as functions); in the statement dual to a given statement, the arrows are reversed. Now, there are other occurences of this which were only arrived at through a consequent application of the dualization strategy to the original situation, namely projective and injective objects in abelian categories, a category and its dual category, or some technically even more involved constructions from Grothendieck's mathematics.

Thus, we find that many concepts of modern mathematics fall under this notion of duality. However, our enumeration of examples actually includes very different types of situations. The first three of them are of the "spaces-functions" type (see below), and are concrete dualities in the sense of Lawvere-Rosebrugh, while the next three are formal dualities. In the following sections, we shall follow up the historical development of category-theoretic dualities, and the epistemological properties of the conceptions at the various stages of development.

4 Mac Lane and "functional" vs. "axiomatic" duality

In [ML50], Saunders Mac Lane makes an attempt inspired by category theory to cope with certain incomplete dualities in group theory.[9] He focuses on group-theoretical notions which can be expressed in terms of arrow composition.[10] To make clear what his aim is, he is led to distinguish between two types of duality:

> For a topological space the duality between homology and cohomology groups with locally compact abelian coefficient groups can be formulated in terms of character groups. Another formulation is suggested by the axiomatic homology theory of Eilenberg and Steenrod. In this formulation, the axioms for a homology theory refer [...] only to certain homomorphisms; the dual statements are exactly the axioms for a cohomology theory. [...]
>
> Duality phenomena also appear in the case of vector spaces. [...] In these instances there is a process assigning to each object a dual object and to each transformation a dual transformation, so that a "functional" duality is present. Similarly, the duality of (plane) projective geometry may be formulated in two ways: *functional*, by assigning to each figure its polar reciprocal with respect to a fixed conic; *axiomatic*, by observing that the axioms for plane projective geometry are invariant under the interchange of "point" with "line".
>
> Even for discrete abelian groups or for discrete (infinite-dimensional) vector spaces, a functional duality does not exist. We aim to provide an axiomatic duality covering such cases. [ML50, 494f]

It seems that the two approaches to projective duality available inspired Mac Lane's efforts to get axiomatic duality in certain categories. Thus, projective duality for us is not only a situation to be compared epistemologically with the situation(s) in category theory but moreover turns out to be historically a source of inspiration for some of the latter.

While for locally compact Hausdorff abelian groups, there is also such a functional duality by Pontrjagin's procedure, a functional duality is lacking in other cases. For example, in both the category of groups and the category of abelian groups,

[9] See [Krö07] section 2.4.3. for a detailed historical account.

[10] Mac Lane is aware that the "formulation of duality in terms of homomorphisms does not suffice to subsume all known 'duality' phenomena" [p.494]; he refers to [Hal40] for phenomena not subsumed.

many constructions and results may be dualized. However, these categories are not self-dual. Mac Lane's axiomatization of the duality present in the category of abelian groups later was modified by Buchsbaum and Grothendieck, yielding the self-dual notion of abelian category. Now the dual of any result which may be proved from the axioms for an abelian category also holds in such a category.

Let us stress that Mac Lane interprets the role Category theory can play in the context of group theory as analogous to the axiomatic way of speaking about projective duality, rather than the functional one (also present in the vector space case). He is quite closely sticking to the idea of replacing terms by others in expressions (namely reverse arrows or rather, reverse the order of the factors in products, interchange the terms monomorphism and epimorphism and so on).

Historically speaking, this axiomatic duality was prepared by the fact that functional duality (where available as with finite-dimensional vector spaces or locally compact Hausdorff abelian groups) happens to come with a contravariance, as we have seen towards the end of the preceding section. Mac Lane isolated this feature to make it the basic ingredient of his axiomatic approach.

Later distinctions of types of dualities are related to Mac Lane's. For instance, exponentiation as occuring in the Lawvere-Rosebrugh distinction is a mapping and so falls under Mac Lane's functional duality.

5 Buchsbaum, Grothendieck, and duality in homological algebra

In categories with algebraic objects, one often has a notion of exact sequence. The basic question is whether a given functor preserves such exactness or not. Homological algebra answers this question by considering an exact sequence as a complex and calculating its cohomology. This yields the "derived functors" of the given functor. This has been done by Cartan and Eilenberg in [CE56] (written in 1953) for categories of modules and by Buchsbaum in 1955 for general exact (abelian) categories.[11] Now, Cartan and Eilenberg stress a latent duality in their work:

> In this chapter we present all the algebraic tools of homology theory [. . .]. The treatment here differs from the standard one in that great care is taken to maintain all symmetries and thus keep the system self-dual at all times. [. . .] The reader will have ample opportunities

[11] See [Krö07] section 3.1.2.2. for a detailed historical account.

> to convince himself that the preservation of this kind of a duality is indispensable. [CE56, 53]

Nevertheless, Cartan and Eilenberg cannot help treating separately right and left derived functors respectively and to distinguish even the different possible variances of the functors. Mac Lane's 1950 paper faced a similar situation, namely the repetition of dual argumentations in the Eilenberg-Steenrod 1952 axiomatization of homology and cohomology theories [ES52].

Buchsbaum provided a solution of the duality problem in the Cartan-Eilenberg situation; in this solution, the concept of dual category plays a role (which Buchsbaum denotes $\mathcal{A}^*$ when the category $\mathcal{A}$ is given). However, this concept allows only to make explicit the *dualization process*, *i.e.*, to explain how the proposition dual to a given proposition is obtained (reverse the arrows); but to avoid dual argumentations, one needs moreover (and more importantly) a *principle of duality*, *i.e.*, a metatheorem establishing under which circumstances the dual proposition so obtained is *valid*. The principle of duality given by Buchsbaum reads: with $\mathcal{A}$, also $\mathcal{A}^*$ is exact (where Buchsbaum's notion of "exact category" vouches for the possibility of speaking about exact sequences, and is very close to the now standard notion of abelian category).

Here are some of Buchsbaum's main results:

> In treating derived functors, it suffices to consider left derived functors of a covariant functor of several variables; all other types needed may then be obtained by a dualization process. [Buc55, 1]

> The axiomatic homology and cohomology theories of Eilenberg and Steenrod [1952] may be defined using an arbitrary exact category $\mathcal{A}$ as the range of values of the theory. Thus, replacing $\mathcal{A}$ by $\mathcal{A}^*$ replaces a homology theory by a cohomology theory, and vice versa. [CE56, 385]

> The Pontrjagin duality for discrete and compact abelian groups readily shows that the category $\mathcal{C}$ of compact abelian groups is the dual of the category $\mathcal{M}$ of discrete abelian groups. Thus we conclude that $\mathcal{C}$ satisfies Axioms V, VI and VI^*. [CE56, 386]

Let us pause to make a first epistemological comparison of category-theoretic and classical dualities from what we have seen so far. Reversion of arrows can be seen as a purely formal exchange of some parts of the language, like in classical dualities. But which are the respective epistemic properties? In the case of projective duality and Boolean duality, the exchange of parts of the language with others is possible *salva veritate*. In the case of projective duality, this is achieved only by the

introduction of ideal elements (points at infinity) which seem to be epistemically less accessible than the other objects of the theory.

On the other hand, reversion of arrows applied to a true theorem does not yield systematically a true theorem. Counterexamples occur as with injective and projective objects.[12] In general, this occurs when the categorical environment is not self-dual.[13] One has to distinguish between the dualization procedure for obtaining the dual statement and duality principles which assert the truth of the dual of a true statement.

Another observation of interest for our epistemological purpose is contained in the following explanation Buchsbaum gives for the fact that his duality theory was outside the scope of Cartan-Eilenberg (where only categories of modules are considered). Let $H(A, B)$ denote the construction of the homology functor in their manner; Buchsbaum says:

> In [the] category [of all left Λ-modules $\mathcal{M}_\Lambda$], $H(A, B) = \mathrm{Hom}_\Lambda(A, B)$. However, the dual category $\mathcal{M}_\Lambda^*$ admits no such concrete interpretation. This explains the fact that the duality principle could not be efficiently used, as long as we were restricted to categories concretely defined, in which the objects were sets and the maps were maps of those sets. [CE56, 382]

To put it in more general terms: when starting from a category C composed of (structured) sets and functions (or, more technically, from a "concrete" category, that is, a category C with an underlying functor $U : C \to Set$),[14] its dual category C^{op} obtained by arrow reversion often is more "ideal" than C itself in that the arrows so obtained need not be set functions (C^{op} need not be concrete). Therefore, Buchsbaum considered the step to pass to axiomatically given categories (not necessarily concrete in this sense) as the crucial step for making use of a duality principle.

We should compare this "ideality" and the ideal elements of projective geometry carefully. In the case stressed by Buchsbaum, the dual objects as a whole are more "ideal" than the original objects. In projective geometry, on the other hand,

[12] And with direct and inverse limits as well; we will however not discuss this case here.

[13] In the case of injective and projective modules, the internal minutes of the Bourbaki meetings relate this to the fact that the category of sets is not autodual. See *La Tribu 56* concerning the Bourbaki *rédaction n.373*. These documents are not among those available online; they can be seen in the "Archives Delsarte" at the *Institut Elie Cartan, Université de Lorraine, Nancy*.

[14] Note that this technical usage of the word "concrete" quite closely corresponds to what Buchsbaum in a still non-technical manner called "categories concretely defined". It is also related to the Lawvere-Rosebrugh conception of "concrete dualities" in that the category in which the objects of the form V^A live often is concrete. We will elaborate on this point further on.

it is not just the dual objects in general but only the ideal ones (the objects dual to parallel lines) which are less accessible. When applied as we suggest to do in Buchsbaum's case, the usage of the term "ideal" seems to be not identical to its usage in classical ontological doctrines like realism, etc. Rather, it concerns whether something is representable as set and structure.

We should add, moreover, that there are many situations where there are dual concrete categories. This occurs when their underlying functors, which throw away the extra structure, are representable, that is, are of the form $U(-) = C(c, -)$, for some object c. See [PT91]. Here the object c is a free object on one free generator. The duality between the category of finite dimensional vector spaces and its opposite is of this form, the base field playing the role of c.

To see the limitations of Buchsbaum's project, let us take up the historical thread where we left it. Grothendieck around 1955 became interested in applying the Cartan-Eilenberg derivation procedure to functors defined on categories of sheaves; the result was the 1957 Tohoku Paper [Gro57]. Buchsbaum's "two theorems for one proof"-strategy in this case plays only a minor role. The reason is that in Cartan-Eilenberg homological algebra, it depends on the kind of nonexactness and on the variance of a given functor whether projective or injective resolutions are to be used in its derivation:

- if one wants to make exact a left exact functor, one is interested in the right derived functors etc.;
- the right derived functors are obtained by using injective resolutions for all covariant variables of the functor and projective resolutions for all contravariant variables; *vice versa* for left derived functors (see [CE56, 84]).

Thus, in view of the properties of the functor Γ studied by him, Grothendieck *couldn't help* looking for injective objects.

Buchsbaum's duality theory makes it possible to restrict the development of the derivation procedure *in its general form* to left derived functors and projective resolutions. By a dualization process, the stipulation of enough injective objects in a given category can be transformed to the stipulation of enough projective objects in its dual category. However, at least one of these two propositions has still to be *proved* for the duality principle to have any effect.

Moreover, the duality principle does not mean that one can choose for *one single* functor whether one wants to work with injective or projective resolutions (such a choice being possible only for so-called *balanced* functors); one can choose simply whether one derives the functor itself or rather its dual functor.

From Grothendieck's Tohoku paper onwards, category-theoretic dualities haven't any longer the aim to get two theorems by one proof; rather, they generalize the situation in vector space theory (study an object by studying its dual). As we will see, there is still an introduction of "ideal elements" in these cases, but in a different way. These ideal elements in Grothendieck's mathematics aren't introduced any longer to obtain a *salva veritate*-duality. So what is the epistemic gain one has in mind instead?

6 Grothendieck, spaces and functions, and the epistemic gain

We take the following as a basic methodological principle of modern mathematics: In order to learn something about an object which could be called a space, one studies the functions defined on that space and having values in a similar but "simpler" space (concrete duality in the sense of Lawvere-Rosebrugh). The elements of dual vector spaces and character groups are representations of certain other spaces or groups with values in some particularly simple space or group. This is similar with cohomology groups at the level of chain complexes being dualised to cochain complexes.

Similar strategies occur in functional analysis, both in the case of the Banach-Alaoglu theorem and of Gelfand's representation theorem for Banach algebras. For instance, Gelfand substituted an algebra of functions on a space for an arbitrary Banach algebra A by defining a mapping $A \to C(X)$, where X is $\mathrm{Spec}(A)$, the compact Hausdorff space of all multiplicative linear forms of A (which can also be interpreted as the set of maximal ideals of A equipped with a certain topology).[15]

The mathematical details of the definition of this topology actually are important for our purpose: Both in the situation of the Banach-Alaoglu theorem and of the Gelfand representation theorem, you have to choose the weak* topology to get the very important property of compactness—and this topology is related from the outset to the concept of dual vector space. Thus, not only is the space X used in the theorem a subset of the space dual to A as a vector space, but it comes equipped with a topology closely related to this dual space.

[15] The mapping $A \to C(X)$ is actually defined as the composite mapping $A \to \mathrm{Spec}(A) = X \to C(X)$; this is an isomorphism which sends $x \in A$ to a map from $X = \mathrm{Spec}(A)$ to $\mathbb{C}$, which is evaluating $f \in \mathrm{Spec}(A)$ at x. $\mathrm{Spec}(A)$ is the space of characters of A, i.e., the set of its continuous characters, that is continuous nonzero linear homomorphisms into the field of complex numbers, and canonically equipped with a so-called spectral topology. Elements of $\mathrm{Spec}(A)$ are a kind of function on A, so can be evaluated against a member of A.

Summing up, we find that there are at least three levels on which a given algebra is made more accessible by representing it as a $C(X)$, and that on at least two of these levels, vector space duality plays a central role:

- The first level is that an "arbitrary" algebra is replaced by a space of functions (we know now what the elements of the algebra are).
- The second level is that we study a complicated object (an element of the algebra) by studying its values under linear forms, and these values are simple objects (elements of the base field);
- But the usefulness of the second idea depends largely on the properties of X (its compactness) furnished by the consideration of topologies related to the dual space.

The study of elements of Banach algebras by studying their values under linear forms actually incorporates a very subtle "duality" (in the sense of an exchange of parts of the language), namely the idea to change the roles in the expression $f(x)$, *i.e.*, keep x fixed and vary f instead.[16] This observation allows us to elaborate a bit on the matter of "simplicity". There would be no point in saying that in $C(X)$, the X is simpler than A, given that it is a space of certain linear forms on A. $\mathbb{C}$ (the dualizing object, the object in which the functions of $C(X)$ take values) is simpler. The idea rather is: replace one complicated object (an element x of A) by *many* simple objects (the values of x under all these linear forms); *this* is the motivation to keep x fixed and vary f instead. Thus, dual objects should not be thought of as being more accessible than original objects (this being true only of the dualizing object); rather, they *make* accessible the original objects—and *this* is the epistemic gain, of course.

Functional analysis up to Gelfand's result constitutes a first stage of our history of how duality of space and function allowed for mathematical progress. But there was a second, overtly category-theoretic stage, when Grothendieck (having begun his mathematical career in the field of functional analysis) started to adapt Gelfand's strategy for use in algebraic geometry. This has nicely been exposed by Cartier in [Car01] p.397. Grothendieck's strategy very closely parallels Gelfand's: he substituted functions for algebraic objects (the elements of an arbitrary commutative ring A) by mapping $A \to \Gamma(X, \mathcal{O})$ where $X = \mathrm{Spec}(A)$, the set of maximal ideals of A equipped with the Zariski topology, $\mathcal{O}$ a sheaf defined on that space and Γ the section functor of that sheaf, yielding a set of functions as values.

[16] See [BML65] p.185, for instance. An application of this fundamental idea in the field of Hopf Algebras used as physical models is described in [Cor03] p.24.

We think that this line of development (transporting ideas from functional analysis to algebraic geometry by stressing the category-theoretic aspect) played the major role in the development of duality as a central theme in structural mathematics while the Mac Lane-Buchsbaum-Tohoku line of development (the axiomatic approach to duality) was far less important. While Mac Lane and Buchsbaum pursued a two theorems by one proof-strategy, Grothendieck pursued the strategy to prove a theorem by working in a dually equivalent framework where the corresponding proof is easier to get.

To understand this point, note that in all cases discussed so far of passage from an original space (considered as an object of some category) to a space of functions, arrows are reversed. The new objects are dual in the sense that they are objects of a dually equivalent category. Here, the epistemic gain seems to occur on the first of the three levels discussed above: instead of studying "remote", "abstract" objects (like arbitrary Banach algebras or arbitrary commutative rings), we have the result that these categories are dually equivalent to categories of function spaces of certain types; thus we can study these more "accessible" objects instead. But a more detailed study of what Grothendieck actually did might show that there are other levels in his case as well.

If Grothendieck's approach to mathematics can be characterized by an overall strategy or method, such a characterization certainly would involve the theme of analogy between different fields of mathematics. It is clear, for instance, that Grothendieck's algebraic geometry heavily relies on Dedekind's idea of an analogy between number and function.[17] In fact, we can find repeatedly that Grothendieck aimed at making analogies complete and therefore enlarged the category he was working in. The example alluded to by Cartier above is the analogy between algebraic geometry and commutative algebra, made complete by the passage from varieties to affine schemes. Another case would be the analogy between Galois theory and the theory of coverings, made complete by the introduction of Grothendieck topologies. The completion of the analogy was in each case achieved by (1) determining first the categories corresponding to the mathematical theories involved and (2) modifying one of them as much as necessary to obtain a pair of (dually) equivalent categories—whereby problems in one category become solvable by transfer to the other category.[18] So we can pass from something more "remote" to something more "accessible" by dual equivalence. And the modification of the cat-

[17] See [Cor03] section 4.3. We might add that in the interaction of the concepts "space" and "function", as present in the study of functions defined on surfaces, the epistemological duality of object and tool is particularly visible. Riemann introduced his surfaces in the study of (abelian) functions while later on functions have been used to study Riemann surfaces.

[18] See [GM96] p.76 for more examples.

egory studied originally in order to obtain the dual equivalence with some "tame" category is the introduction of ideal elements in this case.

Rather as the addition of ideal elements in the projective case led to a geometry with pleasanter features, namely, self-duality, many constructions of Grothendieck were motivated by the idea that rather than work in a category of nice objects, which often itself doesn't possess nice qualities, it is better to 'complete' into a nice category. For example, we embed a category into the category of presheaves on it, which is the free cocompletion. This is an extension of Cayley's theorem, embedding a group G in the category of G-sets, as the group acting on its underlying set. We can recover G from this category of G-sets, and this is part of a very large story of Tannaka duality, whereby one recovers an algebraic entity from a category of 'geometric' representations. This extends even to a duality between theories and their categories of models (see [AF]).

Let us relate the rather vague notions of remoteness, accessibility and niceness ascribed to Grothendieck's strategy in what precedes to the category-theoretic notion of concreteness discussed in the preceding section. The following quote illustrates the idea that there's something more manageable on the concrete side of a duality.[19]

> [...] for a given category A, the existence of a duality with some concrete category B might give considerable additional information about A: if e.g. B has limits – often quite obvious constructions in concrete categories – the category A automatically will have colimits which, moreover, can be described explicitly (for A algebraic usually a difficult task) as S-images of limits in B. [PT91] pp. 111-112

Such features are among those making the category A "nice".

7 Conclusions

We have seen in sections and that in the settings of Mac Lane and Buchsbaum, category theoretic duality was actually used for enhancing (doubling) the set of proved theorems by linguistic exchange. In these cases, dual objects tended to be more ideal (epistemologically more remote) than original ones. On the other hand, section suggests that when category theoretic duality is employed in the "space-functions" way, the key issue seems to be that the dualizing object is less ideal than the original objects.

[19] The S is just one of the adjoints involved in the dual equivalence.

As we mentioned, this paper marks only the first steps towards a treatment of the internal epistemological and ontological features of duality in mathematics. First of all, the paper is meant to be largely descriptive; we didn't intend to criticise in Mik Detlefsen's sense the epistemic status of the knowledge gained by the uses of duality described. And the descriptive work is not finished; for instance, a finer analysis in our opinion should focus on the fact that different identification criteria are used in each case.

Moreover, further work should look beyond the issue of having one side more concrete than the other. For example, duality may relate different structures of the same domain to each other, some of which are easier to work with. For example, in the case of Fourier analysis, the convolution of two functions is transformed into a multiplication. A more modern and involved example occurs with mirror symmetry for Calabi-Yau manifolds, where the Kähler and complex structures are exchanged as one passes between mirror manifolds. It turns out that to perform calculations on one of these structures for a particular manifold, it may be easier to work on the other structure on the mirror.

Bibliography

[AF] Steve Awodey and Forssell. First-order logical duality. http://arxiv.org/abs/1008.3145.

[BG99] James C. Becker and Daniel Henry Gottlieb. A history of duality in algebraic topology. In *[Jam99]*, pages 725–745. 1999.

[BM10] Lise Bioesmath-Martagon. *Eléments d'une biographie de l'espace projectif*. Presses universitaires de Nancy, 2010.

[BML65] Garrett Birkhoff and Saunders Mac Lane. *A survey of modern algebra*. MacMillan, 1965.

[Buc55] David A. Buchsbaum. Exact categories and duality. *Trans. Amer. Math. Soc.*, 80:1–34, 1955.

[Car01] Pierre Cartier. A mad day's work: from Grothendieck to Connes and Kontsevich. the evolution of concepts of space and symmetry. *Bull. Amer. Math. Soc.*, 38(4):389–408, 2001.

[CE56] Henri Cartan and Samuel Eilenberg. *Homological Algebra*. Princeton University Press, 1956.

[Cor03] David Corfield. *Towards a Philosophy of Real Mathematics*. Cambridge Univ. Press, 2003.

[Cor10] David Corfield. Lautman et la réalité des mathématiques. *Philosophiques*, 37(1):95–109, 2010.

[Cox61] H. S. M. Coxeter. *The real projective plane*. Cambridge Univ. Press, 2nd edition, 1961.

[EML45] Samuel Eilenberg and Saunders Mac Lane. General theory of natural equivalences. *Trans. Amer. Math. Soc.*, 58:231–294, 1945.

[ES52] Samuel Eilenberg and Norman E. Steenrod. *Foundations of algebraic topology*. Princeton University Press, 1952.

[GBGL08] Timothy Gowers, June Barrow-Green, and Imre Leader, editors. *The Princeton Companion to Mathematics*. Princeton University Press, 2008.

[GM96] S.I. Gelfand and Y.I. Manin. *Methods in homological Algebra*. Springer, 1996.

[Gro57] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J.*, 9:119–221, 1957.

[Hah27] Hans Hahn. Über lineare Gleichungssysteme in linearen Räumen. *J. Reine Angew. Math.*, 157:214–229, 1927. JFM **53** 369.

[Hal40] P. Hall. Verbal and marginal subgroups. *J. Reine Angew. Math.*, 182:156–157, 1940.

[Jam99] I. M. James, editor. *History of Topology*. North–Holland, Amsterdam, 1999.

[Krö07] Ralf Krömer. *Tool and object. A history and philosophy of category theory*, volume 32 of *Science Network Historical Studies*. Birkhäuser, Basel, 2007.

[LR03] F. William Lawvere and Robert Rosebrugh. *Sets for Mathematics*. Cambridge Univ. Press, 2003.

[ML50] Saunders Mac Lane. Duality for groups. *Bull. Amer. Math. Soc.*, 56:485–516, 1950.

[Nag39] Ernst Nagel. The formation of modern conceptions of formal logic in the development of geometry. *Osiris*, 7:143–224, 1939. JFM **65** 1091, Daub 2261, MR1 S.34, Zbl.022.37803.

[PT91] H.-E. Porst and W. Tholen. Concrete dualities. In H. Herrlich and H.-E. Porst, editors, *Category Theory at Work*, pages 111–136. Heldermann Verlag, Berlin, 1991.