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Uniqueness of Trigonometric Series and Descriptive Set Theory, 1870–1985

ROGER COOKE

Communicated by JEREMY GRAY

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Introduction

It is an interesting phenomenon in the history of mathematics that a technique invented to study a particular problem of seemingly narrow scope sometimes leads to fundamental changes across the entire world of mathematical research, dwarfing in importance the original problem that gave rise to it. Such problems as the solvability of equations by radicals, the vibrating string, and the three-body problem, while important, are of less interest overall than group theory, trigonometric series, and algebraic topology, which received a major impetus from these particular problems. Such is the case with the problem of uniqueness of trigonometric series expansions. The study of this

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topic was begun by E. HEINE (1870) before there was such a thing as set theory. HEINE's invitation to CANTOR to collaborate on what seemed to be a rather restricted question, of interest only to a few specialists, led to the discovery of new concepts of fundamental importance not only in analysis but in geometry, algebra, and all other areas of mathematics. CANTOR's attempt to answer a particular question in the theory of trigonometric series led him to invent a new language that changed forever the way mathematicians thought about their subject. Realizing that the issues he had uncovered were more fundamental than the narrow question of the uniqueness of trigonometric series representations, CANTOR soon abandoned the subject and subsequently devoted his time to the creation of set theory. Yet the set theory that he developed continued to make major contributions to the solution of the original problem. The two subjects touched briefly in 1903, when LEBESGUE used trigonometric series as an application of his integral, and again in 1916, when LUZIN's seminar produced almost simultaneously ALEKSANDROV's proof of the continuum hypothesis for BOREL sets, SUSLIN's discovery of analytic sets, and MEN'SHOV's proof that there are sets of multiplicity having measure zero. Thereafter they developed separately until the 1970's and 1980's, when suddenly descriptive set theory, now enriched by the theory of BOREL games, began to exhibit all uniqueness theorems from a new point of view. The following pages attempt to give an account of this century of mathematical history. In this introductory section I shall summarize the prehistory of this topic in order to provide a picture of the situation in 1870, the year in which the narrative begins. The following synopsis is highly selective and is mostly contained in the more extended account of DAUBEN (1979, pp. 6–46).

Expansions in Trigonometric Series. The early history of trigonometric series expansions is among the best-studied topics in all of mathematics. Although some uses of trigonometric series in the early eighteenth century are mentioned by BURKHARDT (1901), the first use of such series in a major problem of physics was by DANIEL BERNOULLI in 1747 in connection with the problem of the vibrating string. The use of such series by D. BERNOULLI and EULER marked the establishment of a systematic technique of expressing unknown or complicated functions as combinations of simpler functions. Trigonometric series, in particular, became the preferred technique after the appearance of FOURIER's work on heat conduction. The practical problems involved in the use of trigonometric series are of two opposite kinds: 1) Given a function $f(x)$ (of period 2π), find a trigonometric series

$$\tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1}$$

that represents the function. That is, through some procedure of analysis T_a find the coefficients a_m and b_n. (The reason for putting the odd-looking coefficient $\frac{1}{2}$ in front of a_0 is technical and suggests the standard solution to this half of the problem, *cf.* Eq. (3) below.) 2) Given a trigonometric series (1), find a function $f(x)$ that the series represents. In other words, sum the series through some

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procedure T_s. These two practical problems lead to a host of speculative problems. For instance, let us assume that we have techniques, say T_a and T_s for solving the two problems when the data are suitable. The technique T_a will be an algorithm accepting functions $f(x)$ of period 2π as input and yielding the coefficients a_i and b_j, $i = 0, 1, \ldots, j = 1, 2, \ldots$, as output; T_s will accept the series (1) as input and yield a function $f(x)$ of period 2π as output. If we start with a function $f(x)$ of period 2π that is acceptable input for the technique T_a, will the application of T_a yield a trigonometric series that is acceptable data for the technique T_s? If it does, will the application of T_s recover the original function $f(x)$? Conversely, if we are given a trigonometric series to which the technique T_s applies, will the result be a function to which T_a applies; and if so, will the application of T_a yield the original series? The mathematical theory of trigonometric series representation consists of the development of suitable techniques T_a and T_s and establishing theorems about the classes of functions to which such techniques apply. The problems that arise are generated by the interplay of operations that make up the techniques T_a and T_s with the possible data sets to which they can be applied. The two main questions are the following:

1. *Which functions can be represented by a trigonometric series?* The answer, of course, depends on the meaning of the terms *function* and *represent*, and these in turn are the essential elements in defining a technique T_a. Criteria for choosing a "good" technique are derived from the desirability of representing as large a class of functions as possible without making the inverse problem of recovering the function from the series unreasonably difficult, or worse, making it impossible by getting a class of representable functions so large that two different functions have the same series representation. This problem leads naturally to the next question.
2. *How many trigonometric series representations does a given function have?* The possible answers may be divided into three classes: zero, one, more than one. As in the case of the first question, which answer actually applies will depend on the meaning attached to the term *represent*, and this meaning depends on the technique T_a one has in mind. Normally the most desirable answer is one, but this ideal may not be attainable.

Trigonometric Series Representations in the Eighteenth Century and Early Nineteenth Century. Since the purpose of this article is to explore the methods that were developed historically in the attempt to answer these questions and exhibit their connection with descriptive set theory, I shall begin by summarizing the answers given by the predecessors of CANTOR. It is well-known that Question 1 was debated by EULER and D. BERNOULLI, and the history of this controversy has been well-documented by HOBSON (1927), DIEUDONNÉ (1970), GROSS (1992), and many others. It is of interest that BERNOULLI's belief that "any" periodic function has such a representation was based on physical considerations (*cf.* 1755a); he did not have any technique T_a for finding a particular series to represent an arbitrary function, nor, at the time, did EULER (1755).

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Such a technique seems to be implicit in some work of LAGRANGE from this period (1759). As BURKHARDT has pointed out (1901, pp. 33–34), LAGRANGE (Œuvres, I., pp. 164–166) essentially wrote the "Fourier" formulas for the coefficients of a completely general periodic function; but, having another goal in mind, he did not exhibit them explicitly. RIEMANN (Werke, p. 234) suggests that LAGRANGE's attention was diverted by the EULER/D'ALEMBERT polemic. LAGRANGE did, however, consider the basic trigonometric series, $\cos x + \cos 2x + \ldots$, whose partial sums are (essentially) the DIRICHLET kernel. He claimed that the sum of this series is $-\frac{1}{2}$ for all x, a statement to which D'ALEMBERT objected, pointing out that for $x = 45°$ the terms of the series repeat in a cycle of length 8, giving $\frac{1}{\sqrt{2}} + 0 + \left(-\frac{1}{\sqrt{2}}\right) + (-1) + \left(-\frac{1}{\sqrt{2}}\right) + 0 + \frac{1}{\sqrt{2}} + 1 + \ldots$. LAGRANGE was unimpressed with this objection and defended himself by pointing out that a similar objection could be urged by taking $x = 1$ in the series expansion $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$, which everyone granted was a valid relation (Œuvres, Vol. 1, p. 323). JACOBI (1827, p. 2; Werke, Vol. 6, p. 13) and SACHSE (1880a, p. 234; 1880b, p. 47) remark that EULER did write out explicit formulas for the "Fourier coefficients" of an integrable function in 1777, in a paper published in 1798 (I have not seen this paper). The investigations of trigonometric series by EULER and BERNOULLI, however, were mostly concentrated on summing particular trigonometric series such as $\sum \frac{\sin kx}{k^r}$. Such work is of great importance, to be sure, since a vital step at the beginning of any mathematical theory is the accumulation of explicit examples.

A technique for computing a suitable trigonometric series to represent a prescribed function was discovered by FOURIER (1807), and, as is well-known, involved computing the coefficients as integrals. The so-called FOURIER formulas for the coefficients (for an even function $\varphi(y)$ of period 2, vanishing at $+1$ and -1) were given by FOURIER as

$$b_{2i+1} = \int_{-1}^{+1} \varphi(y) \cos(2i+1)\frac{\pi y}{2}\, dy\,. \tag{2}$$

For later use I remark at this point that the FOURIER formulas as they are known today, for functions of period 2π, are

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\, dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx,$$
$$m = 0, 1, \ldots, \quad n = 1, 2, \ldots. \tag{3}$$

The formulas (3) express the standard technique T_a for periodic functions of one variable. Given an integrable function $f(x)$, the series (1) with a_m and b_n given by (3) is called the *Fourier series* of the function $f(x)$. Whether a trigonometric series is a FOURIER series thus depends on the notion of integration one holds;

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whatever concept of integration one has in mind, it should be expected that some trigonometric series are *not* FOURIER series.

The consequences of this work of FOURIER were manifold. First, the method of computing the series meant that the class of representable functions was to consist of integrable functions. At the time no one imagined any serious restrictions on integrability, since the notion of function and integral was (from a modern point of view) rudimentary: a function of a variable was, according to JOHN BERNOULLI (*cf.* MEDVEDEV, 1975, p. 34; 1991, p. 41), "an expression formed in any way from the variable and constants." Integration was performed by seeking a primitive, and it was taken for granted, based on the use of power series representations, that a primitive "existed" for functions as understood in the sense just mentioned. Of course when questions arose about these concepts, the doubts were bound to spill over into the theory of trigonometric series as well. Second, FOURIER's singling out of what we now call the FOURIER series, effectively answered the second question, *i.e.*, it ruled out the possibility that more than one series could represent a given function[1]. Indeed, if term-by-term integration of series is regarded as justified, one can easily deduce that the coefficients must be given by the FOURIER formulas. Consequently when doubts arose about the legitimacy of term-by-term integration, the uniqueness problem was bound to arise also. Third, by providing a feasible technique T_a FOURIER's work focused the attention of mathematicians on the converse problem: to prove that the FOURIER series of a given function actually converges to the function. Another half-century was to pass before the scrutiny of the terms *function* and *integral* was sufficiently close to raise the question whether there might be other trigonometric series that represent a given function. Nevertheless, FOURIER's work indirectly led to the renewal of interest in this question. By producing a technique based on integration (in contrast to the case of power series, where the coefficients are obtained from the function to be represented by differentiation), FOURIER gave a motive for developing a theory of integration independent of differentiation. It was precisely in this context that RIEMANN asked the question, "What then is meant by an integral?" (Werke, p. 239).

The Convergence Problem. It is well-known that DIRICHLET (1829) was the first[2] to give a rigorous proof of the convergence of the FOURIER series. DIRICHLET (1829, pp. 157–158) rejected an earlier attempted proof by CAUCHY (1826), which was resurrected later by RIEMANN (*cf.* Werke, p. 234) and then questioned again by H. A. SCHWARZ (1872, p. 232). In this work DIRICHLET also attempted to extend the concept of the integral to include discontinuous functions and proposed his famous example of an everywhere discontinuous function. His convergence theorem remained the standard convergence theorem for decades, and is still taught to undergraduates today because of the clarity of the

[1] I am indebted to JESPER LÜTZEN for this remark.

[2] However POISSON had earlier (1823) attempted to justify a representation now known as the POISSON integral of a function, equivalent to ABEL summation of the FOURIER series of the function.

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principles on which it is based. The theorem naturally attracted attention to the problem of delineating the *class of functions* for which trigonometric representations are valid. Like all convergence theorems it asserts that the analysis/summation technique (applying the FOURIER formulas (3) as T_a and ordinary convergence of series as T_s) will produce a series from a function, and the sum of the series will equal the function, provided the function assumes only finite values, has only a finite number of discontinuities, and has only a finite number of maxima and minima. (The last condition is nowadays made more restrictive, being phrased as piecewise monotonicity. It guarantees that the function can have only jump discontinuities; DIRICHLET found that the FOURIER series converges to the average of the right-hand and left-hand limits at a discontinuity.)

DIRICHLET's work drew attention to the difficulties with FOURIER series representation: these difficulties arise when the function to be represented tends to infinity at some points, or has infinitely many (jump) discontinuities, or has infinitely many maxima and minima (oscillates infinitely often). In a later paper (1837b) DIRICHLET stated that the existence of points where the function becomes infinite does not affect the overall convergence provided the (improper) integral of the function converges. According to SACHSE (1880a, p. 244; 1880b, p. 59) DU BOIS-REYMOND (1875b) showed that this assertion is correct if the integral converges *absolutely*. The most interesting extension promised (but not delivered) by DIRICHLET was the assertion that a function is representable by its FOURIER series, even if it has infinitely many discontinuities, provided the discontinuities are (as we would now say) nowhere dense, *i.e.*, as he phrased it, between any two points a and b there is an interval (r, s), of positive length, on which the function is continuous. DIRICHLET promised to provide the basic analysis to justify such an assertion in a later paper, but he never published such a paper. The justification of DIRICHLET's assertion is due to LIPSCHITZ, and was achieved only in 1864 (see below).

Riemann's *Habilitationsschrift*. The state of knowledge about trigonometric series was elegantly summed up by RIEMANN when he joined the faculty at Göttingen in 1854[3]. While discussing the history of FOURIER series, RIEMANN noted that the concept of a FOURIER series depends on the concept of the integral. He therefore spent three pages discussing the meaning of integration, summarizing and clarifying the work of CAUCHY, and giving a necessary and sufficient condition for a function to have a meaningful integral over an interval $[a, b]$: the function must be bounded[4] and for any $\sigma > 0$ the sum of the lengths of the intervals of any partition of $[a, b]$ on which the oscillation of the

[3] This address was published by DEDEKIND in 1867, the year following RIEMANN's death. It is remarkable that RIEMANN made no attempt to publish a paper containing so many notable advances in analysis. The explanation, no doubt, lies in the number of unanswered questions raised in the paper.

[4] RIEMANN actually said *finite*, but it is clear from the context and previous statements that he thought of infinity as a value of a function at a point x_0 only when the function *tends to infinity* as x tends to x_0.

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function is greater than σ must tend to zero when the mesh of the partition (length of its longest interval) tends to zero. This property is now stated by saying that the set of points of discontinuity must be of measure zero; of course RIEMANN did not have the concept of measure 0 as we now know it, and he did not explicitly state the connection between this property and continuity. Thus was the *Riemann integral* introduced. RIEMANN did not mention that a continuous function is integrable, though this result is implied by what he did prove. He gave the necessary and sufficient conditions just mentioned for a function to be integrable, then moved on to other things.

After giving a history of the subject of trigonometric series representation, RIEMANN proceeded to give his own statement of the most important outstanding problem in the area (Werke, p. 244):

> Previous works on this subject [trigonometric series representations – RC] have been directed at proving [the validity of representation by] the Fourier series for the cases that occur in Nature; one could therefore begin a proof for a completely arbitrary function and later on subject the behavior of the function to arbitrary restrictions, provided the latter did not interfere with this purpose. For our purposes the proof should be subject only to the hypotheses necessary for the representability of the function; we must therefore first find necessary conditions for representability, from which sufficient conditions will then be chosen. Thus while the previous works have said: if a function has such-and-such properties, then it is representable by a Fourier series, we must start from the opposite question: if a function is representable by a trigonometric series, what thereby follows regarding its behavior, the variation of its value when its argument varies continuously?

Thus while previous mathematicians had sought *sufficient* conditions for the FOURIER series of a function to converge to the function, RIEMANN proposed to look for *necessary* conditions. Note that this formulation frames the question in a certain way: RIEMANN was going to assume that a function is the sum of a convergent trigonometric series and see what restrictions that assumption places on the function. Since this paper contains many theorems and techniques that still lie at the heart of uniqueness theory, it is worthwhile giving a summary of its major parts.

1. RIEMANN began by considering the series (1). (In RIEMANN's notation the letters, a and b denoting the coefficients are reversed; to avoid confusion, we shall adhere to the notation of (1)). He pointed out that if the coefficients a_n and b_n tend to zero, then the terms $A_n(x) = a_n \cos nx + b_n \sin nx$ tend to zero for every value of x, while if the coefficients do not tend to zero, then $A_n(x)$ can tend to zero only for particular values of x. (The second half of this assertion is an imprecise statement of the CANTOR-LEBESGUE theorem, though RIEMANN offered no proof of it.) He proposed to consider the two cases separately and began by considering the case when $a_n \to 0$ and $b_n \to 0$.

2. He formed the function $F(x) = \frac{1}{4} a_0 x^2 - \sum \frac{a_n \cos nx + b_n \sin nx}{n^2}$ obtained by integrating the series formally twice and showed that this function is continuous

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and that the following restricted limit (the ratio α/β remains bounded away from zero and infinity)

$$\lim_{\alpha,\beta\to 0} \frac{F(x+\alpha+\beta)-F(x-\alpha+\beta)-F(x+\alpha-\beta)+F(x-\alpha-\beta)}{4\alpha\beta} \tag{4}$$

equals the sum of the series (*Riemann's first theorem*). He further showed that the ratio

$$\frac{F(x+2\alpha)+F(x-2\alpha)-2F(x)}{2\alpha}$$

tends to 0 as $\alpha \to 0$ (*Riemann's second theorem*). The important implication of this theorem is that, if $F(x)$ has either a right-hand or left-hand derivative at a point, then in fact it has both, the two coincide, and the derivative $F'(x)$ exists at the point. In other words, the curve $y = F(x)$ may fail to have a tangent at a point x, but it cannot have a corner at any point. (It is interesting that RIEMANN invokes this second theorem only to prove one minor point in his argument (Werke, p. 258). The second theorem is an easy corollary of the first provided the original series converges, and might therefore seem to be dispensible. Yet it is the essential element allowing exceptional points in the hypotheses of uniqueness theorems. Without it the only set that is demonstrably a set of uniqueness is the empty set.)

These two theorems are the heart of the classical uniqueness theorem of CANTOR that will be discussed below. To see how much is due to RIEMANN and how much was left undone by him it is instructive at this point to examine the plan of the proof of the classical theorem, which asserts that *if the series* (1) *converges to zero on an interval of length* 2π *except at a finite number of points, then* $a_m = 0 = b_n$, $m = 0, 1, \ldots$, $n = 1, 2, \ldots$. The standard proof proceeds as follows: RIEMANN's first theorem is combined with a lemma of H. A. SCHWARZ, according to which $F(z)$ must be linear on each interval on which the limit (4) is zero. The continuity of $F(z)$ and RIEMANN's second theorem then imply that the function $F(z)$ has no corners, hence must in fact be *globally* linear. It then follows (by separating terms) that a polynomial in x must be equal to a *uniformly convergent* trigonometric series, which is periodic in x. Since the only periodic polynomial is a constant, and it is known that the twice-integrated series must converge uniformly, it follows that all the coefficients must be zero.

As this discussion shows, what RIEMANN's paper did not have was the lemma of SCHWARZ, which asserts that RIEMANN's first theorem (with $\alpha = \beta$ and the sum of the series (1) equal to 0) implies that $F(x)$ is linear. RIEMANN took a different approach to the study of the function $F(x)$, proving that for any interval $[b, c]$ the relation

$$\lim_{\mu\to\infty} \mu^2 \int_b^c F(x)\cos\mu(x-a)\lambda(x)\,dx = 0 \tag{5}$$

holds provided $\lambda(x)$ is any twice-differentiable function on $[b, c]$ whose second derivative has only a finite number of maxima and minima and which satisfies $\lambda(b) = 0 = \lambda(c)$. Twentieth-century mathematicians have frequently seen here

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a prefiguration of the concept of a distribution, the properties of the function $F(x)$ being deduced from its action on the "test functions" $\lambda(x)$ of (5).

3. At this point RIEMANN had found necessary conditions for a function $f(x)$ to be the sum of an everywhere-convergent trigonometric series whose coefficients tend to zero: there must exist a continuous function $F(x)$ such that the limit (4) equals $f(x)$ and (5) holds for the specified class of functions $\lambda(x)$. He then proceeded to show conversely that if such a function $F(x)$ exists, then there exists a trigonometric series with coefficients tending to zero whose sum is $f(x)$ *at each point where the series converges.* (The required series is the formally twice-differentiated FOURIER series of the function $F(x) - Bx - Cx^2$, when B and C are chosen so that this function is periodic. Condition (5) guarantees that the coefficients of the twice-differentiated series will tend to zero, as was later proved by ASCOLI (1874, p. 307) in an extended commentary on this work of RIEMANN.) The gap left at this point is obvious, however: The series is not proved to converge; all that is known is that it cannot converge to a value different from $f(x)$. RIEMANN remarked that if the coefficients of the series tend to zero, then the convergence of the series at a particular point x is determined by the behavior of the function $f(x)$ in an immediate neighborhood of the point[5] (the *Riemann localization principle*).

4. RIEMANN then turned once again to the question of the coefficients, saying, "Now in many cases whether the coefficients of the series eventually become infinitely small will have to be decided not from their expression as definite integrals [he seems to have only Fourier series in mind at this point] but in some other way." He then proceeds to show that in fact the coefficients of the FOURIER series of an integrable function *must* tend to zero (the *Riemann-Lebesgue lemma*).

5. Having finally characterized the functions having everywhere-convergent FOURIER series as those integrable functions that are "generalized second derivatives" (in the sense of (4)) of continuous functions satisfying (5), RIEMANN turned at last to the case of series whose coefficients do not tend to zero. (These are of great importance in modern mathematics, since the FOURIER-STIELTJES coefficients of a measure do not necessarily tend to zero.) For this case RIEMANN operated formally on the series whose sum (if the series converged) would be the function $F(x)$. By considering $G(x, t) = \frac{1}{2}[F(x + t) + F(x - t)]$, which is

[5] In fact, as RIEMANN's argument shows (Werke, p. 253), the behaviour of the series at x is determined by the values of the function $F(x)$ near x, a fact which is important if one starts only with a series having bounded coefficients, assuming nothing about its convergence. The function $F(x)$ is defined as the sum of the twice formally integrated series, whether or not the function $f(x)$ is defined. This *localization principle* for general trigonometric series is a much deeper property than the localization principle for *Fourier* series, which is easily deduced from the RIEMANN-LEBESGUE lemma (see below). For the history of the localization principle see PAPLAUSKAS (1960).

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legitimate if the original series happens to converge at x, he showed that the series could converge only for those values of x for which

$$\lim_{\mu \to \infty} \mu^2 \int_b^c G(x, t) \cos \mu(t - a) \lambda(t)\, dt = 0 . \tag{6}$$

He concluded from this relation that the original series could converge only at values of x about which the points where (5) fails are symmetrically located.

By repeating his earlier analysis, replacing the function $F(x)$ with $G(x, t)$, RIEMANN finally reached a conclusion (Werke, p. 259):

> A function $f(x)$ having only a finite number of maxima and minima can be represented by [*i.e.*, is the sum of – RC] a trigonometric series whose coefficients do not tend to zero only for a finite number of values of x; for the integral $\mu^2 \int_b^c F(x) \cos \mu(x - a) \lambda(x)\, dx$ tends to zero as μ tends to infinity only for a finite number of points. Hence there is no need to consider this case further.

6. Having finished his discussion of convergent FOURIER series, RIEMANN devoted the final section of his paper to an exploration of some interesting counterexamples showing the possible complexity of the relationship between integrability and representability by a trigonometric series.

RIEMANN's *Habilitationsschrift* was written in 1854, but not published until 1867, after RIEMANN's death. In the meantime the work of DIRICHLET was extended by LIPSCHITZ (1864). LIPSCHITZ showed that a FOURIER series converges to the function $f(x)$ that generates it at any point x for which the *Lipschitz condition* $|f(x + \delta) - f(x)| \leqq B|\delta|^\alpha$ holds for some $\alpha > 0$ and all sufficiently small δ. This condition is stronger than ordinary continuity, as SACHSE (1880a, p. 246; 1880b, p. 62) points out. LIPSCHITZ' theorem is not a generalization of the theorem of DIRICHLET in this case. (And in any case LIPSCHITZ relies on DIRICHLET's theorems in the final analysis; his work was concerned with finding ways to make DIRICHLET's arguments apply to a wider class of functions.) Of more interest in LIPSCHITZ' work was his justification of DIRICHLET's claim that a function could be represented by its FOURIER series provided any two of its discontinuities were separated by an interval of continuity. LIPSCHITZ' version of the theorem, relying on the LIPSCHITZ inequality, put no restrictions on the number of maxima and minima the function possessed. In the course of the argument, however, LIPSCHITZ considered several cases according as there exist only a finite number of singular points in a neighborhood of which the function is infinitely oscillating or the function is infinitely oscillating at every point of an interval, or a mixture of these two cases. The first of these cases states a property of the extreme points that is a prefiguration of a property from set theory that was soon to be isolated, given a name, and studied by CANTOR: the *derived set* of the set of extrema is finite.

Our summary of the prehistory of the uniqueness problem is now complete. We have seen that DIRICHLET, RIEMANN, and LIPSCHITZ found sufficient conditions for the representability of a function by its FOURIER series and that RIEMANN explored some examples of integrable functions that are not so representable as well as some non-integrable functions that are nevertheless the sum

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of a trigonometric series at infinitely many points. DIRICHLET seems to have believed throughout his life that a continuous periodic function has an everywhere-convergent FOURIER series, while RIEMANN frequently expressed the view that DIRICHLET's convergence theorem encompassed all the functions needed by physics[6]. The convictions of DIRICHLET and RIEMANN seem to have been widely shared. According to SACHSE (1880a, p. 245; 1880b, p. 60) HARNACK, in a paper "Über die unendlich oft oscillirenden und unstetigen Functionen" published at Tübingen University in 1870, claimed that DIRICHLET, RIEMANN, and LIPSCHITZ had proved that any continuous function was represented by its FOURIER series (I have not seen this paper). This conviction was not refuted until DU BOIS-REYMOND (1875a), after repeated attempts to prove such a theorem, gave a counterexample.

RIEMANN's 1854 *Habilitationsschrift* gave a summary of the state of knowledge about trigonometric series representations in the mid-nineteenth century and focused attention on the main problems connected with them. In the years between the writing and the publication of this paper, however, several foundational questions arose that cast doubt upon some of the principles RIEMANN took for granted. For one thing the connection between uniform convergence and termwise integrability of a series came to be generally recognized. The connection between uniform convergence of a series of continuous functions and continuity of the sum had been recognized much earlier. ABEL (1826, (Œuvres, I., p. 225)) pointed out the example of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin nx}{n} = \begin{cases} \frac{x}{2}, & 0 < x < \pi, \\ \frac{x}{2} - \pi, & \pi < x < 2\pi, \\ 0, & x = 0 \text{ or } x = \pi \text{ or } x = 2\pi, \end{cases}$$

as an exception to CAUCHY's claim that the sum of a convergent series of

[6] See, for example, his doctoral dissertation, "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse," Göttingen, 1851 = Werke, p. 3, where he says, "The property of being defined by a single law of dependence for all values of z within a given interval was formerly ascribed only to a particular type of function (*functiones continuæ* in EULER's notation); recent research, however, has shown that there are analytic expressions by means of which any continuous function can be represented for a prescribed interval," or in his *Habilitationsschrift* (Werke, p. 237), where he says, "In fact for all the cases that occur in Nature, which were the only ones being discussed, [the theoretical foundation given by DIRICHLET] was complete; for despite the magnitude of our ignorance about the ways in which the forces and states of matter vary infinitesimally in space and time, we can nevertheless confidently assume that no functions occur in Nature to which Dirichlet's investigations are not applicable," or on pp. 257–258, where he says, "Indeed, Dirichlet's proof . . . actually leaves nothing more to be desired if one eliminates the unnecessary assumption that the function be continuous." (All three examples cited by SACHSE, 1880a, p. 245; 1880b, p. 60).

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continuous functions is continuous. This connection was investigated by SEIDEL (1847), STOKES (1848), and others (*cf.* MEDVEDEV 1975, pp. 78–102; 1991, pp. 85–110). A similar remark of CAUCHY's that a convergent series could be integrated term-by-term was harder to doubt, since convergent power series and FOURIER series can, in fact, be integrated termwise. WEIERSTRASS seems to have been the first to invoke uniform convergence in this connection, in his lectures at the University of Berlin (noted by L. THOMÉ 1866, p. 334; see also HEINE, 1870, p. 353). This complication was quite serious, since it cast doubt on the entire basis of trigonometric series representation, *i.e.*, the FOURIER formulas for the coefficients. If one could not be sure that these are the only possible coefficients, the possibility arose that a function might be represented by more than one trigonometric series. This possibility was disquieting because of the large number of applications such series had received. One of particular importance (noted by SACHSE, 1880a, p. 261; 1880b, p. 97) was a description by JACOBI (1827) of certain sums of the roots of an algebraic equation in terms of integrals that give the FOURIER coefficients. In his discussion of these formulas, JACOBI remarked (1827, p. 1; Werke, Vol. 6, p. 3), that, since the FOURIER coefficients of a function are given by the integrals (3), a trigonometric series expansion for the function, if it can be found without using integration, provides a way of evaluating such integrals. This assertion, of course, is wrong if there is more than one trigonometric series expansion. Although it appears that JACOBI used only absolutely convergent series in this paper, the possibility that work based on this principle might have to be re-examined was a real one. The problem of *uniqueness* of a trigonometric series representation thus finally appeared as an important theoretical question.

These new complications set the stage for the drama that is the main subject of the present article. As the curtain rises in this drama, the first scene is the work of HEINE and CANTOR in the year 1870.

§1. The 1870's

The problem of uniqueness of a FOURIER series representation arose when it came to be realized that the derivation of the formulas for the FOURIER coefficients was subject to doubt. No condition other than uniform convergence of a series was known to guarantee the validity of termwise integration, and it had been known since the work of STOKES in 1848 that uniform convergence was impossible if the function represented was discontinuous. The first attempts to fill this gap involved looking for a replacement for uniform convergence, and the first replacement found was given the rather clumsy name of *generally uniform convergence* (*Konvergenz im Allgemeinen in gleichem Grad*). This term was invented to cover the case of a discontinuous function that is the sum of a series of continuous functions; it meant that the series converged uniformly on the set remaining after small neighborhoods of the discontinuities were removed. The first in the field was E. HEINE (1870), who gave this definition and proved that a function equal to zero except at a finite number of points could

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not be represented by a generally uniformly convergent trigonometric series (the only possible candidate is the series all of whose coefficients are zero, and it of course does not converge to the function where the function is nonzero). It was HEINE who stated the question of uniqueness explicitly (1870, p. 353), saying that the works of DIRICHLET, RIEMANN, and LIPSCHITZ had shown that a function could under some conditions be represented by a trigonometric series, but had not shown in *how many ways* this could be done.

It follows from HEINE's theorem that a function cannot be represented by more than one generally uniformly convergent trigonometric series, since the difference of two such series would be a generally uniformly convergent series representing the identically zero function, hence would have to be the zero series. If it were known that a FOURIER series converges generally uniformly, it would then follow that if a trigonometric series converges generally uniformly to an integrable function, it must be the FOURIER series of that function. In the absence of such a theorem, however, uniqueness theorems divide into two types: "easy" theorems, which assert that the only trigonometric series converging (or summable) to zero (with possible exceptions at certain points) is the zero series, and "hard" theorems, which assert that the only trigonometric series converging (or summable) to an integrable function (with possible exceptions at certain points) is the FOURIER series of that function. Only for functions whose FOURIER series are known to be convergent (or summable) can the second type of theorem be deduced from the first.

Cantor's Work. CANTOR reported (Werke, p. 71) that HEINE had made him acquainted with the work on uniqueness just described. As a result, CANTOR set about to remove the restrictions on the type of convergence needed in HEINE's uniqueness theorem. His first step was to show that the assumption that the coefficients of a trigonometric series tend to zero is actually a consequence of the assumption that the terms tend to zero at all points of an interval. He reached this goal in his first paper (1870a). The theorem is a special case of a more general result now known as the CANTOR-LEBESGUE theorem. CANTOR required eight pages of rather delicate reasoning to establish this result and in the end perhaps suffered more than the result was worth to him, although it turns out to be a very important theorem indeed (*cf.* COOKE 1979). As shown above in the discussion of RIEMANN's paper, only two things were lacking for a complete proof of the uniqueness theorem. The first was the proof that the coefficients tend to zero, *i.e.*, the CANTOR-LEBESGUE theorem, and the second was the lemma of SCHWARZ that a function $F(x)$ for which the limit (4) is zero on an interval must be linear on that interval. CANTOR's hard work in establishing the first of these results was almost immediately shown to be unnecessary by KRONECKER (*cf.* CANTOR 1871a, p. 294; Werke, p. 84; however KRONECKER's remark had been made earlier by RIEMANN, *cf.* RIEMANN's Werke, p. 255). Indeed, if

$$G(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx = A_0 + \sum_{n=1}^{\infty} A_n(x) = 0$$

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at every point x, one can consider the function $H(x, u) = \frac{1}{2}[G(u + x) + G(u - x)] = 0$, which is given by

$$H(x, u) = \sum_{n=0}^{\infty} A_n(u) \cos nx \ .$$

With u fixed this series converges to zero for all x, and its coefficients necessarily tend to zero, since the series $\sum_{n=0}^{\infty} A_n(u)$ converges (to 0). The weaker uniqueness theorem proved *assuming* that the coefficients tends to zero is therefore applicable, and one can conclude that $A_n(u) = 0$ for all n and all u, whence it easily follows that $a_m = b_n = 0$, $m = 0, 1, \ldots, n = 1, 2, \ldots$.

Despite the apparent absence of any need for this theorem, CANTOR nevertheless worked out a simplified proof of it (1871b). Some years later he was annoyed by a note of APPELL (1879) claiming a simpler proof. APPELL's "proof" is clearly fallacious, since it assumes that the *maximum* of $A_n(x)$ on an interval $[a, b]$ tends to zero, which is of course much stronger than the assumption that $A_n(x)$ tends to zero at each point. CANTOR (1880a, b) pointed out this fact, and remarked with some asperity that, "In any case I have given a somewhat simpler presentation of my theorem, which makes *no use* of the *assumption* of uniform convergence, in [1871b]. In my opinion the nature of the subject precludes any further simplification."

After this digression, let us return to our discussion of CANTOR's work on the problem of uniqueness. CANTOR proved three basic uniqueness theorems, in increasing generality. The first (1870b) asserted that the only trigonometric series that converges to zero at every point is the identically zero series. It was here that CANTOR communicated the proof of the lemma of SCHWARZ mentioned above. In a footnote CANTOR pointed out that the proof relies essentially on a theorem frequently used and proved by WEIERSTRASS: *a continuous function on a finite closed interval assumes its maximum value.* It is now known that this lemma of SCHWARZ did not arise spontaneously from SCHWARZ' own research. CANTOR had posed it as a conjecture in a letter to SCHWARZ of 17 February 1870 (*cf.* PURKERT 1987, p. 21), pointing out that the absence of this result was the only barrier preventing him from completing the proof of his uniqueness theorem. With the lemma proved, the uniqueness theorem in this case follows easily, as shown above, and required only half a page to complete.

In his second uniqueness theorem (1871a) CANTOR communicated the result of KRONECKER mentioned above, making it possible to dispense with the CANTOR-LEBESGUE theorem, then proceeded to weaken the hypothesis of the uniqueness theorem, assuming only that the series converges to zero for all but a finite number of points on $[0, 2\pi]$, at which no assumption is made as to the behavior of the series. This did not require any new argument, only a reference to RIEMANN's second theorem. Though it was a small generalization in itself, however, it seems to have caused CANTOR to reflect on the geometry of the line. In particular, supposing that the exceptional points form a convergent sequence, he must have noticed that RIEMANN's second theorem "presses the wrinkles" out of the function $F(x)$ at all the points of the sequence, *and hence, since the function $F(x)$ is continuous, at the limit of the sequence also.* The way was

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therefore open to consider *infinitely many* exceptional points, provided they "clustered" at only a finite number of points[7]. The crucial concept that needed to be isolated was the notion that a sequence of points *clusters* at certain points. (Even to express this idea without using the language of set theory is difficult, so familiar have CANTOR's concepts become to mathematicians nowadays.)

CANTOR's third (and final) uniqueness theorem (1872) led him out of analysis and into the creation of set theory. What began as an attempt to extend the uniqueness theorem to infinitely many exceptional points raised so many new questions that CANTOR never came back to the original problem. The reason is easy to discern from his preface to this paper (1872, p. 123; Werke, p. 92):

> The extension under discussion *here* asserts that the theorem remains valid even when the assumption of the convergence of the series or the [value of the limit] is eliminated for an *infinite* number of values of x in the interval $(0 \ldots 2\pi)$.
>
> However, to achieve this end I have been obliged to preface a discussion, albeit one consisting mostly of indirect references, that sheds light on certain relations that arise whenever a finite or infinite number of numerical quantities are given; I am thereby led to certain definitions that are included here only in order to present the theorem under discussion in the clearest possible form.

This passage is the preamble to set theory. It is followed by the now-classical construction of the real numbers from CAUCHY sequences of rational numbers, complicated by the fact that equality of the limits of two convergent sequences of rational numbers is not the same thing as the identity of the sequences themselves. Repeating the limiting process thus led CANTOR to a larger class of objects, namely to the concept of a λ-fold sequence of rational numbers rather than immediately to the real numbers. This result, however, coincided perfectly with the geometry of the exceptional points he wished to consider, and he said that a point of a line (on which a unit of length has been chosen) is determined when its displacement from the origin of the scale is given as a rational number or a limit of type λ (limit of a λ-fold sequence). At this point the first definitions of set theory enter the mathematical literature (1872, p. 128; Werke, p. 97):

> For the sake of brevity I call a given finite or infinite number . . . of points a *point set*. . .
>
> If a point set is given in a finite interval, a second point set is generally given along with it, and with the latter a third, etc., which are essential to understanding the first set.
>
> In order to define these point sets, we must begin with the concept of a *limit point ["cluster point"] of a point set*. I define a "limit point of a point set P" to be a point of the line situated is such a way that each neighborhood of it contains *infinitely* many points of P, and it may happen that the point itself belongs to the set.

---

[7] The property of set theory involved here was formulated somewhat less precisely by LIPSCHITZ (1864, p. 308; *cf* also HAWKINS 1970, pp. 14–16). Although CANTOR must have known this work, he never refers to it, and his isolation of the concept of a derived set is undoubtedly independent of LIPSCHITZ' work.

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> By a "neighborhood of a point" I mean any interval that has the point *in its interior*. It is easy to prove that a ["bounded"] point set consisting of an infinite number of points has at least *one* limit point. Every point of the line is now in a definite relation to a given set P, either being a limit point of P or not, and thereby along with the point set P the set of limit points of P is *conceptually* given, a set which I wish to denote by P' and call the first *derived set of* P.
>
> Unless the point set P' contains only a finite number of points, it also has a derived set P'', which I call the *second derived set of* P. By ν such transitions one obtains the concept of the νth derived set $P^{(\nu)}$ of P . . . .
>
> It may happen – and this is the case we are exclusively interested in at present – that after ν transitions the set $P^{(\nu)}$ consists of a finite number of points, and hence has no derived set; in this case we wish to call the original point set P a set *of type* ν, so that P', P'', . . . are of types $\nu - 1$, $\nu - 2$, ...

Seen in this context (rather than in the usual unmotivated classroom setting of pointset topology) the concept of a limit point and derived set are completely natural, almost inevitable, results of the attempt to decide the question of uniqueness of trigonometric series representations. CANTOR had explicitly given the definition of a set and made the first attempt to classify sets according to their complexity; this is the essence of *descriptive set theory*, in contrast to what the Russians call *metric set theory* (measure theory), which studies the *size* of sets that are not too complicated in the sense of descriptive set theory. Note that CANTOR has not invented any name for the empty set. When a derived set is empty, he says simply that it does not exist.

With these concepts CANTOR found his theorem easy to extend: supposing the set of exceptional points is of type r, the whole proof depends on proving that the function $F(x)$ is linear on any interval containing only a finite number of points of each derived set $P^{(k)}$, $1 \leqq k \leqq r$. This proof, on the other hand, can be carried out by induction. For $k = 1$, the argument was given above, and is simply an application of RIEMANN's second theorem and the lemma of SCHWARZ mentioned above. The induction step uses the simple fact that if an interval (a, b) contains only a finite number of points of $P^{(k+1)}$, say $x_0, \ldots, x_s$, $a < x_0 < x_1 < \ldots < x_s < b$, then any interval (c, d) with $x_{j-1} < c < d < x_j$, $j = 1, \ldots, s$, contains only a finite number of points of $P^{(k)}$. The function $F(x)$ is therefore linear on (c, d), hence, since c and d are arbitrary, on $[x_{j-1}, x_j]$ also. But now, since $F(x)$ is continuous and has no corners, it is linear on the entire interval (a, b). The induction step is now complete. Since the entire interval $(0, 2\pi)$ contains only a finite number of points of $P^{(r)}$ (because $P^{(r)}$ is a finite subset of $(0, 2\pi)$), it now follows that $F(x)$ is a linear function, and the proof proceeds as before.

This result was CANTOR's last contribution to the theory of uniqueness of trigonometric series. Evidently he realized that in the concept of derived sets he was near the answer to problems of a more central and fundamental character than the relatively special topic of uniqueness. From this time on he devoted himself to the development of this new topic. Nevertheless his first papers on set theory (1874, 1878, 1879a) do not give any further development of the concept of a derived set. Instead they study more general properties of sets such as

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cardinality, one-to-one correspondences, and the like. Only in a series of five papers published in the *Mathematische Annalen* (1879b, 1880c, 1882, 1883, 1884a) did he return to this topic. In the first of these papers (1879b) he extends his notion of type slightly, defining a set to be *of first kind* if one of its derived sets $P^{(n)}$ is finite in every finite interval, "so that $P^{(n)}$ has *no* limit points, and hence no derived set" – he still apparently had not invented the empty set – and *of second kind* if $P^{(n)}$ is infinite for every n. He also made the crucial remark that $P^{(n)}$ is contained in P', though P' is not necessarily contained in P, *i.e.*, that, in our notation, $P' \supseteq P'' \supseteq \ldots \supseteq P^{(n)} \supseteq P^{(n+1)} \supseteq \ldots$ as far as the sequence can be continued.

It was in the second of these five papers, where he introduced a great deal of the terminology of modern set theory, that he finally made use of the empty set (1880c, p. 356; Werke, p. 146):

> It is also useful to have a notation to express the absence of points. For this purpose we choose the letter O; thus $P \equiv O$ means that the set P contains *no* points, and strictly speaking does not exist at all. To give an example, a point set is of *first kind* and type n if $P^{(n+1)} \equiv O$ while $P^{(n)}$ is different from O.

It was through the successive derived sets that CANTOR discovered infinite ordinal numbers. Because, as mentioned above, the derived sets are nested, it was natural to define $P^{(\infty)}$ as the intersection of all the finite derived sets $P^{(n)}$; but then CANTOR realized that the derived set could once again be extracted, leading to $P^{(\infty+1)}$, and eventually to $P^{(n_0\infty^\nu + n_1\infty^{\nu-1} + \ldots + n_\nu)}$ and beyond. He thus opened the way to the study of transfinite ordinal numbers. It is perhaps understandable, with so many new questions to be explored, that CANTOR never took the time to ponder whether the uniqueness theorem holds when the exceptional set P satisfies $P^{(\nu)} \equiv O$ for a *transfinite* ν. It does (*cf.* LEBESGUE 1903, p. 466), and the proof is an easy and elegant application of the HEINE-BOREL theorem, the essential principle of which had been used by WEIERSTRASS (1880) in his work on analytic function theory (*cf.* MEDVEDEV 1975, pp. 88–90; 1991, pp. 96–97).

At this point we leave CANTOR, a man described by DAUBEN (1979, p. 49) as haunted by the mysteries of the continuum and the transfinite numbers. He was not the last mathematician to be haunted in this way. A prominent example in the first half of the twentieth century was N. N. LUZIN, as we shall see.

Other Work During the 1870's. Both CANTOR and HEINE took RIEMANN's *Habilitationsschrift* as the basis of their investigations. They were not alone in doing so. All the investigations of trigonometric series during the decade from 1870 to 1880 cited this work as point of departure. We note here just two works of considerable importance. The first is a long commentary on RIEMANN's work by ASCOLI (1874). ASCOLI first investigated in detail RIEMANN's counter-examples, then discussed the uniqueness question from the "hard" point of view (in the sense defined above). That is, he considered a function $f(x)$ that was RIEMANN-integrable on any interval from which small neighborhoods of a fixed finite set of points was removed and was the sum of a trigonometric series

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except at a "ν-fold infinite" set of points (1874, p. 43, a set of first kind and type ν in CANTOR's notation; he also used the phrase "system of points of order ν," 1874, p. 68). Considering only piecewise continuous functions that do not have an infinite number of maxima and minima, ASCOLI showed that if such a function is the sum of a trigonometric series except on a set of first kind and type ν, then the coefficients are given by the integrals (3), the latter being interpreted as principal values at points where the integrand is infinite.

Two other important contributions to the problem of uniqueness during the 1870's were made by DU BOIS-REYMOND. First, by an ingenious manipulation of the RIEMANN function $F(x)$ he was able to show (1875b) that if a bounded integrable function $f(x)$ is the sum of a trigonometric series except at the points of a set of finite CANTOR type, then the coefficients are necessarily given by the FOURIER formulas (3). Thus at most one trigonometric series can represent an integrable function[8]. This result was fundamental, and portions of the argument were reproduced by SACHSE (1880a, pp. 268–271; 1880b pp. 105–109) and GIBSON (1893, pp. 162–164). The work of ASCOLI and DU BOIS-REYMOND has been discussed in somewhat more detail by PAPLAUSKAS (1961).

The possibility remains that the number of trigonometric series representing a given function is zero, and in his other memoir (1875a) DU BOIS-REYMOND raised the possibility that this case might actually occur, even for continuous functions. To be specific, he gave an example of a continuous function whose FOURIER series fails to converge at certain points.

In summary, the decade of the 1870's saw the proof of the basic uniqueness theorems of both "easy" and "hard" types. A picture began to emerge of the kinds of functions for which a trigonometric (FOURIER) representation exists. A by-product of the work was the creation of set theory, which immediately began to grow independently of its connection with the uniqueness problem. A full century was to pass before set theory, now grown into an enormous body of knowledge at the very frontier of mathematical abstraction, revisited its origins and once again shed light on the problem of uniqueness of trigonometric series. The way in which this separation and reunion came about is the subject of the remainder of the present paper.

§2. The Years 1880–1916

In the years between 1880 and World War I a number of new developments in mathematics exerted influence on the general subject of trigonometric series expansions, but these developments had no influence on the uniqueness theory at the time. Worthy of note, though somewhat peripheral to our present topic, is a paper of HARNACK (1880) investigating the trigonometric series representation of functions whose squares are integrable (in the sense of RIEMANN, of

[8] It should be noted that DU BOIS-REYMOND's understanding of the integral differed from that of RIEMANN, *cf.* SACHSE (1880a, pp. 250–251; 1880b, p. 84).

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course). HARNACK actually developed a great deal of what we now call the L_2-theory of FOURIER series, including BESSEL's inequality and a version of the RIESZ-FISCHER theorem that was inexact because the space of RIEMANN-square-integrable functions is not complete. In doing so he introduced names for some new concepts that sounded very much like the names introduced by HEINE and CANTOR. He said that a series $\sum_{k=1}^{\infty} \varphi_k(x)$ converged to a function $f(x)$ "generally uniformly" (*im Allgemeinen gleichmässig* – note the subtle difference from HEINE's *im Allgemeinen in gleichem Grad*) if the partial sums $S_n(x) = \sum_{k=1}^{\infty} \varphi_k(x)$ satisfy

$$\lim \int [f(x) - S_n(x)]^2 \, dx = 0 .$$

This is, of course, precisely what we now call square-mean convergence. Referring to the remainder $R_n(x) = \sum_{k=n+1}^{\infty} \varphi_n(x)$ when the series converges in this sense, HARNACK says (1880, p. 126):

> There may still be infinitely many places at which R_n lies outside a certain distance (defined by δ) which can even be infinite; but such places are distributed in every arbitrarily small interval in such a way that they exert no influence on the value of the integral; they form a set of first kind.

HARNACK had defined a set of first kind in a different way from CANTOR's. According to HARNACK (1880, p. 128) a collection of points is of first kind if the points can be enclosed in a collection of intervals whose total length can be made arbitrarily small. Thus he had defined what is now known as a set of measure zero or of content zero, according as the number of covering intervals is infinite or finite (HARNACK clearly intended a finite covering; for closed bounded sets the two concepts coincide). This paper thus introduces many important concepts taken for granted nowadays, such as the negligibility of "small" sets, square-mean convergence, *etc.*, but was ultimately unsuccessful because of the limitations of the RIEMANN integral, as HARNACK himself came to realize. For a more thorough discussion, see MEDVEDEV (1975, pp. 120–128; 1991, pp. 128–137). We have included this brief mention as a way of introducing the large number of topics in set theory that needed to be clarified before the subject of uniqueness of trigonometric series representation could rise above the high-water mark reached by DU BOIS-REYMOND in 1875.

In particular descriptive set theory had to be advanced to the point where measurable sets could be defined, after which the relations between smallness in the sense of cardinality, topology, and measure had to be sorted out. At the same time it was necessary to clarify the meaning of the term *function* by setting out the permissible principles of definition that could be used to define one variable as a function of another. A great deal of this work was accomplished between 1898 and 1905 by the French mathematicians. As this work has been thoroughly studied by HAWKINS (1970), MEDVEDEV (1976), and others, I give only a brief summary of the parts relevant to our inquiry.

First, both CANTOR and HARNACK had introduced sets of first kind as a type of negligible set. For CANTOR these sets (now called sets of finite CANTOR-BENDIXSON rank) could be neglected in uniqueness theorems for trigonometric

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series representations. For HARNACK they could be neglected in integration. If uniqueness theory was to be extended in the direction of finding the largest possible exceptional sets, it would be necessary to be quite clear about the relations among the various kinds of "negligibility." Second, as the functions allowed in analysis became less and less "regular" in their behavior and the deficiencies of the RIEMANN integral became more and more apparent, it became necessary to formulate a clear statement of the allowable principles upon which a satisfactory theory of functions, including integration could be based. CANTOR, for instance, was using transfinite operations quite freely, and, as we now know, there are dangers in too casual an application of such operations. But even without the paradoxes that were soon to appear in set theory, judging from the amount of work carried out to make set theory more intuitive, there must have been a feeling on the part of many mathematicians that the meaning of such operations was not quite so clear as it might be.

Measure Theory. One obvious respect in which a set on the line may be small is geometric – its total length may be zero. This property is implicitly contained in the criterion for integrability given by RIEMANN and is explicitly defined by HARNACK as the concept of *first kind.* From the notion of length zero, however, to the general study of sets for which a length can reasonably be defined is a long journey (see HAWKINS 1970). We are not going to trace this history of this topic, but merely point out one place at which some concepts of use in the present investigation occur explicitly. The one point we wish to discuss in some detail occurs in lectures BOREL gave at the École Normale in the spring of 1897 (BOREL 1898). In constructing a CANTOR set in [0, 1] BOREL wished to show rigorously that after all the complementary open intervals, whose total length was less than 1, were removed, there remained points other than the endpoints of the closed intervals that were left at each stage. To this end he explicitly stated and proved the HEINE-BOREL theorem – which, however, he had used in previous work (1895). BOREL then defined the general notion of a measurable set as one for which a measure could be defined. He defined these sets in such a way that the measure of a countable union of open intervals (contained in [0, 1]) is the sum of the lengths of the intervals, the measure of the complement of a measurable set is 1 minus the measure of the set, and the measure of a countable disjoint union of measurable sets is the sum of the measures of the individual sets. He then stated explicitly (1898, p. 47) that "the fundamental theorem just proved [that the interval [0, 1] cannot be covered by a countable union of intervals of total length less than 1 – RC] assures us that these definitions will never contradict one another," and in a footnote, "At least it is easy to obtain this result by procedures entirely similar to those used to establish this theorem." BOREL was right on this point, of course; rigorous proofs were soon written out and published (*cf.* LEBESGUE 1904, pp. 103–110).

BOREL's description of the measurable sets (1898) was laconic, to say the least. In his thesis (1902, p. 240) and in his lectures at the Collège de France in 1902 LEBESGUE (1904, pp. 108–109) attempted to be more explicit about these

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sets, describing them as sets obtained from intervals by a *finite* number of applications of two operations: countable union, and countable intersection. BOREL himself made a similar restriction as far as *explicit* measurable sets are concerned (BOREL, 1905, p. 17), saying that, "It may be noticed that any set we can *effectively* form is obtained by carrying out the operations represented by the symbols $(E_1 + E_2 + \ldots)$ [countable union – RC] and $(E_1, E_2, \ldots)$ [countable intersection – RC] a finite number of times. Thus they are all measurable." LEBESGUE called sets obtained in this restricted way *B-measurable* sets, though BOREL's statement clearly implies that he did not rule out the existence of "non-effectively definable" measurable sets. LEBESGUE stated explicitly that there are measurable sets (in the sense defined in his thesis) that are not *B*-measurable sets.

It is interesting that after developing the aspects of measure theory in set theory LEBESGUE almost immediately brought it back to its roots in the theory of trigonometric series. He described the situation as follows (1903, p. 456):

> Among the methods that have been employed to study trigonometric series the only one that can be applied to [non-Riemann-integrable] functions is the method of Riemann. But up to now this method has produced only two or three general properties and in particular the theorem of Cantor on the impossibility of two expansions for the same function. Whenever the problem arose of obtaining precisely sufficient conditions for the possibility of a trigonometric expansion, Riemann, along with all the authors who have studied the question, restricted himself to the study of Fourier series. This is already a very vast area, for it follows from the works of Dini and Ascoli that a continuous function cannot be represented by any trigonometric series except its Fourier series. P. Du Bois-Reymond, using methods that are perhaps not entirely safe from criticism, has extended this theorem to Riemann-integrable functions. The search for sufficient conditions for the possibility of trigonometric expansions of functions which are not Riemann integrable has not yet begun because of the current state of ignorance as to the form of the coefficients of the expansion of such a function.

LEBESGUE proved (1903, pp. 456–467) that if a *bounded* function, or more generally, a function bounded except in a neighborhood of a countable nowhere dense set is representable by a trigonometric series that converges to it except at the points of a (possibly different) countable nowhere dense set, then that series is necessarily the FOURIER series of the function, the integrals (3) now being interpreted as LEBESGUE integrals. In doing so he extended CANTOR's theorem that the coefficients of a trigonometric series convergent on an interval must tend to zero, proving the same result in the case of a series converging on any set of positive measure. As a result this theorem is now known as the CANTOR-LEBESGUE theorem. Having established the uniqueness of the representation, LEBESGUE then went on to formulate his own very general sufficient conditions for the convergence of a FOURIER series. Thus for the first time since its creation set theory returned to the area in which it was born and contributed to the understanding of the problem that gave rise to the theory in the first place. Note, however, that the exceptional set was a countable nowhere dense set, *i.e.*, it was described in terms of its topology rather than in terms of its measure. As

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mentioned above, the transfinite procedures being used to define ever more complicated functions and sets of ever more complicated topological structure had begun to demand clarification many years earlier, and the clarification process had already begun.

Category and Continuity. RENÉ BAIRE shed some light on the difficulty in 1898 with his dissertation, which contains the famous BAIRE classification of functions. Starting with a "Class 0" consisting of the continuous functions (the simplest class, since WEIERSTRASS had shown that such functions could be approximated uniformly on any bounded interval by polynomials, indeed even polynomials with rational coefficients, about which one could feel quite confident), BAIRE proceeded to functions of Class 1, those discontinuous functions that are pointwise limits of sequences of continuous functions (and hence pointwise limits of polynomials with rational coefficients), then to Class 2, the limits of sequences of functions of Class 0 or 1 that are not themselves in either of these classes (an example is the DIRICHLET function equal to a on the rational numbers and b on the irrational numbers), *etc.* A function that is a limit of a sequence of functions of finite BAIRE class, without itself belonging to any finite BAIRE class, is of Class ω, after which one can form class $\omega + 1$, *etc.*, and so on for all countable ordinal numbers. It is obvious that no new classes emerge after the countable ordinals are exhausted. The union of the BAIRE classes is huge, and human ingenuity is exhausted giving examples of effectively computable functions that fail to belong to one of the first few classes. Clearly, it seems, a field of operations has been provided that must be adequate for any conceivable purposes of analysis. Yet it is possible to define functions that do not belong to any BAIRE class.

In seeking criteria for a function to belong to BAIRE class 1, BAIRE (1899) was led to formulate the now-standard concept of *category*, a name HAUSDORFF criticized as "colorless". A set is *of first category* if it is the union of a countable number of nowhere dense sets. BAIRE could then prove his famous theorem: *a function that is the limit of a sequence of continuous functions on an interval is continuous at the points of a set that is dense in the interval. In fact its discontinuities form a set of first category.* The notion of category also proved useful in formulating a property sufficiently general to include all the sets anyone was likely to need, yet restrictive enough to rule out the worst cases of pathology that can be generated by the axiom of choice. This property, known as the BAIRE property of a set E, is that in every open interval I there is a subinterval J such that either $J \cap E$ or $J \setminus E$ is of first category in J.

BAIRE's arguments made free use of the axiom of choice (equivalent to the well-ordering principle or the use of transfinite ordinal numbers). The controversy over this principle led BOREL and LEBESGUE to distinguish the classes of objects that could be defined without using it as objects known "effectively" (*effectivement*), or defined *analytically*, or "nameable" (*nommable*). As an example, BOREL (1903) showed how to construct explicitly a sequence of continuous functions $f_n(x)$ converging to a function $f(x)$ whose discontinuities form a countable nowhere dense set.

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The relation between the concepts of measure and category took some time to clarify. We have already remarked on LEBESGUE's characterization (1904, p. 109) of B-measurable sets as those obtained from the intervals using a *finite* number of countable unions and countable intersections. On the other hand he defined B-measurable *functions* in terms of the BAIRE classification. This seeming inconsistency actually called attention to the close relationship between the two objects. A set is BOREL-measurable in the sense used today if and only if its characteristic function belongs to the BAIRE classification. By taking the more restrictive definition of Borel-measurable sets, however, LEBESGUE lost the "if" part of this equivalence. Moreover, though he had stated confidently that there exist measurable sets that are not Borel sets, he wrote in a footnote (1904, p. 112) "I do not know if it is possible to *name* [*i.e.*, define in terms of standard operations of analysis without using the axiom of choice – RC] *even one* function that is not B-measurable; I do not know if there exist nonmeasurable functions."

LEBESGUE's approach raised the question of the relationship between the BOREL-measurable functions and the BAIRE functions. What is more curious, however, is that neither BAIRE nor BOREL noticed this relationship, even though they simultaneously published monographs on the subject (BAIRE, 1905, BOREL 1905). At the same moment LEBESGUE attempted to show what generality in defining functions could be achieved using constructive procedures. In his memoir (1905, p. 165) LEBESGUE explicitly abandons the restriction that countable unions and intersections be applied only a finite number of times, saying that "this restriction never entered into any of my arguments; I kept it because it seemed to me that it led to a less remote family of sets than the one considered by M. Borel in his *Leçons sur la Théorie des Fonctions*." By taking this step LEBESGUE regained the equivalence of BOREL sets with sets whose characteristic functions are BAIRE functions. Moreover he introduced a natural hierarchy, classifying the BOREL sets according to their complexity: A set E is of class α if there is an interval $[a, b]$ and a BAIRE function $f(x)$ of class α such that $E = f^{-1}[a, b]$, but no such representation of E is possible with a BAIRE function of lower class. The functions LEBESGUE called *analytically representable* proved to be precisely the BOREL-measurable functions.

LEBESGUE's long memoir (1905) is an important contribution to the development of descriptive set theory because of its classification of measurable sets, but also, ironically, because it contains one of the most fruitful mistakes in all the history of analysis. On p. 192 LEBESGUE wrote, "the portion of this set for which we have $\alpha \leqq y_1 \leqq \beta$ is thus B-measurable, *hence so is its projection on the manifold* . . . ". The phrase emphasized here (there is no emphasis in the original) is LEBESGUE's mistake: the projection of a BOREL set is not necessarily a BOREL set. One of the first great achievements of the Moscow analysts in the seminar of N. N. LUZIN was the systematic study of the consequences of this interesting fact.

The year 1905 was of crucial importance for the questions of set theory that are involved in the present study. In that year a crisis that had been building for some time came to a climax. Because of the antinomies and paradoxes that

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were appearing in set theory there was an intense debate starting in 1905, with arguments for and against such principles as the axiom of ZERMELO being contributed by RUSSELL, HADAMARD, F. BERNSTEIN, BOREL, BAIRE, and LEBESGUE (*cf.* MEDVEDEV 1975, p. 152, 1991; p. 162, and 1982, pp. 246–265; LUZIN, 1933, pp. 509–512). This controversy does not directly affect our main theme, however, which is the connection between set theory and the uniqueness problem for trigonometric series. Thus in 1905 the situation stood as follows:

1. The uniqueness theorems that had been proved (up to 1905) strongly suggested that it would be instructive to search for necessary and sufficient conditions under which a set is "negligible" in the sense that the only trigonometric series that converges to zero on the complement of the set is the identically zero series. One would expect that such sets are in some sense "small."

2. There are three senses in which a subset of an interval may be considered "small." (A set is then "large" if its complement is small; one should expect to find sets that are neither large nor small):

Cardinality. The set is countable (or finite, which is an extreme form of smallness).

Topology. The set is of first category (or nowhere dense, an extreme form of smallness).

Geometry (*Measure*). The set is of measure zero.

The relations between these three forms of smallness and their relation to the uniqueness problem needed to be clarified. (There is still another way of "sizing" sets, namely by the theory of HAUSDORFF dimension. I do not know of any results relating this theory to uniqueness. It may be that such results are contained in known results on category and measure.)

As for the relations among the various kinds of smallness, it seems the three forms of smallness have very little in common beyond the obvious fact that a countable set is small in all three senses[9]. Between topological and geometric smallness there is no necessary relationship. A set may be topologically small and geometrically large. (And by consideration of complements, it may be geometrically small and topologically large.) To construct such a set start with

[9] Even this minimal relation is lost in abstract compact metric spaces. By considering a space consisting of a CANTOR set together with the midpoints of the complementary intervals, one finds a compact space containing an uncountable nowhere-dense closed set (the CANTOR set) whose complement is a countable dense open set. Thus a set may be small in the sense of cardinality and large in the sense of topology; its complement will then be large in cardinality and topologically small. By defining the measure of the midpoint of a complementary interval to be the length of the complementary interval and the measure of the remainder to be zero, one finds a regular BOREL measure in which every non-empty open set has positive measure and for which there is a small set (in the sense of cardinality) that is large in the sense of measure (and of course, by considering complements, a set that is large in the sense of cardinality and small in the sense of measure).

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a CANTOR set C_0 of positive measure $\alpha_0 > 0$, place a copy of it in each of its complementary intervals, resulting in a new CANTOR set C_1 of larger measure $\alpha_1 = \alpha_0 + \alpha_0(1 - \alpha_0)$, then place a copy of C_0 in each of the intervals complementary to the new set, *etc.* By repeating these operations one will produce closed nowhere dense sets $C_0 \subset C_1 \subset \ldots \subset C_n \ldots$ of measures $\alpha_0 < \alpha_1 < \ldots < \alpha_n \ldots$ respectively satisfying $\alpha_k = \alpha_{k-1} + \alpha_0(1 - \alpha_{k-1})$, $k = 1, \ldots, n$. The union E of all these sets will be of first category and have measure α satisfying $\alpha = \lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \alpha_{k-1} + \alpha_0(1 - \alpha_{k-1}) = \alpha + \alpha_0(1 - \alpha)$, *i.e.*, $\alpha_0(1 - \alpha) = 0$, and since $\alpha_0 > 0$, it follows that $\alpha = 1$. (It is easy to see, however, that a *nowhere dense* set cannot be the complement of a set of measure zero, since its complement contains open intervals.)

As for the relations between the three kinds of smallness and the uniqueness problem, partial answers existed only for the measure-theoretic questions in 1905. It was known from LEBESGUE's work of 1903 that a "negligible" set must be of measure zero. (Every set of positive measure contains a closed set of positive measure; the FOURIER series of the characteristic function of a closed set of positive measure converges to zero on the complement of the set by the RIEMANN localization principle for FOURIER series, yet has non-zero coefficients). It was not known, however, whether *every* set of measure zero (for example, the CANTOR set) was negligible. Even less was known for the other two categories of smallness. Although the only known negligible sets were countable, they were also nowhere dense, and it was not known whether a negligible set *had* to be countable or of first category; it was also not known whether every countable set is negligible.

Beyond these facts it was also known that the CANTOR-LEBESGUE theorem applies to sets that are *not* small in the geometric sense, *i.e.*, if a trigonometric series converges on such a set, its coefficients must tend to zero; it was not known, however, whether a CANTOR-LEBESGUE theorem could be proved for sets that are uncountable or of second category.

Finally within set theory itself the question of "middle-sized" sets (in a stricter sense than implied by the definitions just given) was still open: CANTOR's famous continuum hypothesis that the cardinality of the continuum is precisely the cardinality of the set of countable ordinals. It is equivalent to the assertion that any uncountable subset of the real numbers can be placed in one-to-one correspondence with the points of the interval [0, 1]. CANTOR had proved this to be true for *perfect* subsets in a letter to MITTAG-LEFFLER published in *Acta Mathematica* (CANTOR 1884b; the restriction to perfect sets does not seem to be in the German translation of DINI's textbook of real analysis; *cf.* DINI-LÜROTH 1892, p. 27).

Some answers or partial answers to these questions were achieved in the next few years by a number of individual mathematicians working for the most part in isolation from one another.

FATOU (1906) studied the series obtained by formally integrating a trigonometric series *once* and showed that a trigonometric series is the FOURIER series of a LEBESGUE-integrable function if and only if the termwise integrated series converges to an absolutely continuous function. (It follows as a corollary that

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termwise integration of any FOURIER series is legitimate, even if the series diverges at every point!)

W. H. & G. C. YOUNG (1906, p. 64) showed that the continuum hypothesis is true for an "inner limiting" set (a countable intersection of open sets, though defined in a rather complicated manner). That is, they proved that if such a set is uncountable, it has the cardinality of the continuum. This result was also proved by HAUSDORFF, who called such a set a G_δ-set (for *Gebiete* and *Durchschnitt*, *cf.* Hausdorff 1914, p. 305) and extended the result to a countable intersection of a countable union of G_δ-sets, which he called a $G_{\delta\sigma\delta}$-set.

Two years later F. BERNSTEIN (1908) analyzed the proofs of uniqueness to learn the extent to which the various parts of the proofs could be generalized. He found that the main principle was the lemma of SCHWARZ mentioned above. Now this lemma extended easily, in the way already indicated, to intervals with a countable nowhere dense set removed. But even more turned out to be true: the lemma remains valid for an interval with any countable set whatsoever removed (*cf.* DINI-LÜROTH 1892, p. 122). Relying on a result of HÖLDER (1884, p. 183), BERNSTEIN proved that the negligible sets for the lemma of SCHWARZ are precisely those that contain no perfect subsets. (He called them *totally imperfect sets* and proved that uncountable sets with this property exist, thereby incidentally proving that BOREL non-measurable sets exist.) By showing that the lemma of SCHWARZ holds for totally imperfect sets BERNSTEIN thus proved that these sets are negligible for purposes of the uniqueness theorem.

The following year W. H. YOUNG (1909) proved that the CANTOR-LEBESGUE theorem holds when the hypothesis of positive measure ("not being small in measure") is replaced by topological largeness, that is, if a trigonometric series converges except on a set of first category, then its coefficients tend to zero. He deduced also that every countable set is negligible and commented on BERNSTEIN's work (which had not yet appeared in print when he wrote but had been communicated to him in a letter) saying that sets containing no perfect subset, "are, from their very nature, incapable of being defined by any simple law, so that they are unlikely to arise, except in the form of countable sets." He then proceeded to show that the complement of such a set had to be of second category. In all this he was absolutely right as far as applications to trigonometric series go. The set of points at which a trigonometric series either converges to a non-zero limit or diverges is in fact a $G_{\delta\sigma\delta}$ set; hence by HAUSDORFF's theorem mentioned above, if it is uncountable it must contain a perfect subset. Thus BERNSTEIN's uniqueness theorem is not really any more general than YOUNG's.

Yet another year later YOUNG (1910) showed, using the technique of FATOU, that a trigonometric series whose pointwise sequences of partial sums are bounded above and below by LEBESGUE-integrable functions that are finite except on a countable set of points is a FOURIER series.

The last of the results that I wish to cite before discussing the contributions of the Moscow school of analysts is a very fundamental result discovered by VALLÉE-POUSSIN (1916) that is still of use today: *A trigonometric series is a Fourier series if its coefficients tend to zero and the upper and lower limits of its*

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pointwise sequence of partial sums are Lebesgue-integrable functions that are finite except on a set containing no perfect subset. (For uses see KECHRIS & LOUVEAU, 1987, p. 45; as in the case of BERNSTEIN's theorem, the exceptional set, being a $G_{\delta\sigma\delta}$-set, can only be countable.)

§3. The Moscow Analysts

Nearly all the Russian mathematicians of the nineteenth century who were well-known in the West were associated with St. Petersburg; these include P. L. CHEBYSHEV, A. M. LYAPUNOV, V. Ya. BUNYAKOVSKII, A. A. MARKOV, and others. Yet in the years 1900–1910 the University of Moscow possessed a thriving school of differential geometers, the great applied mathematician N. E. ZHUKOVSKII, and many lesser-known mathematicians such as N. V. BUGAEV, B. K. MLODZEEVSKI, and P. A. NEKRASOV, who were of a philosophical bent, and inclined toward mystical associations of mathematical concepts with metaphysical concepts.

For example BUGAEV attempted to construct a theory of discontinuous functions. He called this theory *arithmology* and referred to his own metaphysical system as *evolutionary monadology*; *cf.* DEMIDOV (1985). BUGAEV's son, the poet ANDREY BELYI, mentioned this theory in some beautiful, but nearly untranslatable verses included in his reminiscences of his father (1931, p. 19):

Ты говорил: «Летящие монады
В эонных волнах плещущих времен –
Не существуем мы; и мы – громады,
Где в мире мир трепещущий зажжен . . . »
Твои глаза и радостно, и нежно
Из-под очков глядели на меня.
И там, и там – над нивой безбережной
Лазуриласъ пучина бытия.

The meaning, but not the emotion, of these lines can be rendered as follows:

You were saying, "Fleeting monads
Awash in the primeval waves of the ages –
We do not exist, yet we are the colossi,
The place where the trembling world was kindled . . . "
Your eyes, full of joy and tenderness,
Were peering out at me from beneath your glasses.
And there, and there, over the boundless wheat field,
Arched the azure deep of Being.

A few pages later BELYI wrote that his father was driven from room to room because of his preoccupation with mathematics: "Some men live with friends, some with relatives, some with their wives; my father lived with mathematics."

The philosophical inclination of the Moscow school is well illustrated by the metaphysical views of BUGAEV (*cf.* 1889), who associated discontinuity with freedom of the will. An association between randomness and freedom of the will

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was made by P. A. NEKRASOV, who for many years engaged in an acrimonious polemic with MARKOV on the metaphysical interpretation of probability, filling hundreds of pages of the *Matematicheskii Sbornik* with his own metaphysical views (NEKRASOV 1902). NEKRASOV also wrote a defense of the politically and philosophically conservative Moscow mathematicians (1904), replete with self-adulation.

D. F. EGOROV enrolled as a student at the University in 1887 and eventually took a Ph.D. under BUGAEV's direction, after which he joined the faculty. (For more information and references on the philosophical aspects of the Moscow school see FORD 1991.) We shall say a few preliminary words about the Moscow school of mathematicians in the years 1900–1914 as background for the more technical discussion that follows.

These mathematicians took a keen interest in the philosophical aspects of CANTOR's set theory. Most of the classical works on foundational questions written by BOLZANO, DEDEKIND, CANTOR, KRONECKER, and others were translated into Russian before 1900. MLODZEEVSKII gave a course in real analysis at the University of Moscow as early as 1901. As he continued to teach the course in real functions, MLODZEEVSKII began to bring it more and more up to date. Using DINI-LÜROTH (1892) as the primary text and TANNERY (1904) as a supplement at first, by 1907 he was recommending the monographs by BOREL, LEBESGUE, and BAIRE cited above (*cf.* MEDVEDEV 1963, pp. 51–53).

A book by I. I. ZHEGALKIN on transfinite numbers was published in 1907. In that same year appeared the first Russian textbook on set theory, called *Structure and Measure* (*Строение и Мера*), written by V. L. NEKRASOV[10]. Both of these works were master's theses (*cf.* PHILLIPS 1978, p. 281).

Such was the general atmosphere at the University when N. N. LUZIN enrolled in 1901. LUZIN was already inclined somewhat toward mysticism and mathematics, and he was encouraged in this direction by his friend P. A. FLORENSKII, who had enrolled a year earlier. Both were keenly interested in the current state of set theory. In a letter of 1904 LUZIN gave FLORENSKII a second-hand account of a scene from the International Congress in Heidelberg, as told by A. A. VOLKOV (DEMIDOV, PARSHIN, POLOVINKIN & FLORENSKII 1989, pp. 130–131):

> . . . Cantor announced that he had succeeded in mapping an interval without endpoints onto an interval with endpoints, but he hasn't given the proof yet. Then some other mathematician, a young man, rose and said that the possibility of this mapping [apparently a homeomorphism is meant, though it seems strange that Cantor didn't realize such a mapping is impossible – RC] entails paradoxes and gave arguments in support of this. Cantor, who had listened to him in great agitation, announced that these were the most critical moments of his life, and he would think further . . .

---

[10] A copy of this book was sent to W. H. & G. C. YOUNG and now belong to I. GRATTAN-GUINNESS, to whom I wish to express thanks for a photocopy of the opening pages.

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LUZIN went to the West in 1906, and letters from his mentor EGOROV at this time give an indication of what he did there (ALEKSANDROFF, MEDVEDEV, YUSHKEVICH 1980, p. 340):

> ... What you write about your work in differential equations interests me very much. Do you know an article, by some German, I believe, that also constructs an algorithm for computing the integral of a differential equation ...

In the Central State Historical Archives in Moscow, *fond* 418, *opis'* 513, there is a paper by LUZIN (*edinitsa khraneniya* 4967) that consists of an algorithm for solving a differential equation by successive approximations. The method has some features of the method of CHAPLYGIN (1919). The successive approximations are not assumed integrable; their upper and lower DARBOUX integrals are used to give a recursive definition of the sequence of approximations to the solution. (These sequences converge monotonically.) The peculiar feature of LUZIN's method is that it takes the upper and lower limits of each sequence of successive approximations as initial data for another application of the algorithm, thereby becoming a *transfinite* process. LUZIN showed that this process could terminate at any countable ordinal, though he did not show that the termination would result in a solution. Although this item is not dated, it bears no references later than 1900, and it may be that it is this work that EGOROV is referring to. It is clear from LUZIN's own words that he valued this work because it would be a practical, computational use of transfinite numbers. That his interest lay in this direction is clear from both his correspondence and his diaries and notes, in which he made repeated attempts to settle the continuum hypothesis. However, he did not confine his attention entirely to logic and set theory. In a letter of 1910 to FLORENSKII he wrote (DEMIDOV, PARSHIN, POLOVINKIN, & FLORENSKII 1989, p. 161):

> I'm beginning to work a little in the area of trigonometric series, which for me are only symbolic of some kind of general laws that can be sensed now, but are not yet written down. But in trigonometric series themselves there is much that is interesting. Thus for example the series $\sum_{n=1}^{\infty} \frac{\sin nx}{\log n}$ converges everywhere and consequently represents a function $F(x)$. But it turns out that it isn't a Fourier series. Now until recently it was thought that if a function $F(x)$ can be expanded in a trigonometric series, then that series is in fact its Fourier series, i.e., the coefficients are obtained as follows:
>
> $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} F(x)\,dx\ , \quad a_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\cos nx\,dx\ , \quad b_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\sin nx\,dx\ ,$$
>
> if the series is
>
> $$F(x) = a_0 + (a_1\cos x + b_1\sin x) + \ldots + (a_n\cos nx + b_n\sin nx) + \ldots .$$
>
> The reason for the error, of course, is that in expressing the coefficients of the trigonometric series by the Fourier formulas one assumes *implicite* the possibility of certain operations on a series that one does not have the right to assume in general.
>
> There are other equally interesting things in trigonometric series.
>
> I have discovered the distressing fact that in studying *Mengenlehre* I have fallen behind in other areas; in many of them I really don't understand anything at all and

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in others I understand, but, as they say, execrably. I plan to correct this deficiency over the next two years.

LUZIN's mentor EGOROV, partly because of personal psychological problems LUZIN was experiencing at this time, partly in order to keep him away from revolutionary activity in Russia, sent LUZIN abroad for extended study, so that LUZIN became closely acquainted with the prominent mathematicians of France and Germany. An important epoch in their work, and in the history of analysis as well, occurs in 1910–1911, and a record of it is preserved in EGOROV's letters to LUZIN from this time (*cf.* ALEKSANDROV, MEDVEDEV, & YUSHKEVICH 1980). On 12/25 December 1910 (we give both the Julian date used by EGOROV and the Gregorian date to which it corresponds) EGOROV writes:

My dear Nikolai Nikolaevich,

Of course it would be very good if the theorem happens to be true.

In regard to this, take a look at a paper of Weyl [1909 – RC] in the *Mathematische Annalen*, Vol. 67, where necessary and sufficient conditions are given for *wesentlich gleichmässige Convergenz* of a series in orthogonal functions. Here, it turns out, it is necessary that the series $\sum c_n^2 \sqrt{n}$ converge. But if we give up the uniformity (*wesentlich* means except for a set of measure zero) then maybe your theorem is true.

But for some reason I doubt it. To be specific, it seems very probable to me that a convergent Fourier series (even in [general] orthogonal functions) must also converge uniformly except on a set of measure zero, and then Weyl's result comes into effect.

As for the mechanism expressing a function given by a series $\sum c_n \varphi_n(x)$ in orthogonal functions when $\sum c_n^2$ converges, it was given by Riesz, and after him by Weyl (*Mathematische Annalen*, **69**; Riesz gives the literature), and Weyl gives a rather simple proof. It turns out that the series of sums S_n of the series $\sum c_n \varphi_n(x)$ will converge *im Mittel* and from it one can extract a sequence

$$S_{n_1}, S_{n_2} \ldots,$$

converging to $F(x)$!

The conjecture EGOROV is discussing is almost certainly LUZIN's famous conjecture, made in his dissertation (1915, p. 219): *The Fourier series of a square-integrable function converges almost everywhere.* What EGOROV seems to have in mind is the following argument: Suppose $\sum c_n^2 < +\infty$. By the RIESZ-FISCHER theorem $\sum c_n e^{inx}$ is then the FOURIER series of an L_2-function. If LUZIN's conjecture is correct, then this series converges almost everywhere. Then, if EGOROV's conjecture is correct, the series actually converges *wesentlich gleichmässig.* Hence by WEYL's theorem $\sum c_n^2 \sqrt{n}$ converges. Thus $\sum c_n^2 < +\infty \Rightarrow \sum c_n^2 \sqrt{n} < +\infty$, which is absurd. LUZIN was right, however. His conjecture was finally confirmed fifty years later by L. CARLESON (1966).

Most interesting of all is the letter of 9/22 January 1911:

My dear Nikolai Nikolaevich,

I did prove the proposition I told you about, but not quite the way I stated it. To be specific: *If a sequence of functions converges at every point of an interval, one*

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can extract from it a [sub]sequence that converges essentially uniformly on the entire interval.

Since *essentially uniformly* means *uniformly almost everywhere*, the alleged proposition is false. EGOROV had made a small mistake here, assuming that $\lim \mu(E_n) = 0$ implies $\mu(\limsup E_n) = 0$, and was led into wonderland. He roamed happily in it for nearly three weeks, drawing marvelous conclusions (including a "proof" of LUZIN's conjecture, since it followed from his false theorem that convergence of a sequence in measure implies uniform convergence of a subsequence almost everywhere). As he said in a letter of 22 January/4 February, "In general, it seems, one can obtain good results [from this]." Then, the same day, having apparently received a request from LUZIN for permission to communicate this result, he wrote again, saying that it would be perfectly all right, since he had already sent two notes to the *Comptes Rendus*. In a postscript he laments that a set of measure zero is "still a rather unpleasant thing." It was not until 28 January/10 February that he wrote to LUZIN to point out his "unforgivable mistake." The result, changing essential uniform convergence to almost uniform convergence, is of course what we now call EGOROV's theorem (though in fact both BOREL and LEBESGUE had proved it earlier, *cf.* MEDVEDEV 1975, pp. 105–106; 1991, p. 113).

Against this background, let us take up the contributions of LUZIN and his students to descriptive set theory and the uniqueness problem.

The Luzin Seminar, 1915–1916. LUZIN presented his work *Integration and the Trigonometric Series* as a master's thesis in 1914, but EGOROV found it so impressive that he submitted it as LUZIN's doctoral dissertation, for which it was accepted. In the following year LUZIN, EGOROV, & SIERPIŃSKI (who, as an Austrian citizen, was under arrest in Moscow because of the war, but was allowed to be at liberty and participate in the academic life of the university) conducted seminars on various questions. In one of the seminars the subject was the structure of BOREL-measurable sets. ALEKSANDROV, by an ingenious analysis of the BOREL hierarchy, succeeded in proving that each uncountable BOREL set contains a non-empty perfect subset, hence must have cardinality of the continuum. ALEKSANDROV's proof (1916) involved what was called the A-operation, which labeled each BOREL set with a sequence of pairs of positive integers and suggested the universal space N^N that became the central object in descriptive set theory. SUSLIN, who was also in the seminar at the time, noticed that the A-operation actually led to a larger class of sets than the BOREL sets – the *analytic sets*. It followed that an uncountable analytic set contains a non-empty perfect subset. SUSLIN's main result was that a set is a BOREL set if and only if the set and its complement (called a *co-analytic set*) are both analytic. SUSLIN (and apparently LUZIN) thought of the A-operation as a way of obtaining BOREL sets *effectively* in the sense used by BOREL and LEBESGUE. At the time they were not conscious of the heavy use that was made of the axiom of choice in nearly all areas of mathematics. (When SIERPIŃSKI in 1918 compiled a list of propositions that depend on this result, LUZIN was horrified by the

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length of it.) These results pushed descriptive set theory to the utmost limit that its axioms could reach: the question whether an uncountable co-analytic set contains a non-empty perfect subset turns out to be undecidable on the basis of the axioms. In fact GÖDEL (1938) showed that set theory remains consistent with the simultaneous adjunction of four new axioms, three of which are the axiom of choice, the continuum hypothesis, and the existence of an uncountable co-analytic set containing no perfect subset.

The results of ALEKSANDROV and SUSLIN were published in the *Comptes Rendus* in February 1916 and January 1917, the latter with a supplement by LUZIN stating that any two disjoint analytic sets could be covered by two disjoint BOREL sets and that analytic sets are LEBESGUE-measurable. The articles by SUSLIN and LUZIN did not mention the fact that SUSLIN had been led to his results by having constructed a BOREL set in the plane whose projection onto the line was not a BOREL set, thereby discovering the error in LEBESGUE's memoir of 1905 on analytic representation of functions. In 1930, when LUZIN's monograph on analytic functions was published, LEBESGUE gallantly pointed out this fact in a preface to the work, saying that he considered it the most fortunate error he had ever made.

But LUZIN had by no means lost interest in questions about trigonometric series. While writing his dissertation he had compiled a list of unsolved problems in this area, 52 of which were included as an appendix when his dissertation was published in 1951. Among these was the problem listed as No. 29 by the editors (BARI & MEN'SHOV): *Construct a trigonometric series with coefficients tending to zero that is summable by Fejér's method with sum* 0 *except on a set of measure* 0. In his dissertation (§§ 79–87) LUZIN had given a very thorough review of all the uniqueness results known up to that time and had posed the problem of finding a "canonical" trigonometric series to represent a given function (whenever such representations existed). The conjecture to which the series of Problem 29 would be a counterexample was framed as follows in LUZIN's dissertation (1915, pp. 244–246):

> **83.** At the beginning of this chapter (§ 79) we proved that for every measurable function $f(x)$ there exists a *class* $\{T\}_f$ of trigonometric series that represent the function. It is easy to see that this existence theorem does not solve the Fourier problem posed above, since this problem by its nature requires that a given function be represented not by a *class* of trigonometric series, but by a *single* trigonometric series, and moreover not by just *any* series, but by a completely definite one . . .
>
> **85.** . . . Let us define a *null-series* to be any trigonometric series τ that represents zero, i.e.,
>
> $$0 \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos nx + \beta_n \sin nx \ . \qquad (\tau)$$
>
> . . . Now if we wish to make a choice of a series from the class $\{T\}_f$, we must have some criterion χ, expressed in finite form without using the axiom of choice, which will be satisfied by only one series in the class $\{T\}_f$; let us denote this series by T^{χ}. If we also want some analogy between T^{α} and the Fourier series in the sense of Cauchy, we must subject the criterion χ to the following two axioms:

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I. The only series in the class $\{T\}_0$ of null-series that satisfies the criterion χ is the series all of whose coefficients are zero.

II. If series T_1 and T_2 satisfy the criterion χ, then their difference $T_1 - T_2$ also satisfies χ.

Conversely, each criterion χ subject to these axioms is a *criterion for uniqueness* . . .

86. Likely criteria for uniqueness. Consider a class $\{T\}_f$. Among the series of this class it is natural to consider those that *converge* to $f(x)$ as representing the function $f(x)$ more closely than the divergent series of this class, which are only summable to the function $f(x)$. Hence it is natural to choose a unique series in the class $\{T\}_f$ only from the convergent series of the class, provided there are any such series. Such a choice is all the more natural in that probably there exists only one series in the class $\{T\}_f$ that converges to $f(x)$ almost everywhere. Indeed if there were two distinct series of this kind in the class $\{T\}_f$, their difference would give a series that converges to zero almost everywhere and has nonzero coefficients. But this is improbable by the following propositions:

Cantor's Theorem: *If a trigonometric series converges to zero everywhere, all of its coefficients are zero.*

Young's Theorem: *If a trigonometric series converges to zero everywhere except perhaps at the points of a countable set, then all of its coefficients are zero.*

If this proposition remains true when the set of possible exceptional points is any set of measure zero, then there is only one series in the class $\{T\}_f$ that converges almost everywhere; in this case the choice of a series can be made in a very natural manner . . .

87. If there are no convergent series in the class $\{T\}_f$, then it is natural to choose the series from among those whose coefficients vary most regularly as n tends to ∞. Such series are those with coefficients tending to zero. This criterion of smallness is a generalization of the Riesz-Fischer criterion [for mean-square convergence to a square-integrable function – RC].

It is interesting to note that one can easily exhibit a null-series with *bounded* coefficients: for example the series

$$\tfrac{1}{2} + \cos x + \cos 2x + \ldots + \cos nx + \ldots$$

is such a series. In contrast, no one has yet constructed a null-series with coefficients *tending to zero.* It may be that there are no such series at all (except $a_0 = 0$, $a_n = b_n = 0$). If such is the case, then there exists only one series in the class $\{T\}_f$ with coefficients tending to zero; in this case the choice of such a series is most natural . . .

The conjecture in §86 was settled by MEN'SHOV in a most ingenious manner: he constructed a singular measure concentrated on the CANTOR-type set $C = E_{\frac{1}{4}\frac{2}{6}\frac{3}{8}\frac{4}{10}\ldots}$ constructed by leaving the first and last $\frac{1}{4}$ of an interval (removing the middle half), then leaving the first and last $\frac{2}{6}$ (removing the middle third) in the two intervals that remain, then leaving the first and last $\frac{3}{8}$ (removing the middle fourth) in the four intervals that remain, *etc.* The resulting nowhere dense perfect set gives rise to a continuous non-decreasing function $F(x)$ on the interval from which the set was constructed by excising its complement, as follows: Define $F(x) = \frac{1}{2}$ on the middle half of the interval (which was removed

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at the first stage of the construction of the set); then on the two "middle-third" intervals removed at the second stage set $F(x) = \frac{1}{4}$ and $F(x) = \frac{3}{4}$, *etc.* In this way $F(x)$ is eventually defined as a non-decreasing function on the complement of the set and can be extended to the set by continuity.

MEN'SHOV showed that the coefficients of the FOURIER-STIELJES series of this function, *i.e.*, the numbers

$$c_n = \int_0^1 e^{-2\pi inx}\, dF(x), \quad n = 0, \pm 1, \pm 2, \ldots$$

in the case of the interval $[0, 1]$, tend to zero. (Because of the relation $e^{ix} = \cos x + i \sin x$, any trigonometric series (1) can be written as a series of exponentials and conversely, so that one may use either form of the series.) When these coefficients tend to zero, it is possible to prove that the FOURIER-STIELTJES series

$$\sum_{n=-\infty}^{+\infty} c_n e^{2\pi inx}$$

converges to zero on the intervals on which $F(x)$ is constant, *i.e.*, intervals complementary to C.

Until MEN'SHOV's example was constructed uniqueness theory had really been working in a vacuum, since no one knew of any examples of non-uniqueness when the exceptional sets were "small" (in any of the three senses described above). MEN'SHOV's example was the first non-obvious example of non-uniqueness, and it made the subject at once more difficult and more interesting. It is clear from YOUNG's theorem that a set that is "small" in cardinality is negligible, but MEN'SHOV's result shows that a set that is "small" in the sense of measure might not be negligible for the purposes of trigonometric series representations. MEN'SHOV's results raised some interesting questions: Is cardinality less than the continuum the necessary and sufficient condition for negligibility? Is the ordinary CANTOR set negligible? What role does category play in this problem? (MEN'SHOV's result showed that being of first category – indeed, even being nowhere dense – was not sufficient for negligibility. On the other hand, there are pathological decompositions of the real line into a pair of complementary sets, neither of which contains a perfect subset. By BERNSTEIN's uniqueness theorem each of these sets is negligible. To exclude such pathology, it seems reasonable to ask whether a negligible set having the BAIRE property is of first category.)

MEN'SHOV's result was communicated in the *Comptes Rendus* in October of 1916. From that time on for the next 40 years there was steady progress on the uniqueness problem.

§4. The Years 1920–1955

MEN'SHOV's result showed that although some kinds of smallness are sufficient for a set to be negligible, merely having measure zero is not sufficient. Moreover it pointed out a standard way of proving that a set is not negligible,

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namely by constructing a non-zero FOURIER-STIELTJES series whose coefficients tend to zero and which converges to zero on the complement of the set. Sets whose non-negligibility can be proved in this way are now called *sets of strict multiplicity* (*cf.* KECHRIS & LOUVEAU 1987, p. 3). The question that immediately arose concerned the nature of the standard CANTOR "middle-thirds" set: Was it negligible or not?

The answer to this question came indirectly from LUZIN's seminar. After the war SIERPINSKI returned to Poland (reconstituted as a nation for the first time since the eighteenth century) to participate in the intellectual life of his homeland. This Polish revival was particularly brilliant in the area of mathematics we are now considering. It was A. RAJCHMAN (1922) who gave the answer to the question just posed: Any set of HARDY-LITTLEWOOD-STEINHAUS type is negligible, and the standard CANTOR set is a set of this type. A set of HARDY-LITTLEWOOD-STEINHAUS type, or *H-set* for short, is a set E for which there is an infinite strictly increasing sequence of positive integers $n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$ such that the union (mod 2π) $\bigcup_{k=1}^{\infty} n_k E$ is not everywhere-dense in the interval $(0, 2\pi)$. (This property was first discussed in a paper by HARDY & LITTLEWOOD (1914, Theorem 1.40) on Diophantine approximation. For the standard CANTOR set one can take $n_k = 3^k$ to prove that this CANTOR set is an H-set. RAJCHMAN's result made the subject of uniqueness still more curious, since it showed that cardinality less than the continuum as a property of the exceptional set was not a necessary condition for uniqueness. It showed, moreover, that the line of demarcation between negligible and non-negligible exceptional sets cuts across the nowhere-dense perfect sets. Clearly the difference between the two cases was somehow bound up with the nature of the middle interval removed at each stage. Was it simply a matter of the *size* of this interval, or did the algebraic nature of its endpoints enter into the picture. The latter seemed likely, since, for example, $\sin x$ and $\cos x$ are algebraic numbers if x is a rational multiple of π.

RAJCHMAN was the first of the Polish mathematicians to contribute to this area, and was soon joined by A. ZYGMUND, with whom he published several joint papers. ZYGMUND (1923a) contributed to the clarification of the method of proving uniqueness by connecting RIEMANN's method of summation with that of POISSON (ABEL summation). To be specific, he showed (1923a, p. 523) that if the ABEL/POISSON "partial sums" of a trigonometric series whose coefficients tend to zero are denoted $p(r, x)$, *i.e.*,

$$p(r, x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n,$$

and $F(x)$ is the RIEMANN function, *i.e.*,

$$F(x) = \tfrac{1}{4} a_0 x^2 - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2},$$

then

$$\liminf_{r \uparrow 1} p(r, x) \leqq \limsup_{h \downarrow 0} \frac{F(x+h) + F(x-h) - 2F(x)}{h^2}$$

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and

$$\liminf_{h \downarrow 0} \frac{F(x+h) + F(x-h) - 2F(x)}{h^2} \leqq \limsup_{r \uparrow 1} p(r, x).$$

It results from RAJCHMAN's multiplier method (formally convoluting a trigonometric series with a smooth function) that a trigonometric series with bounded coefficients that is ABEL/POISSON summable to zero uniformly on compact subsets of an open interval actually converges to zero uniformly on compact subsets of that interval. (*Cf.* KECHRIS & LOUVEAU 1987, pp. 62–65.) KECHRIS & LOUVEAU state several conditions equivalent to the vanishing of such a trigonometric series on an open interval, among them the condition that the RIEMANN function $F(x)$ obtained by formally integrating the series twice be a linear function on the interval. It is clear from the proof that another equivalent condition is that the series be Abel summable to zero uniformly on compact subsets of the interval.) The RAJCHMAN multiplier theory is an essential element in the later reformulation of conditions for closed sets to be negligible. This reformulation, using the language of functional analysis, was carried out by PYATETSKII-SHAPIRO (1952, 1954), and was a crucial element in the fundamental result of SALEM & ZYGMUND (1955) on the negligibility and nonnegligibility of sets of CANTOR type.

It seems to have been ZYGMUND (1923b) who first used the phrase *set of uniqueness* to describe a set having the property that the only trigonometric series convergent to zero on its complement is the zero series. Since this name is more euphonious than the name *negligible set* that has been used up to this point, it will be used exclusively from now on (though it might seem more logical to define the *complement* of such a set as a set of uniqueness). ZYGMUND considered two types of sets of uniqueness, which he labeled U_1 and U_2. A U_1-set is a set of uniqueness, as just defined, and a U_2-set is a set E for which only the zero trigonometric series has the properties that 1) the RIEMANN function $F(x)$ is continuous; 2) $p(r, x) = o\left(\frac{1}{1-r}\right)$ as $r \uparrow 1$; 3) the series is ABEL summable to zero on the complement of E. ZYGMUND stated that a countable union of U_2-sets is a U_2-set, and that a 1-point set is a U_2-set.

ZYGMUND's longer memoir on this subject (1926a, b), in which the RAJCHMAN theory of multipliers was used, was crucial for the reformulation of this work by PYATETSKII-SHAPIRO in the 1950's. Of particular importance was the Tauberian-type theorem (1926b, p. 87) mentioned above which states that if a trigonometric series with coefficients tending to zero is ABEL summable to zero on an open interval, then it converges to zero on that interval and uniformly on any closed subinterval of it.

Meanwhile LUZIN's seminar, nicknamed *Lusitania* by its enthusiastic participants, was continuing its work. One of LUZIN's students, N. BARI (1923, 1927) established that the union of countably many closed sets of uniqueness is again a set of uniqueness. She went on to show that there are sets of uniqueness whose intersection with every interval has cardinality C. This result can be deduced from the following facts: 1) the ordinary Cantor set is a set of unique-

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ness; 2) the union of countably many closed sets of uniqueness is a set of uniqueness; 3) any set obtained by translating and/or dilating a set of uniqueness is a set of uniqueness (the MARCINKIEWICZ-ZYGMUND theorem; *cf.* KECHRIS & LOUVEAU, 1987, p. 71). Finally, she showed that there exist perfect sets of uniqueness that are not H-sets, thus removing any easy conjectures about the location of the dividing line between sets of uniqueness and sets of multiplicity among the nowhere-dense perfect sets.

Since there seemed to be no easy measure-theoretic or topological criterion for distinguishing the perfect sets of uniqueness from perfect sets of multiplicity, it began to seem reasonable to consider the algebraic properties of such sets. BARI was the first to do so (1936) and became a strong proponent of the view that the dividing line between uniqueness and multiplicity would have to be drawn using the language of number theory. Her paper (1936) was based on the observation that when a CANTOR set E_η with constant dissection ratio η is constructed and a singular function $\Phi(x)$ is built on this set, the coefficients of the FOURIER-STIELTJES series of $\Phi(x)$ can be expressed in terms of the function $\psi_\varepsilon(t)$ equal to the number of fractional parts in the sequence $\{t\theta^1\}$, $\{t\theta^2\}, \ldots \{t\theta^n\}, \ldots$ satisfying the inequality $\{t\theta^n\} \geqq \varepsilon$. (Here $\theta = 1/\eta$ and $\{x\}$ is the distance between x and the nearest integer to x.) A consequence was that if there is a value of ε for which $\psi_\varepsilon(t) \to \infty$ as $t \to \infty$ (roughly speaking, if the numbers $\{t\theta^m\}$ are badly approximated by integers), then the coefficients of the FOURIER-STIELTJES series tend to zero, and so the set is a set of multiplicity. BARI noted that η has this property if it is rational and has a numerator different from 1 when expressed in lowest terms. The end result of all this was that the only sets of uniqueness E_η with η rational were those with $\eta = \frac{1}{3}, \frac{1}{4}, \ldots$ *etc.*

BARI's result was brilliant; what was needed to give it still further applications was a study of the function $\{t\theta^n\}$ for more general numbers θ. Intuitively it seems very unlikely that the function $\psi_\varepsilon(t)$ can be finite for very many values of ε and t, especially in view of the KRONECKER-WEYL theorem, which asserts that the fractional parts mod 1 of the multiples of any irrational number are uniformly distributed in [0, 1]. It therefore comes as a surprise that this situation can arise for a rather extensive class of numbers θ, those that are now called *Pisot* numbers. PISOT himself (1938) called them ρ-numbers and characterized them as algebraic integers all of whose conjugates lie inside the unit disk in the complex plane. PISOT's memoir is large and occupied with many different aspects of the subject, but he does (p. 238) give the characterization of these numbers as the numbers θ such that for some real number λ the series $\sum\{\lambda\theta^n\}^2$ converges. (It turns out that when this series does converge, the number λ is in the field generated by θ over the rational numbers, and if λ is an algebraic integer, the exponent 2 can be reduced to 1; *cf.* KECHRIS & LOUVEAU 1987, p. 85).

PISOT was not the only one to encounter this class of numbers. ERDÖS (1939), studying the convolution of an infinite number of symmetric BERNOULLI distribution functions $\beta(\alpha^{-n}x)$, remarked that the set of values of a for which this function was absolutely continuous seemed to be oddly distributed, containing $a = (\frac{1}{2})^{1/n}, n = 1, 2, \ldots$, but not containing the golden section number

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$\frac{\sqrt{5}-1}{2}$, which lies in the midst of them. In the light of the work of BARI and PISOT this result becomes more comprehensible, since the reciprocal of the number, $\frac{\sqrt{5}+1}{2}$, is a PISOT number. Any CANTOR set is homeomorphic to the space $\{-1, +1\}^N$, which therefore inherits a natural measure from the FOURIER-STIELTJES measure concentrated on the CANTOR set. At the same time an element of $\{-1, +1\}^N$ can obviously be regarded as a sequence of BERNOULLI trials.

The PISOT numbers were rediscovered independently of PISOT a year later by VIJAYARAGHAVAN (who started his research from approximately the same sources as PISOT). VIJAYARAGHAVAN's first paper (1940) dealt with a function similar to the function $\psi_\varepsilon(t)$ considered by BARI. In his second paper (1941) he defined a class S of numbers identical to PISOT's ρ-numbers, showed that it contains an abundance of members, and gave a characterization of it very similar to that given by PISOT.

These results of number theory were linked with trigonometric series by SALEM over a period of several years (1942, 1943, 1944). In the first of these papers SALEM proved that, in a definite sense, almost all CANTOR-type sets are sets of multiplicity. He subsequently looked more deeply into the exact conditions under which the set was a set of multiplicity. Although there were some slips in this work, they were eventually rectified using the work of PYATETSKII-SHAPIRO (1952, 1954). The result is the SALEM-ZYGMUND theorem (SALEM & ZYGMUND 1955): *A Cantor-type set $P(\xi; \eta_1, \eta_2, \ldots \eta_k)$ is a set of uniqueness (in fact an $H^{(n)}$-set) if $1/\xi$ is a Pisot number and $\eta_1, \ldots, \eta_k$ belong to the field generated by ξ over the rational numbers. Conversely, if $1/\xi$ is not a Pisot number or if one of the numbers η_i, $1 \leqq i \leqq k$, is not in the field generated by ξ over the rational numbers, then $P(\xi; \eta_1, \ldots, \eta_k)$ is a set of multiplicity; in fact it is a set of strict multiplicity*, i.e., *there is a non-zero Fourier-Stieltjes series whose coefficients tend to zero and which converges to zero on the complement of $P(\xi; \eta_1, \ldots, \eta_k)$.*

The notation of this theorem is as follows: The numbers η_i and ξ are arbitrary except for the restrictions $0 < \eta_1 < \ldots < \eta_k < 1$ and $\xi = 1 - \eta_k < \min(\eta_1 - 0, \ldots, \eta_k - \eta_{k-1})$. The set $P(\xi; \eta_1, \ldots, \eta_k)$ is formed by the basic operation of replacing a closed interval [a, b] by $k+1$ subintervals each of length $\xi(b-a)$, having left-hand endpoinds at 0, $(1-\eta_1)a + \eta_1 b$, $(1-\eta_2)a + \eta_2 b, \ldots, (1-\eta_k)a + \eta_k b$. (The condition $\xi = 1 - \eta_k$ guarantees that the point b will be the right-hand endpoint of the last interval.) An $H^{(n)}$-set is a generalization of an H-set defined above and is defined as a set E such that $\bigcup_{\vec{n}} \vec{n}E$ is not dense in the n-dimensional torus T^m for some set of vectors with integer coordinates $\vec{n}$ that is "too thick" to be contained in any finite union of hyperplanes.

Since MEN'SHOV's 1916 example, mathematicians had been concentrating their attention on what seemed to be the decisive battleground in the struggle to characterize sets of uniqueness, namely the nowhere-dense perfect sets, in particular, the sets obtained by regular dissections of CANTOR type. Here at last in the SALEM-ZYGMUND theorem MEN'SHOV's technique seemed to have been

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pushed to its limits. For the nowhere dense perfect sets constructed by a regular dissection technique and the associated singular measures, nearly complete information was now available[11] as to the nature of such a CANTOR set and its associated FOURIER-STIELTJES series. The answer seemed to be a vindication of BARI's belief that it is the arithmetic properties of sets that determine their character as sets of uniqueness or of multiplicity. Yet closer examination will show that this conclusion is justified only for CANTOR-type sets, and even for a very special class of these sets, namely those in which the dissection procedure is the same at each stage. This is very far from a general characterization even of CANTOR sets. In fact a result of MEYER (1972) showed that for a simple repeated dissection with the interval $[a, b]$ at the i^{th} stage being replaced by $[a, a + \xi_i(b - a)] \cup [b - \xi_i(b - a), b]$ the resulting CANTOR set is a set of uniqueness if $\sum \xi_i^2 < \infty$, so that no *general* characterization of even simple CANTOR sets as sets of uniqueness or multiplicity in terms of arithmetic properties is possible.

It should not be concluded, however, that the SALEM-ZYGMUND theorem was merely the end of a chapter. In fact it is an essential tool making it possible to construct a curve $\gamma(t)$ in the space of closed subsets of the circle (see below) such that $\gamma(t)$ is a set of uniqueness when t is rational and a set of multiplicity when t is irrational (*cf.* KECHRIS & LOUVEAU, 1987, pp. 125–126). This curve, in turn, is a key element in proving a theorem about the class of sets of uniqueness, as our final section will show.

Before concluding the story with the latest contributions of descriptive set theory to the problem, let us summarize what has been discussed so far.

1. Every countable set is a set of uniqueness, but there are uncountable sets of uniqueness.
2. Every set of uniqueness is of measure zero, but there are sets of measure zero that are sets of multiplicity.
3. The "borderline" cases are CANTOR sets obtained by repeatedly replacing closed intervals with unions of their closed subintervals in such a way that no endpoints are ever discarded. For the geometrically simplest of these, those in

[11] I say "nearly complete" for the following reason: The proof of the part of the SALEM-ZYGMUND theorem that asserts multiplicity involves the construction of a measure whose FOURIER–STIELTJES coefficients tend to zero. There are two cases, according as ξ is not the reciprocal of a PISOT number or some η_i is not in the extension of the rational numbers generated by ξ. In the latter case the theorem is proved by showing that actually $P(\xi; \eta_i, 1 - \xi)$ is a set of multiplicity, and this in turn is done by constructing a measure $\mu_{\eta i, \xi}$ concentrated on $P(\xi; \eta_i, 1 - \xi)$ whose FOURIER-STIELTJES coefficients tend to zero unless ξ^2 is the reciprocal of a PISOT number and η_i belongs to the field generated by ξ. This proof thus leaves open the question of the behavior of these coefficients if ξ^2 is the reciprocal of a PISOT number but ξ is not (*e.g.* $\xi = 1/\sqrt{2}$). Thus the SALEM-ZYGMUND theorem does not decide the behavior of all FOURIER-STIELTJES *series* constructed by MEN'SHOV's technique, though it does settle the basic question of uniqueness or multiplicity of the *sets* involved.

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which the dissection process is the same at each stage, a complete characterization is given in terms of the number-theoretic properties of the parameters that define the procedure.
4. A countable union of closed sets of uniqueness is a set of uniqueness.
5. The only non-pathological *examples* of uncountable sets of uniqueness are CANTOR sets, and these are all $H^{(n)}$-sets. PYATETSKII-SHAPIRO (1954) showed, however, that there are perfect sets of uniqueness that are not of this type.
6. The only examples of closed sets of multiplicity of measure zero are CANTOR sets, and in every case the fact that the set is of multiplicity is deduced from the existence of a measure that is concentrated on the set and whose FOURIER-STIELTJES coefficients tend to zero. A set of this type is said to be of *strict multiplicity*, as mentioned above. Conversely a set, possibly not of uniqueness, is called a set of *extended uniqueness* if no such measure exists.
7. Except for point 1, above, we know almost nothing about *non-closed* sets in terms of uniqueness or multiplicity.

In view of all these facts, we seem to have almost no intuitive insight into the structure of sets of uniqueness. What we have is a mixture of measure-theoretic and topological criteria, together with a criterion from number theory (the SALEM-ZYGMUND theorem) that, in view of MEYER's result, cannot possibly provide a general characterization of the closed sets of uniqueness. What we need to do is look at our results from a different perspective to see if they are leading us toward a solution of the characterization problem we wish to solve. To achieve this perspective it would seem prudent for mathematicians 1) to confine their attention to *closed* sets until they have all been characterized as sets of uniqueness or multiplicity (of course they will have to be nowhere dense perfect sets to be of any interest); 2) to begin by finding characterizations of these sets as sets of strict multiplicity or extended uniqueness. It turns out that when step 1) is taken, descriptive set theory comes to our aid and provides the perspective we were seeking. The answer, it is true, is not comforting, but at least we have a clear idea of where we stand. The final section of this narrative is intended to show how this situation came about over the last two decades.

§5. The Years 1970–1985

Restricting attention to closed subsets of the circle, as suggested in the preceding section, opens up new possibilities for the application of descriptive set theory, since the set of closed subsets of a compact metric space forms the basic object of study in descriptive set theory: a *Polish space* (a separable topological space whose topology can be defined by a metric in which the space is complete). The fundamental Polish space is the space of sequences of positive integers N^N (mentioned above in connection with the work of ALEKSANDROV & SUSLIN). In the case of a compact metric space the closed subsets form another compact metric space with the metric known as the *fractal* metric (the distance between two sets is the smallest positive number t such that each point

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of either set is no farther than t from at least one point of the other). In the context of a Polish space all of the standard results on the BOREL hierarchy, analytic sets, co-analytic sets, *etc.*, carry over from LUZIN's monograph of 1930. The advantage of the new context is that it enables us to consider a *class* of subsets, for example, the closed sets of uniqueness or the closed sets of extended uniqueness, as a *subset* of this new space. We can then ask what kind of set it is, *i.e.*, where it lies in the BOREL or projective hierarchy. This approach leads to metatheorems that say things like, "If such-and-such a condition were necessary and sufficient for a closed set to be a set of uniqueness, then the closed sets of uniqueness would form a BOREL set in the Polish space of closed subsets of the circle." If it can be proved that the sets of uniqueness do not form a BOREL set, it will follow that the condition is not necessary and sufficient. For example, we know that H-sets are sets of uniqueness. It can be shown that the closed H-sets form a set of type $G_{\delta\sigma}$ in the space of closed subsets of the circle. If the sets of uniqueness do not form a BOREL set in this space, there must be sets of uniqueness that are not H-sets. This type of reasoning, in more subtle and elaborate form, is the new way of looking at the uniqueness problem that has recently attracted attention in several centers of research.

Recent Results. As described by KECHRIS & LOUVEAU (1987, pp. 8–9), the problem of characterizing the closed sets of uniqueness was mentioned to KECHRIS in a conversation in 1982. (In a letter to me KECHRIS said that the conversation was with S. PICHORIDES.) Since it is easy to prove that the closed sets of multiplicity form an analytic set, the complexity of the set of closed sets of uniqueness hinges entirely on whether or not it is itself analytic. If it is analytic, which seemed unlikely, it would be a BOREL set, and then the problem would be to find its rank in the BOREL hierarchy. KECHRIS conjectured that it was not analytic, and was proved right in 1983 by R. SOLOVAY[12], who further proved that the extended sets of uniqueness are also not analytic. Thus the class of closed sets of uniqueness is co-analytic (its complement is the continuous image of the space N^N) but not analytic. By SUSLIN's theorem, this set is not a BOREL set in the Polish space of closed subsets of the circle.

The same result was reached by R. KAUFMAN (1984), following another route and independently of SOLOVAY's work. This result, according to KECHRIS & LOUVEAU (1987, p. 9), "rules out the possibility of any characterization of when an arbitrary closed (or perfect) set is [a set of uniqueness] or [a set of strict multiplicity] in terms of geometric, analytic, or number-theoretic properties of the closed set which are of 'explicit enough' form so that they can be expressed by countable operations, given any reasonable specification of the set, like *e.g.*, in terms of its contiguous intervals." Thus the *set-theoretic* complexity of the object being studied implies that no explicitly computable characterization of it is likely to be achieved. This result is not good news, but at least it prevents

[12] I have been made unable to find SOLOVAY's proof of this result. KECHRIS & LOUVEAU (1987) cite only a private conversation for it.

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mathematicians from believing that the problem can be answered in terms of standard concepts and techniques.

Although the proof of the SOLOVAY-KAUFMAN theorem is too long to be discussed in any detail (*cf.* KECHRIS & LOUVEAU 1987, pp. 123–127), it is worth noting that the proof consists of three main parts. First, the SALEM-ZYGMUND theorem is used to construct a "curve" in the space of closed subsets of the torus, *i.e.*, a continuous image of the interval [0, 1] such that the image of each rational number is a set of uniqueness and the image of each irrational number is a set of multiplicity. (To be specific, the image of the point x is the CANTOR trisection set $E[\frac{1}{4}; \frac{3}{8} + \frac{x}{9}, \frac{3}{4}]$.) Second, the recently developed technique of BOREL games is used[13] to prove that the class of compact subsets of the rational numbers in [0, 1] is $\underset{\sim}{\Pi}^1_1$-complete, which by definition means that 1) the set is co-analytic and 2) every co-analytic set in any Polish space whatever is the inverse image of it under some BOREL-measurable function. (Such a set obviously cannot be analytic.) Finally, the curve constructed is used to define a continuous mapping under which the set of compact subsets of the rational numbers in [0, 1] is the inverse image of the set of closed sets of uniqueness. It follows that the closed sets of uniqueness also form a $\underset{\sim}{\Pi}^1_1$-complete set, which is therefore co-analytic but not analytic. The use of the SALEM-ZYGMUND theorem here is a beautiful example of the elegant use of a particular, seemingly special, case to deduce a general result about an entire class of objects. To that extent it mirrors the theme with which I opened the introduction to this paper.

Despite the pessimistic conclusions resulting from the application of descriptive set theory to uniqueness, there are positive aspects of this knowledge as well. In a letter to me A. KECHRIS has pointed out that knowing the exact position of the class of closed sets of uniqueness (co-analytic, but not analytic) in the projective hierarchy is a useful result and has compared it to the theorem that the effectively enumerable sets of positive integers are exactly the Diophantine sets, another positive result with negative implications (it precludes the possibility of finding an algorithm for solving Diophantine equations).

More progress has been made recently on the topic of uniqueness, and indeed some of this progress is precisely due to the application of descriptive set theory. Let us conclude by listing these results.

1. In 1981 N. N. KHOLSHCHEVNIKOVA generalized (in a certain sense) BARI's result on countable unions of closed sets of uniqueness, proving that it is consistent with and independent of the axiom system ZFC to assume that any union of less than a continuum of closed sets of uniqueness is a set of uniqueness. (In particular any set of cardinality less than the continuum is a set of uniqueness, which is just BERNSTEIN's theorem.) At the same time she proved that the union of any two *disjoint* G_δ-sets of uniqueness is a set of uniqueness, and that if A is a set of uniqueness and B is both a G_δ- and an F_σ-set of

[13] A. KECHRIS has pointed out to me that the use of Borel games is not essential to prove this result, which had been proved earlier by HUREWICZ.

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uniqueness, then $A \cup B$ is a set of uniqueness. Both of these last two results turn out to be corollaries of a later result of CARLET & DEBS (1984) that generalizes BARI's result. In the CARLET-DEBS theorem the sets of uniqueness need not be closed absolutely; it suffices that they be relatively closed in their union (*cf.* KECHRIS & LOUVEAU 1987, p. 44).

2. DEBS & SAINT RAYMOND (1987) proved that any set of uniqueness having the BAIRE property is of first category. Thus, except for set-theoretic pathology, a set of uniqueness must be small in both measure and topology.

In these last few paragraphs the historical perspective has necessarily been lost because of the nearness of the events reported. Nevertheless one very broad conclusion does seem justified: Descriptive set theory is more than a technique for studying the structure of sets. It acts also in the metamathematical realm to provide a guide to what can and what cannot be proved. In this connection its future collaboration with uniqueness theory seems assured. After all, mathematicians have hardly begun the daunting task of proving uniqueness theorems for *multiple* trigonometric series.

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$$f(x) = \sum_{p=0}^{p=\infty} (a_p \cos \cdot px + b_p \sin px)$$

die Werthe

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\alpha f(\alpha), \quad a_p = \frac{1}{\pi} \int_{-\pi}^{+\pi} d\alpha f(\alpha) \cos \cdot p\alpha, \quad b_p = \frac{1}{\pi} \int_{-\pi}^{+\pi} d\alpha f(\alpha) \sin \cdot p\alpha$$

haben, jedesmal wenn diese Integrale endlich und bestimmt sind," *Abhandlungen der Königlichen Bayerischen Akademie der Wissenschaften*, Zweite Klasse, **XII**, Erste Abteilung, 117–166.

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$$\sin\cdot\phi^{\lambda} + \sin\cdot 2\phi^{\lambda} + \sin\cdot 3\phi^{\lambda} + \cdots + \sin\cdot n\phi^{\lambda}$$

$$\cos\cdot\phi\lambda + \cos\cdot 2\phi^{\lambda} + \cos\cdot 3\phi^{\lambda} + \cdots + \cos\cdot n\phi^{\lambda},"$$

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