Continuity and effectiveness in topoi

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PLEASE NOTE THAT THE BIBLIOGRAPHY IS STILL MISSING. CHECK THE FILE IN A WEEK OR TWO TO GET A COMPLETE VERSION. I HAVE NOT CHECKED THE TEXT THROUGH YET.

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Introduction

The categorical approach to recursion theory has produced very interesting results in such areas as the study of higher-type structure [Hyl77b, Sco80a, LS83] (the last contains also references to earlier work) and the study of models of computation [Sco82, Plo85]. The theory concentrates on the idea of "computation" but often discards "effectiveness", although this notion should be represented in the theory: any mechanizable computation is certainly effective. Consequently, it is also necessary to represent the possibility of failure: a computation may diverge on some input. Therefore, it seems to be an inescapable conclusion that the theory must include a notion of partiality.

At the present time, the development of the categorical approach to effectiveness presents two major directions: one focuses on Eršov's category of enumerated sets and the study of its full subcategories (cf. [Smy77, Vis80, LM84c]). The essential structure of effectivity is determined by the maps rather than by the objects: a map $f : \{a_n\}_{n\in\mathsf{N}} \longrightarrow \{b_n\}_{n\in\mathsf{N}}$ between two enumerated sets is a function between the indexed sets such that the equation

$$f(a_n) = b_{g(n)}$$

holds for some recursive function g. Many elementary facts of recursion theory can be expressed as properties of particular objects of the category. But, in order to study particular structures, one often has to refer back to the set N of indices, resulting in a rather heavy use of notation, and the main effort in proofs is usually concentrated in showing that some constructed set is an enumerated set.

The other approach rejects altogether the idea that objects have to be enumerated in order to be given some effective structure, and it insists instead on having effectiveness as an internal feature of the whole universe. Rather than structuring sets with enumerations in the classical universe, one considers sets in an intuitionistic universe where a property such as "all functions on the set of natural numbers are recursive" may hold. The first example of this idea is the realizability model of [Kle45] (cf. [Tro73, McC84]), but it found its category-theoretic setting in [Hyl82] where the effective topos $\mathcal{F}$ is introduced.

The effective topos has many desirable properties: the functions on the natural numbers are all recursive (Church's Thesis), and the ω-Axiom of Choice and Markov's Principle hold. Also, the category of enumerated sets is fully embedded in the effective topos as is the category PER of partial equivalence relations on the applicative structure on N. Martin Hyland and Eugenio Moggi announced that the interrnal full subcategory on stable quotients on N (the internal version of PER) is closed under internal arbitrary limits, a result suggesting very interesting developments in the theory of polymorphic types.

Concerning the abstract theory of partiality, categories of partial maps appeared as examples of monoidal categories which were iatroduced by [EK66], but the main interest of the theory concerned categories of modules over a ring, and no real development of a theory for partiality came at that time. Much later [PH86] reconsidered the problem. In a series of lectures in Siena in 1984, they presented dominical categories in order to give a basic category-theoretic setting for an abstract study of recursion theory. At the same time, Gordon Plotkin made explicit the reason why Scott-continuous functions between complete partial orders are partial: his method was simply to remove the bottom element (which represents the "undefined" va]ue) from each complete partial order, and consider what is left of the continuons functions between them. A first acconnt of a general category-theoretic treatment of this idea was considered by [LM84a], who drew categorical intuitions from the work of DiPaola and Heller.

We are convinced that in a treatment of computations it is convenient and instructive to have an intuitionist's attitude. Support for this approach rests on many works on intuitionistic mathematics. The work of this thesis is particularly influenced by the approach advocated by Dana Scott and his collaborators

[Sco79, Sco80b, Fou80, Gra79, McC84, Hyl82]. Our approach makes strong use of topos theory, which we think is the best expression of the link between category theory and intuitionism. Category theory has to do with closure properties of families of maps and with the characterization of maps and spaces in terms of mapping properties. Definability of these map is often important. Intuitionistic logic treats existence in terms of explicit constructions; hence, definability is again important. The conservative translation of category theory into intuitionistic logic makes the harmony between these theories precise. As we remarked, intuitionistic logic has models in which maps on the natural numbers are always computable (again, this is a kind of definability), so we combine under one point of view both algebraic explicitness and computational notions.

An important question that has to be answered concerns the treatment of partiality in an intuitionistic world. We found an answer when we proved a representation theorem for dominical categories: every dominical category is a full subcategory of a category of partial maps on a topos [Ros85]. One can paraphrase this result as follows: every dominical category is a category of sets and partial functions. In the proof of that result it became clear how an intuitionist could choose to define partial functions: select a subset Σ of the set Ω of truth-values such that the truth-value true $\in \Sigma$, and accept as partial functions those whose domains of definition are classified by Σ. By this we mean that a partial function $\varphi : X \longrightarrow\!\!\!\!\!\!\circ\ \ Y$ is accepted if and only if

$$\forall x \in X.|x \in \mathrm{dom}\,\varphi| \in \Sigma.$$

In order to obtain a category of partial functions, Σ must satisfy the further condition that

$$p \in \Sigma \wedge [p = \mathrm{true} \rightarrow q \in \Sigma] \rightarrow (p \wedge q) \in \Sigma$$

With a questionable choice of name, we call such an object a dominance, because it classifies the domains of the partial functions. A final word about specializing these considerations to the classical case: in the topos $\mathcal{S}$ of classical sets, the two possible choices for a dominance give rise eactly to the two categories of total functions and of partial functions on sets.

The interesting choices of dominance are found in non-Boolean topoi. For instance, in the effective topos $\mathcal{F}$ we can consider the set of truth-values which answer a recursive question:

$$\Sigma = \{p \in \Omega : \exists f \in \mathsf{N}^{\mathsf{N}}.p \leftrightarrow (\exists z \in \mathsf{N}.f(z) = O)\}.$$

This is a dominance. In $\mathcal{F}$, all functions on the natural numbers are recursive, and the request $f(z) = O$ represents a generic recursive statement. (We first saw this definition in a lecture by Dana Scott in Pisa in the fall of 1983).

We argue beginning in Chapter 5 that properties of Σ make the relation between continuity and effectiveness of functions apparent. We discuss the study of the known models for computation and elaborate on their mutual relationships, in order to discuss the usefulness of having a universal notion of "recursively enumerable". In Chapter 8 we gather ideas from the results we obtain about the models to explain a possible synthetic approach to the theory of computation.

To sustain our arguments in favour of such an approach, we must first explain the elementary theory for partial maps. To the best of our knowledge, what we present is original, and we compare it to the literature already available. In particular, in the generality of our treatment, we can answer some open questions presented in [PH86]. We then continue by presenting the general theory of dominance. We do not wish to claim that this is the way it has to be done. But, it is certainly very simple; it includes all the other theories; it has straightforward applications to some important categories of partial maps. We already explained that we are convinced that the use of this notion when looking at the models considerably helps intuition. And there is a very clear sense in which a dominance gives rise to continuity of functionals.

In Chapter 1 we review the logical expressive power in a topos and discuss connected problems in Grothendieck topoi. In Chapter 2 we introduce p-categories as an abstraction of categories of partial maps and prove a completeness result for these. We compare our definition to others from the literature, consider a few common constructions of category theory in the context of p-categories, and solve some problems posed in [PH86] about stability of ranges and coproducts.

In Chapter 3 we produce a canonical presentation of a category of partial maps in a topos by means of a dominance . We prove there is an algebraic characterization for dominance and produce various embedding

theorems of an arbitrary p-category into categories of partial maps defined by means of dominances. Regarding the p-category as sitting inside a topos simplifies the study of various constructs which are essential for an abstract approach to recursion. In particular, we prove a general continuity principle for topological spaces generated by the families of subets classified by Σ.

In Chapter 4 we study higher-type objects and Turing morphisms in a topos and in a general p-category, inproving on results in the literature. We also present a λ-calculus for partial functions (which is related to the ideas presented by [Plo85]) and prove a completeness theorem for such theories.

In Chapter 5 the abstract theory of dominance and partial maps is applied to partial recursive functions. Our approach is to produce an abstract theory of computation which can be applied to various topoi, and we prove that in this setting all maps between ω-algebraic posets are Scott-continuous.

In Chapter 6 we study two topoi which are strongly related to recursion theory: the effective topos $\mathcal{F}$ (studied by [Hyl82]); and the recursive topos $\mathcal{R}$ (introduced by [Mul81]). The first is in essence Kleene's realizability interpretation. The second is constructed in a way which resembles a topological category. They both embed Eršov's category of enumerated sets, which is also a very good enviromnent for a treatment of computability (cf. [Erš73]). We give new presentations for the recursive topos which enable us to characterize the theory it classifies, and we produce a functor from the effective topos into the recursive topos which is useful in connecting the two different models of constructive mathematics. Finally, we use all this machinery to attack the problem of higher-order functionals, producing a new, very simple, proof of a theorem of [LM84c]. We are convinced that our approach sheds much light on the subject, as we try to explain in our exposition.

In Chapter 7 we study topological applications of the general theory of Chapter 3, showing that there is a topos in which all maps from a certain, non-trivial subobject of $\mathrm{P}\,\omega$ into itself are Scott-continuous. Finally, we give a new proof of Fourman's result that there is a topos where all real maps of many real variables are continuous.

In Chapter 8 supported by the various examples, and sustained by some key suggestions from Dana Scott, we introduce and develop some ideas for a synthetic theory of computation, based on an object of r.e. truth-values. The idea is to make the dominance the central core of the theory: the definitely more direct approach to effectiveness is an encouragement to do so. Also, one would like to restrict the universe (=the topos, the p-category) and give a straightforward interpretation of the constructs useful for the theory of computation, such as recursive definitions and fixed-point operators. Dana Scott suggested to concentrate the attention on the notion of σ-set as those objects which are completely determined by their Σ-classified subsets. These special objects carry a canonical partial-order structure. Moreover, the effective complete partial orders are σ-sets, and many of their properties can be generalized to σ-sets.

Chapter 1

The logic of a topos

In this chapter we intend to review some of the concepts of category theory that are needed to present our research. Particular attention is to be devoted to topoi: their relationship with intuitionistic logic will be an essential tool for some of our work.

The literature on category theory is extensive, and the reader is referred to books like [Mac71], or [Mit65]. There are also many references for topos theory itself, for instance [Tie73], [Wra75], [Joh77], [BW85], as well as references for the study of the connections between topos theory and intuitionistic logic, see [Law72a], [Osi75], [MR77], [FS79], [Fou77], [BJ81]. Rather than choosing some among the many references, we believe it is better to gather in one chapter all the various facts we shall need, and to establish notation. We do not discuss the basic notions of category theory: a clear and quick review of basic category theory is chapter 1 of [BW85], extensive presentations are in [Mac71] and [HS73]. We refer to any of these for basic definitions such as *category*, *functor*, *natural transformation*, *monomorphism*, *epimorphism*, *isomorphism*, *subobject*, *limit*, *colimit*, and *adjoint functors*. The notations that we adopt for these concepts are standard: we use double-tailed arrows $\rightarrowtail$ for monomorphisms and thick-pointed arrows $\twoheadrightarrow$ for epimorphisms. We shall also conform with the standard confusion about subobjects and identify a subobject $[m : A \rightarrowtail B]$ with one of its representative monomorphisms. We often indicate such a subobject simply as $A \rightarrowtail B$.

We begin with the definition of *topos*, and discuss some elementary properties about powerobjects. we explain the interplay with intuitionistic logic introducing the logical theory of the topos, and discuss it by comparing logical constructs and their categorical counterparts. Internal topologies are then considered: the construction of the topos of sheaves is sketched briefly. We concentrate on transfer properties for closed subobjects and separated objects as presented in [Hyl82]. In doing this we emphasize no particular examples, but the reader may always check the example of the category $\mathcal{S}$ of sets and functions (preferably in intuitionistic logic!). Examples abound in the second part where we discuss logical properties of particular classes of topoi, such as Grothendieck topoi and subcanonical topologies. In particular, we recall the definition of sheaf on a site, and the forcing relation, and discuss properties of particular subcanonical topologies.

1.1 Basic notions of topos theory

Categorical discussion. The notion of elementary topos was introduced by F.W. Lawvere and M. Tierney in their attempt to study sheaf theory axiomatically. The definition of topos they introduced has been refined during the years, so that the one we give appears, at first sight, weaker. One thing worth mentioning is that topoi first appeared in the work in algebraic geometry of Artin, Grothendieck, and others, see [GV72]; it may seem surprising that they were usefully related to intuitionistic logic. And our main goal is to use them in relation with recursion theory. In the bibliography, we listed references to some of the important works in topos theory though we shall not use them in this brief description.

Definition. A *topos* $\mathcal{E}$ is a category which has finite limits and powerobjects.

A powerobject of an object A is an object $\mathrm{P}(A)$ with a relation

$$\in_A \rightarrowtail A \times \mathrm{P}(A),$$

which is universal, in the sense that for every relation $R \rightarrowtail A \times B$, there is a unique map $r\colon B \longrightarrow \mathrm{P}(A)$ such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{s} & \in_A \\ \downarrow & & \downarrow \\ A \times B & \xrightarrow{r} & A \times \mathrm{P}(A) \end{array}$$

is a pullback for an appropriate, forcedly unique, map $s\colon R \longrightarrow \in_A$.

A particular instance of powerobject is $\mathrm{P}(1)$. As we know that for every object A, it is the case that $1 \times A \simeq A$, the universal property can be restated as follows: for all $R \rightarrowtail A$, there is a unique map $r\colon A \longrightarrow \mathrm{P}(1)$ such that

$$\begin{array}{ccc} R & \xrightarrow{s} & \in_1 \\ \downarrow & & \downarrow \\ A & \xrightarrow{r} & \mathrm{P}(1) \end{array}$$

for some appropriate $s\colon R \longrightarrow \in_1$. It is easy to prove that $1 \simeq \in_1$. Indeed, for every object A and every map $s\colon A \longrightarrow \in_1$, the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{s} & \in_1 \\ \mathrm{id}\downarrow & & \downarrow\epsilon \\ A & \xrightarrow{\epsilon \circ s} & \mathrm{P}(1) \end{array}$$

is a pullback. By the property of the powerobject $\mathrm{P}(1)$ there must be exactly one such. So $\in_1$ is a terminal object. There is a conventional notation for the particular powerobject $\mathrm{P}(1)$, and we shall follow the convention and denote it by Ω. Also, as $1 \simeq \in_1$, it is confusing not to mention the name of the monic map: we call it $\mathrm{true}\colon 1 \longrightarrow \Omega$, or "true", and often confuse the map with its domain denoting it as $\mathrm{true} \rightarrowtail \Omega$

A last remark is about the natural transformation $\mathcal{E}(-,\Omega) \stackrel{\cdot}{\rightarrow} \mathrm{Sub}$ which is the family of functions

$$\begin{array}{rcl} \mathcal{E}(A,\Omega) & \rightarrow & \mathrm{Sub}(A) \\ r\colon A \longrightarrow \Omega & \mapsto & r^{-1}(\mathrm{true}) \end{array}$$

where $\mathrm{Sub}(A)$ is the set of subobjects of A. The property of the powerobject states exactly that that transformation is an isomorphism. Given a subobject $R \rightarrowtail A$, the unique map from A to Ω corresponding to it will be denoted by χ_R and called the *characteristic map* of $R \rightarrowtail A$. For this reason, the object Ω with the map $\mathrm{true}\colon 1 \longrightarrow \Omega$ is called the *subobject classifier*.

Logical notation. The canonical example of a topos is the category $\mathcal{S}$ of sets and functions. And, like the canonical example, it is possible to introduce a notation that helps in understanding the sense in which a topos is a universe of intuitionistic set theory.

Suppose T is a type theory whose types are the objects of $\mathcal{E}$, and terms are built as follows:

(a) o is a term of type 1;

(b) a variable $x \in A$ of type A is a term of type A;

(c) if t is of type A and v of type B, then $\langle t, v\rangle$ is of type $A \times B$;

(d) if w is of type $A \times B$, then $p(w)$ is of type A and $q(w)$ is of type B;

(e) if t, v are of type A, then $|t = v|$ is of the type Ω;

(f) if τ is of type Ω, then $\{x \in A : \tau\}$ is of type $\mathrm{P}(A)$, with the variable $x \in A$ *bound* in $\{x \in A : \tau\}$;

(g) if τ is of type $\mathrm{P}(A)$ and t is of type A, then $|t \in \tau|$ is of type Ω;

(h) if t is of type A and $f\colon A \longrightarrow B$ is a map in $\mathcal{E}$, then $f(t)$ is a term of type B.

Condition (iv) is a particular case of (viii), but we preferred to spell it out. A *free* occurrence of the variable x is one which is not bound. The vertical bars in $|t = s|$ are just used as delimeters: we may use them to enhance readability.

Notation. Abbreviations are introduced as follows:

(a) $\mathrm{true} = \{x \in 1 : x = \mathrm{o}\}$;

(b) $\sigma \wedge \tau = |\langle \sigma, \tau \rangle = \langle \mathrm{true}\,, \mathrm{true}\,\rangle|$.

(c) $\sigma \rightarrow \tau = |\sigma \wedge \tau = \sigma|$;

(d) $\forall x \in A.\ \phi = |\{x \in A : \phi\} = \{x \in A : x = x\}|$

(e) $\sigma \vee \tau = \forall x \in \Omega.\ [\sigma \rightarrow x] \wedge [\tau \rightarrow x] \rightarrow x$;

(f) $\bot = \forall x \in \Omega.\ x = \mathrm{true}\,$;

(g) $\exists x \in A.\ \phi = \forall z \in \Omega.\ \forall x \in A[\phi \rightarrow z] \rightarrow z = \mathrm{true}\,$.

where σ, τ and ϕ are of the type Ω.

The canonical interpretation. We are going to show that there is an interpretation of the terms into maps in the topos $\mathcal{E}$, and that this satisfies the axioms of intuitionistic set theory. A term t of type A will be interpreted with respect to a string of variables $\mathbf{x} = x_1, \ldots, x_n$ containing all the free variables of t, and its interpretation will be a map $t_{\mathrm{x}}\colon A_1 \times \ldots \times A_n \longrightarrow A$ where $A_1, \ldots, A_n$ is the string of types of the variables in $\mathbf{x}$.

1.1.1 **Definition** The interpretation is given by induction on the formation of a term t:

(i) $\mathrm{o}_{\mathrm{x}}\colon \mathrm{A}_1 \times \ldots \times \mathrm{A}_{\mathrm{n}} \longrightarrow 1$;

(ii) $(x_i)_{\mathrm{x}} = p_i\colon A_1 \times \ldots \times A_n \longrightarrow A_i$ is the i-th projection, if A_i is the type of x_i;

(iii) $\langle\rangle \mathbf{x}t, v = \langle t_{\mathrm{x}}, v_{\mathrm{x}}\rangle\colon A_1 \times \ldots \times A_n \longrightarrow A \times B$;

(iv) $p(w)_{\mathrm{x}} = p \circ w_{\mathrm{x}}\colon A_1 \times \ldots \times A_n \longrightarrow A \times B \rightarrow A$,
$q(w)_{\mathrm{x}} = q \circ w_{\mathrm{x}}\colon A_1 \times \ldots \times A_n \longrightarrow A \times B \rightarrow B$;

(v) $(t = r)_{\mathrm{x}} = \chi_{\Delta} \circ \langle t_{\mathrm{x}}, r_{\mathrm{x}}\rangle\colon A_1 \times \ldots \times A_n \longrightarrow A \times A \rightarrow \Omega$, where $\Delta = \langle \mathrm{id}\,, \mathrm{id}\,\rangle\colon A >\!\!\longrightarrow A \times A$;

(vi) $\{x \in A : \tau\}_{\mathrm{x}}\colon A_1 \times \ldots \times A_n \longrightarrow \mathrm{P}(A)$ is the unique map determined by the relation $(\tau_{x+\mathrm{x}})^{-1}(\mathrm{true}\,) >\!\!\longrightarrow A \times (A_1 \times \ldots \times A_n)$;

(vii) $|t \in \tau| = \chi_{\in} \circ \langle t_{\mathrm{x}}, \tau_{\mathrm{x}}\rangle\colon A_1 \times \ldots \times A_n \longrightarrow A \times \mathrm{P}(A) \rightarrow \Omega$;

(viii) $f(t)_{\mathrm{x}} = f \circ t_{\mathrm{x}}\colon A_1 \times \ldots \times A_n \longrightarrow A \rightarrow B$.

The interpretation defined above is a particular instance of a much more general case of interpretation of a type theory in a category, cf. [Rey74, Joh77, MR77]. In defining it we have preferred to use characteristic maps instead of a more intuitive notation employing subobjects which would resemble more closely the example of the category of sets. We chose to do so because categories are an abstraction of functions, so it is appropriate to use arrows and composition. Besides, the category-theoretic notion of subobjects is unpleasant as one has to deal with equivalent classes, and the natural isomorphism

$$\mathrm{Sub}(A) \simeq \mathcal{E}(A, \Omega),$$

somehow solves the problem by giving a "canonical" notation for the equivalence classes. In fact we shall often confuse a characteristic map and the subobject it classifies.

Another confusion will be allowed with global elements $a\colon 1 \longrightarrow A$. Namely, the term $a(\mathrm{o})$ is of type A, but it will be denoted simply as a. With the same frame of mind, for a given $R \rightarrowtail A$, the map $\{x \in A : \chi_R\}\colon 1 \longrightarrow \mathrm{P}(A)$ will be confused with R. A term of the type Ω will be called a *formula* or *truth value*. Insisting in our confusion, we will often identify τ_{x} with its extensional value $(\tau_{\mathrm{x}})^{-1}(\mathrm{true}) \rightarrowtail A$.

Terms and formulae would be of no use if we give no axioms or rules. The list of axioms that follows can be read as a set of basic properties of the constructs we have introduced, when interpreted in the topos $\mathcal{E}$. We say that a formula ϕ *holds* in $\mathcal{E}$, and write $\mathcal{E} \Vdash \phi$, if, considered the sequence $\mathbf{x} = x_1, \ldots, x_n$ of the free variables in ϕ, the subobject

$$(\phi_{\mathrm{x}})^{-1}\,\mathrm{true} \rightarrowtail A_1 \times \ldots \times A_n$$

is the largest element in $\mathrm{Sub}(A_1 \times \ldots \times A_n)$.

Logical properties. The interpretations of the logical constructs satisfy all the axioms of a typed intuitionistic set theory. This is very useful when working inside a topos, as one can directly apply set-theoretic intuition instead of, sometimes very settled, categorical arguments.

1.1.2 **Proposition** *The following formulae hold in $\mathcal{E}$:*

(i) $\phi \to [\psi \to \phi] \qquad [\phi \to [\psi \to \theta]] \to [[\phi \to \psi] \to [\phi \to \theta]]$;

(ii) $\mathrm{true}\,, \qquad \phi \wedge \psi \to \phi, \qquad \phi \wedge \psi \to \psi$;

(iii) $\phi \to [\psi \to [\phi \wedge \psi]], \qquad \forall x \in A.\ \phi \to \phi$;

(iv) $x = x, \qquad x = y \leftrightarrow y = x, \qquad \phi \wedge x = z \to \phi(x/z)$;

(v) $x = \mathrm{o}$;

(vi) $z = \langle x, y\rangle \leftrightarrow p(z) = x \wedge q(z) = y$;

(vii) $X = \{x \in A : \tau\} \leftrightarrow \forall x \in A[x \in X \leftrightarrow \tau]$.

Moreover the following rules produce true formulae from others:

$$\frac{\phi \quad \phi \to \psi}{\psi}$$

provided the free variables in ϕ appear free in the conclusion

$$\frac{\phi \to \psi}{\phi \to \forall x \in A.\ \psi}$$

provided x is not free in ϕ

$$\frac{\phi}{\phi(x/t)}$$

provided t is free for x.

Proof Is trivial. A property about interpretation, which is useful for the proof, is the following:

$$s(y/t)_{\mathrm{x}} = s_y \circ t_{\mathrm{x}},$$

where x is the only free variable occurring in s. The conclusion is easy. □

It is possible to state also a completeness theorem, but we do not need it for our purposes. As the interpretation is quite straightforward, from now on we shall not distinguish a term t from its interpretation t_{x} in $\mathcal{E}$ and use t for both.

Categorical meaning. We give a brief presentation of categorical properties of the interpretation of the new terms. For sake of simplicity, we reduce the number of free variables possibly occurring in a formula to one.

(1) true $= \chi_\Delta \Delta : 1 \longrightarrow \Omega$ is true (this is one of the reason to confuse subobjects and characteristic map).

(2) $(\sigma \wedge \tau)^{-1}(\text{true}) \rightarrowtail A$ is the intersection of $\sigma^{-1}(\text{true})$ and $\tau^{-1}(\text{true})$. Indeed, by the characteristic property, $(\sigma \wedge \tau)^{-1}\,\text{true}$ must be the pullback

$$\begin{array}{ccc} R & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \langle \text{true}, \text{true} \rangle} \\ A & \xrightarrow{\langle \sigma, \tau \rangle} & \Omega \times \Omega \end{array}$$

because a map $x : X \longrightarrow A$ factors through R and only if

$$\sigma x = \text{true} = \tau x,$$

which is equivalent to $x : X \longrightarrow A$ factors through $\sigma^{-1}(\text{true})$ and through $\tau^{-1}(\text{true})$.

(3) $(\sigma \rightarrow \tau)^{-1}\,\text{true} \rightarrowtail A$ is the Heyting implication $\sigma^{-1}(\text{true}) \rightarrow \tau^{-1}(\text{true})$ in the poset $\text{Sub}(A)$. Take $x : X \rightarrowtail A$. Then $X \leq (\sigma \rightarrow \tau)^{-1}(\text{true})$ if and only if $(\sigma \wedge \tau)x = \sigma x$. In other words, the intersection $(\sigma \wedge \tau)^{-1}(\text{true}) \cap x$ is $\sigma^{-1}(\text{true}) \cap x$. But

$$(\sigma \wedge \tau)^{-1}(\text{true}) = \sigma^{-1}(\text{true}) \cap \tau^{-1}(\text{true}),$$

so the previous condition is equivalent to

$$x \cap \sigma^{-1}(\text{true}) \leq \tau^{-1}(\text{true}).$$

(4) $(\forall x \in A.\ \phi)^{-1}(\text{true}) \rightarrowtail B$ is the largest subobject $R \rightarrowtail B$ such that

$$q^{-1}(R) \leq \phi^{-1}(\text{true})$$

for $q : A \times B \longrightarrow B$ the projection map. Indeed $x : X \rightarrowtail B$ factors through R if and only if

$$\phi(\text{id} \times x) = \text{true},$$

which is equivalent to

$$q^{-1}(X) \leq \phi^{-1}(\text{true})$$

because

$$\begin{array}{ccc} A \times X & \xrightarrow{q} & X \\ \downarrow{\scriptstyle \text{id} \times x} & & \downarrow{\scriptstyle x} \\ A \times B & \xrightarrow{q} & B \end{array}$$

is a pullback.

To check the properties of the last three constructs, the following principle is useful: suppose ϕ and ψ has one free variable of the type $A \times B$; then

$$\mathcal{E} \Vdash\ \forall x \in A.\ \phi \rightarrow \psi$$

if and only if for all $X \rightarrowtail A \times B$

$$\text{if } X \leq \text{id} \times \phi^{-1}(\text{true}),\ \text{then } X \leq \text{id} \times \psi^{-1}(\text{true}).$$

The proof of the principle follows from the previous calculations.

(5) $(\sigma \vee \tau)^{-1}(\text{true}) \rightarrowtail A$ is the union of $\sigma^{-1}(\text{true})$ and $\tau^{-1}(\text{true})$ in the poset $\text{Sub}(A)$.

(6) $\bot^{-1}(\text{true}) \rightarrowtail A$ is the least subobject of A.

(7) $(\exists x \in A.\ \phi)^{-1}(\text{true}) \rightarrowtail B$ is the image of $\phi^{-1}(\text{true}) \rightarrowtail A \times B$ along $q: A \times B \longrightarrow B$, i.e. the least subobject of B through which the map

$$\phi^{-1}(\text{true}) \rightarrowtail A \times B \xrightarrow{q} B$$

factors, the factorization map being epic.

By means of the usual set-theoretic constructions, it is possible to show now that the topos $\mathcal{E}$ has all finite colimits and exponentials, the initial object is strict, epic maps are quotients of their equivalence relations, coproducts are disjoint, the inverse image functions

$$f^{-1} : \text{Sub}(B) \rightarrowtail \text{Sub}(A)$$

have a right adjoint and a left adjoint. For instance, the exponential B^A is given by

$$\{f \in \text{P}(A \times B) : \forall x \in A\ \exists! y \in B.\ \langle x, y\rangle \in f\}.$$

We will use the logic interpretation freely, working with object of $\mathcal{E}$ as if they were sets, but carefully checking that our way of reasoning is intuitionistic.

1.2 Topologies

It is fair to say that the object Ω of a topos $\mathcal{E}$ controls great part of the logic of the topos. So one may try to produce new topoi from $\mathcal{E}$ by means of "appropriate" modal operators on Ω. The correct operators to use are *topologies*. Historically, their first appearance was in the particular case of a topos of "presheaves", which are presented in a subsequent section.

Definition. A *topology* on the topos $\mathcal{E}$ is a map $j: \Omega \longrightarrow \Omega$ such the following equalities hold in $\mathcal{E}$:

(a) $\text{true} = j\,\text{true}$

(b) $jjw = jw$

(c) $j(w \wedge z) = jw \wedge jz$.

The first condition is stronger than it looks: as $w = |w = \text{true}|$, it follows that

$$w = |w = \text{true}| \leq |jw = \text{true}| = jw.$$

There are many interesting algebraic properties of topologies, but we will not present them. For a clear, complete exposition, we can safely refer the reader to [FS79].

Sheaves. It is possible to *change* the logic of $\mathcal{E}$ by shifting the values by j. As far as Ω is concerned, the change is simple: take $\Omega_j = \{z \in \Omega : jz = z\}$, notice that $\text{true} \in \Omega_j$ and Ω_j is closed under meet. Also the inclusion $\Omega_j \rightarrowtail \Omega$ has a right adjoint as, for $z \in \Omega_j$ and $w \in \Omega$,

$$\mathcal{E} \Vdash [z \rightarrow w] \leftrightarrow [z \rightarrow jw].$$

So Ω_j is closed under implication and join is defined on it as $j(z \vee w)$.

Nothing has to be changed for products, as their constructs do not involve Ω, but things get a bit more complicated at higher orders: naively, we would like to take only *j-separated* objects X, which are by definition those satisfying

$$\mathcal{E} \Vdash j|x = y| \rightarrow x = y$$

and *j-closed* subobjects $R \rightarrowtail X$, which are defined as those such that

$$\mathcal{E} \Vdash j|x \in R| \rightarrow x \in R.$$

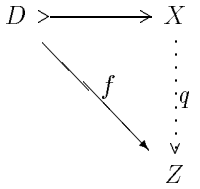
The straightforward approach cannot work because then what about the monomorphism

$$x \mapsto \{z \in X : z = x\} : X \longrightarrow \mathrm{P}(X) \ ?$$

The right idea is not too far away. Say a subobject $D \rightarrowtail X$ *j-dense* if

$$\mathcal{E} \Vdash j|x \in D| = \mathrm{true}\,,$$

and define an object Z in $\mathcal{E}$ to be a *j-sheaf* if for every j-dense subobject $D \rightarrowtail X$ and every map $f : D \longrightarrow Z$ there is a unique $g : X \longrightarrow Z$ such that



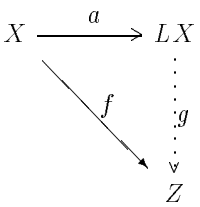
commutes. Let $\mathcal{E}_j$ be the full subcategory of $\mathcal{E}$ of j-sheaves. It is possible to prove $\mathcal{E}_j$ is a topos where Ω_j is the classifier. We do not give a proof—there are many references where it can be found—, but we want to mention the construction of the *associated sheaf*: for X an object in $\mathcal{E}$, let LX be the object obtained as quotient of the object

$$QX = \{s \in \mathrm{P}(X) : \forall x, y \in X.\ x \in s \wedge y \in s \rightarrow j|x = y| \wedge j(\exists x \in X.x \in s)\}$$

by the equivalence relation $j|s = t|$. The object LX is a j-sheaf and there is a map

$$a : x \mapsto \{s \in QX : j|x \in s|\} : X \longrightarrow LX,$$

such that for every map $f : X \longrightarrow Z$ from X into a j-sheaf Z,



commutes for a unique $g : LX \longrightarrow Z$. The argument skecthed above can be summarized in the following statement.

1.2.1 **Theorem** *Let $j : \Omega \longrightarrow \Omega$ be a topology, let $\mathcal{E}_j$ be the full subcategory of j-sheaves of $\mathcal{E}$. Then $\mathcal{E}_j$ is a topos, and if $I : \mathcal{E}_j \longrightarrow \mathcal{E}$ is the inclusion functor, then there is a functor $L : \mathcal{E} \longrightarrow \mathcal{E}_j$ such that*

$$L \dashv I : \mathcal{E}_j \longrightarrow \mathcal{E},$$

is a geometric morphism.

We should recall that a pair of adjoint functors $f^* \dashv f_* : \mathcal{E} \longrightarrow \mathcal{G}$ form a *geometric morphism* from the topos $\mathcal{E}$ to the topos $\mathcal{G}$, if f^* is left exact. One can prove that the composition $f_* f^* : \mathcal{G} \longrightarrow \mathcal{G}$ is the associated-sheaf functor for an appropriate topology on $\mathcal{G}$.

Finally, the logic of the topos $\mathcal{E}_j$ can be related to that of $\mathcal{E}$, by means of the topology operator. Define the *translation* of a term t in the theory of $\mathcal{E}_j$ to a term t_j in the theory of $\mathcal{E}$, by inductively substituting subterms of the form $|t = v|$ by $j|t = v|$ and subterms of the form $|t \in r|$ by $j|t \in r|$.

1.2.2 **Theorem** *Suppose ϕ is a formula of the theory of $\mathcal{E}_j$. Then it is also a formula in the theory of $\mathcal{E}$, and*

$$\mathcal{E}_j \Vdash \phi \text{ if and only if } \mathcal{E} \Vdash \phi_j .$$

Moreover, the following equalities hold in $\mathcal{E}$:

$$(\phi \wedge \psi)_j = \phi_j \wedge \psi_j ; \qquad (\phi \to \psi)_j = \phi_j \to \psi_j ;$$

$$(\phi \vee \psi)_j = j(\phi_j \vee \psi_j); \qquad \bot_j = j\bot ;$$

$$(\forall x \in X.\ \phi)_j = \forall x \in X.\ \phi_j ; \qquad (\exists x \in X.\ \phi)_j = j(\exists x \in X.\ \phi_j).$$

Transfer properties. Our main interest about topologies, however, is not in j-sheaves, but rather in transfer properties from j-sheaves back to objects in $\mathcal{E}$. This because, in all the topoi appearing in chapter 4 and 7, we know perfectly well the sheaves and the associated-sheaf functor, but do not have much information about the larger topos. Here we are going to repeat the argument of §5 of [Hyl82].

1.2.3 **Lemma** *A subobject $R \rightarrowtail X$ is j-closed if and only if the commutative diagram*

$$\begin{array}{ccc} R & \xrightarrow{a'} & LR \\ \downarrow & & \downarrow \\ X & \xrightarrow{a} & LX \end{array}$$

is a pullback.

Proof By definition of L,

$$\mathcal{E} \Vdash s \in LR \leftrightarrow j|\exists x \in R.x \in s|.$$

Therefore

$$\mathcal{E} \Vdash y \in a^{-1}(LR) \leftrightarrow j|\exists x \in R.x \in \{y\}|.$$

But

$$\begin{aligned} \mathcal{E} \Vdash \ \exists x \in R.\ x \in \{y\} &\leftrightarrow \exists x \in R.\ x = y \\ &\leftrightarrow y \in R. \end{aligned}$$

So

$$\mathcal{E} \Vdash y \in a^{-1}(LR) \leftrightarrow j|y \in R|.$$

And $R = a^{-1}(LR)$ if and only if R is j-closed. □

While little is preserved by the inclusion of $\mathcal{E}_j$ into $\mathcal{E}$, a lot can be said about j-closed subobjects, because (in Hyland's words) a j-closed subobject $R \rightarrowtail X$ *agrees with its meaning in* $\mathcal{E}_j$ in the sense that

$$a^{-1}(LR \rightarrowtail LX) = R \rightarrowtail X.$$

The following proposition states some transfer properties of closed subobjects.

1.2.4 **Proposition** *Suppose $R \rightarrowtail X$ is j-closed.*

(i) *If $f: Z \longrightarrow X$, then $f^{-1}R \rightarrowtail Z$ is closed and $L(f^{-1}R) = (LF)^{-1}(LR)$.*

(ii) *Suppose $S \rightarrowtail X$ is also j-closed.*
Then $R \cap S \rightarrowtail X$ is j-closed, and $L(R \cap S) = LR \cap LS$.

(iii) *Suppose $A \rightarrowtail X$ is any subobject.*
Then $(A \to R) \rightarrowtail X$ is j-closed, and $L(A \to R) = LA \to LR$.

(iv) *Suppose $S \rightarrowtail X \times Y$ is j-closed.*
Then $L(\forall x \in X.\ S) \rightarrowtail Y$ is j-closed and $L(\forall x \in X.\ S) = \forall z \in LX.\ LS$.

Proof (i) Trivial, by 1.2.3. (ii) Trivial, by the property of the topology j and 1.2.3. (iii) Let $x \in X$, and check that

$$\mathcal{E} \Vdash j(x \to jz) = x \to jz,$$

because

$$\mathcal{E} \Vdash x \wedge j(x \to jz) \leq j[x \wedge (x \to z)] \leq jjz = jz.$$

Also $L(A \to R) = LA \to LR$, because any subobject $S \rightarrowtail LX$ is of the form $L(a^{-1}S)$. (iv) Similar to the previous one. □

Suppose σ is a formula of the theory of $\mathcal{E}$ built from formulae $\beta_1, \ldots, \beta_n$ by means of $\wedge, \to, \forall$, binding, say, $y_1, \ldots, y_k$ of type $Y_1, \ldots, Y_k$ respectively. Suppose moreover the basic formulae $\beta_1, \ldots, \beta_n$ have a j-closed interpretation. Say the free variables occurring in σ have type X and let σ_L be the formula obtained by substituting in σ the variables of type X by variables of type LX, and the bound variables $y_1, \ldots, y_k$ of type $Y_1, \ldots, Y_k$ with variables of type $LY_1, \ldots, LY_K$. Then the interpretation of σ in $\mathcal{E}$ is the restriction to X of the interpretation of σ_L in the sense that

$$\sigma = a^{-1}\sigma_L.$$

In simple terms, the interpretation of σ is the *restriction* of its corrected interpretation in $\mathcal{E}_j$, corrected because the objects $X, Y_1, \ldots, Y_k$ may very well be out of $\mathcal{E}_j$. As another *restriction* principle, the following one about function spaces with a j-separated basis will be very useful.

1.2.5 **Proposition** *Suppose Y is a j-separated object in $\mathcal{E}$. Then Y^X is j-separated, and the evaluation map* $\mathrm{ev}: X^Y \times X \longrightarrow Y$ *is the unique map such that the diagram*

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{\mathrm{ev}} & Y \\ {\scriptstyle b \times a}\downarrow & & \downarrow \\ LY^{LX} \times LX & \xrightarrow{a} & LY \end{array}$$

commutes, where

$$b : f \mapsto [[s] \mapsto [fs]]: Y^X \longrightarrow LY^{LX}.$$

Proof The first part is trivial, by the definition of equality of maps. The second part follows from the fact that $a: Y \longrightarrow LY$ is monic, as the diagram trivially commutes. □

1.3 Presheaves

As we warned at the beginning of the chapter, there have not been many examples yet. The subject in the next sections will recover from this lack. The first examples are topoi of *presheaves* which are, in a way, "powers" of the topos $\mathcal{S}$ of sets and functions: this is not quite correct, but one may think of presheaves as indexed sets, and the limit structure of the topos is induced pointwise from that of $\mathcal{S}$. But after that, the definition of powerobjects is forced upon.

Presheaves. Let C be a small category, i.e. a category such that the collection of all objects in C is a set and all families $\mathsf{C}(A, B)$ of maps from A to B in C are sets.

A *presheaf* over C is a contravariant functor

$$F: \mathsf{C}^{\mathrm{op}} \longrightarrow \mathcal{S}.$$

For A an object in C and $a \in FA$, and $f: B \longrightarrow A$ in C, we will often write $a|_f$ for $Ff(a)$ and read it *a restricted along f*. The category $\mathcal{C}$ consists of presheaves over C and natural transformations as maps. Better than indexed sets, presheaves can be thought of as variable sets: the objects of C as the stages of variation, where the passages from one stage to another are the restriction functions $|_f$. We next discuss a few examples of presheaves.

Representable presheaf. Let A be a fixed object in C. The assignment

$$\begin{array}{ccc} B & \mapsto & \mathsf{C}(B,A) \\ f\uparrow & \mapsto & \downarrow -\circ f \\ C & \mapsto & \mathsf{C}(C,A) \end{array}$$

is a functor which will be denoted by $\mathcal{Y}A$ and called the *representable* presheaf of A. Moreover, given a map $f\colon A \longrightarrow B$, define a natural transformation $\mathcal{Y}f = f\circ -\colon \mathcal{Y}A \stackrel{\cdot}{\longrightarrow} \mathcal{Y}B$ as

$$\begin{array}{rcl} \mathcal{Y}f_C\colon \mathcal{Y}A(C) & \to & \mathcal{Y}B(C) \\ g & \mapsto & f\circ g. \end{array}$$

Then $\mathcal{Y}\colon \mathsf{C} \longrightarrow \mathcal{C}$ is a functor.
Constant presheaf. Let S be a set. Let ΔS be the functor defined as follows:

$$\begin{array}{ccc} A & \mapsto & S \\ f\uparrow & \mapsto & \downarrow \mathrm{id}_S \\ B & \mapsto & S; \end{array}$$

for $f\colon S \longrightarrow T$ a function from S to T, let $\Delta f\colon \Delta S \stackrel{\cdot}{\longrightarrow} \Delta T$ be the natural transformation

$$\begin{array}{rcl} (\Delta f)_A\colon \Delta S(A) & \to & \Delta T(A) \\ s & \mapsto & f(s). \end{array}$$

Then the assignment $\Delta\colon \mathcal{S} \longrightarrow \mathcal{C}$ is the *constant* presheaf functor. Moreover, any map $g\colon \Delta S \stackrel{\cdot}{\longrightarrow} \Delta T$ in $\mathcal{C}$ is obviously of the form Δf, where f is one, hence every, component of g. In other words, the functor Δ is full.
Product presheaf. Let F and G be two presheaves over C. Then the functor $F\times G$ pointwise defined by the product as follows:

$$\begin{array}{ccc} A & \mapsto & FA\times GA \\ f\uparrow & \mapsto & \downarrow Ff\times Gf \\ B & \mapsto & FB\times GB \end{array}$$

is the product of F and G in the category $\mathcal{C}$, where the two natural transformations $p\colon F\times G \stackrel{\cdot}{\longrightarrow} F$ and $q\colon F\times G \stackrel{\cdot}{\longrightarrow} G$ are defined at A as the first and second projection from $FA\times GA$, respectively,.
Coproduct presheaf. Let F and G be two presheaves over C. Then the functor $F+G$ pointwise defined by the coproduct as follows:

$$\begin{array}{ccc} A & \mapsto & FA+GA \\ f\uparrow & \mapsto & \downarrow Ff+Gf \\ B & \mapsto & FB+GB \end{array}$$

is the coproduct of F and G in the category $\mathcal{C}$, where the two natural transformations $i\colon F \stackrel{\cdot}{\longrightarrow} F+G$ and $j\colon G \stackrel{\cdot}{\longrightarrow} F+G$ are defined at A as the first and second injection into $FA+GA$.
Image presheaf. Let $t\colon F \stackrel{\cdot}{\longrightarrow} G$ be a natural transformation in $\mathcal{C}$. For A in C, let

$$IA = \{y\in GA : \exists x\in FA.\ y = t_A(x)\} = \mathrm{im}\, t_A$$

be the image of the function $t_A\colon FA \longrightarrow GA$, and, for a map $f\colon B \longrightarrow A$, let

$$If = Gf|_{IA}$$

which takes IA into IB since the diagram

$$\begin{array}{ccc} FA & \xrightarrow{t_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow{t_B} & GB \end{array}$$

commutes; let $e_A : FA \longrightarrow\!\!\!\!\triangleright\ IA$ and $i_A : IA \rightarrowtail GA$ be the surjection and the inclusion of the canonical factorization of $t_A : FA \longrightarrow GA$. Then $i : I \rightarrowtail G$ is the smallest subfunctor of G, through which t factors, and $t = i \circ e$ for the epimorphic natural transformation $e : F \stackrel{\cdot}{\longrightarrow} I$.

Function-space presheaf. Let F and G be presheaves over C. For A an object of C, let $(G^F)A$ be the set of all the families

$$s = (s_f : FB \longrightarrow GB)_{f : B \longrightarrow A},$$

indexed by the maps $f : B \longrightarrow A$ with codomain A (both domain and map vary), such that for all $g : C \longrightarrow B$ and $f : B \longrightarrow A$, the diagram

$$\begin{array}{ccc} FB & \xrightarrow{Fg} & FC \\ {\scriptstyle s_f}\downarrow & & \downarrow{\scriptstyle s_{fg}} \\ GB & \xrightarrow{Gg} & GC \end{array}$$

commutes. A map $f : D \longrightarrow A$ induces the function

$$(G^F)f : s \mapsto (s_{fg})_{g : C \longrightarrow D} : (G^F)A \longrightarrow (G^F)D$$

Then (G^F) is the *function space presheaf* of F and G. One can easily prove that for any three presheaves F, G and H and any natural transformation $t : F \times K \stackrel{\cdot}{\longrightarrow} G$ there is a unique natural transformation $u : K \stackrel{\cdot}{\longrightarrow} G^F$ such that

$$t = \mathrm{ev}\,(\mathrm{id} \times u),$$

where $\mathrm{ev} : F \times G^F \stackrel{\cdot}{\longrightarrow} G$ is the natural transformation defined as

$$\begin{array}{rcl} \mathrm{ev}_A : FA \times G^F(A) & \to & GA \\ \langle x, s \rangle & \mapsto & s_{\mathrm{id}_A}(x). \end{array}$$

Think of an element of $G^F(A)$ as a map from the variable set F to the variable set G, from stage A onward.

Subfunctor classifier. Let Ω be the functor defined as

$$\begin{array}{ccc} A & \mapsto & \mathrm{Sub}(\mathcal{Y}A) \\ f\uparrow & \mapsto & \downarrow\mathcal{Y}f^{-1} \\ B & \mapsto & \mathrm{Sub}(\mathcal{Y}B) \end{array}$$

The natural transformation $\mathrm{true} : 1 \stackrel{\cdot}{\longrightarrow} \Omega$ picks out the largest subobject in each $\Omega(A)$:

$$\begin{array}{rcl} \mathrm{true}_A : 1(A) & \to & \Omega(A) \\ \mathrm{o} & \mapsto & \mathrm{id} : A \rightarrowtail A \end{array}$$

It is simple to show that these assignments define the classifier in $\mathcal{C}$.

Power presheaf. Let F be a presheaf over C. For an object A of C, let $\mathrm{P}(F)(A)$ be the set of all families

$$R = (R_f \subset FB)_{f : B \longrightarrow A}$$

of subsets R_f indexed by the maps $f: B \longrightarrow A$ (both domain and map vary) such that for $g: C \longrightarrow B$ and $f: B \longrightarrow A$,

$$Fg[R_f] \subset R_{fg}.$$

The action of $\mathrm{P}(F)$ on maps is defined as in the example of the function space:

$$\mathrm{P}(F)(f): R \mapsto (R_{fg})_{g:C} \longrightarrow {}_D: \mathrm{P}(F)(A) \longrightarrow \mathrm{P}(F)(D).$$

It is easy to prove that $\mathrm{P}(F)$ is the power presheaf. The subfunctor $\in \; \rightarrowtail F \times \mathrm{P}(F)$ is

$$\in (A) = \{\langle a, R\rangle \, : a \in R_{\mathrm{id}\,_A}\}.$$

An element of $\mathrm{P}(F)(A)$ can be tought of as a variable subset of F existing only from stage A. Notice that the construction we have performed is word by word that of the function-space presheaf Ω^F.

1.3.1 **Proposition** *The category $\mathcal{C}$ is a topos. Moreover, the functor $\Delta: \mathcal{S} \longrightarrow \mathcal{C}$ is part of a geometric morphism*

$$\Delta \dashv \Gamma: \mathcal{C} \longrightarrow \mathcal{S}.$$

Proof The first part is easy after the examples given above. As for the second part, by definition of products and equalizers in $\mathcal{C}$, Δ preserves finite limits. Then define $\Gamma: \mathcal{C} \longrightarrow \mathcal{S}$ as

$$\begin{array}{ccc} A & \mapsto & \mathcal{C}(1, A) \\ f\uparrow & \mapsto & \downarrow f \circ - \\ B & \mapsto & \mathcal{C}(1, B). \end{array}$$

Then $\Gamma\Delta S \simeq S$; therefore, as Δ is full,

$$\begin{aligned} \mathcal{C}(\Delta S, F) &\simeq \mathcal{S}(\Gamma\Delta S, \Gamma F) \\ &\simeq \mathcal{S}(S, \Gamma F). \; \square \end{aligned}$$

Forcing. The idea about variable sets is made more precise when reading the interpretation of intuitionistic set theory for the topos $\mathcal{C}$. This can be done by introducing a Kripke-like semantics in the model $\mathcal{C}$ based on a notion of *forcing*, with the objects of C as stages.

Suppose ϕ is a formula with free variables of types $x_1, \ldots, x_n$. So $\phi \rightarrowtail X_1 \times \ldots \times X_n$, and we aim to describing the situation when $\langle s_1, \ldots, s_n\rangle \in X_1(A) \times \ldots \times X_n(A)$ is in $\phi(A)$. In a sort of reverse notation, we are going to describe the relation

$$s \in \phi(A)$$

in terms of a forcing

$$A \Vdash \phi(s).$$

1.3.2 **Definition** The relation of *forcing* $A \Vdash \phi(s)$ is given for A an object in $\mathcal{C}$ and s an assignment of the free variables of ϕ such that

$$s(x) \in X(A) \quad \text{if } x \text{ has type } X.$$

$A \Vdash x \in U(s)$ if $\langle s(x), \mathrm{id}\,_A\rangle \in s(U)(A)$;

$A \Vdash x = y(s)$ if $s(x) = s(y)$;

$A \Vdash \psi \wedge \theta(s)$ if $A \Vdash \psi(s)$ and $A \Vdash \theta(s)$;

$A \Vdash \psi \vee \theta(s)$ if $A \Vdash \psi(s)$ or $A \Vdash \theta(s)$;

$A \Vdash \psi \rightarrow \theta(s)$ if, for all $f: B \longrightarrow A$ in C, whenever $B \Vdash \psi(s|_f)$, also $B \Vdash \theta(s|_f)$;

$A \Vdash \exists z \in F.\ \psi(s)$ if there is a $b \in F(A)$ such that $A \Vdash \psi(s, b/z)$;

$A \Vdash \forall z \in F.\ \psi(s)$ if for all $f: B \longrightarrow A$ and for all $b \in F(B)$, $B \Vdash \psi(s, b/z)$.

The proof of the following criterion is long, tiresome, and straightforward.

1.3.3 **Proposition** *Let ϕ be a formula of typed theory of the topos $\mathcal{C}$. Then $\mathcal{C} \Vdash \phi$ holds if and only if $A \Vdash \phi(s)$ holds for all objects A in C and all possible assignments s of the free variables in ϕ.*

Representable functors. Now we turn to study the representable functors, first by stating the Yoneda lemma.

1.3.4 **Lemma** (Yoneda) *Let A be an object of C and let F be a presheaf over C. Then there is a natural bijection*

$$\begin{array}{rcl} \mathcal{C}(\mathcal{Y}A, F) & \simeq & FA \\ t: \mathcal{Y}A \dashrightarrow F & \mapsto & t_A(\mathrm{id}_A) \end{array}$$

from the set of natural transformations from $\mathcal{Y}A$ into F onto the set FA.

Proof Suggestion: the diagram

$$\begin{array}{ccc} \mathcal{Y}A(A) & \xrightarrow{t_A} & FA \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle Ff} \\ \mathcal{Y}A(B) & \xrightarrow{t_B} & FB \end{array}$$

commutes. □

As a consequence, C can be embedded into $\mathcal{C}$ in a very nice way.

1.3.5 **Corollary** *Let A and B be objects in C and let $t: \mathcal{Y}A \dashrightarrow \mathcal{Y}B$ be a natural transformation. Then t is of the form $f \circ - = \mathcal{Y}f$ for a unique map $f: A \longrightarrow B$ in C. So the functor $\mathcal{Y}: \mathsf{C} \longrightarrow \mathcal{S}$ is a full, left exact embedding of categories.*

Proof Let $f = t_A(\mathrm{id}_A)$. The remaining part of the statement follows immediately. □

In the sequel we will often confuse C with its image through $\mathcal{Y}$. The following two useful principles for representable functors were suggested by M. Fourman in [Fou80].

1.3.6 **Lemma** *Suppose C has finite products. Let A and B be objects of C and let F be a presheaf over C. If $\psi(x, y)$ is a formula, whose free variables are at most x and y of type F and B respectively, and $a \in FA$, then*

$$A \Vdash \forall y \in B.\ \psi(a, y) \text{ if and only if } A \times B \Vdash \psi(a|_p, q)$$

where p and q are the two projections from $A \times B$.

Proof Note that $q \in B(A \times B)$; so if

$$A \Vdash \forall y \in B.\ \psi(a, y),$$

then

$$A \times B \Vdash \psi(a|_p, q)$$

by 1.3.2. Suppose

$$A \times B \Vdash \psi(a|_p, q);$$

let $f: C \longrightarrow A$ be any map and $b \in B(C)$. Therefore $b: C \longrightarrow B$ by definition, and $\langle f, b\rangle: C \longrightarrow A \times B$. Restricting along $\langle f, b\rangle$, we obtain

$$C \Vdash \psi(a|_f, b)$$

since

$$(a|_p)|_{\langle f,b\rangle} = a|_f \quad \text{and} \quad q|_{\langle f,b\rangle} = b.$$

Thus

$$A \Vdash \forall y \in B.\ \psi(a, y)$$

by 1.3.2. □

1.3.7 **Theorem** (Choice principle for representable functors) *Let A be an object of C and let F, G be presheaves over C. If $\psi(x,y,z)$ is a formula whose free variables are among x, y and z respectively of type A, F and G, then*

$$\mathcal{C} \Vdash \forall z \in G[\forall x \in A\ \exists y \in F.\ \psi(x,y,z) \to \exists f \in A^F\ \forall x \in A.\ \psi(x, fx, z)].$$

Proof Let C be any object of C, let $a \in GC$ and suppose

$$C \Vdash \forall x \in A\ \exists y \in F.\ \psi(x,y,a).$$

Then by 1.3.6,

$$C \times A \Vdash \exists y \in F.\ \psi(q, y, a|_p)$$

where now p and q are the projections from $C \times A$. By definition of forcing, there is $b \in F(C \times A)$ such that

$$C \times A \Vdash \psi(q, b, a|_p).$$

But by 1.3.4, we have

$$c \mapsto \widehat{c} = ([g \mapsto c|_{\langle f,g\rangle}])_{f:D} \longrightarrow_C : F(C \times A) \simeq F^A(C).$$

In particular

$$ev\langle \widehat{b}|_p, q\rangle = b|_{\langle p,q\rangle} = b.$$

Thus

$$C \times A \Vdash \psi(q, \widehat{b}|_p(q), a|_p)$$

and by 1.3.6

$$C \Vdash \forall x \in A.\ \psi(x, \widehat{b}(x), a).$$

Therefore by 1.3.2

$$C \Vdash \exists f \in F^A\ \forall x \in A.\ \psi(x, fx, a),$$

that was to be proved. □

1.4 Sheaves

Sites. As we already mentioned, topologies were first used on topoi of presheaves, and the definition was obviously different, depending drastically on the category of presentation C. As for the forcing relation, it will be useful to have a different view on topologies for topoi of presheaves: so we recall the other definition of topology.

Definition. Let C be a small category and J a function with associates to any object A of C of a family $J(A)$ of sets of maps with codomain A. We say that J is a *topology* on C if, for every A in C, the family $J(A)$ satisfies the following conditions:

(a) $\{\mathrm{id}: A \longrightarrow A\}$ is in $J(A)$;

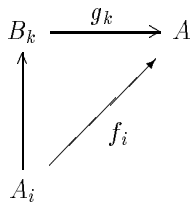
(b) if $\{f_i : A_j \longrightarrow A : i \in I\}$ is in $J(A)$ and $\{f_{ij} : A_{ij} \longrightarrow A : j \in K_i\}$ is in $J(A_i)$ for every $i \in I$, then $\{f_i f_{ij} : A_{ij} \longrightarrow A : i \in I, j \in K_i\}$ is in $J(A)$;

(c) if $\{f_i : A_i \longrightarrow A : i \in I\}$ is in $J(A)$ and $f : B \longrightarrow A$, then there is a family $\{g_k : B_k \longrightarrow B : k \in K\}$, in $J(B)$ such that for every $k \in K$ there is $i \in I$ for which the diagram

$$\begin{array}{ccc} B_i & \xrightarrow{g_i} & B \\ \downarrow & & \downarrow f \\ A_i & \xrightarrow{f_i} & A \end{array}$$

commutes;

(d) if $\{g_k : B_k \longrightarrow A : k \in K\}$ is a family of maps, and $\{f_i : A_i \longrightarrow A : i \in I\}$ is in $J(A)$, and for $i \in I$ there is a $k \in K$ such that the triangle



commutes, then $\{g_k : B_k \longrightarrow A : k \in K\}$ is in $J(A)$.

The families in $J(A)$ are called *coverings* of A and the pair (C, J) is called a *site*.

A topology on C induces a topology $j : \Omega \longrightarrow \Omega$ in $\mathcal{C}$ by defining

$$g : B \longrightarrow A \in j_A(R \rightarrowtail A)(B)$$

if and only if there exists $F \in J(A)$ such that every $f : C \longrightarrow A$ in F is in RC and factors through g. It is very simple to prove that j satisfies the condition for a topology, and is uniquely determined by J.

Examples. Let $J(A) = \{\mathrm{id} : A \longrightarrow A\}$ for A in C. Then J is a topology on C called the *trivial* topology.

Suppose C has finite limits. Let $J(A) = \{\{f : B \longrightarrow A\} : f \text{ is monic}\}$. Then J is a topology on C. Let also

$$H(A) = \{\{f : A \longrightarrow B\} : f \text{ and its pullbacks are epic}\}.$$

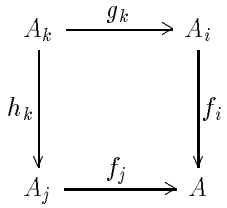
Then H is a topology on C.

Definition. Let C be a small category with finite limits and let J_1 and J_2 be topologies on C. We say that J_1 is *finer than* J_2 if for every object A in C

$$J_1(A) \supset J_2(A).$$

Often a topology is given by a basis of coverings which satisfy only conditions (i)-(iii): it is obvious how to generate the topology from those. We will always present a topology by a generating basis. Note that, given any collection of families of maps with common codomains, it is always possible to take the smallest (= the least fine) topology, such that the given families are coverings in it.

Compatibility. Let (C, J) be a fixed site. Consider a covering $\{f_i : A_i \longrightarrow A : i \in I\}$. By conditions (ii) and (iii), it is always possible to produce a set K, a function $u : K \longrightarrow I \times I\mathrm{K}$ and a family of maps $g_k : A_k \longrightarrow A_i$, $h_k : A_k \longrightarrow A_j$, for $u(k) = \langle i, j \rangle$, such that the diagram



commutes, and the families

$$\{g_k : A_k \longrightarrow A_i : u(k) = \langle i, j \rangle\}, \quad j \in J,$$

and

$$\{h_k : A_k \longrightarrow A_j : u(k) = \langle i, j \rangle\}, \quad i \in I$$

are coverings (we will refer to such a situation in the next section, as a *grid* on the covering $\{f_i : A_i \longrightarrow A\}$. Let now F be a presheaf over C. A family of elements $a_i \in F A_i$ is *compatible* with respect to the covering $\{f_i : A_i \longrightarrow A : i \in I\}$ if

$$a_i|_{g_k} = a_j|_{h_k}, \qquad \text{for } u(k) = \langle i, j \rangle.$$

Sheaves. We say that F is a *sheaf over* the site (C, J) if, given an object A in C, any covering $\{f_i\colon A_i \longrightarrow A : i \in I\}$ of A and any compatible family of elements $a_i \in FA_i$, there exists a unique element $a \in FA$ such that

$$a_i = a|_{f_i},$$

for every $i \in I$.

It is possible to prove that this notion of sheaf corresponds precisely to that of j-sheaf in $\mathcal{C}$. Note that in case the empty family is a covering of A, a sheaf F must satisfy the condition

$$FA = \{\mathrm{o}\}.$$

Also, in case J is the trivial topology, any presheaf over C is a sheaf.

Grothendieck topoi. We will write $\mathsf{Sh}(\mathsf{C}, J)$, or simply $\mathsf{Sh}(\mathsf{C})$, for the full subcategory of $\mathcal{C}$ whose objects are the sheaves for the site (C, J). In other words,

$$\mathsf{Sh}(\mathsf{C}, J) = \mathcal{C}_j\,.$$

A category of the form $\mathsf{Sh}(\mathsf{C}, J)$ is called a *Grothendieck topos*. In particular, when J is the trivial topology, the topos $\mathsf{Sh}(\mathsf{C}, J)$ is $\mathcal{C}$, proving that a presheaf category is a Grothendieck topos.

By composing the two geometric morphisms $L \dashv I\colon \mathsf{Sh}(\mathsf{C}, J) \longrightarrow \mathcal{C}$ and $\Delta \dashv \Gamma\colon \mathcal{C} \longrightarrow \mathcal{S}$, we obtain a geometric morphism from $\mathsf{Sh}(\mathsf{C}, J)$ to the topos $\mathcal{S}$ of sets:

$$\Delta \dashv \Gamma\colon \mathsf{Sh}(\mathsf{C}, J) \longrightarrow \mathcal{S}.$$

We followed the usual notation for this morphism, as it can be proved that, if there is a geometric morphism into $\mathcal{S}$, then this is unique.

Forcing revisited. The forcing relation can be adapted to fit a Grothendieck topos $\mathsf{Sh}(\mathsf{C}, J)$.

1.4.1 **Definition** All the defining conditions for $A \Vdash \psi(s)$ are the same as in 1.3.2, except for disjunction and the existential quantifier which are changed into:

$A \Vdash \psi \vee \theta(s)$ if there is a covering $\{f_i\colon A_i \longrightarrow A : i \in I\}$ of A such that for every $i \in I$, either $A_i \Vdash \psi(s|_{f_i})$ or $A_i \Vdash \theta(s|_{f_i})$.

$A \Vdash \exists x \in F.\ \psi(s)$ if there is a covering $\{f_i\colon A_i \longrightarrow A : i \in I\}$ of A such that for every $i \in I$, there exists $s_i \in FA_i$ and $A_i \Vdash \psi(s|_{f_i}(s_i/x))$.

1.4.2 **Lemma** *Let (C, J) be a site.*

(i) *If $A \Vdash \psi(s)$ and $f\colon B \longrightarrow A$, then $B \Vdash \psi(s|_f)$.*

(ii) *If there is a covering $\{f_i\colon A_i \longrightarrow A : i \in I\}$ of A such that $A \Vdash \psi(s|_{f_i})$ for every $i \in I$, then $A \Vdash \psi(s)$.*

Proof By induction on the complexity of ψ. □

The first property stated in the lemma above is reminiscent of *persistence* in the Kripke-Beth semantics for intuitionistic logic, and the second property was appropriately given the name of *local* character of forcing. The next result is now obvious.

1.4.3 **Proposition** *Suppose ϕ is a formula in the theory of $\mathsf{Sh}(\mathsf{C})$. Then $\mathsf{Sh}(\mathsf{C}) \Vdash \phi$ holds if and only if $A \Vdash \phi(s)$ hold for all object A in C and all possible assignment s to the free variables in ϕ.*

1.5 Subcanonical topologies

In his Curry's Festschrift paper [Sco81b], Scott uses the Yoneda embedding of a cartesian closed category into the topos of presheaves on it, in order to produce a model of intuitionistic set theory which extends a typed λ-calculus. Some further applications of that idea were carried on in [Ros85]. If a non-trivial topology J on a category C has the property that representable functors are sheaves for it, then it is possible maintain the embedding of C into the topos $\mathsf{Sh}(\mathsf{C}, J)$ of sheaves, which is certainly a different model of sets from the topos $\mathcal{C}$ of presheaves. A topology J on C is called *subcanonical* if every representable functor over C is a sheaf for J.

Characterization. It is possible to characterize subcanonical topologies. In order to do that, some categorical notions are needed.

Definition. Let C be a small category with finite limits. Let $\{f_i: A_i \longrightarrow A : i \in I\}$ be a family of maps with codomains A.

(a) The family $\{f_i : i \in I\}$ is *epimorphic* if for every $f, g: A \longrightarrow B$, $ff_i = gf_i$ for every $i \in I$, implies $f = g$.

(b) The family $\{f_i : i \in I\}$ is *effective epimorphic* if it is epimorphic and whenever $\{g_i: A_i \longrightarrow B : i \in I\}$ is a family of maps such that $g_i f_{ij} = g_j f_{ji}$ for all $i, j \in I$ (where $f_{ij}: A_i x_A A_j \longrightarrow A$ is the projection), then there exists a map $g: A \longrightarrow B$ such that $g_i = gf_i$. Notice that such a map g is unique since the family is epimorphic.

(c) The family $\{f_i\} : i \in I$ is *stable effective epimorphic* if for every $f: B \longrightarrow A$, there is an effective epimorphic family $\{g_i: B_i \longrightarrow B\}$ such that the diagrams

$$\begin{array}{ccc} B_i & \xrightarrow{g_i} & B \\ \downarrow & & \downarrow f \\ A_i & \xrightarrow{f_i} & A \end{array}$$

commute.

1.5.1 **Proposition** *Let C be a small category with finite limits. If $Z(A)$ is the set of all stable effective epimorphic families of maps with codomain A, then Z is a subcanonical topology on C. Furthemore it is the finest subcanonical topology on C and is called the canonical topology.*

Proof A representable presheaf $\mathcal{Y}A$ is a sheaf for a topology J if and only if J is generated by effective epimorphic families. And an effective epimorphic family is a covering only if it is stable. □

1.5.2 **Proposition** *Let (C, J) be a site with J subcanonical. Then the functor $\mathcal{Y}: \mathsf{C} \longrightarrow \mathcal{C}$ factors through $\mathsf{Sh}(\mathsf{C}, J)$ preserving all limits which exist in C. Moreover, if $\{f_i: A_i \longrightarrow A : i \in I\}$ is a covering, any grid on it, $g_k: A_k \longrightarrow A_i$, $h_k: A_k \longrightarrow A_j$, for $u(k) = \langle i, j\rangle$, induces a diagram in $\mathsf{Sh}(\mathsf{C}, J)$ for which $\mathcal{Y}A$ is a colimit.*

Proof Follows immediately from 1.3.4. The second part is a rewriting of the sheaf condition. □

Properties of canonical coverings. While the proposition says nothing new with respect to the presheaf case, the next one shows that there are actual gains in passing to the topos of sheaves.

1.5.3 **Proposition** *Let (C, J) be a site with J subcanonical and let A, B and C be objects in C.*

(i) *Suppose $f: A \longrightarrow B$. If $\{f: A \longrightarrow B\}$ is in $J(B)$, then*

$$\mathsf{Sh}(\mathsf{C}) \Vdash \forall y \in B\ \exists x \in A.\ y = fx.$$

(ii) *Suppose* $\{f\colon A \longrightarrow C, g\colon B \longrightarrow C\}$ *is in* $J(C)$. *Then*

$$\mathsf{Sh(C)} \Vdash \forall z \in C\ [\exists x \in A.\ z = fx \vee \exists y \in B. z = gy].$$

(iii) *Let* ψ *be a formula with two free variables* x *and* y. *If*

$$\mathsf{Sh(C)} \Vdash \forall x \in A\ \exists! y \in B.\ \psi(x, y),$$

then there exists $f\colon A \longrightarrow B$ *such that*

$$\mathsf{Sh(C)} \Vdash \forall x \in A.\ \psi(x, fx).$$

Proof (i) Suppose $f\colon A \longrightarrow B$ is in $J(B)$. For any object C and any $g\colon C \longrightarrow B$, there must be a diagram

$$\begin{array}{ccc} D & \xrightarrow{h} & C \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & B \end{array}$$

so that $\{h\colon D \longrightarrow C\}$ is in $J(C)$. It is

$$fk = g|_h.$$

Therefore, by 1.4.1.

$$\mathsf{Sh(C)} \Vdash \forall y \in B\ \exists x \in A.\ y = fx.$$

(ii) Similar to the previous one. (iii) Suppose

$$\mathsf{Sh(C)} \Vdash \forall x \in A\ \exists y \in B.\ \psi(x, y).$$

In particular

$$A \Vdash \exists y \in B.\ \psi(\mathrm{id}\,, y),$$

thus there exists a covering $\{f_i\colon A_i \longrightarrow A : i \in I\}$ of A and maps $g_i\colon A_i \longrightarrow B$ for $i \in I$ such that

$$A_i \Vdash \psi(\mathrm{id}\,|_{f_i}, g_i), \text{ and } A \Vdash \forall y, z \in B.\ [\psi(\mathrm{id}\,, y) \wedge \psi(\mathrm{id}\,, z) \to y = z].$$

Consider now the diagram

$$\begin{array}{ccc} A_{ij} & \xrightarrow{f_{ij}} & A_i \\ \downarrow & & \downarrow{\scriptstyle f_i} \\ A_j & \xrightarrow{f_j} & A \end{array}$$

By 1.3.2,

$$A_{ij} \Vdash \psi(f_i|_{f_{ij}}, g_i|_{f_{ij}})$$

and

$$A_{ij} \Vdash \psi(f_i|_{f_{ji}}, g_j|_{f_{ji}}).$$

But

$$f_i|_{f_{ij}} = f_j|_{f_{ji}} \quad \text{and } f_i|_{f_{ij}} = \mathrm{id}\,|_{f_i f_{ij}},$$

from the uniqueness condition we get that

$$g_i f_{ij} = g_j f_{ji}.$$

Therefore, as J is subcanonical, there exists $f\colon A \longrightarrow B$ such that

$$g_i = f f_i .$$

Thus

$$A_i \Vdash \psi(\mathrm{id}\,|_{f_i}, f|_{f_i})$$

for every $i \in I$. Then, by 1.4.1,

$$A \Vdash \psi(\mathrm{id}\,, f).$$

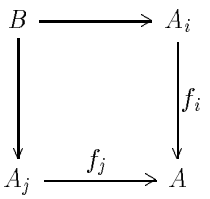
Since C has a terminal object, by 1.3.2 we obtain

$$1 \Vdash \forall x \in A.\ \psi(x, fx)$$

which is equivalent to the desired conclusion. □

As the existence hypothesis in the choice principle of 1.3.7 does not come with uniqueness, it is difficult that such a principle may extend to $\mathsf{Sh}(\mathsf{C}, J)$ for J subcanonical. But under strong hypotheses, one can prove that local existence implies global existence.

Let K be the *partition topology* generated by families $\{f_i\colon A_i \longrightarrow A\}$ which are effective epimorphic and disjoint in the sense that if, for $i \neq j$, the diagram



commutes, then B is the strict initial object of C. Note that the condition is satisfied if there is no commutative diagram with f_i and f_j, for $i \neq j$. In particular, if C has a strict initial object 0, then the empty family of maps is a disjoint effective epimorphic covering of 0.

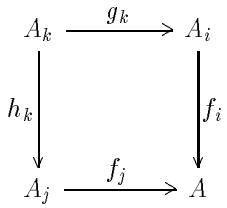
1.5.4 **Lemma** *Let J be a topology on C. Suppose the partition topology K is finer than J. Let ϕ be a formula of the theory of $\mathcal{C}$. Then*

$$\mathcal{C} \Vdash j[\exists x \in F.\ \phi(x)] \rightarrow \exists x \in F.\ \phi(x).$$

Proof Suppose $A \Vdash j[\exists x \in F.\ \phi(x)]$. By 1.2.2 and 1.3.2, there are a covering $\{f_i\colon A_i \longrightarrow A\}$ and a family $s_i \in FA_i$ such that

$$A_i \Vdash \phi(s_i).$$

Suppose a diagram



can be completed. Then A_k is the strict initial object, and

$$s_i|_{f_{ij}} = \mathrm{o} = s_j|_{f_{ji}},$$

because the empty family covers 0. So there is a unique element $s \in FA$ such that

$$A \Vdash \phi(s).\ \square$$

1.5.5 **Corollary** *Let J be a topology on C. Suppose the partition topology K is finer than J. Let A be an object of C and let F, G be sheaves over C. If $\psi(x, y, z)$ is a formula whose free variables are among x, y and z respectively of type A, F and G, then*

$$\mathcal{C} \Vdash \forall z \in G[\forall x \in A\ \exists y \in F.\ \psi(x, y, z) \to \exists f \in A^F\ \forall x \in A.\ \psi(x, fx, z)].$$

Changing site. The final result of this chapter is part of the folklore for Grothendieck topoi: it produces a way to reduce a site of definition of a topos of sheaves. It is a corollary of a general result (Theorem 1.3.10 in [MR77] and Theorem 2 in sct. 7.3 in [BW85]). We cannot state, nor prove that general theorem, as we have not developped the special category-theoretic machinery. But we can give a direct proof of the particular result that we need.

Some notation to begin: suppose C is a category and J a subcanonical topology. Suppose also that B is a full subcategory of C. Let J' be the topology on B generated by those coverings of J on objects in B which consist of maps in B.

1.5.6 **Theorem** *Let C, B and J as above. Suppose that all coverings in J' are coverings in J. Moreover, suppose that for each object C in C there is a covering $\{f_i : B_i \longrightarrow C\}_{i \in I}$ in J such that all objects B_i are in B. Then there is a commutative diagram of functors*

$$\begin{array}{ccc} \mathsf{B} & \subset & \mathsf{C} \\ {\scriptstyle \mathcal{Y}}\downarrow & & \downarrow{\scriptstyle \mathcal{Y}} \\ \mathsf{Sh}(\mathsf{B}, J') & \simeq & \mathsf{Sh}(\mathsf{C}, J) \end{array}$$

commutes.

Proof First consider the covering $\{f_i : B_i \longrightarrow C\}_{i \in I}$, and notice that a grid for it can produced consisting all of maps in B, as the latter is a full subcategory of C. So, by 1.5.2, $\mathcal{Y}C$ is a colimit in $\mathcal{A}$ of a diagram in B. Let $Y : \mathsf{B} \longrightarrow \mathsf{C}$ be the inclusion functor. Set $\mathcal{A}$ for the topos $\mathsf{Sh}(\mathsf{C}, J)$ and $\mathcal{B}$ for the topos $\mathsf{Sh}(\mathsf{B}, J')$. Define the functor $R : \mathcal{A} \longrightarrow \mathcal{B}$ as

$$\begin{array}{ccc} F & \mapsto & FK \\ f\downarrow & \mapsto & \downarrow fK \\ G & \mapsto & GK, \end{array}$$

so that R restricts a sheaf on C to B. Because all coverings in J' are in J, the presheaf FK is a sheaf for J'. Let B be an object in B. Then

$$R\mathcal{Y}B = \mathcal{Y}B \circ K = \mathcal{Y}B.$$

Again, as the coverings of J' are in J, R preserves colimits. Thus, for C an object in C, the functor $R\mathcal{Y}C$ is a colimit of a diagram in B (identified now with its image along the Yoneda functor in $\mathcal{B}$). Finally, we produce an inverse for R. Let G be a sheaf in $\mathcal{B}$. Define a presheaf TG on C as follows:

$$\begin{array}{ccc} C & \mapsto & \mathcal{B}(R\mathcal{Y}C, G) \\ f\uparrow & \mapsto & \downarrow - \circ R\mathcal{Y}f \\ D & \mapsto & \mathcal{B}(R\mathcal{Y}D, G). \end{array}$$

Because any covering of C produces a colimit diagram in B for which C is a colimit, the functor TG is a sheaf on C. Trivially, T is lifted to a functor from $\mathcal{B}$ to $\mathcal{A}$. By Yoneda lemma,

$$KTG \simeq G,$$

and, by using again the fact that an object in C is a colimit of a diagram in B,

$$TKF \simeq F. \ \square$$

Chapter 2

Categories of partial maps

Categories of partial maps are nothing new in category theory, but the remarkably simple notion of dominical category introduced by [PH86] has sparked new interest. They expressly introduced dominical categories, as a basic tool for a novel abstract formulation of recursion theory. Dominical categories are intended to capture axiomatically the idea of a "category of partial maps". It was after proving a completeness result about the theory in [Ros80], that it became clear to us how completeness could still be obtained with some weakenings in the definition. Not only this, the completeness result can be read as a abstract characterization of the categories of partial maps.

The new notion which we introduce is that of p-category: they are structured categories resembling very closely dominical categories. A reader familiar with the work of [PH86], will recognize the many similarities. The completeness theorem states that an arbitrary p-category is a full subcategory of a category of partial maps, the embedding being as tight as possible. The results in section 2.3 should be useful also for the new approach to the theory of domains of [Sco80a], suggested by [Plo85]. In the more general setting, we are also able to improve various results in the literature. In particular, we show that ranges and coproducts are stable, solving a problem posed by [PH86]. Stability will also be useful for the embedding theorems of chapter 3.

2.1 p-categories

Dominions. The notion we are going to treat is an attempt to axiomatize categories of partial maps defined over categories with binary products. So we must first explain what we mean by a partial map: suppose A is a given category with binary products, and $\mathcal{M}$ a family of monos of A closed under identities and composition, with the property that any pullback of a monic in $\mathcal{M}$ exists in A and a representative of it belongs to $\mathcal{M}$. We shall call such a family a *dominion*. Even though it is not necessary, it will simplify the treatment to suppose dominions are closed under isomorphic monos, and we shall do so. If B is another category with finite products and a dominion $\mathcal{N}$, a functor $F\colon \mathsf{A} \longrightarrow \mathsf{B}$ which takes monos in $\mathcal{M}$ to monos in $\mathcal{N}$ is said to *preserve dominions*.

Trivial examples of dominions are the collection of isomorphisms in an arbitrary category, and the family of all monomorphisms in a category with finite limits.

Definition. Given a dominion $\mathcal{M}$ in A, a partial map $X \xleftarrow{m}\!\!\prec A \xrightarrow{\phi} Y$ from X to Y is said to be *defined in* $\mathcal{M}$, if $m \in \mathcal{M}$. Two partial maps $X \xleftarrow{m}\!\!\prec A \xrightarrow{\phi} Y$ and $X \xleftarrow{n}\!\!\prec B \xrightarrow{\psi} Y$ are *equivalent* if there is an isomorphism $i\colon A \longrightarrow B$ making the two obvious diagrams commute. Sometimes, we may refer to a partial map as $\phi\colon X \multimap Y$, if there is no confusion about the mono $m\colon A \rightarrowtail X$.

The conditions on $\mathcal{M}$ are essential in order to define a category of partial maps: the category $\mathsf{P}\,(\mathsf{A},\mathcal{M})$ of partial maps has the same objects as A, and arrows are equivalence classes of partial maps defined in $\mathcal{M}$. Composition of two maps whose representatives are $X \xleftarrow{m}\!\!\prec A \xrightarrow{\phi} Y$ and $Y \xleftarrow{n}\!\!\prec B \xrightarrow{\psi} Z$ is represented

by $X \xleftarrow{k} C \xrightarrow{\vartheta} Z$ where $k = m \circ n'$, $\vartheta = \psi \circ \phi'$, and

$$\begin{array}{ccc} C & \xrightarrow{\phi'} & B \\ {\scriptstyle n'}\downarrow & & \downarrow{\scriptstyle n} \\ A & \xrightarrow{\phi} & Y \end{array}$$

is a pullback.

Main property. The next proposition summarizes the fundamental property of a category of partial maps: it contains the base category A as the subcategory of *totally defined* maps, and also inherits part of the cartesian structure. Properties of the latter will be the essential feature of the axiomatization which will immediately follow. Instead totally defined maps will be recaptured later.

2.1.1 **Proposition** *Let A be a category with finite products, $\mathcal{M}$ a dominion. Then there is an embedding $J\colon \mathsf{A} \longrightarrow \mathsf{P}(\mathsf{A}, \mathcal{M})$. Moreover, the product defined on the image of A can be extended as a bifunctor to all of $\mathsf{P}(\mathsf{A}, \mathcal{M})$.*

Proof Define

$$JX = X \quad \text{and} \quad J(f\colon X \longrightarrow Y) = X \xleftarrow{\mathrm{id}} X \xrightarrow{f} Y.$$

J is clearly a functor. It is faithful: indeed, given $f, g\colon X \longrightarrow Y$ in A, suppose $X \xleftarrow{\mathrm{id}} X \xrightarrow{f} Y$ and $X \xleftarrow{\mathrm{id}} X \xrightarrow{g} Y$ are equivalent. Then the existing isomorphism $i\colon X \longrightarrow X$ is the identity, and $f = g$ in A. The extension of the product is accomplished as follows: for X, Y objects, it is $X \times Y$; and for partial maps $X \xleftarrow{m} A \xrightarrow{\phi} Z$ and $Y \xleftarrow{n} B \xrightarrow{g} W$, it is $X \times Y \xleftarrow{m \times n} A \times B \xrightarrow{f \times g} Z \times W$—note that $m \times n$ must be in $\mathcal{M}$, because it is the pullback of m along $p(\mathrm{id} \times n)$. □

There is no need to require that A has products to obtain a category $\mathsf{P}(\mathsf{A}, \mathcal{M})$, but A will be so in all the cases we shall consider. Actually, "product" is the central feature of in the axiomatic counterpart of the situation in 2.1.1. The attention is concentrated on the *algebraic* properties of the "product" bifunctor.

2.1.2 **Definition** A *p-category* is a category C endowed with a bifunctor $\times\colon \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$ which will be called *product*, a natural transformation $\Delta\colon \mathrm{Id} \dashrightarrow \times \circ \langle \mathrm{Id}, \mathrm{Id} \rangle$ which will be called *diagonal* and for each object X in C natural transformations $p_{-,X}\colon - \times X \dashrightarrow \mathrm{Id}$, and $q_{X,-}\colon X \times - \dashrightarrow \mathrm{Id}$ called *projections*, such that

$$\begin{array}{ll} p_{X,X}\Delta_X = \mathrm{id}_X = q_{X,X}\Delta_X, & (p_{X,Y} \times q_{X,Y})\Delta_{X \times Y} = \mathrm{id}_{X \times Y}, \\ p_{X,Y}(\mathrm{id}_X \times p_{Y,Z}) = p_{X,Y \times Z}, & p_{X,Z}(\mathrm{id}_X \times q_{Y,Z}) = p_{X,Y \times Z}, \\ q_{X,Z}(p_{X,Y} \times \mathrm{id}_Z) = q_{X \times Y,Z}, & q_{Y,Z}(q_{X,Y} \times \mathrm{id}_Z) = q_{X \times Y,Z}; \end{array}$$

and the isomorphisms α and μ for associativity and commutativity defined by

$$\alpha_{X,Y,Z} = ((\mathrm{id}_X \times p_{Y,Z}) \times q_{Y,Z}q_{X,Y \times Z})\Delta_{X \times (Y \times Z)}\colon X \times (Y \times Z) \longrightarrow (X \times Y) \times Z$$

and

$$\mu = (q_{X,Y} \times p_{X,Y})\Delta_{X \times Y}\colon X \times Y \longrightarrow Y \times X$$

are natural in all variables—notice that their components need not be so. A *p-functor* $F\colon \mathsf{C} \longrightarrow \mathsf{D}$ between p-categories is a functor preserving products, projections and diagonals up to a natural isomorphism.

Example. For A a category with finite products, $\mathcal{M}$ a dominion, the category $\mathsf{P}(\mathsf{A}, \mathcal{M})$ is a p-category. In particular, the category A itself is a p-category.

A few comments about the various requirements of the definition above. There s no question about the first two identities in 2.1.2: if $\times$ is to behave like a cartesian product, then these identities must hold. The other four are slightly redundant with the commutativity isomorphism. All four are needed to prove that α

and μ are isomorphisms, but the fact that μ is a natural isomorphism yields one pair of identities from the other. The reader will be more convinced about the four last identities in 2.1.2 after a look at the *domains*.

Domains. Given a map $\phi\colon X \longrightarrow Y$ in the p-category C we write $\mathrm{dom}\,\phi$ for the map

$$p_{X,Y}(\mathrm{id} \times \phi)\Delta_X\colon X \longrightarrow X$$

and call it a *domain*. The definition of domain is in the lines of that given by [PH86] as well as the treatment which follows. A list of properties of domain maps follows here, after a useful technical lemma. We must warn the reader that, from now on, we will deliberately omit the indices.

2.1.3 **Lemma** *Suppose* C *is a p-category, and* ϕ *and* ψ *are maps in* C. *The following equalities hold:*

(i) $p(\mathrm{id} \times \Delta) = p,\ q(\Delta \times \mathrm{id}) = q$;

(ii) $\alpha(\mathrm{id} \times \Delta) = (\mathrm{id} \times \Delta)$;

(iii) $p(\mathrm{id} \times \phi) = p(\mathrm{id} \times \mathrm{dom}\,\phi)$;

(iv) $(\mathrm{id} \times \mathrm{dom}\,\phi)\Delta = \Delta\,\mathrm{dom}\,\phi,\ (\mathrm{dom}\,\phi \times \mathrm{id})\Delta = \Delta\,\mathrm{dom}\,\phi$;

(v) *if* $p\phi = p\psi$ *and* $q\phi = q\psi$, *then* $\phi = \psi$.

Proof (i) As $p\Delta = \mathrm{id}$, it follows that $(\mathrm{id} \times p)(\mathrm{id} \times \Delta) = \mathrm{id}$, hence

$$p(\mathrm{id} \times \Delta) = p(\mathrm{id} \times p)(\mathrm{id} \times \Delta) = p.$$

The other identity is proved by applying the natural isomorphism μ:

$$q(\Delta \times \mathrm{id}) = p\mu(\Delta \times \mathrm{id}) = p(\mathrm{id} \times \Delta)\mu = p\mu = q.$$

(ii) Notice

$$(\mathrm{id} \times p)(\mathrm{id} \times \Delta) = \mathrm{id} \quad \text{and} \quad qq(\mathrm{id} \times \Delta) = q.$$

Therefore

$$\alpha(\mathrm{id} \times \Delta)\Delta = (\mathrm{id} \times q)\Delta = \Delta \times \mathrm{id}.$$

(iii) As $q\Delta = \mathrm{id}$ and $p(\mathrm{id} \times q) = p(\mathrm{id} \times p)$, it is

$$\begin{aligned} p(\mathrm{id} \times \phi) &= p(\mathrm{id} \times \phi)(\mathrm{id} \times q)(\mathrm{id} \times \Delta) \\ &= p(\mathrm{id} \times q)(\mathrm{id} \times (\mathrm{id} \times \phi))(\mathrm{id} \times \Delta) \\ &= p(\mathrm{id} \times p)(\mathrm{id} \times (\mathrm{id} \times \phi))(\mathrm{id} \times \Delta) \\ &= p(\mathrm{id} \times \mathrm{dom}\,\phi). \end{aligned}$$

(iv) By (ii),

$$\begin{aligned} (\mathrm{id} \times \mathrm{dom}\,\phi)\Delta &= (\mathrm{id} \times p)(\mathrm{id} \times (\mathrm{id} \times \phi))(\mathrm{id} \times \Delta)\Delta \\ &= (p \times \mathrm{id})\alpha(\mathrm{id} \times (\mathrm{id} \times \phi))(\mathrm{id} \times \Delta)\Delta \\ &= (p \times \mathrm{id})((\mathrm{id} \times \mathrm{id}) \times \phi)\alpha(\mathrm{id} \times \Delta)\Delta \\ &= p(\mathrm{id} \times \phi)(\Delta \times \mathrm{id})\Delta \\ &= p(\Delta \times \mathrm{id})(\mathrm{id} \times \phi)\Delta \\ &= \Delta\,\mathrm{dom}\,\phi. \end{aligned}$$

The second identity is obtained by composing with μ. (v) The map $(p \times q)\Delta\colon X \times Y \longrightarrow X \times Y$ is the identity. So $\phi = (p \times q)\Delta\phi = (p\phi \times q\phi)\Delta$; also $(p\psi \times q\psi)\Delta = \psi$. Hence the assertion follows. □

2.1.4 **Proposition** *The domain operator has the following properties:*

(i) $\mathrm{dom}\ \mathrm{id} = \mathrm{id},\ \mathrm{dom}\,p = \mathrm{id},\ \mathrm{dom}\,q = \mathrm{id},\ \mathrm{dom}\,\Delta = \mathrm{id}$;

(ii) $\mathrm{dom}(\psi\phi) = \mathrm{dom}((\mathrm{dom}\,\psi)\ \phi)$;

(iii) $(\mathrm{dom}\,\psi)\phi = \phi\,\mathrm{dom}(\psi\phi)$;

(iv) $\mathrm{dom}(\phi \times \psi) = \mathrm{dom}\,\phi \times \mathrm{dom}\,\psi$;

(v) $\mathrm{dom}\,\phi \circ \mathrm{dom}\,\vartheta = \mathrm{dom}\,\vartheta \circ \mathrm{dom}\,\phi$.

Proof (i) Trivial. (ii) Follows immediately from 2.1.3(iii). (iii) Just recall the definition of p-category to check that

$$\begin{aligned}(\mathrm{dom}\,\psi)\phi &= p(\mathrm{id}\,\times\psi)\Delta\phi\\ &= p(\phi\times\psi\phi)\Delta\\ &= p(\phi\times\mathrm{id}\,)(\mathrm{id}\,\times\psi\phi)\Delta\\ &= \phi p(\mathrm{id}\,\times\psi\phi)\Delta\\ &= \phi\,\mathrm{dom}(\psi\phi).\end{aligned}$$

(iv) Notice that the map

$$((p\times p)\times(q\times q))\Delta\colon (X\times Y)\times(Z\times W)\longrightarrow (X\times Z)\times(Y\times W)$$

can be obtained as a composition of isomorphisms for associativity and commutativity, hence it is natural in all variables. Then

$$\begin{aligned}\mathrm{dom}(\phi\times\psi) &= p((\mathrm{id}\,\times\mathrm{id}\,)\times(\phi\times\psi))\Delta\\ &= (p\times p)((\mathrm{id}\,\times\phi)\times(\mathrm{id}\,\times\psi))(\Delta\times\Delta)\\ &= (p(\mathrm{id}\,\times\phi)\Delta)\times(p(\mathrm{id}\,\times\psi)\Delta)\\ &= \mathrm{dom}\,\phi\times\mathrm{dom}\,\psi\end{aligned}$$

(v) Use 2.1.3(iv) to show

$$(\mathrm{dom}\,\phi\times\mathrm{dom}\,\vartheta)\Delta = \Delta\,\mathrm{dom}\,\phi\,\mathrm{dom}\,\vartheta = \Delta\,\mathrm{dom}\,\vartheta\,\mathrm{dom}\,\phi.$$

Composing with p on the left gives the result. □

2.1.5 **Corollary** *The following hold:*

(i) $\mathrm{dom}(\mathrm{dom}\,\phi) = \mathrm{dom}\,\phi$;

(ii) $\mathrm{dom}\,\phi \circ \mathrm{dom}\,\phi = \mathrm{dom}\,\phi$;

(iii) $\mathrm{dom}(\mathrm{dom}\,\phi \circ \mathrm{dom}\,\psi) = \mathrm{dom}\,\phi \circ \mathrm{dom}\,\psi$;

(iv) *composition on the set of domain maps on a given object defines an infsemilattice structure with a greatest element.*

It should already be clear how a domain map enjoys properties which would be typical of an actual domain of definition. Indeed in a category of partial maps that is exactly what they are.

2.1.6 **Proposition** *Let* A *be a category with finite products,* $\mathcal{M}$ *a dominion. In the category* $\mathsf{P}(\mathsf{A},\mathcal{M})$ *the map* $\mathrm{dom}(X \xleftarrow{m} A \xrightarrow{\phi} Y)$ *is the equivalence class determined by the pair* $X \xleftarrow{m} A \xrightarrow{m} X$.

Proof Recall that the composition defining a domain is built with first projection and diagonal which are total maps. So we need to evaluate the composition of the two partial maps $X \times X \xleftarrow{\mathrm{id}\,\times m} X \times A \xrightarrow{p} X$ and $X \xleftarrow{\mathrm{id}} X \xrightarrow{\Delta} X \times X$. But

$$\begin{array}{ccc} A & \xrightarrow{\langle m,\mathrm{id}\rangle} & X\times A\\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\,\times m}\\ X & \xrightarrow{\Delta} & X\times X\end{array}$$

is a pullback and $m \in \mathcal{M}$, and the result follows. □

Total maps. The leading example then suggests that in a p-category a "total" map ought to be one whose domain is the identity. So call a map $\phi\colon X \longrightarrow Y$ in the p-category C *total* if $\mathrm{dom}\,\phi = \mathrm{id}_X$. The last four identities in 2.1.2 attest projections are total maps.

2.1.7 **Proposition** *For all objects X and Y in a p-category C, the maps id_X, $p_{X,Y}$, $q_{X,Y}$ and Δ_X are total. If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are total maps, then their composition $gf\colon X \longrightarrow Z$ is total.*

Proof Follows directly from 2.1.4. □

2.1.8 **Corollary** *The total maps form a subcategory C_t with the same objects as C; moreover it has binary cartesian products.*

Proof The only thing to prove is that C has products. By 2.1.4, C_t is closed under the product functor. Projection maps from $X \times Y$ are exactly $p_{X,Y}$ and $q_{X,Y}$. For any pair of total maps $f\colon Z \longrightarrow X$ and $g\colon Z \longrightarrow Y$, let $h = (f \times g)\Delta\colon Z \longrightarrow X \times Y$. It is total since a composition of total maps, and $ph = p(f \times g)\Delta = p(f \times \mathrm{id})(\mathrm{id} \times g)\Delta = fp(\mathrm{id} \times g)\Delta = f\,\mathrm{dom}\,g = f$ and, similarly, $qh = g$. Uniqueness follows from 2.1.3(v). □

Thus the situation of 2.1.1 has indeed been recovered axiomatically. It seems as if enough properties of products have been put forward to ensure handyness—recall chapter 7 in [Mac71]. Even more: looking at 2.1.5(iv), one would think that intersections have been replaced by composition; so what about pullbacks in general? If the axiomatization is correct, it should express inverse images by means of composition, at least, for domains.

Meanwhile, note that the product on C is a true cartesian product on a subcategory, and this may be a simple way to check if the identities of 2.1.2 hold in a given category. The result of this section can be summarised to give the following criterion.

2.1.9 **Theorem** *Let C be a category endowed with a bifunctor $\times\colon \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$ and natural transformations $\Delta\colon \mathrm{Id} \stackrel{\cdot}{\longrightarrow} \times \circ \langle \mathrm{Id}, \mathrm{Id} \rangle$ and $p_{-,X}\colon - \times X \stackrel{\cdot}{\longrightarrow} \mathrm{Id}$, $q_{X,-}\colon X \times - \stackrel{\cdot}{\longrightarrow} \mathrm{Id}$ for each X, object of C. The category C is a p-category if and only if there is a subcategory B of C with the same objects as C, such that the bifunctor $\times$ and the natural transformations define a cartesian product on it and the derived associativity and commutativity isomorphisms are natural with respect to maps in C.*

2.2 Two comparisons

Dominical categories. After mentioning so many times other axiomatic approaches to the problem of partial maps, we should compare the notion we introduced with the others. The first comparison is between the notion of p-category and that of dominical category.

Let us review the definition of dominical category given by [PH86]: let C be a *pointed* category, i.e. a category with distinguished maps $0_{X,Y}\colon X \longrightarrow Y$, stable under composition (that is, $0_{X,Y}\phi = 0_{Z,Y}$ and $\psi 0_{X,Y} = 0_{X,W}$). Call a map $f\colon X \longrightarrow Y$ in C *weakly total* if it satisfies the property that,

$$\text{for every } \phi\colon Z \longrightarrow X, \text{ if } f\phi = 0, \text{ then } \phi = 0.$$

Notice that composition of weakly total maps is weakly total, and identities are weakly total. Set C' for the subcategory of C consisting of weakly total maps. Finally the category C is *dominical* if it is endowed with a bifunctor $\times\colon \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$ which applies $\mathsf{C}' \times \mathsf{C}'$ into C', defining a cartesian product on that subcategory, such that the induced commutativity and associativity isomorphisms are natural with respect to maps in C and the following conditions hold: for all ϕ, ψ in C,

$$\phi \times \psi = 0 \quad \text{if and only if} \quad \text{either } \phi \text{ or } \psi \text{ is } 0;$$

$$p(\psi \times \mathrm{id}) = \phi p \quad \text{and} \quad (\phi \times \phi)\Delta = \Delta\phi.$$

The following result is an immediate consequence of 2.1.9.

2.2.1 **Proposition** *A dominical category C is a p-category. Moreover, the categories C' and C_t coincide.*

Proof The first assertion follows immediately from 2.1.9. Suppose now $f\colon X \longrightarrow Y$ is weakly total. Then $\mathrm{dom}\,f = p(\mathrm{id} \times f)\Delta = p\Delta = \mathrm{id}$, as $\times$ is a cartesian product on C'. So f is total. Last, if $\mathrm{dom}\,g = \mathrm{id}$, for $g\colon X \longrightarrow Y$, and $\phi\colon Y \longrightarrow Z$ is such that $g\phi = 0$, then $\phi = (\mathrm{dom}\,g)\phi = p(\mathrm{id} \times g)\Delta\phi = p(\phi \times g\phi)\Delta = 0$. Thus g is weakly total. □

2.2.2 **Remark** We note that in an arbitrary pointed p-category where the zero maps satisfy the condition $\phi \times 0 = 0 \times \phi = 0$, a total map is weakly total. To prove it, repeat the last part of the proof above.

An inverse to 2.2.1 cannot hold: take A any category, and let $\mathcal{I}$ be the dominion consisting of the isomorphisms of A. Then the p-category $\mathsf{P}(\mathsf{A},\mathcal{I})$ cannot be dominical because it is not pointed. A more interesting counterexample was suggested by Franco Montagna: the *syntactic* p-category T from the theory PA of arithmetic. The category T has one object and maps are provable-equivalence classes of Σ_1-formulae $F(x,y)$ of PA, for which PA proves functionality in y. The composition of two maps $F(x,y)$ and $G(y,z)$ is defined by the formula

$$\exists y.\ F(x,y) \wedge G(y,z).$$

It is a p-category.

Given a map $F(x,y)$, its domain $D(x,x')$ is defined by the composition $\exists y.\ F(x,y) \wedge x = x'$. So the map is total exactly if

$$\mathrm{PA} \vdash \forall x\ \exists y.\ F(x,y).$$

On the other hand, a weakly total map $G(x,y)$ must only satisfy, for all $H(z,x)$ in T,

$$\text{if } \mathrm{PA} \vdash \forall x,y,z\ \neg[G(x,y) \wedge H(z,x)], \quad \text{ then } \mathrm{PA} \vdash \forall x,z.\ \neg H(z,x).$$

It is possible to prove that there are maps satisfying the condition for weak totality which cannot be total. Note that a Π_2-formula K such that $\mathrm{PA} + K$ has the same Π_1-consequences as PA, defines a weakly total map in T: say K has the form $\forall x \exists y.\ J(x,y)$ where J is Δ_0. The formula J' defined as

$$J(x,y) \wedge \forall z \leq y.\ \neg J(x,y)$$

is Σ_1 and provably functional. Moreover

$$\mathrm{PA} \vdash \forall x\ \exists y.\ J(x,y) \leftrightarrow \exists y.\ J'(x,y).$$

Take any provably functional Σ_1-formula $F(z,x)$ and suppose $J'F = 0$, that is to say

$$\mathrm{PA} \vdash \forall z,y \neg \exists x.\ F(z,x) \wedge J'(x,y).$$

Therefore

$$\mathrm{PA} \vdash K \rightharpoonup \forall z,x.\ \neg F(z,x),$$

and the hypothesis on K yields

$$\mathrm{PA} \vdash \forall z,x.\ F(z,x),$$

as this last formula is Π_1. So $F = 0$ in T, and J' is weakly total.

There is a Δ_0-formula K such that K and $\neg K$ satisfy the previous property of Π_1-conservativeness over PA. So they both induce weakly total maps in T, but they cannot be both total. Therefore T cannot be a dominical category with the natural structure.

Dht-categories. The second comparison is between p-categories and *dht*-symmetric categories studied by Hoenke and his school, see [Hoe77], [Hoe72], [Vog79]. A *dht*-symmetric category is a symmetric monoidal closed category $(\mathsf{C},\otimes,I,a,r,l,c)$—as in [EK66], the structure $(\mathsf{C},\otimes,I,a,r,l,c)$ consists of a bifunctor $\otimes\colon \mathsf{C}\times\mathsf{C} \longrightarrow \mathsf{C}$, an object I, and natural isomorphisms:

$$\begin{aligned} a_{X,Y,Z}\colon X\otimes(Y\otimes Z) &\longrightarrow (X\otimes Y)\otimes Z \\ r_X\colon X\otimes I &\longrightarrow X, \\ l_X\colon I\otimes X &\longrightarrow X, \\ c_{X,Y}\colon X\otimes Y &\longrightarrow Y\otimes X, \end{aligned}$$

satisfying some *obvious* coherent conditions, see [Mac71]—with some further structure: a natural transformation $d\colon \mathrm{Id} \dot{\longrightarrow} \otimes\langle \mathrm{Id},\mathrm{Id}\rangle$ and a family of maps $\{t_X\colon X \longrightarrow I | X \in \mathsf{C}\}$, and a distinguished map $o\colon I \longrightarrow O$,

such that $X \otimes O = O$, the maps $o \circ t_X : X \longrightarrow O$ and $r_X(\mathrm{id}_X \otimes t_O) : O = X \otimes O \longrightarrow X$ are the only ones with source or target O. Moreover the following conditions are satisfied:

$$r_X(\mathrm{id}_X \otimes t_X)d_X = l_X(t_X \otimes \mathrm{id}_X)d_X = \mathrm{id}_X,$$

$$(r_X(\mathrm{id}_X \otimes t_Y) \otimes l_Y(t_X \otimes \mathrm{id}_Y))d_{X\otimes Y} = \mathrm{id}_{X\otimes Y}.$$

If one substitutes $p_{X,Y}$ and $q_{X,Y}$ for the maps $r_X(\mathrm{id}_X \otimes t_Y)$ and $l_Y(t_X \otimes \mathrm{id}_Y)$ in the previous equations, they resemble the first two conditions of 2.1.2 very closely. We think that Hoenke had an intuition about maps like these resembling projections, when he introduced the category $\widehat{\mathsf{C}}$ which is the clone generated by all the maps a, r, l, c, t.

2.2.3 **Proposition** *A dht-symmetric category which satisfies the further conditions that*

$$t_{I\otimes I} = r_I, \qquad t_{X\otimes I}(\mathrm{id}_X \otimes t_Y) = t_{X\otimes Y},$$

is a p-category. Moreover, the category $\widehat{\mathsf{C}}$ is contained in C_t, and the object I is a terminal object in C_t.

Proof Set $p_{X,Y}$ and $q_{X,Y}$ for the maps $r_X(\mathrm{id}_X \otimes t_Y)$ and $l_Y(t_X \otimes \mathrm{id}_Y)$. Set $\Delta = d$. Therefore, Δ, $p_{-,X}$ and $q_{X,-}$ are natural transformations. We already notice that the first two equations in 2.1.2 hold. As for the third,

$$\begin{aligned} p(\mathrm{id} \otimes p) &= r(\mathrm{id} \otimes t)(\mathrm{id} \otimes r(\mathrm{id} \otimes t)) \\ &= r(\mathrm{id} \otimes r(t \otimes \mathrm{id})(\mathrm{id} \otimes t)) \\ &= r(\mathrm{id} \otimes t(t \otimes \mathrm{id})(\mathrm{id} \otimes t)) \\ &= r(\mathrm{id} \otimes t(\mathrm{id} \otimes t)) \\ &= r(\mathrm{id} \otimes t) \\ &= p. \end{aligned}$$

The other three are checked similarly. To prove that α and μ are natural in all variables, we first show that each of their "components" coincide with the homonymous one of a and c respectively, then deduce they coincide by means of the identity $\mathrm{id}_{X,Y} = (p \times q)\Delta$, which already holds for *dht*-symmetric categories. So let us evaluate the two components of $a : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z$:

$$\begin{aligned} pa &= r(\mathrm{id} \otimes t))a \\ &= ra(\mathrm{id} \otimes (\mathrm{id} \otimes t) \\ &= (\mathrm{id} \otimes r)(\mathrm{id} \otimes (\mathrm{id} \otimes t) \\ &= \mathrm{id} \otimes (r(\mathrm{id} \otimes t) \\ &= \mathrm{id} \otimes p \end{aligned}$$

$$\begin{aligned} qa &= l(t \otimes \mathrm{id})a \\ &= l(l(t \otimes t) \otimes \mathrm{id})a \\ &= l(l \otimes \mathrm{id})a(t \otimes (t \otimes \mathrm{id})) \\ &= l(t \otimes l(t \otimes \mathrm{id})) \\ &= l(t \otimes \mathrm{id})l(t \otimes \mathrm{id}) \\ &= qq. \end{aligned}$$

The proof for c and μ is similar. It follows now that I is a terminal object in C_t, as $\otimes$ is a cartesian product in C_t and $p : X \otimes I \longrightarrow X$ is an isomorphism for all objects X. □

2.2.4 **Remark** The first papers by Hoenke displayed a non-equational, definition for *dht*-symmetric categories. [Sch84] showed that it could be reduced to an equational one. Unfortunately we were unable to obtain the original source *loc.cit.* ; we only found references to that in [Vog79], where the author, quoting from [Sch84], states that the category $\widehat{\mathsf{C}}$ has a cartesian product. If that is so, then a *dht*-symmetric category would be a p-category with no further conditions. We were not able to prove $\widehat{\mathsf{C}}$ has a cartesian product, and we suspect that there is a misprint in the definition as given by [Vog79]. The interested reader should check the original bibliography on *dht*-symmetric categories.

2.3 The category of domains

We want to carry on the intuition about domain maps, and exploit the metaphor of $\mathrm{dom}\,\phi$ as being the domain of definition of ϕ. Think of domains as a sort of inclusion maps: we shall build a category on these maps as objects.

The definition. Let $\mathsf{Dom}(C)$ denote the *category of domains* of C. Its objects are the maps in C of the form $\mathrm{dom}\,\phi$ and its arrows $\gamma\colon \mathrm{dom}\,\phi \longrightarrow \mathrm{dom}\,\psi$ are the maps in C which are defined on he whole $\mathrm{dom}\,\phi$ and take values in $\mathrm{dom}\,\psi$, in the sense that

$$\mathrm{dom}\,\gamma = \mathrm{dom}\,\phi \quad \text{and} \quad \gamma = \mathrm{dom}\,\psi \circ \gamma.$$

Identity maps are the domains themselves and composition of γ and β in $\mathsf{Dom}(C)$ is $\beta \circ \gamma$ as defined in C: indeed, if $\gamma\colon \mathrm{dom}\,\phi \longrightarrow \mathrm{dom}\,\psi$ and $\beta\colon \mathrm{dom}\,\psi \longrightarrow \mathrm{dom}\,\vartheta$, then $\mathrm{dom}(\beta \circ \gamma) = \mathrm{dom}((\mathrm{dom}\,\beta)\gamma) = \mathrm{dom}((\mathrm{dom}\,\psi)\gamma) = \mathrm{dom}\,\gamma = \mathrm{dom}\,\phi$, and $\beta \circ \gamma = \mathrm{dom}\,\vartheta \circ \beta \circ \gamma$.
First properties. The category C_t of total morphisms in C is embedded into $\mathsf{Dom}(C)$ by a functor H which takes an object to its identity map and a map in C to itself. We shall show that the category $\mathsf{Dom}(C)$ is the natural completion of C with respect to domains. In order to do this we need the following

2.3.1 **Lemma** *Suppose C is a p-category.*

(i) *Let $\phi\colon X \longrightarrow Y$ be a map in C; then ϕ induces a map $\phi\colon \mathrm{dom}\,\phi \longrightarrow \mathrm{id}_Y$ in $\mathsf{Dom}(C)$.*

(ii) *Let $\vartheta\colon \mathrm{dom}\,\phi \longrightarrow \mathrm{dom}\,\psi$ be a map in $\mathsf{Dom}(C)$; then $\vartheta = \mathrm{dom}\,\psi \circ \vartheta \circ \mathrm{dom}\,\phi$.*

(iii) *Suppose $\mathrm{dom}\,\vartheta\colon \mathrm{dom}\,\phi \longrightarrow \mathrm{dom}\,\psi$ in $\mathsf{Dom}(C)$; then $\mathrm{dom}\,\vartheta = \mathrm{dom}\,\phi$ and it is a mono in $\mathsf{Dom}(C)$.*

Proof (i) Follows from 2.1.4. (ii) As $\mathrm{dom}\,\vartheta = \mathrm{dom}\,\phi, \vartheta = \vartheta \circ \mathrm{dom}\,\phi$ and the result follows from the very definition of a map in $\mathsf{Dom}(C)$. (iii) By definition, $\mathrm{dom}\,\phi = \mathrm{dom}(\mathrm{dom}\,\vartheta) = \mathrm{dom}\,\vartheta$. Hence, for any β with target $\mathrm{dom}\,\phi$ in $\mathsf{Dom}(C)$, $\mathrm{dom}\,\vartheta \circ \beta = \beta$: this yields the assertion. □

A map as in (iii) will be called a *subdomain* of $\mathrm{dom}\,\phi$. The set of subdomains of an object A will be denoted by $\mathrm{Subd}(A)$. We can now give a more detailed picture of C_t in $\mathsf{Dom}(C)$.

2.3.2 **Theorem** *The embedding H of C_t in $\mathsf{Dom}(C)$ is full and faithful, and preserves products.*

Proof Full-and-faithfulness is trivial. As to preservation of products, take $X \times Y$ in C_t and suppose $\vartheta\colon \mathrm{dom}\,\phi \longrightarrow \mathrm{id}_X$ and $\psi\colon \mathrm{dom}\,\phi \longrightarrow \mathrm{id}_Y$. Then we want to prove that the map $\gamma = (\vartheta \times \psi)\Delta$ is the unique map from $\mathrm{dom}\,\phi$ into $\mathrm{id}_X \times \mathrm{id}_Y$ induced by ϑ and ψ such that

$$p\gamma = \theta \quad \text{and} \quad q\gamma = \psi.$$

To show that γ is defined on $\mathrm{dom}\,\phi$, first notice

$$\mathrm{dom}\,\phi = \mathrm{dom}((\mathrm{dom}\,\phi \times \mathrm{dom}\,\phi)\Delta)$$

by 2.1.3 and 2.1.4. Then

$$\begin{aligned}\mathrm{dom}\,\gamma &= \mathrm{dom}((\vartheta \times \psi)\Delta)\\ &= \mathrm{dom}(\mathrm{dom}(\vartheta \times \psi)\Delta)\\ &= \mathrm{dom}((\mathrm{dom}\,\phi \times \mathrm{dom}\,\phi)\Delta)\\ &= \mathrm{dom}\,\phi.\end{aligned}$$

Also, $p\gamma = \vartheta p(\mathrm{id} \times \psi)\Delta = \vartheta\,\mathrm{dom}\,\psi = \vartheta$; and, similarly, $q\gamma = \psi$. Finally, 2.1.3 gives the uniqueness of γ. □
Description of $\mathsf{Dom}(C)$. We shall identify C_t with its image in $\mathsf{Dom}(C)$, and call its objects *total*: upper case letters $X, Y, \ldots$ will denote total objects, while $A, B, \ldots$ will vary over arbitrary objects of $\mathsf{Dom}(C)$, unless actual names from C are used. In particular, we will write very often $\varepsilon >\!\!\longrightarrow A$ to mean $\varepsilon\colon \varepsilon >\!\!\longrightarrow A$ is a subdomain: no confusion can arise. We must also point out, once and for all, that the same name can be used for different arrows in $\mathsf{Dom}(C)$, if it happens that they are defined by the same map in C.

2.3.3 **Theorem** *The category $\mathsf{Dom}(C)$ has binary products and inverse images of subdomains. The subdomains are closed under composition. Thus the family $\mathcal{D}$ of subdomains is a dominion and there is an embedding of p-categories $I\colon \mathsf{C} \longrightarrow \mathsf{P}(\mathsf{Dom}(C), \mathcal{D})$.*

Proof The product $\mathrm{dom}\,\phi \times \mathrm{dom}\,\psi$ is just $\mathrm{dom}(\phi \times \psi)$: the proof of this follows from the fact that it embeds naturally into $X \times Y$, if these are the total objects on which ϕ and ψ are given. Indeed $\mathrm{dom}\,\phi \rightarrowtail X$ and $\mathrm{dom}\,\psi \rightarrowtail Y$ are subdomains, and so is $\mathrm{dom}\,\phi \times \mathrm{dom}\,\psi \rightarrowtail X \times Y$. Thus the only thing to prove is that, if $\gamma\colon A \longrightarrow X$ factors through $\mathrm{dom}\,\phi$ and $\beta\colon A \longrightarrow Y$ factors through $\mathrm{dom}\,\psi$, then $\langle \gamma, \beta \rangle$ factors through $\mathrm{dom}\,\phi \times \mathrm{dom}\,\psi$. But, if $\gamma = \mathrm{dom}\,\phi\,\gamma$ and $\beta = \mathrm{dom}\,\psi\,\beta$, it is

$$\langle \gamma, \beta \rangle = \langle \mathrm{dom}\,\phi\,\gamma, \mathrm{dom}\,\psi\,\beta \rangle = (\mathrm{dom}\,\phi \times \mathrm{dom}\,\psi)\langle \gamma, \beta \rangle .$$

Composition of subdomains is a subdomain by 2.1.4. The inverse image of a subdomain $\varepsilon \rightarrowtail B$ along ϑ is $\mathrm{dom}(\varepsilon\vartheta) \rightarrowtail A$, as the diagram

$$\begin{array}{ccc} \mathrm{dom}(\varepsilon\vartheta) & \xrightarrow{\varepsilon\vartheta} & \varepsilon \\ {\scriptstyle \mathrm{dom}(\varepsilon\vartheta)}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ A & \xrightarrow{\vartheta} & B \end{array}$$

is a pullback, for it obviously commutes:

$$\varepsilon\varepsilon\vartheta = (\mathrm{dom}\,\varepsilon)\vartheta = \vartheta\,\mathrm{dom}(\varepsilon\vartheta),$$

and if $\gamma\colon C \longrightarrow \varepsilon$, $\beta\colon C \longrightarrow A$ are such that $\varepsilon\gamma = \vartheta\beta$, then

$$\beta = \beta\,\mathrm{dom}\,\beta = \beta\,\mathrm{dom}(\varepsilon\gamma) = \beta\,\mathrm{dom}(\varepsilon\vartheta\beta) = \mathrm{dom}(\varepsilon\vartheta)\beta;$$

so $\beta\colon C \longrightarrow \mathrm{dom}(\varepsilon\vartheta)$ and $\varepsilon\vartheta\beta = \varepsilon\varepsilon\gamma = \gamma$. The embedding I is defined by taking a map $\phi\colon X \longrightarrow Y$ in C to the partial map $X \longleftarrow \mathrm{dom}\,\phi \xrightarrow{\phi} Y$. □

This should already give a clear picture of $\mathsf{Dom}(C)$: a map represents a map in C defined on its domain possibly corestricted to a domain, as long as the corestriction does not influence it. Note that restriction and corestriction of a map to a subdomain are obtained by simple composition on the left and on the right, respectively.

2.3.4 **Corollary** *The image of the embedding $I\colon \mathsf{C} \longrightarrow \mathsf{P}(\mathsf{Dom}(C), \mathcal{D})$ is the full subcategory of total objects. Therefore the embedding I is an equivalence if and only if in C the domains split.*

2.3.5 **Corollary** *The preorder on $\mathrm{Subd}(X)$ of a given total object X in $\mathsf{Dom}(C)$ is isomorphic to the infsemilattice of domains of X in C. Moreover, given a map $\phi\colon X \longrightarrow Y$ in C, the inverse image function $\phi^{-1}\colon \mathrm{Subd}(Y) \longrightarrow \mathrm{Subd}(\mathrm{dom}\,\phi)$ preserves meets and top element.*

2.3.6 **Corollary** *Let C be p-category. In $\mathsf{Dom}(C)$,*

(i) *if $\varepsilon \rightarrowtail A$ is a subdomain, then $\varepsilon^{-1} = \varepsilon \cap -$ and $(\varepsilon \circ -) \dashv \varepsilon^{-1}\colon \mathrm{Subd}(A) \longrightarrow \mathrm{Subd}(\varepsilon)$;*

(ii) *let $p\colon A \times B \longrightarrow B$ be a projection; then $p^{-1} = - \times \mathrm{id}_B$.*

Proof (i) Obvious, as $\mathrm{Subd}(\varepsilon) = \{\gamma \in \mathrm{Subd}(A) : \gamma \leq \varepsilon\}$. (ii) Trivial. □

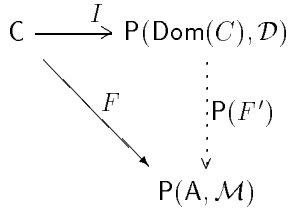
We are now in a position to prove a completeness result for p-categories . First we show how to recover A from $\mathsf{P}(\mathsf{A}, \mathcal{M})$.

2.3.7 **Theorem** *Let A be a category with finite products, $\mathcal{M}$ a dominion. Then the functor $HJ\colon \mathsf{A} \longrightarrow \mathsf{Dom}(P(A, \mathcal{M}))$ which is full, faithful and representative. Therefore it is an equivalence of categories.*

Proof Given an object X in A, $HJ(X)$ is the equivalence class represented by $X \xleftarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X$. As both functors are faithful, so is their composition. Fullness is trivial. As to representativity, recall from 2.1.6 that an object of $\mathsf{Dom}(\mathsf{P}(\mathsf{A}, \mathcal{M}))$ is represented by a pair $X \xleftarrow{m} A \xrightarrow{m} X$, with $m \in \mathcal{M}$. This object is isomorphic to $HJ(A)$. □

A suggestive way to describe $\mathsf{Dom}(P(A, \mathcal{M}))$ is to think of it as a category of superobjects of A .

2.3.8 **Theorem** *Let* C *be a p-category and* I *the embedding functor into* $\mathsf{P}(\mathsf{Dom}(C), \mathcal{D})$. *Suppose* A *is a category with finite products, and* $\mathcal{M}$ *is a dominion. Given a functor of p-categories* $F\colon \mathsf{C} \longrightarrow \mathsf{P}(\mathsf{A}, \mathcal{M})$, *then there is a unique functor* $F'\colon \mathsf{Dom}(C) \longrightarrow \mathsf{A}$ *such that the diagram*



commutes up to a natural isomorphism, preserves products, dominions and pullbacks of subdomains.

Proof In virtue of 2.3.7 there is a functor $K\colon \mathsf{Dom}(P(A, \mathcal{M})) \longrightarrow \mathsf{A}$ which picks out a representative of a subobject in $\mathcal{M}$. Also F preserves domains. Thus

$$F(\mathrm{dom}\,\phi\colon X \longrightarrow X) = (FX \xleftarrow{m} A \xrightarrow{m} FX).$$

Define

$$F'(\mathrm{dom}\,\phi) = K(m, m).$$

The definition of F' on arrows is forced:

$$F'(\vartheta\colon \mathrm{dom}\,\phi \longrightarrow \mathrm{dom}\,\psi) = g\colon F'(\mathrm{dom}\,\phi) \longrightarrow F'(\mathrm{dom}\,\psi),$$

where g is the unique map which factors the map $KF(\vartheta)$ through the monic

$$KF(\mathrm{dom}\,\psi\colon \mathrm{dom}\,\psi \longrightarrow Y).$$

The conclusion is then straightforward. □

We suspect it is possible to give a description of C in terms of properties of an operator dom as in 2.1.4, see [Car84]. Also, it is possible to develop the interpretation of theories expressed by Horn sequents in the category $\mathsf{Dom}(C)$, provided relations are interpreted by subdomains. We do not do this here, because it will be obtained as a byproduct of results in the next chapter.

2.4 Ranges

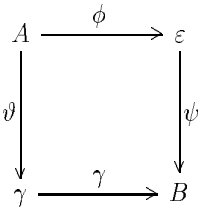
So far the *equational* study of p-categories has proved to be successful: not only there are explicit expressions for such categorical constructs as inverse images and intersections, but also one can recognize a real category of partial maps by means of 2.3.4—a second-order statement, though.

From the point of view of categorical logic, it would be interesting to see if a definition of image or of coproduct can be introduced consistently in p-categories: consistently, in the sense of some transfer principle from a p-category to its category of domains. To do this, we shall borrow definitions from [PH86] and discuss them in the light of 2.3.3. First we study the notion of range of a map.

Definition. A *range* of a map $\phi\colon X \longrightarrow Y$ in C is a domain map $\varepsilon\colon Y \longrightarrow Y$ such that $\varepsilon\phi = \phi$ and for all $\psi, \vartheta\colon Y \longrightarrow Z$ if $\psi\phi = \vartheta\phi$, then $\psi\varepsilon = \vartheta\varepsilon$.

As $\varepsilon = \mathrm{dom}\,\varepsilon$, it will appear both as an object and a map in $\mathsf{Dom}(C)$. Besides, as $\phi\varepsilon = \phi$, in $\mathsf{Dom}(C)$ we have $\phi\colon X \longrightarrow \varepsilon$. One may suspect that it is an *image* of $\phi\colon \mathrm{dom}\,\phi \longrightarrow Y$, that is the least subobject of Y through which ϕ factors, its factorization being an epic map. But this cannot happen. Take, for instance, C to be the monoid Rec of partial recursive functions defined on recursive subsets of N. The domains are the recursive sets, but the subobjects of N in $\mathsf{Dom}(Rec)$ are all the r.e. subsets and hence not all maps can have a domain image. So it is necessary to know which properties maps like $\phi\colon X \longrightarrow \varepsilon$ enjoy. Call a map $\phi\colon \mathrm{dom}\,\phi \longrightarrow \mathrm{dom}\,\psi$ in $\mathsf{Dom}(C)$ a *range map* if $\mathrm{dom}\,\psi$ is a range of ϕ in C.

2.4.1 **Proposition** *Let* C *be a p-category. Suppose in* Dom(C) *the map* $\phi: A \longrightarrow \varepsilon$ *is a range map, and* $\gamma: \gamma \rightarrowtail B$ *is a subdomain. Given any commutative diagram*



there is a unique map, namely $\psi: \varepsilon \longrightarrow \gamma$, *making the two triangles commute.*

Proof We intend to show that ψ is defined into γ. Indeed, by definition $\gamma\psi\phi = \gamma\gamma\vartheta = \gamma\vartheta = \psi\phi$. Then $\gamma\psi\varepsilon = \psi\varepsilon$, as ε is a range of ψ. But $\varepsilon = \mathrm{dom}\,\psi$, so $\gamma\psi = \psi$ and $\psi: \varepsilon \longrightarrow \gamma$ in Dom(C). It makes the lower triangle commute; hence also the upper one, as γ is monic. And it is unique with this property. □
Definition. Maps ϕ and γ, satisfying the conclusion of 2.4.1, are said to be *orthogonal.*

2.4.2 **Corollary** *Suppose* C *is a p-category. In* Dom(C)

(i) *a range map which is also a subdomain, is an iso;*

(ii) *a range map is epic.*

Proof (i) Immediate. (ii) Let $\phi: A \longrightarrow \varepsilon$ be a range map, and suppose $\psi\phi = \vartheta\phi$. Then $\psi = \psi\varepsilon = \vartheta\varepsilon = \vartheta$. □

For a map $\phi: A \longrightarrow B$, we say a subdomain $\varepsilon \rightarrowtail B$ is a *domain image* of ϕ if ε is the least subdomain of B through which ϕ factors.

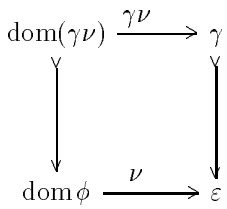
2.4.3 **Theorem** *Let* $\phi: X \longrightarrow Y$ *be a map in* C. *If the domain* ε *is a range of* ϕ *in* C, *then* $\varepsilon: \varepsilon \rightarrowtail Y$ *is the domain image of* $\phi: \mathrm{dom}\,\phi \longrightarrow Y$ *in* Dom(C), *and its factorization is an epic map.*

Proof Suppose $\phi = \gamma\nu$ for $\gamma \rightarrowtail B$ a subdomain. Apply 2.4.1 with ε substituted for ψ and ν for ϑ. □

2.4.4 **Corollary** *Range, when they exist, are unique.*

Characterization. Denote the range of ϕ by $\mathrm{ran}\,\phi$. What one would want to have is a set of properties of range maps which completely characterize them in Dom(C).

2.4.5 **Theorem** *Let* $\phi: X \longrightarrow Y$ *be a map in* C. *A domain* ε *is the range of* ϕ *in* C *if and only if* ε *is the domain image of* ϕ *and the factor map is epic and stable under pullback along subdomains in* Dom(C), *i.e. for* $\gamma \rightarrowtail \varepsilon$ *subdomain, in the pullback*

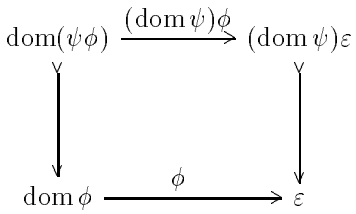


the map $\gamma\nu$ *is epic, in fact a range map.*

Proof Suppose ε is the range of ϕ. As $\gamma^{-1}(\phi) = \gamma\phi: \mathrm{dom}(\gamma\phi) \longrightarrow \gamma$, one must prove that $\gamma = \mathrm{ran}\,\gamma\phi$, because of 2.4.3. First notice $\gamma\gamma\phi = \gamma\phi$. Then, if $\psi\gamma\phi = \vartheta\gamma\phi$, by hypothesis $\psi\gamma\varepsilon = \vartheta\gamma\varepsilon$; and the assertion follows as $\gamma\varepsilon = \gamma$. Suppose now ε is a stable domain image. We must prove is that $\varepsilon = \mathrm{ran}\,\phi$ in C. As ϕ factors through ε, $\phi = \varepsilon\nu$ for some $\nu: \mathrm{dom}\,\phi \longrightarrow \varepsilon$. By definition $\nu = \varepsilon\nu$. So $\nu = \phi$ and $\phi = \varepsilon\phi$. Suppose ϕ equalizes $\psi, \vartheta: Y \longrightarrow Z$ in C . Then by 2.1.4,

$$(\mathrm{dom}\,\psi)\phi = \phi\,\mathrm{dom}(\psi\phi) = \phi\,\mathrm{dom}(\vartheta\phi) = (\mathrm{dom}\,\vartheta)\phi.$$

As the diagram



is a pullback, the map

$$(\mathrm{dom}\,\psi)\phi\colon \mathrm{dom}(\psi\phi) \longrightarrow\!\!\!\!\triangleright\ (\mathrm{dom}\,\psi)\varepsilon$$

is epi and $(\mathrm{dom}\,\psi)\varepsilon$ is its domain image. Similarly, $(\mathrm{dom}\,\vartheta)\varepsilon$ is the domain image of $(\mathrm{dom}\,\vartheta)\phi$. Therefore $(\mathrm{dom}\,\psi)\varepsilon = (\mathrm{dom}\,\vartheta)\varepsilon$. It is trivial now:

$$\phi\varepsilon(\mathrm{dom}\,\psi)\phi = \psi(\mathrm{dom}\,\psi)\varepsilon\phi = \vartheta(\mathrm{dom}\,\vartheta)\varepsilon\phi = \vartheta\varepsilon(\mathrm{dom}\,\vartheta)\phi.$$

As $(\mathrm{dom}\,\psi)\phi$ is epi, it follows that $\psi\varepsilon = \vartheta\varepsilon$. □

2.4.6 **Corollary** *Suppose* C *is a p-category. Then in* $\mathsf{Dom}(C)$

(i) *composition of range maps is a range map;*

(ii) *identities are range maps.*

Ranges and limits. If in a p-category C all maps have ranges, in which case we will say that C *has ranges*, then it is possible to produce some well known categorical constructs. Of course, if a p-category C has ranges, then by 2.4.1, any map in $\mathsf{Dom}(C)$ can be factored uniquely as a range map and a subdomain.

2.4.7 **Proposition** *In* $\mathsf{Dom}(C)$*, a monomorphism which is orthogonal to epis is a domain.*

Proof Let $\nu\colon A \rightarrowtail B$ be such a mono. Factor it as $\nu\colon A \longrightarrow \mathrm{ran}\,\nu$ and $\mathrm{ran}\,\nu\colon \mathrm{ran}\,\nu \rightarrowtail B$ and form a square with these two maps and $\mathrm{id}\colon A \longrightarrow A$ and $\nu\colon A \rightarrowtail B$. As $\nu\colon A \longrightarrow \mathrm{ran}\,\nu$ is epic, it follows by orthogonality that $\mathrm{ran}\,\nu \leq \nu$. But $\nu \leq \mathrm{ran}\,\nu$ by construction; therefore it is a domain. □

Note that the property above, together with 2.4.1, is the first category-theoretical characterization we can give for domains in $\mathsf{Dom}(C)$.

2.4.8 **Proposition** *Suppose* C *is a p-category with ranges. Let* $\phi\colon A \longrightarrow B$*. The infsemilattice homomorphism* $\phi^{-1}\colon \mathrm{Subd}(B) \longrightarrow \mathrm{Sub}(A)$ *has a left adjoint* $\exists\phi \dashv \phi^{-1}$*, defined as* $\exists\phi(\varepsilon) = \mathrm{ran}(\phi\varepsilon)$*. Hence* $\exists\phi$ *preserves joins. Moreover,* $\exists\phi\psi = \exists\phi\exists\psi$*; and if* $\varepsilon \rightarrowtail \mathrm{dom}\,\phi$ *is a domain, then* $\exists\varepsilon = \varepsilon \circ -$*.*

Proof An application of 2.4.3 and 2.3.6. □

2.4.9 **Proposition** *Let* C *be a p-category with ranges. Any retract in* $\mathsf{Dom}(C)$ *is (isomorphic to) a domain. Therefore, the graph of a map*

$$\langle \mathrm{id}\,, \phi\rangle\colon \mathrm{dom}\,\phi \longrightarrow \mathrm{dom}\,\phi \times \mathrm{dom}\,\psi$$

is a subdomain.

Proof Suppose $\nu\colon \mathrm{dom}\,\nu \rightarrowtail \mathrm{dom}\,\phi$ has a retraction ρ. then $\rho\nu = \mathrm{id}$. So $(\rho\,\mathrm{ran}\,\nu)\nu = \mathrm{id}$. Thus the epic map $\nu\colon \mathrm{dom}\,\nu \longrightarrow \mathrm{ran}\,\nu$ has a left inverse, and is iso. □

2.4.10 **Corollary** *The category* $\mathsf{Dom}(C)$ *has equalizers, hence all finite limits.*

Proof A diagonal Δ is the graph of an identity. Thus, given $\phi, \psi\colon A \longrightarrow B$, the pullback $(\phi \times \psi)^{-1}\Delta$ exists as it is the inverse image of a domain, and is the equalizer of ϕ and ψ. □

2.5 Stability of range maps

This section is devoted to the proof of stability under pullback of range maps with mild hypotheses on the p-category C: it has to have ranges and the operator ran must commute with products. When this is the case, we shall say that C is an *r-category*.

The following result is essentially folklore.

2.5.1 **Proposition** *Let C be a p-category with ranges. Ranges commute with products in C if and only if range maps are stable under pullback along projections in $\mathsf{Dom}(C)$.*

Proof Let $\phi\colon A \longrightarrow \varepsilon$ be a range map, let $p\colon \varepsilon \times B \longrightarrow \varepsilon$ be a projection. Then the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\phi \times \mathrm{id}} & \varepsilon \times B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{\phi} & \varepsilon \end{array}$$

is a pullback; and the assertion follows from 2.4.6. □

The main theorem of this section says that Beck-Chevalley condition holds in $\mathsf{Dom}(C)$, when C is an r-category. The statement is the following.

2.5.2 **Theorem** *Let C be an r-category. Suppose the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \zeta} \\ C & \xrightarrow{\vartheta} & D \end{array}$$

is a pullback in $\mathsf{Dom}(C)$. Given a subdomain $\varepsilon \rightarrowtail B$, then

$$\vartheta^{-1}\exists\zeta(\varepsilon) = \exists\psi(\phi^{-1}(\varepsilon)).$$

The theorem will be proved in various steps. As the property stated in 2.5.2 for the particular pullback is not symmetric, in case we need to be specific, we shall indicate at which vertex subdomains must be taken.

Notice that $\phi = q\langle \mathrm{id}\,, \phi\rangle$, so a map is a composition of a projection and a subdomain. As projections and subdomains are stable under pullback, we need to prove three particular cases of 2.5.2:

2.5.3 **Lemma** *The diagram of projections*

$$\begin{array}{ccc} A \times B \times C & \xrightarrow{q \times \mathrm{id}} & B \times C \\ {\scriptstyle \mathrm{id}\, \times p}\downarrow & & \downarrow{\scriptstyle p} \\ A \times B & \xrightarrow{q} & B \end{array}$$

is a pullback which satisfies Beck-Chevalley condition for subdomains of $A \times B$.

2.5.4 **Lemma** *Suppose $\delta\colon A \rightarrowtail C$ is a subdomain. The diagram*

$$\begin{array}{ccc} A \times B & \xrightarrow{p} & A \\ {\scriptstyle \delta \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \delta} \\ C \times B & \xrightarrow{p} & C \end{array}$$

is a pullback which satisfies Beck-Chevalley condition for subdomains of A.

2.5.5 **Lemma** *Suppose $\delta: A \rightarrowtail C$ is a subdomain and $\phi: B \longrightarrow C$ a map. The diagram*

$$\begin{array}{ccc} \phi^{-1}(A) & \xrightarrow{\delta\phi} & A \\ \downarrow & & \downarrow \\ B & \xrightarrow{\phi} & C \end{array}$$

is a pullback which satisfies Beck-Chevalley condition for subdomains of B.

With the previous remark it is straightforward to obtain the proof of the general case by attacking three pullbacks together. So let us prove the three lemmata.

Proof of 2.5.3. The diagram is obviously commutative and for every pair $f: Z \longrightarrow A \times B$, $g: Z \longrightarrow B \times C$ with $qf = pg$, the map $h = \langle pf, qf, qg \rangle: Z \longrightarrow A \times B \times C$ is unique such that $(\mathrm{id} \times p)h = f$ and $(q \times \mathrm{id})h = g$. So it is a pullback. Let then $\varepsilon \rightarrowtail A \times B$ be a subdomain and evaluate:

$$\begin{aligned} p^{-1}\exists q(\varepsilon) &= \exists q(\varepsilon) \times \mathrm{id} \\ &= \mathrm{ran}(q\varepsilon) \times \mathrm{ran}(\mathrm{id}) \\ &= \mathrm{ran}(q\varepsilon \times \mathrm{id}) \\ &= \mathrm{ran}((q \times \mathrm{id})(\varepsilon \times \mathrm{id})) \\ &= \exists(q \times \mathrm{id})((p \times \mathrm{id})^{-1}(\varepsilon)), \end{aligned}$$

by means of 2.3.6 and 2.4.8. □

Proof of 2.5.4. The diagram is a pullback by 2.3.6. So let $\varepsilon \rightarrowtail A$ be a subdomain of A:

$$\begin{aligned} p^{-1}\exists\delta(\varepsilon) &= \exists\delta(\varepsilon) \times \mathrm{id} \\ &= \delta\varepsilon \times \mathrm{id} \\ &= (\delta \times \mathrm{id})(\varepsilon \times \mathrm{id}) \\ &= \exists(\delta \times \mathrm{id})(p^{-1}(\varepsilon)), \end{aligned}$$

by using 2.3.6 and 2.4.8. □

Proof of 2.5.5. Let $\gamma \rightarrowtail B$ be a subdomain. Recall 2.4.5 and compute:

$$\begin{aligned} \delta^{-1}\exists\phi(\gamma) &= \delta^{-1}(\mathrm{ran}(\phi\gamma)) \\ &= \mathrm{ran}(\delta\phi\gamma) \\ &= \mathrm{ran}(\delta\phi\,\mathrm{dom}(\delta\phi)\gamma) \\ &= \exists\delta\phi((\phi^{-1}(\delta))^{-1}(\gamma)). \end{aligned}$$ □

2.5.6 **Corollary** *Suppose $\phi: B \longrightarrow C$ is a map and $\delta: A \rightarrowtail C$ and $\gamma \rightarrowtail B$ subdomains. Then*

$$\delta \cap \exists\phi(\gamma) = \exists\phi(\phi^{-1}(\delta) \cap \gamma).$$

2.5.7 **Proposition** *Let C be an r-category. For every object A in $\mathsf{Dom}(C)$ there is a closure operator $^{-}: \mathrm{Sub}(A) \longrightarrow \mathrm{Sub}(A)$, defined by $\overline{\nu : dom\nu \rightarrowtail A} = \mathrm{ran}\,\nu$. The closed subobjects are exactly the subdomains. Moreover, given $\phi: A \longrightarrow B$,*

$$\phi^{-1}(\bar{\mu}) = \overline{\phi^{-1}(\mu)} \text{ and } \exists\phi(\bar{\nu}) = \exists\phi(\nu).$$

Proof As ν factors through $\bar{\nu}$, the operator is exhaustive. By definition, $\bar{\nu} = \exists\nu(\mathrm{id})$: so it is idempotent and closed subobjects are the subdomains. By 2.5.2, $\phi^{-1}(\bar{\mu}) = \overline{\phi^{-1}(\mu)}$. In particular, the bar operator commutes with meets. The last two identities follow from 2.5.2 and 2.4.8 respectively. □

Graphs. As the major emphasis in chapter 3 will be on sites with subcanonical topologies, it seems appropriate here to see when range maps, which are stable epimorphisms, may be effective. We can prove that these are effective under a further assumption on the r-category. To explain the way in which we

approach the problem, we look at the way the graph $(\mathrm{id} \times \phi)\Delta\colon X \longrightarrow X \times Y$ determines the partial map $\phi\colon X \longrightarrow Y$ itself. As usual, we shall shift the problem into $\mathsf{Dom}(C)$.

Definition. A subdomain $\rho \rightarrowtail X \times Y$ of $\mathsf{Dom}(C)$ is *single-valued* if

$$\rho \times \rho \cap (p \times p)^{-1}(\Delta_X) \leq \Delta_{X\times Y}.$$

It is an obscure definition for the familiar concept of single-valuedness.

2.5.8 **Remark** The graph of a partial map is a single-valued domain; the graph of an identity is the equality relation, and the graph of a composition is the relational composition of the graphs of the factors.

Given now a single-valued domain $\rho \rightarrowtail X \times Y$, let $\delta = \exists p(\rho)$ be the domain of definition of ρ. By definition, there is a range map

$$p\rho\colon \rho \longrightarrow\!\!\!\!\triangleright\ \delta$$

in $\mathsf{Dom}(C)$ and single-valuedness yields that it is also monic. Indeed, suppose $\phi, \psi\colon A \longrightarrow \rho$ are such that $p\rho\phi = p\rho\psi$. Then the map $\langle \rho\phi, \rho\psi\rangle\colon A \longrightarrow \delta$ factors through $\rho \times \rho$ trivially, and through $(p \times p)^{-1}(\Delta)$ because

$$(p \times p)\langle \rho\phi, \rho\psi\rangle = \langle p\rho\phi, p\rho\psi\rangle = \Delta p\rho\phi,$$

by hypothesis. So $\langle \rho\phi, \rho\psi\rangle$ factors through $\Delta_{X\times Y}$ and

$$\phi = \rho\phi = \rho\psi = \psi.$$

In order to define a map from δ into Y, we need to know if the monic range map $p\rho\colon \rho \longrightarrow\!\!\!\!\triangleright\ \delta$ is a domain, in order to apply 2.4.2 and produce an inverse map. Meanwhile we have proved the following.

2.5.9 **Proposition** *A domain $\rho\colon X \times Y \longrightarrow X \times Y$ which is single-valued is the graph of a map $\phi\colon X \longrightarrow Y$ if and only if $p\rho$ is isomorphic to a domain.*

Stable effective ranges. A way to make sure that the last condition holds is to require that C satisfies the weak axiom of choice. In a way which is similar to that of [PH86], we say that C satisfies the *weak axiom of choice* if every partial monomorphism has a section,where a map ν in C is a *partial monomorphism* if, for all ψ, ϑ such that $\nu\psi = \nu\vartheta$, it is $(\mathrm{dom}\,\nu)\psi = (\mathrm{dom}\,\nu)\vartheta$.

2.5.10 **Proposition** *A map $\nu\colon X \longrightarrow Y$ is a partial monomorphism in C if and only if $\nu\colon \mathrm{dom}\,\nu \longrightarrow Y$ is a mono in $\mathsf{Dom}(C)$. Moreover, C satisfies the weak axiom of choice if and only if every mono in $\mathsf{Dom}(C)$ is isomorphic to a subdomain.*

Proof The first part is immediate. The second follows from 2.4.9. □

So, if C satisfies the weak axiom of choice, single-valued domains are exactly the graphs of maps. It follows then that every range map is a coequalizer: the quotient of its kernel pair. A proof of this can be found in [MR77] or in [Gri71]. The idea of the proof is to mimic the construction of a map on a quotient, by defining it on representatives independently of the particular choice. Finally, we can draw the conclusion.

2.5.11 **Proposition** *Suppose C is an r-category satisfying the weak axiom of choice. Then range maps are the epimorphisms of $\mathsf{Dom}(C)$, and they are stable effective.*

2.6 Zero maps and coproducts

Zero maps. In this section we look at zero maps and coproducts in a p-category. As long as zero maps are considered, our interest will be essentially concentrated on giving a characterisation of those p-categories which are dominical. So recall from section 2.2 that a category C is *pointed* if for any pair of objects X, Y there is a map $0_{X,Y}\colon X \longrightarrow Y$ such that for any ϕ in C, $\phi 0 = 0 = 0\phi$. We say that a p-category C *has zero maps* if it is pointed and

$$\phi \times 0 = 0 = 0 \times \phi.$$

The second definition contains a condition of stability under product. In fact, its counterpart in $\mathsf{Dom}(C)$ is straightforward.

2.6.1 **Theorem** *A p-category* C *is non-empty and has zero maps if and only if* $\mathsf{Dom}(C)$ *has a strict initial object and the maps* $0 \rightarrowtail \mathrm{dom}\,\phi$ *are subdomain If this is the case, then the preorder* $\mathrm{Subd}(A)$ *has a least element, for any object* A *of* $\mathsf{Dom}(C)$.

Proof Suppose C is a non-empty p-category with zero maps. Then the strict initial object is any fixed zero map $0\colon X \longrightarrow X$. It is a domain as $\mathrm{dom}\,0 = 0$. Given any object A in $\mathsf{Dom}(C)$, the appropriate map $0_{X,Y}$ is defined on 0 and takes value in A, trivially. Any other map $\phi\colon 0 \longrightarrow A$ must satisfy $\mathrm{dom}\,\phi = 0$; so

$$\phi = \phi\,\mathrm{dom}\,\phi = \phi 0 = 0.$$

Thus 0 is an initial object in $\mathsf{Dom}(C)$. It is strict because, if $\psi\colon \delta \longrightarrow 0$, then

$$\psi = 0\psi = 0, \quad \text{and} \quad \delta = \mathrm{dom}\,\psi = 0.$$

The other implication is trivial. □

It is compulsory to give a characterisation of dominical categories. Recall from chapter 1 that in an infsemilattice, $\neg a$ denotes the largest element, if it exists, which has null meet with a.

2.6.2 **Theorem** *A non-empty p-category* C *with zero maps is dominical if and only if in* $\mathsf{Dom}(C)$ *product satisfies the cancellation property: if* $A \times B = 0$, *then either* $A = 0$ *or* $B = 0$; *and, for all total objects* X *of* $\mathsf{Dom}(C)$, *if a subdomain* $\delta \rightarrowtail X$ *is not the identity, then there is a non-zero map* $\phi\colon A \longrightarrow X$ *such that* $\phi^{-1}\delta = 0$. *Moreover, if* C *has ranges, the second condition can be weakened to the following: if* $\neg\delta = 0$, *then* $\delta = \mathrm{id}_X$.

Proof The first part is trivial, when noticing that $\psi = 0$ if and only if $\mathrm{dom}\,\psi = 0$. The second part follows, as $\phi^{-1}\delta = 0$ implies $\exists\phi(\mathrm{id}\,) \cap \delta = 0$. □

Coproducts. The last categorical construct we will treat is that of coproduct. It may appear strange that, after being so fussy about products, we will plainly introduce coproducts in a p-category C in the usual way, saying that a p-category C *has coproducts* if, for each pair objects X and Y in C, there is an object $X + Y$ and two maps, called injections, $i\colon X \longrightarrow X + Y$ and $j\colon Y \longrightarrow X + Y$, universal with this property. The map induced by universality from a pair ϕ, ψ is denoted by $\phi + \psi$, and $\phi + \psi$ stands for the map $\phi i + \psi j$. As for the case of ranges, it will turn out that in a p-category coproducts enjoy many unexpected properties.

2.6.3 **Proposition** *Let* C *have coproducts. Then for* $\phi\colon X \longrightarrow Z$ *and* $\psi\colon Y \longrightarrow Z$,

$$\mathrm{dom}\,\phi + \psi = \mathrm{dom}\,\phi + \mathrm{dom}\,\psi.$$

Proof The following three identities holds

$$\Delta = i \times i + j \times j(\Delta + \Delta),$$

$$\mathrm{id} \times \phi + \psi i \times i + j \times j = i \times \mathrm{id} + j \times \mathrm{id}\,(\mathrm{id} \times \phi + \mathrm{id} \times \psi),$$

$$pi \times \mathrm{id} + j \times \mathrm{id} = p + p.$$

Then

$$\begin{aligned} \mathrm{dom}\,\phi + psi &= p\,(\mathrm{id} \times \phi + \psi)\,\Delta \\ &= (p + p)(\mathrm{id} \times \phi + \mathrm{id} \times \psi)(\Delta + \Delta) \\ &= \mathrm{dom}\,\phi + \mathrm{dom}\,\psi. \end{aligned}$$ □

2.6.4 **Corollary** *Suppose* C *is a p-category.*

(i) *If* f *and* g *are total, then* $f + g$ *is total.*

(ii) *Let* $\delta \rightarrowtail A + B$ *be a subdomain. Then* $\delta \cap -$ *preserves sums of subdomains.*

(iii) *Injections are total.*

Proof (i) Trivial. (ii) By 2.3.5,

$$\delta \cap (\varepsilon + \gamma) = \delta(\varepsilon + \gamma) = \delta\varepsilon + \delta\gamma = \delta \cap \varepsilon + \delta \cap \gamma$$

. (iii) As $\mathrm{id} = i + j$, it is

$$\mathrm{id} = \mathrm{dom}\, i + \mathrm{dom}\, j$$

and

$$\mathrm{id}\, i^{-1}(\mathrm{dom}\, i + \mathrm{dom}\, j) = \mathrm{dom}((\mathrm{dom}\, i + \mathrm{dom}\, j)i) = \mathrm{dom}(i\, \mathrm{dom}\, i) = \mathrm{dom}\, i. \ \square$$

2.6.5 **Theorem** *The category* C *has coproducts if and only if the category* $\mathsf{Dom}(C)$ *has coproducts and domain maps are closed under sums.*

Proof Immediate from 2.6.3. $\square$

Other properties of coproducts can be proved with the assumption that C has zero maps. They were pointed out to us by Eugenio Moggi.

2.6.6 **Proposition** *Suppose* C *is a p-category with zero maps and coproducts. Then in* $\mathsf{Dom}(C)$

(i) *injections are subdomains;*

(ii) *for* $\varepsilon \rightarrowtail A$, $\gamma \rightarrowtail B$ *subdomains,* $\varepsilon + \gamma$ *is the join of* $\varepsilon + 0$ *and* $0 + \gamma$ *in* $\mathrm{Subd}(A + B)$.

Proof (i) We prove it just for total objects, as the general result follows immediately. Let X and Y be total objects. In C we have $\mathrm{id} + 0 \colon X + Y \longrightarrow X$. We want to show the map $i \colon X \longrightarrow X + Y$ is isomorphic to $\mathrm{dom}\mathrm{id} + 0 \colon \mathrm{dom}\mathrm{id} + 0 \longrightarrow X + Y$ in $\mathsf{Dom}(C)$. First notice $\mathrm{dom}\mathrm{id} + 0 = \mathrm{id} + 0$. Moreover, $i \colon X \longrightarrow \mathrm{dom}\mathrm{id} + 0$, as $(\mathrm{id} + 0)i = i$ and $\mathrm{id} + 0 \colon \mathrm{dom}\mathrm{id} + 0 \longrightarrow X$ by definition. Finally, these two maps are inverse of each other:

$$\mathrm{id} + 0i = \mathrm{id} \quad \text{and} \quad i\mathrm{id} + 0 = i + 0 = \mathrm{id}\, i + 0j = \mathrm{id} + 0 = \mathrm{dom}\mathrm{id} + 0,$$

the last map appearing in the chain of equalities being the identity on the object $\mathrm{dom}\mathrm{id} + 0$ in $\mathsf{Dom}(C)$. (ii) By (i), $\varepsilon + \delta$ is the sum of $\varepsilon + 0$ and $0 + \delta$. So it is also their join. $\square$

2.6.7 **Corollary** *Suppose* C *is a p-category with zero maps and coproducts.*

(i) *Injections are disjoint.*

(ii) *The operator* ran *commutes with coproducts.*

Proof (i) $(\mathrm{id} + 0)(0 + \mathrm{id}) = 0 + 0 = 0$. (ii) Let $\phi \colon A \longrightarrow C$ and $\psi \colon B \longrightarrow D$. By definition,

$$\exists(\phi + \mathrm{id})(\mathrm{id} + 0) = \exists\phi(\mathrm{id}) + 0$$

and

$$\exists(\mathrm{id} + \psi)(\mathrm{id} + 0) = \mathrm{id} + 0.$$

By 2.4.8 and the property of coproducts,

$$\exists(\phi + \psi) = \exists(\phi + \mathrm{id})\exists(\mathrm{id} + \psi) = \exists(\mathrm{id} + \psi)\exists(\phi + \mathrm{id}).$$

So, by 2.6.6 and 2.4.8,

$$\begin{aligned}
\mathrm{ran}(\phi + \psi) &= \exists\phi + \psi(\mathrm{id} + 0 \cup 0 + \mathrm{id}) \\
&= \exists(\phi + \mathrm{id})\exists(\mathrm{id} + \psi)(\mathrm{id} + 0) \cup \exists(\mathrm{id} + \psi)\exists(\phi + \mathrm{id})(0 + \mathrm{id}) \\
&= \exists\phi + \mathrm{id}\,(\mathrm{id} + 0) \cup \exists\mathrm{id} + \psi(0 + \mathrm{id}) \\
&= \exists\phi(\mathrm{id}) + 0 \cup 0\exists\psi(\mathrm{id}) \\
&= \mathrm{ran}\,\phi + \mathrm{ran}\,\psi. \ \square
\end{aligned}$$

Putting images and coproducts together one gets joins of domains.

2.6.8 **Proposition** *Suppose* C *has ranges, zero maps and coproducts. For* A *an object of* $\mathsf{Dom}(C)$*, the preorder* $\mathrm{Subd}(A)$ *is a bounded distributive lattice.*

Proof Let ε, δ, be given subdomains of A, and set $s = \mathrm{id} + \mathrm{id} : A + A \longrightarrow A$. By 2.6.6, the map $\varepsilon + 0 : \varepsilon + 0 \longrightarrow \varepsilon$ is an isomorphism of $\mathsf{Dom}(C)$. So

$$\exists s(\varepsilon + 0) = \exists \varepsilon + 0 (\varepsilon + 0) = \varepsilon.$$

Similarly $\exists s(0 + \delta) = \delta$. As $\exists s$ preserves joins, $\varepsilon \cup \delta$ exists and equals $\exists s(\varepsilon + \delta)$, by 2.6.6. To prove the lattice is distributive, recall 2.5.6 and evaluate:

$$\begin{aligned} \varepsilon \cap (\delta \cup \gamma) &= \varepsilon \cap \exists s(\delta + \gamma) \\ &= \exists s((\varepsilon + \varepsilon) \cap (\delta + \gamma)) \\ &= \exists s((\varepsilon \cap \delta) + (\varepsilon \cap \gamma)) \\ &= (\varepsilon \cap \delta) \cup (\varepsilon \cap \gamma). \ \square \end{aligned}$$

Stability. The results we have proved about coproducts all strengthen those of [PH86]: they hold in a more general setting than dominical categories. Notice in particular that no further assumption about the behaviour of coproducts like "a sum of total maps is total" or "stability of coproducts with respect to products" has been made, differently from *loc.cit.* . The fact that a sum of total maps is total follows directly from 2.6.3, while preservation of coproducts by products is needed only in order to prove stability under pullbacks, similarly to what happened fror stability of ranges. Recall that coproducts are *preserved by products* if the maps

$$\mathrm{id} \times i + \mathrm{id} \times j : X \times Y + X \times Z \longrightarrow X \times (Y + Z)$$

are isomorphisms.

2.6.9 **Proposition** *Let* C *be a p-category with zero maps. Coproducts are stable under product in* C *if and only if coproducts are stable under product in* $\mathsf{Dom}(C)$*.*

Proof The inverse image map from $\mathrm{Subd}(X \times (Y + Z))$ into $\mathrm{Subd}(X \times Y + X \times Z)$ induced by

$$\mathrm{id} \times i + \mathrm{id} \times j : X \times Y + X \times Z \longrightarrow X \times (Y + Z)$$

is an isomorphism and takes $\mathrm{id} \times (\varepsilon + \delta)$ to $\mathrm{id} \times \varepsilon + \mathrm{id} \times \delta$. Hence the conclusion by 2.3.6. $\square$

The next result will be a condition for stability of coproducts in $\mathsf{Dom}(C)$. We prove it in the same way as 2.5.2.

2.6.10 **Theorem** *Suppose* C *has ranges and coproducts stable under products. Then* $\mathsf{Dom}(C)$ *has coproducts stable under pullback.*

Proof Let $\phi : A \longrightarrow B + C$. As $\phi = q\langle \mathrm{id}, \phi \rangle$, it is a composition of a subdomain and a projection, by 2.4.9. By 2.3.6 and 2.6.9, $q^{-1}(B + C)$ is a coproduct; by 2.3.6 and 2.6.4, $\langle \mathrm{id}, \phi \rangle^{-1}(q^{-1}(B + C))$ is a coproduct. $\square$

Chapter 3

Dominance

Within the powerful framework of a topos, it is possible to give a general internal construction to produce p-categories. The central notion for this will be a subobject of Ω which is used to *classify* the domains of the partial maps which we want to look at. Moreover, any p-category can be fully embedded into one built in such a way, which is a strenghtening of 2.3.3. The essential feature of the internal construction we are going to give is that the basic notions for p-categories are represented by objects in the topos. More concretely, it will be possible to define a dominion on a topos $\mathcal{E}$, starting from a subobject Σ of the classifier: it will be the collection of those *truth values* which are domains, and will be the classifier of the domain monos.

3.1 Classes of subobjects

Let $\mathcal{E}$ be a fixed topos. Let $\Sigma \rightarrowtail \Omega$ be a subobject of truth values, say $\sigma\colon \Sigma \rightarrowtail \Omega$. Say a mono $m\colon A \rightarrowtail E$ is *classified by* Σ, if its characteristic map factors through Σ. Therefore $m\colon A \rightarrowtail E$ is classified by Σ if and only if

$$\mathcal{E} \Vdash \forall x \in E.\ |\exists y \in A.\ x = my| \in \Sigma.$$

Say a subobject is *classified* by Σ if one, and hence each, of its representatives is classified by Σ, and let $\mathrm{T}_\Sigma E$ be the set of subobjects of E classified by Σ. Often we will just say that the subobject $A \rightarrowtail E$ is Σ, to mean that it is classified by Σ. By definition of subobject classifer, a map $f\colon D \longrightarrow E$ induces a map $\mathrm{T}_\Sigma f\colon \mathrm{T}_\Sigma E \longrightarrow \mathrm{T}_\Sigma D$ by inverse image. Therefore $\mathrm{T}_\Sigma\colon \mathcal{E} \longrightarrow \mathcal{S}$ is a functor and

$$\mathrm{T}_\Sigma \simeq \mathcal{E}(-, \Sigma).$$

Let $\mathcal{T}$ denote the collection of Σ subobjects in $\mathcal{E}$.

The idea is now to use the family of monos classified by Σ as a dominion on $\mathcal{E}$; two facts derived from previous experience with categories of partial maps in chapter 2 support the idea. First, such a family of monos is automatically closed under pullbacks. Secondly, as a partial converse, in the topos $\mathcal{A}$ of presheaves on a category A, a dominion $\mathcal{M}$ on A induces a subobject $\Upsilon \rightarrowtail \Omega$ which classifies at least all the monos in $\mathcal{M}$. This is stated precisely in the next proposition. Remember a convention from chapter 1 that the category A is identified with its image in $\mathcal{A}$ through the Yoneda embedding.

3.1.1 **Proposition** *Let A be a category with finite products, let $\mathcal{M}$ be a dominion on A. Set $\mathcal{A}$ for the topos of presheaves on A and let Υ be the presheaf defined as*

$$\Upsilon(X) = \{m\colon A \rightarrowtail X : m \in \mathcal{M}\} \quad \text{and} \quad \Upsilon f = f^{-1}.$$

Then $\Upsilon \rightarrowtail \Omega$. Moreover, a subobject $D \rightarrowtail E$ is classified by Σ if and only if for all object A in A and for all $e\colon A \longrightarrow E$, the mono $e^{-1}D \rightarrowtail A$ is in $\mathcal{M}$. In particular, the Σ subobjects of an object A in A are exactly those in $\mathcal{M}$. Thus

$$\Upsilon^A \simeq \mathrm{Subd}(- \times A)$$

is a natural isomorphism.

Proof As $\Upsilon X \subset \Omega X$, to prove Υ is a presheaf and a subfunctor of Ω, one just need to check that for $A \rightarrowtail X \in \Upsilon X$ and $f: Y \longrightarrow X$ in A, $f^{-1}A \rightarrowtail Y$ is in ΥY. This is so because the Yoneda embedding is full and preserves all limits: in particular, the inverse image $f^{-1}A \rightarrowtail Y$ exists in A, as $\mathcal{M}$ is a dominion and $A \rightarrowtail X$ is in $\mathcal{M}$. Suppose now $D \rightarrowtail E$ is a subobject of an object E in $\mathcal{A}$. Its characteristic map is $\chi: E \longrightarrow \Omega$ defined by

$$\chi_Z(e) = e^{-1}D \rightarrowtail Z \in \Omega(Z)$$

for Z an object in A. Hence, $D \rightarrowtail E$ is Σ if and only if for each $e: Z \longrightarrow E$, the subobject $e^{-1}D \rightarrowtail Z$ is in $\mathcal{M}$. The last assertion follows by applying the previous characterization to $e = \mathrm{id}$. □

Let us go back to the general case $\Sigma \rightarrowtail \Omega$. As we said, the family of Σ subobjects comes with closure under pullbacks for free: it is obviously closed under identities if and only if $\mathrm{true}: 1 \longrightarrow \Sigma$, in other words

$$\mathcal{E} \Vdash \mathrm{true} \in \Sigma.$$

As the identities constitute the minimal core of a dominion, from now on we assume that $\mathrm{true} \in \Sigma$.

3.1.2 **Example** Suppose $j: \Omega \longrightarrow \Omega$ is a topology, and consider the object of j-dense truth values:

$$\Xi = \{x \in \Omega : jz = \mathrm{true}\}.$$

then $\Xi \rightarrowtail \Omega$ and

$$\mathcal{E} \Vdash \mathrm{true} \in \Xi.$$

Representing partial maps. In order to discuss closure under composition for $\mathcal{T}$, we have to study representation of partial maps defined on Σ subobjects. Such a partial map will be indicated simply as Σ-*defined*.

There is no way one can define a p-category of the form $\mathsf{P}(\mathsf{A},\mathcal{T})$ unless $\mathcal{T}$ is closed under composition, but it is nonetheless possible to speak of partial maps defined on Σ subobjects. Let $\mathsf{P}(E, D)$ denote the set of equivalence classes of partial maps $E \xleftarrow{m} A \xrightarrow{\phi} D$ where $m \in \mathcal{T}$.

The problem of *representing* Σ-defined partial maps is stated as follows: given an object D, find an object $\widehat{D}$, such that, for all objects E in $\mathcal{E}$,

$$\mathsf{P}(E, D) \simeq \mathcal{E}(E, \widehat{D}).$$

In other words, $\widehat{D}$ must have *enough more* elements than D so that, given a Σ-defined partial map $E \xleftarrow{m} A \xrightarrow{\phi} D$, ϕ can be extended to $\overline{\phi}: E \longrightarrow \widehat{D}$ by sending just the elements of A outside E to the new elements of $\widehat{D}$. Stating this precisely: there must be a map $s: D \rightarrowtail \widehat{D}$ such that for every Σ-defined partial map $E \xleftarrow{m} A \xrightarrow{\phi} D$, there is a unique $\overline{\phi}: E \longrightarrow \widehat{D}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & D \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle s} \\ E & \xrightarrow{\overline{\phi}} & \widehat{D} \end{array}$$

is a pullback.

Take then D in $\mathcal{E}$. Let

$$\widehat{D} = \{t \subset D : [\forall x, y \in D.\ x \in t \wedge y \in t \rightarrow x = y] \wedge |\exists x \in D.\ x \in t| \in \Sigma\}.$$

The object $\widehat{D}$ consists of those subsets of D with at most one element for which the value of "being inhabited" is in Σ.

The theory of existence of [Sco82] may help in understanding this phenomenon. The construction is very similar to that of $\widetilde{D}$, see [Joh77], [KW71] or [Fou77]. The map $\widehat{D} \rightarrowtail \widetilde{D}$ is the inverse image of $\Sigma \rightarrowtail \Omega$ along $d: \widetilde{D} \longrightarrow \Omega$, the characteristic map of $D \rightarrowtail \widetilde{D}$. Never mind: the proof that $\widehat{D}$ represents Σ-defined partial maps into D will make no reference to that.

3.1.3 **Proposition** *If* $\mathrm{true} \in \Sigma$, *then the mono* $s\colon D \rightarrowtail \Omega^D$ *is classified by* Σ.

Proof Define $s(y) = \{y\}$. Then

$$\mathcal{E} \Vdash \forall x, z \in D.\ x \in s(y) \wedge z \in s(y) \to x = z.$$

Also $y \in s(y)$; therefore

$$\mathcal{E} \Vdash |\exists x \in D.\ x \in s(y)| = \mathrm{true} \in \Sigma.$$

The map $s\colon D \longrightarrow \widehat{D}$ is trivially monic. To prove it is classified by Σ, let $t \in \widehat{D}$. Then

$$\begin{aligned} \mathcal{E} \Vdash t \in s(D) &\leftrightarrow \exists x \in D.\ \{x\} = t \\ &\leftrightarrow \exists x \in D.\ x \in t. \end{aligned}$$

Hence

$$\mathcal{E} \Vdash |t \in s(D)| = |\exists x \in D.\ x \in t| \in \Sigma,$$

by definition. Thus the characteristic map of s takes values in Σ. □

We are now in a position to prove the reprepsentation theorem for Σ-defined partial maps.

3.1.4 **Theorem** *Suppose* Σ *is a subobject of* Ω *such that* $\mathrm{true} \in \Sigma$. *Given an object* D *of* $\mathcal{E}$, *let* $\widehat{D}$ *and* $s\colon D \rightarrowtail \widehat{D}$ *be defined as above. Then, for all* Σ*-defined partial maps* $E \xleftarrow{m} A \xrightarrow{\phi} D$, *there is a unique map*

$$\overline{\phi}\colon x \mapsto \{y : x \in A \wedge y = \phi x\}\colon A \longrightarrow \widehat{D}$$

whose corestriction to D *is* ϕ.

Proof Consider $\overline{\phi}$ defined as in the statement of the theorem: to show it takes values in $\widehat{D}$, notice

$$\mathcal{E} \Vdash z \in \overline{\phi}(x) \leftrightarrow x \in A \wedge z = \phi(x).$$

Then if $z, w \in \overline{\phi}(x)$, then $z = \phi(x) = w$ and $z = w$. Moreover

$$\mathcal{E} \Vdash \exists z \in D.\ z \in \overline{\phi}(x) \leftrightarrow x \in A,$$

hence $|\exists z \in D.\ z \in \overline{\phi}(x)| = |x \in A| \in \Sigma$, because A is Σ. As

$$\begin{aligned} \mathcal{E} \Vdash \overline{\phi}(x) \in D &\leftrightarrow \exists y \in D.\ \overline{\phi}(x) = \{y\} \\ &\leftrightarrow \exists y \in D.\ y \in \overline{\phi}(x) \\ &\leftrightarrow x \in A, \end{aligned}$$

the conclusion follows by applying 3.1.3. □

Theorem 3.1.4 will be very useful. In particular, to determine a Σ-defined partial map from E to D, we can give first a map $f\colon E \longrightarrow \widehat{D}$, and then corestrict it to D. The object $\widehat{D}$ is called the Σ-*extension* of D, or just the extension of D when it is clear to which Σ it refers.

3.1.5 **Remark** Notice that $\widehat{1} \simeq \Sigma$, as, for $D = 1$, the first clause defining $\widehat{D}$ is satisfied by all $p \subset 1$ and the second one states that $p \in \Sigma$. Therefore, a monic $m\colon A \rightarrowtail E$ is classified by Σ if and only if there is a map $\overline{m}\colon E \longrightarrow \Sigma$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ & & \mathrm{true} \\ E & \xrightarrow{\overline{m}} & \Sigma \end{array}$$

is a pullback. As the characteristic map of $A \rightarrowtail E$ is $\sigma\overline{m}$, and is uniquely determined by $\overline{m}$ as $\sigma\colon \Sigma \rightarrowtail \Omega$ is monic, we may confuse $\overline{m}\colon E \longrightarrow \Sigma$ with the characteristic map of $A \rightarrowtail E$.

Dominance. The monomorphisms $\mathrm{true}: 1 \rightarrowtail \Sigma$ and $s: \Sigma \rightarrowtail \Sigma$ are composable. Is their composition $s\,\mathrm{true}: 1\Sigma \rightarrowtail \widehat{\Sigma}$ a mono classified by Σ?

3.1.6 **Proposition** *Suppose* $\mathrm{true} \in \Sigma$. *The mono* $s\,\mathrm{true}: 1 \rightarrowtail \widehat{\Sigma}$ *is classified by* Σ *if and only if the class of monos classified by* Σ *is closed under composition.*

Proof Necessity is obvious, as s and true are classified by Σ. As to sufficiency, suppose $m: A \rightarrowtail B$ and $n: B \rightarrowtail C$ are classified by Σ. Let $\chi: B \longrightarrow \Sigma$ be the characteristic map of m, and $\overline{\chi}: B \longrightarrow \widehat{\Sigma}$ the extension of $C \xleftarrow{n} B \xrightarrow{\chi} \Sigma$. Therefore the diagram

$$\begin{array}{ccccc} A & \xrightarrow{m} & B & \xrightarrow{n} & C \\ \downarrow & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \overline{\chi}} \\ 1 & \xrightarrow[\mathrm{true}]{} & \Sigma & \xrightarrow[s]{} & \widehat{\Sigma} \end{array}$$

is a pullback as composition of two pullbacks. Hence $nm: A \rightarrowtail C$ is Σ. □

3.1.7 **Corollary** *The family* $\mathcal{T}$ *of monos classified by* $\Sigma \rightarrowtail \Omega$ *is a dominion on* $\mathcal{E}$ *if and only if* $\mathrm{true} \in \Sigma$ *and* $\{\{\mathrm{true}\}\} \subset \widehat{\Sigma}$ *is classified by* Σ.

In case $\mathrm{true} \in \Sigma$ and Σ satisfies one of the equivalent conditions in 3.1.7, we say Σ is a *dominance*. Therefore the monomorphisms classified by the dominance Σ form a dominion on $\mathcal{E}$. One comment on 3.1.6 is that $s \circ \mathrm{true}$ determines the behaviour of the dominion with respect to composition, being the *generic* composition of Σ-subobjects.

3.1.8 **Example** Let $j: \Omega \longrightarrow \Omega$ be a topology, and $\Xi \rightarrowtail \Omega$ the subobject of j-dense truth values. Then Ξ is a dominance. It is a very particular case of dominance, as it satisfies the further property that the characteristic map of $\Xi \rightarrowtail \Omega$ is idempotent.

The next result show an easy way to build examples of dominances in topoi. It is the basis for the further development in section 3.3.

3.1.9 **Proposition** *Suppose* A *is a category with finite products, and* $\mathcal{M}$ *is a dominion on* A. *Let* $\mathcal{A}$ *be the topos of presheaves on* A, *let* $\Upsilon \rightarrowtail \Omega$ *be the subobject of* Ω *induced by* $\mathcal{M}$. *Then* Υ *is a dominance. Moreover, if* $\mathcal{T}$ *denotes the class of monomorphisms classified by* Υ, *the Yoneda embedding of* A *into* $\mathcal{A}$, *lifts to a full embedding of p-categories* $\mathsf{P}\mathcal{Y}: \mathsf{P}(\mathsf{A}, \mathcal{M}) \longrightarrow \mathsf{P}(\mathcal{A}, \mathcal{T})$.

Proof Suppose $m: C \rightarrowtail D$ and $n: D \rightarrowtail E$ are monomorphisms classified by Σ in $\mathcal{A}$. For A in A, and any $e: A \longrightarrow E$, consider the pullback

$$\begin{array}{ccc} D' & \xrightarrow{d} & D \\ \downarrow{\scriptstyle n'} & & \downarrow{\scriptstyle n} \\ A & \xrightarrow{e} & E \end{array}$$

where the left-most arrow n' is in $\mathcal{M}$ by 3.1.1. Therefore also the inverse image $m': C' \rightarrowtail D'$ of $m: C \rightarrowtail D$ is a mono in $\mathcal{M}$. So is their composition is so. But

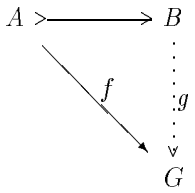
$$\begin{array}{ccccc} C' & \xrightarrow{m'} & D' & \xrightarrow{n'} & A \\ \downarrow & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ C & \xrightarrow[m]{} & D & \xrightarrow[n]{} & E \end{array}$$

is the pullback of $nm: C \rightarrowtail E$ along e. As e is arbitrary, this proves that $\mathcal{T}$ is closed under composition. By 3.1.7, Υ is a dominance, and the conclusion follows from 3.1.1. □

3.2 Σ-injective objects

Completion. A particular attention is worth to those objects on which every Σ-defined partial map can be extended to a total one.

Definition. An object G of $\mathcal{E}$ is Σ-*injective*, if for any mono $A \rightarrowtail B$ classified by Σ, and any $f: A \longrightarrow G$ there is $g: B \longrightarrow G$ such that the triangle



commutes.

This notion has already appeared in the literature under different names. As we said already, it is not necessary to have the complete structure of a topos to deal with partial maps, the instance of injective groups being enough to support the statement. Another example has been extensively studied in recursion theory. It appeared under the name of *complete* objects in [Erš75], [Vis80], and of *partial* object in [LM84c], where it is pointed out how injectivity with respect to "r.e. subsets" plays an essential role in the proof of the recursion theorem (we shall come back to this in the next chapter).

We forgot to mention the simplest of all examples of dominance: pointed sets!

3.2.1 **Example** The object $\Delta = \{\bot, \mathrm{true}\} \subset \Omega$ of decidable truth values is a dominance. In fact, for E an object of $\mathcal{E}$, the Δ-extension $\widehat{E}$ is $E + 1$ because

$$\begin{aligned}\mathcal{E} \Vdash |\exists x \in E.\ x \in t| \in \Delta &\leftrightarrow |\exists x \in E.\ x \in t| = \bot \vee |\exists x \in E.\ x \in t| = \mathrm{true} \\ &\leftrightarrow \forall x \in E.\ x \notin t \vee \exists x \in E.\ x \in t \\ &\leftrightarrow t = \emptyset \vee t \in s(E).\end{aligned}$$

The Δ-injective objects G are exactly the objects which have a global element $1 \to G$.

The following theorem is a useful criterion to detect Σ-injective objects. The main equivalence below was proved by Eugenio Moggi.

3.2.2 **Theorem** *The following are equivalent:*

(i) *G is Σ-injective;*

(ii) *there is a retraction pair from G to $\widehat{G}$;*

(iii) *$s: G \rightarrowtail \widehat{G}$ has a retraction.*

Proof Trivially, (i) implies (ii), by extending the identity map $\mathrm{id}: G \longrightarrow \widehat{G}$ along $s: G \longrightarrow \widehat{G}$. Also (iii) implies (i) by 3.1.4. As to (ii) implies (iii), suppose the pair $m: G \longrightarrow \widehat{G}$, $r: \widehat{G} \longrightarrow G$ is a retraction. Apply 3.1.4 to define the map $n: \widehat{G} \longrightarrow \widehat{G}$ as $n(t) = \{m(g) : g \in t \wedge g \in \mathrm{dom}\, m\}$. Then $r \circ n: \widehat{G} \longrightarrow G$ and $rn(\{g\}) = g$ for all $g \in G$. □

Condition (iii) does not say very much about which properties a retraction pair (m, r) from condition (ii) may have. Anyway it is sufficient to state the extension property more categorically: the inclusion

$$\mathcal{E}(B, G) \rightarrowtail \mathsf{P}(B, G)$$

has a retration.

3.2.3 **Example** Let $j\colon\Omega \longrightarrow \Omega$ be a topology, and $\Xi \rightarrowtail \Omega$ the dominance of j-dense truth values. The object D is a j-sheaf if and only if it is Ξ-injective and j-separated.

Properties. The topos structure seemed to have played an essential role. That is not so: as long as a construct like $\widehat{D}$ with the representation property exists in a p-category, all that has been proved about it can be performed in full generality. It is an application of 3.1.9, but we give a more detailed treatment of it in chapter 4.

3.2.4 **Proposition**

(i) *If G is Σ-injective, then G^E is Σ-injective for any E.*

(ii) *If G and K are Σ-injective, then $G \times K$ is Σ-injective.*

(iii) *A retract of a Σ-injective object is Σ-injective.*

(iv) *If Σ is a dominance, then $\widehat{E}$ is Σ-injective for all E.*

Proof (i) - (iii) are trivial using 3.1.3 and 3.2.2. (iv) Consider the composition $E \rightarrowtail \widehat{E} \rightarrowtail \widehat{\widehat{E}}$, which is classified by Σ, and the partial map $\mathrm{id}\colon E \longrightarrow E$ from $\widehat{\widehat{E}}$ to E. Then $\overline{\mathrm{id}}$ is a retraction for $\widehat{s}\colon\widehat{E} \longrightarrow \widehat{\widehat{E}}$. Indeed, the two squares in the diagram

$$\begin{array}{ccccc} E & \xrightarrow{\mathrm{id}} & E & \xrightarrow{\mathrm{id}} & E \\ {\scriptstyle s}\downarrow & & \downarrow & & \downarrow{\scriptstyle s} \\ \widehat{E} & \xrightarrow[\widehat{s}]{} & \widehat{\widehat{E}} & \xrightarrow[\overline{\mathrm{id}}]{} & \widehat{E} \end{array}$$

are pullbacks; so their composition is a pullback. But

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{id}} & E \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ \widehat{E} & \xrightarrow{\mathrm{id}} & \widehat{E} \end{array}$$

is a pullback, too. By uniqueness, $s \circ \overline{\mathrm{id}} = \mathrm{id}$. $\square$

Characterizations of dominance. In case Σ is a dominance, Σ is itself injective as $\Sigma \simeq \widehat{1}$ and a retraction for $\widehat{s}\colon p \mapsto \{\mathrm{true}\ : p\}\colon\Sigma \longrightarrow \widehat{\Sigma}$ is the map $s'\colon t \mapsto |\,\mathrm{true} \in t|\colon\widehat{\Sigma} \longrightarrow \Sigma$. As we noticed before, 3.2.2 reduces the information on the retraction quite drastically, by replacing $\widehat{s}$ with $s\colon\Sigma \longrightarrow \widehat{\Sigma}$. The problem is that s embeds the poset Σ, flat into the poset $\widehat{\Sigma}$, losing all the information about the partial order; on the contrary, $\widehat{s}$ and s' are order-preserving; moreover $\widehat{s} \dashv s'$, as

$$\begin{aligned} \mathcal{E} \Vdash \{\mathrm{true}\ : p\} \subset t &\leftrightarrow [p = \mathrm{true} \rightarrow \mathrm{true} \in t] \\ &\leftrightarrow p \leq |\,\mathrm{true} \in t|. \end{aligned}$$

Actually this property is yet another characterization of dominance.

3.2.5 **Theorem** *If $\widehat{s}\colon\Sigma \longrightarrow \widehat{\Sigma}$ has a right adjoint s', then Σ is a dominance.*

Proof We must prove $\{\mathrm{true}\}\colon 1 \longrightarrow \widehat{\Sigma}$ is classified by Σ. But, for $t \in \Sigma$,

$$\begin{aligned} \mathcal{E} \Vdash t = \{\mathrm{true}\} &\leftrightarrow \{\mathrm{true}\ : \mathrm{true}\} \subset t \\ &\leftrightarrow \mathrm{true} \leq s'(t) \\ &\leftrightarrow s'(t) = \mathrm{true}\,. \end{aligned}$$

Therefore $\{\{\text{true}\}\} = s'^{-1}(\{\text{true}\})$, hence the conclusion. □

It follows very easily from the definition of dominion that Σ-subobjects are closed under finite intersections. Here is a syntactic proof for the dominance Σ.

3.2.6 **Corollary** *Suppose Σ is a dominance. Then Σ is closed under finite intersections.*

Proof Let $p, q \in \Sigma$ and consider

$$\begin{aligned}\mathcal{E} \Vdash s'\{p : q\} &= |\,\text{true} \in \{p : q\}\,| \\ &= |\,\text{true} \in p \wedge \text{true} \in q\,| \\ &= p \wedge q.\end{aligned}$$

So $p \wedge q = s'\{p : q\} \in \Sigma$. □

Yet the previous characterization of dominance does not seem to be of much help as it requires a higher order construction as $\widehat{\Sigma}$. But we can use it to give another, more algebraic characterization of dominance.

3.2.7 **Theorem** *Suppose $\Sigma \rightarrowtail \Omega$ contains* true*. Then Σ is a dominance if and only if it satisfies*

$$[p \in \Sigma \wedge p \to |q \in \Sigma|] \to p \wedge q \in \Sigma.$$

Proof Suppose Σ is a dominance, and p, q satisfy the hypotheses of the condition. Then $\{q : p\} \in \widehat{\Sigma}$, as

$$\begin{aligned}\exists x \in \Sigma.\ x \in \{q : p\} &\leftrightarrow p \wedge \exists x \in \Sigma.\ x = q \\ &\leftrightarrow p\end{aligned}$$

by hypothesis. So $p \wedge q = s'\{q : p\} \in \Sigma$. Conversely, suppose Σ satisfies the condition, and let $s \in \widehat{\Sigma}$. We need to prove that $|\,\text{true} \in s\,| \in \Sigma$. First notice that for $w \in \Sigma$,

$$\begin{aligned}\mathcal{E} \Vdash w \in s &\to [w = \text{true} \leftrightarrow \text{true} \in s] \\ &\to w = |\,\text{true} \in s\,|\end{aligned}$$

as s has at most one element. Therefore

$$\begin{aligned}\mathcal{E} \Vdash \exists x \in \Sigma.\ x \in s &\to |\,\text{true} \in s\,| \in s \\ &\to |\,\text{true} \in s\,| \in \Sigma,\end{aligned}$$

as $s \subset \Sigma$. So

$$|\,\text{true} \in s\,| = |\,\text{true} \in s\,| \wedge |\exists x \in \Sigma.\ x \in s| \in \Sigma.\ \square$$

3.3 Topoi and p-categories

The warning from 3.2.1 is that we must not look at $\mathcal{S}$ for interesting examples. So in the lines of 3.1.1, we will try to exploit the case of sheaves on a category of the form $\mathsf{Dom}(C)$, where C is a p-category. Putting 2.3.3 and 3.1.9 together, we have

3.3.1 **Proposition** *Let C be a p-category. Let $\mathcal{C}$ be the topos of presheaves on $\mathsf{Dom}(C)$. Then there is a dominance $\Upsilon \rightarrowtail \Omega$ induced by the family $\mathcal{D}$ of domain monos, and there is a full embedding of p-categories $\mathcal{Y}\colon \mathsf{C} \longrightarrow \mathsf{P}(\mathcal{C}, \mathcal{T})$ where $\mathcal{T}$ is the family of monos classified by Υ.*

In particular, all equalities that can be proved about a family of Σ-defined partial maps in intuitionistic logic must hold in all p-categories. Moreover, the result can also be strengthened so as to treat ranges and coproducts.

3.3.2 **Theorem** *Let $\mathcal{E}$ be a topos, let j be a topology on $\mathcal{E}$ and Σ a dominance in $\mathcal{E}$ which is a j-sheaf. Then Σ is a dominance in $\mathsf{Sh}(\mathcal{E}, j)$.*

Proof By hypothesis, Σ is a j-sheaf, so a subobject of Ω_j. Let Σ' denote the extension of Σ in $\mathsf{Sh}(\mathcal{E}, j)$. By definition,

$$\begin{aligned}\mathcal{E} \Vdash \Sigma' = \{t \subset \Sigma : t \ j\text{-closed} \wedge j(\exists x \in \Sigma.\ x \in t) \in \Sigma \wedge \\ \forall x, y \in \Sigma.\ j(x \in t) \wedge j(y \in t) \rightarrow j(x = y)]\}.\end{aligned}$$

Notice that, for $t \subset \Sigma$ closed, $j(x \in t) \leftrightarrow x \in t$. Also as Σ is a sheaf, $j(x = y) \leftrightarrow x = y$. Moreover for $t \in \Sigma'$,

$$\mathcal{E} \Vdash \exists x \in \Sigma.\ x \in t \leftrightarrow \exists! x \in \Sigma.\ x \in t;$$

therefore $j(\exists x \in \Sigma.\ x \in t) = |\exists x \in \Sigma.\ x \in t|$. So $\Sigma' = \widehat{\Sigma} \cap (\Omega_j)^{\Sigma}$, and the characteristic map of $\{\text{true}\}: 1 \longrightarrow \Sigma'$ is s' restricted to Σ'. It must then take values in Σ, as Σ is a dominance in $\mathcal{E}$. □

3.3.3 **Corollary** *Let C be a p-category, and set $\mathcal{C}$ and Υ as in 3.3.1. Let j be a subcanonical topology on $\mathsf{Dom}(C)$ such that Υ is a sheaf for it, and let $\mathcal{E}$ be the topos $\mathsf{Sh}(\mathcal{C}, j)$. Then Υ is a dominance in $\mathcal{E}$ and the embedding $\mathcal{Y}: \mathsf{C} \longrightarrow \mathsf{P}(\mathcal{C}, \mathcal{T})$ takes values into $\mathsf{P}(\mathcal{E}, \mathcal{T}')$ where $\mathcal{T}'$ is the family of Σ-classified monos in $\mathcal{E}$. Moreover, for any representable object A in $\mathcal{E}$, the object $\widehat{A}$ as constructed in $\mathcal{C}$ is a j-sheaf. Therefore $\widehat{A}$ represents Σ-defined partial maps in $\mathcal{E}$. Also*

$$\Upsilon^A \simeq \mathrm{Subd}(- \times A).$$

Proof The first part follows immediately from 3.3.2. Consider now a representable object A. We must prove $\widehat{A}$ is a sheaf. Let $(f_i: B_i \longrightarrow B)$ be a covering. Let $(\overline{\phi_i}: B_i \longrightarrow \widehat{A})$ be a family of elements of $\widehat{A}$, satisfying the compatibility condition that if

$$\begin{array}{ccc} B_{ij} & \xrightarrow{f_{ji}} & B_j \\ {\scriptstyle f_{ij}}\downarrow & & \downarrow{\scriptstyle f_j} \\ B_i & \xrightarrow{f_i} & B \end{array}$$

is a pullback, then

$$\overline{\phi_i} f_{ij} = \overline{\phi_j} f_{ji}. \tag{3.1}$$

The maps $\overline{\phi_i}: B_i \longrightarrow \widehat{A}$ are uniquely determined by partial maps $B_i \longleftarrow\!\prec C_i \xrightarrow{\phi_i} A$ and the condition (3.1) becomes

$$f_{ij}{}^{-1}C_i = f_{ji}{}^{-1}C_j, \quad \phi_i g_{ij} = \phi_j g_{ji},$$

where

$$\begin{array}{ccc} f_{ij}{}^{-1}C_i & \xrightarrow{g_{ij}} & C_i \\ \downarrow & & \downarrow \\ B_{ij} & \xrightarrow{f_{ij}} & B_i \end{array}$$

are all pullbacks of domains.

Then the family $(C_i \rightarrowtail B_i)$ is a compatible family of elements of M. Hence there is a domain $C \rightarrowtail B$ such that $f_i{}^{-1}C = C_i$. Also the family $(g_i: C_i \longrightarrow C)$ is a covering, as pullback of the given one along $C \rightarrowtail B$ and the elements $\phi_i: C_i \longrightarrow A$ are compatible. As A is a sheaf, there is a unique $\phi: C \longrightarrow A$. Obviously, $\overline{\phi}: B \longrightarrow A$ induced by $B \longleftarrow\!\prec C \xrightarrow{\phi} A$ is the unique glueing of the family $(\overline{\phi_i}: B_i \longrightarrow \widehat{A})$. □

For sake of simplicity, the proof has been performed under the hypotheses that $\mathsf{Dom}(C)$ has pullbacks, but the more general case can be dealt with in the same way, remembering that $\mathsf{Dom}(C)$ has always pullbacks of domains. In any case we shall apply 3.3.3 to the category of domains of an r-category, possibly with coproducts; so that $\mathsf{Dom}(C)$ will always have all finite limits.

3.3.4 **Proposition** *Suppose* C *is an r-category satisfying the weak axiom of choice. Then there is a topos* $\mathcal{E}$ *with a dominance* Υ *such that* C *is fully embedded into* $\mathsf{P}(\mathcal{E},\mathcal{T})$ *by a functor which transforms the range operator* ran *into the image operator* im*.*

Proof Take $\mathcal{C}$ as in 3.3.3 and let j be the topology generated by range maps. Let $\mathcal{E}$ be the topos $\mathsf{Sh}(\mathsf{C},j)$. The weak axiom of choice ensures that every range map is effective. By 2.5.11, range maps are stable. As they are epic, the topology they generate is subcanonical. Therefore range maps are surjections in $\mathcal{E}$. Consider now Υ as in 3.3.1. We need to prove that Υ is a sheaf in order to apply 3.3.3: suppose $F \rightarrowtail \gamma$ is a subobject of γ in $\mathcal{E}$, for γ an object of $\mathsf{Dom}(C)$, such that for a certain range map $\phi\colon \varepsilon \longrightarrow\!\!\!\!\triangleright\ \gamma$, the inverse image $\alpha = \phi^{-1}F$ is a domain of $\mathcal{E}$. As ϕ is a range map in $\mathsf{Dom}(C)$, it is surjective in $\mathcal{E}$. So

$$F \simeq \exists\phi(\phi^{-1}F) \simeq \exists\phi(\alpha)$$

is a domain. Trivially now ranges are images. □

The interesting fact about this second embedding theorem is that ranges become images, and the embedding preserve them. Any theorem $\Gamma \vdash \theta$ written with $\exists\wedge$-sequents about domains and maps between them, which can be proved in intuitionistic logic holds in an arbitrary r-category satisfying the weak axiom of choice.

3.3.5 **Theorem** *Suppose* C *is a p-category with zero maps and coproducts stable under products. Then there is a topos* $\mathcal{E}$ *with a dominance* Υ *such that* C *is fully embedded into* $\mathsf{P}(\mathcal{E},\mathcal{T})$ *by a functor of p-categories preserving zero maps and coproducts.*

Proof Take the topos $\mathcal{E}$ of sheaves on $\mathsf{Dom}(C)$ with the finite partition topology which is subcanonical by 2.6.9. As domains are closed under coproducts by 2.6.3, Υ as in 3.3.1 is a sheaf. The conclusion is trivial. □

3.3.6 **Proposition** *Suppose* C *is a p-category with ranges, zero maps and coproducts stable under products. Suppose also that* C *satisfies the weak axiom of choice. Then there is a topos* $\mathcal{E}$ *with a dominance* Υ *such that* C *is fully embedded into* $\mathsf{P}(\mathcal{E},\mathcal{T})$ *by a functor of p-categories preserving all the additional structure. Moreover* Υ *is closed under finite unions.*

Proof This time the hypotheses on C make sure that the finite cover topology is subcanonical. The proof is in all similar to the previous one. □

This is the strongest embedding theorem we can prove: any theorem written with geometric sequents about domains and maps which is provable in intuitionistic logic holds for any r-category satisfying the hypotheses above. So not only p-categories are a good axiomatization for categories of partial maps, but also in a large number of cases, it is possible to think of them as category of partial functions in an appropriate intuitionistic universe of sets. Not only that: the domains of the given partial maps are defined, once and for all, by a "set of truth-values": that on which their characteristic maps take values. In other words, the notion of dominion can be made into a logical datum of the universe.

3.4 A simple criterion for continuity

As Σ encodes the total information about a particular class of subobjects, objects such as Σ^E represents Σ-subobjects of E, in the sense that Σ-subobjects are in one-one correspondence with global elements $1 \to \Sigma^E$ of Σ^E. It is thus possible to speak of arbitrary Σ-subobjects internally in the category, and of the object of all Σ-subobjects of E. It is convenient to use the notation $A \subset_\Sigma E$ for $A \in \Sigma^E$, and extend our conventions calling A a Σ-*subset* of E.

Suppose Σ is a dominance; let E be any given object in $\mathcal{E}$. The object $\Sigma^E \rightarrowtail \mathrm{P}(E)$ is closed under finite intersections, because Σ is. It can be taken as a basis for a topology σ_E on E. Trying not to be misleading, we point out that, in this case, we mean topology in the sense of a family of subsets of an object E satisfying the familiar conditions: a subobject $\tau \rightarrowtail \mathrm{P}(E)$ such that

$$\mathcal{E} \Vdash \forall U, V \in \tau.\ U \cap V \in \tau \wedge\ \forall X \subset \tau.\ \bigcup X \in \tau.$$

Therefore, the topology σ_E is $\{U \in \mathrm{P}(E) : \exists X \subset_\Sigma E.\ U = \bigcup X\}$. And the following result is trivial.

3.4.1 **Proposition** *Let Σ be a dominance in the topos $\mathcal{E}$. Let E, F be objects in $\mathcal{E}$ and let (E, σ_E) and (F, σ_F) be the topological spaces constructed as above on E and F respectively. Then*

$$\mathcal{E} \Vdash \forall f{:}E \longrightarrow F\ \forall V \in \sigma_F.\ f^{-1}V \in \sigma_E.$$

Therefore it is internally valid that all maps from E to F are continuous from (E, σ_E) to (F, σ_F).

Proof Remember: proofs are always suggestions for formal ones. Let $f \in F^E$; it will suffice to show $f^{-1}A \in \sigma_E$ for a basic open $A \subset_\Sigma F$. But $f^{-1}A = A \circ f$ which is then in Σ^E, thus in σ_E. □

It is difficult to comment on the criterion above: on one side it is quite powerful, giving at once a continuity principle for as large a class of topological spaces as $\mathcal{E}$ itself, instantiating once more the idea that the objects of a topos ought to be thought of as sets and topological spaces at the same time. On another side, it says that continuity of the maps is a property inherent of their simple status of functions, henceforth it adds no further information, as in the well known cases of discrete or coarsest topologies in $\mathcal{S}$, determined by the two trivial dominances in 2. A way to overcome the second point would be to give, in particular cases, a different description of the topology σ_E, so that 3.4.1 would change to a continuity principle about other, possibly more interesting, topological spaces. This will be done in chapters 6 and 7.

Chapter 4

Partial functionals

The results of the previous two chapters support the idea that the category-theoretic approach to a theory of partial functions can be fruitful, and that intuitionistic logic can be conveniently applied to it. Our leading example is that of a topos with a dominance, and in the sequel we shall study particular constructs, which are of interest in recursion theory, and then check how they behave with respects to the embeddings of 3.3.

In this chapter we shall try to pursue a detailed study of a possible calculus of partial functions. We gathered ideas from various sources: the main ones are [LM84c], [Plo85], and [Sco82].

4.1 Spaces of partial maps

Suppose $\mathcal{E}$ is a topos and $\Sigma \rightarrowtail \Omega$ a dominance. Let D denote the p-category $\mathsf{P}(\mathcal{E}, \mathcal{T})$. In 3.1.4 we proved that for objects D, E in $\mathcal{E}$, one has $\mathsf{P}(D,E) \simeq \mathcal{E}(D, \widehat{E})$. Therefore, taken any object F,

$$\begin{aligned}\mathsf{P}(D \times F, E) &\simeq \mathcal{E}(D \times F, \widehat{E}) \\ &\simeq \mathcal{E}(F, \widehat{E}^D).\end{aligned}$$

P-exponentials. As the isomorphisms are natural, the previous statement may be taken as the internal definition of the "object of partial maps". This construction was first considered by [LM84c] where there is an interesting study of its abstract properties. The abstract presentation here follows theirs in the main lines, but is more directed toward a topos-theoretic application.

The ingredients of the left-hand side in the isomorphism above, are already in the language of p-categories, and the topos $\mathcal{E}$ is the category D_t of total maps of D. So, given two objects X, Y in a p-category C, call the *p-exponential* of X at Y an object ${}^X Y$ such that the set-valued functors $\mathsf{C}(X \times -, Y)$ and $\mathsf{C}_t(-, {}^X Y)$ are naturally isomorphic.

4.1.1 **Remark** If ${}^X Y$ exists for all Y, then the construct can be extended to a functor ${}^X(-)\colon \mathsf{C} \longrightarrow \mathsf{C}_t$ which is right adjoint to $X \times -\colon \mathsf{C}_t \longrightarrow \mathsf{C}$.

Relationship between the general case and that in a topos. The general case is closely connected to that of a topos: all the embeddings appearing in 3.3 preserve p-exponentials when they exist. So their properties are exactly those of sets of partial maps in an intuitionistic theory. This claim follows essentially from the fact that representable functors have a unique representation.

4.1.2 **Theorem** *Suppose* C *and* B *are p-categories, and* $F\colon \mathsf{C} \longrightarrow \mathsf{B}$ *is a full embedding of p-categories. Let* X *and* Y *be objects in* C*. If the p-exponential* ${}^{FX} FY$ *exists in* B*, and there is an object* Z *in* C *such that* $FZ \simeq {}^{FX} FY$*, then* Z *is the p-exponential of* X *at* Y *in* C*.*

Proof Take W an arbitrary object in C and evaluate

$$\begin{aligned}\mathsf{C}(W,Z) &\simeq \mathsf{B}(FW,FZ)\\ &\simeq \mathsf{B}(FW,{}^{FX}FY)\\ &\simeq \mathsf{B}(FX\times FW,FY)\\ &\simeq \mathsf{B}(F(X\times W),FY)\\ &\simeq \mathsf{C}(X\times W,Y).\ \square\end{aligned}$$

In conclusion, working in a topos freeds from requiring the existence of p-exponentials, and allows to think of them as set-theoretic constructs. The next proposition is immediate from 4.1.2, using the results of chapter 1.

4.1.3 **Proposition** *In the following statements, whenever the object appearing on one side of the isomorphism sign exists in* C*, so does the other:*

(i) ${}^{0}0 \simeq 1$,

(ii) ${}^{1}X \simeq \widehat{X}$,

(iii) ${}^{1}1 \simeq \Sigma$,

(iv) ${}^{X+Y}Z \simeq {}^{X}Z \times {}^{Y}Z$,

(v) ${}^{X}Y$ *is* Σ*-injective,*

(vi) ${}^{X\times Y}Z \simeq ({}^{Y}Z)^{X}$,

(vii) ${}^{X}Y \simeq \widehat{Y}^{X}$.

As an example of the embedding principles of 3.3, we show that $\mathsf{Dom}(C)$ may turn out to be cartesian closed.

4.1.4 **Proposition** *Suppose* C *is a p-category with ranges and p-exponentials. Suppose moreover* C_t *has a strict initial object. Then* $\mathsf{Dom}(C)$ *is cartesian closed.*

Proof By 4.1.3, $\mathsf{Dom}(C)$ has a terminal object. From now on, we work in the topos $\mathcal{C}$ as in 3.3.1. Suppose $\delta \rightarrowtail X$, $\varepsilon \rightarrowtail Y$ in $\mathsf{Dom}(C)$; let $\mathrm{dom}\colon {}^{X}Y \longrightarrow \Sigma^{X}$ be the map

$$\phi \mapsto \{x \in X : \phi(x) \in Y\}.$$

The exponential ε^{δ} is the subobject

$$\{y \in {}^{X}Y : \mathrm{dom}\, y = \delta \wedge y = \varepsilon y\},$$

which is in $\mathsf{Dom}(C)$ as this category is closed under limits. $\square$

Adopting a common convention, for $\phi \in {}^{X}Y$ in a topos, we write at times $\phi(x)\downarrow$ to mean $\phi(x) \in Y$. In particular, if $\phi\colon X \longrightarrow Y$ is a partial map, then

$$\mathrm{dom}\,\phi = \{x \in X : \phi(x)\downarrow\}.$$

4.2 A λ-calculus for partial maps

As an intrinsic feature of mechanical computations is non-convergence of the program at some inputs, it should not be too surprising to find that computations provide an example of p-category. Recalling the equivalence between total computations and cartesian closed categories shown in [Lam80] and [Sco81b], one would then be interested in knowing how far this can be stretched in the case of general computations and, say, p-categories with p-exponentials. The work of [LM84c], and of [Gun85], gave insights toward this. And the theory presented by [Plo85] is an attempt to give a syntactic theory of partial computations.

The theory. Most of what we present in this section is a slight variation of the content of *loc.cit.* . In particular, the typed theory P below is a part of the *metalanguage* introduced there. The types of P consist of a countable set of ground types, and of compound types built as follows:

(a) 1 is a type;

(b) if S and T are types, then $S \times T$ is a type;

(c) if S and T are types, then ${}^S T$ is a type.

It is intended that the notation should look familiar. The *terms* are built from a countable number of variables for each type as follows:

(a) o is a term of type 1;

(b) if s and t are terms of type S and T respectively, then $\langle s, t\rangle$ is of type $S \times T$;

(c) if r is of type $S \times T$, then pr is of type S and qt is of type T;

(d) if t is of type T and x is a variable of type S, then $\lambda x.t$ is a term of type ${}^S T$, with the variable x *bound* in $\lambda x.t$;

(e) if s is a term of type S and r is a term of type ${}^S T$, then $r(s)$ is a term of type T.

An occurence of a variable is *free* in a term if it is not bound.

Comments. Though we believe that the language is already well explained by the similarity with the categorical notation of section 4.1, it may be a good policy to compare this language with that of [Plo85] now, even before giving the axioms. We did not introduce sums and recursive types. Both can be very simply introduced on top of the given language: but while the first would produce no intrinsic difficulty, the latter are quite special of particular models. We did not introduce an undefined symbol as a primitive, because this would force the theory to be *essentially* different from the usual typed λ-calculus, while in our generality, for instance, the equivalence of typed λ-calculus and cartesian closed categories is just a particular case of the equivalence that we will produce. Anyway, undefined symbols can be introduced themselves on top of the given language with no particular difficulty. Finally, by using the type 1, we set

$$\widehat{S} = {}^1 S, \qquad [t] = \lambda u.t, \qquad (\text{let } x \text{ be } r \text{ in } s) = (\lambda x.s)(r(\mathrm{o})),$$

where S is a type, t is a term of type S and s is any term, u and x are variables of type 1 and S respectively.

The axioms and the rules that the theory must satisfy are written in the theory of existence of [Sco82], using an existence predicate E and an equivalence relation $\equiv$ for each type. Its main feature is that of considering *partial elements*, so that it makes sense to substitute terms for variables without restriction. This requires changes in the axiom and rule which deal with the universal quantificatiers. Consequently, one has to be careful when equating a provable formula with its universal closure: it is *not* true that a formula is provable in the theory if and only if its universal closure is. Rather, the relation becomes

$$\mathsf{P} \vdash \forall x.\ \Phi(x) \text{ if and only if } \mathsf{P} \vdash \mathsf{E}x \to \Phi(x).$$

As the theory has clearly algebraic features, we prefer to give the axioms using open formulae. Indeed, we restrict ourselves to treating that fragment G of the logic of existence which suits the particular case, namely open formulae Θ which are implications $\Phi \to \Psi$ of conjunctions of atomic formulae.

The axioms of the theory P are

$$\begin{array}{lll}
\Phi \to \Phi & \Phi \wedge \Psi \to \Phi & \Phi \wedge \Psi \to \Psi \\
x \equiv x & x \equiv y \to y \equiv x & \Phi(z/x) \wedge x \equiv y \to \Phi(z/y) \\
\mathsf{E}\mathrm{o} & & \mathsf{E}u \to u \equiv \mathrm{o} \\
\mathsf{E}\langle x, y\rangle \to \mathsf{E}x \wedge \mathsf{E}y & & z \equiv \langle pz, qz\rangle \\
\mathsf{E}pz \to \mathsf{E}z & \mathsf{E}z \to \mathsf{E}pz & \mathsf{E}y \to x \equiv p\langle x, y\rangle \\
\mathsf{E}qz \to \mathsf{E}z & \mathsf{E}z \to \mathsf{E}qz & \mathsf{E}x \to y \equiv q\langle x, y\rangle \\
\mathsf{E}z(x) \to \mathsf{E}x \wedge \mathsf{E}z & & \mathsf{E}z \to z \equiv \lambda x.z(x) \\
\mathsf{E}\lambda x.t & & \mathsf{E}x \to t \equiv (\lambda x.t)(x)
\end{array}$$

The rules are

$$\frac{\Phi \to \Psi \quad \Psi \to \Theta}{\Phi \to \Theta} \qquad \frac{\Phi \to \Psi \quad \Phi \to \Theta}{\Phi \to \Psi \wedge \Theta}$$

$$\frac{\Phi \wedge \mathsf{E}t \to t \equiv s \quad \Phi \wedge \mathsf{E}s \to t \equiv s}{\Phi \to t \equiv s}$$

$$\frac{\Phi \to \Psi}{\Phi(x/t) \to \Psi(x/t)} \text{ provided } t \text{ is free for } x \qquad \frac{\Phi \wedge \mathsf{E}x \to t \equiv s}{\Phi \to \lambda x.t \equiv \lambda x.s} \text{ provided } x \text{ is not free in } \Phi.$$

The models. Suppose C is a fixed p-category with p-exponentials, and such that C_t has a terminal object 1. An *interpretation* of P will be given by first assigning objects of C to the types. For any ground type α, let $M\alpha$ be the assignment of an object in C. Then M must satisfy:

$$M1 \simeq 1, \qquad M(S \times T) \simeq MS \times MT, \qquad M({}^{S}T) \simeq {}^{MS}MT.$$

We assume for simplicity that these isomorphisms are always identity maps.

Notation. It will be useful in working with the interpretation to have names for some maps in the p-category C. Set $e\colon \widehat{X} \simeq X \times {}^{1}X \longrightarrow X$ for the composition of the canonical isomorphism and the evaluation map, and set $d\colon X \longrightarrow \widehat{X}$ for the adjoint map of the isomorphism $1 \times X \xrightarrow{\sim} X$: note that d is a monomorphism by 4.1.2. Also let

$$\Lambda\colon \mathsf{C}(X \times -, Y) \simeq \mathsf{C}_t(-, {}^{X}Y)$$

denote the natural isomorphism and set

$$\mathrm{ev} = \Lambda(\mathrm{id})\colon X \times {}^{X}Y \longrightarrow Y.$$

The interpretation of a term s of type S in P is given *with respect to* a string of different variables $\mathbf{x} = (x_1, \ldots, x_n)$ which must contain the free variables in s. Say the types of $x_1, \ldots, x_n$ are $T_1, \ldots, T_n$. As we said previously, variables range over partial elements. So the possible values that the variable x_i can assume should be the *elements* in

$${}^{1}MT_i = \widehat{MT_i}.$$

Consistently, the term s will be interpreted in a map in C from $\prod_i \widehat{MT_i}$ into MS.

(a) $M_{\mathbf{x}}\mathrm{o} = 1, \quad M_{\mathbf{x}}x_i = e \circ p_i,$

(b) $M_{\mathbf{x}}\langle s, t\rangle = (M_{\mathbf{x}}s \times M_{\mathbf{x}}t)\Delta,$

(c) $M_{\mathbf{x}}(pr) = p \circ M_{\mathbf{x}}r, \quad M_{\mathbf{x}}(qr) = q \circ M_{\mathbf{x}}r,$

(d) $M_{\mathbf{x}}(r(s)) = \mathrm{ev} \circ M_{\mathbf{x}}\langle s, r\rangle, \quad M_{\mathbf{x}}(\lambda x.t) = \Lambda(M_{x+\mathbf{x}}t \circ d),$

where $x + \mathbf{x}$ is the string $\mathbf{x}$ with the variable x rewritten or added at the beginning, depending on whether x already appeared in $\mathbf{x}$ or not.

By looking at the form of the axioms and rules, it is clear that in order to define a notion of *model* of the theory P, it is necessary to say only when a formula in the fragment G holds in the interpretation M. To do this we first define *satisfaction* for conjunctions of atomic formulae in the old-fashioned way, cf. [Rey74], then *truth* for an implication in G. It all has a very close resemblance with forcing in 1.3.2.

Definition. Suppose that Φ is a conjunction of atomic formulae in P, and $x_1, \ldots, x_n$ are the free variables in Φ, of types $T_1, \ldots, T_n$ respectively. By induction, we define when a map $\varepsilon : X \longrightarrow \prod_i \widehat{MT_i}$ *satisfies* the formula Φ in the interpretation M, and write $\models_M \Phi(\varepsilon)$.

(a) $\models_M \mathsf{E}t(\varepsilon)$ if and only if $\mathrm{dom}(M_{\mathbf{x}}t) \circ \varepsilon = \varepsilon$;

(b) $\models_M [t \equiv s](\varepsilon)$ if and only if $M_{\mathbf{x}}t \circ \varepsilon = M_{\mathbf{x}}s \circ \varepsilon$;

(c) $\models_M [\Phi \wedge \Psi](\varepsilon)$ if and only if $\models_M \Phi(\varepsilon)$ and $\models_M \Psi(\varepsilon)$.

Finally, we say that $\Phi \to \Psi$ *holds* in M, and write $\models_M \Phi \to \Psi$ if, for all maps $\varepsilon : X \longrightarrow \prod_i \widehat{MT_i}$ such that $\models_M \Phi(\varepsilon)$, it is also $\models_M \Psi(\varepsilon)$.

In 4.2.4, we shall show that every interpretation M is a model for the theory P. Rather than proving it directly, we apply 4.1.2 again, because we think that interpretations in topoi are always more suggestive than the general case.

4.2.1 **Lemma** *Suppose an interpretation M of P in the p-category C is given. Let Φ, Ψ be a conjunction of atomic formulae with free variables $x_1, \ldots, x_n$. Let $\varepsilon : X \longrightarrow \prod_i \widehat{MT_i}$.*

(i) *If $\models_M \Phi(\varepsilon)$ and $\beta : Y \longrightarrow X$, then $\models_M \Phi(\varepsilon \circ \beta)$.*

(ii) *If the domains split in C, and $X \xrightarrow{a'} A \xrightarrow{a} X$ is the splitting of $\mathrm{dom}\,\varepsilon$, then*

$$\models_M \Phi(\varepsilon) \text{ if and only if } \models_M \Phi(\varepsilon \circ a).$$

(iii) *If the domains split in C, then $\models_M \Phi \to \Psi$ if and only if, given an arbitrary total map $f : X \longrightarrow \prod_i \widehat{MT_i}$,*

$$\text{if } \models_M \Phi(f) \text{ then } \models_\Psi (f).$$

Proof (i) Trivial. (ii) Follows from (i), as $\varepsilon = \varepsilon \circ a \circ a'$. (iii) Immediate from (ii). □

4.2.2 **Proposition** *Let C be a p-category with p-exponentials. Let $\mathcal{C}$ be the topos of presheaves built on the category $\mathsf{Dom}(C)$, and $\mathcal{Y} : \mathsf{C} \longrightarrow \mathsf{P}(\mathcal{C}, \mathcal{T})$ the full embedding of p-categories. Then there is an interpretation N of P into $\mathsf{P}(\mathcal{C}, \mathcal{T})$ defined as*

$$N\alpha = \mathcal{Y}M\alpha,$$

such that, given any formula Θ in G,

$$\models_M \Theta, \text{ if and only if } \models_N \Theta.$$

Proof The assignment N defines an interpretation, as $\mathcal{Y}$ preserves products and p-exponentials. Suppose now Φ, Ψ are conjunctions of atomic formulae in G. First, as $\mathcal{Y}$ is a full embedding, given $\varepsilon : X \longrightarrow M\mathbf{T}$ in C, it follows that

$$\models_M \Phi(\varepsilon) \text{ if and only if } \models_N \Phi(\mathcal{Y}\varepsilon).$$

Next, if $\models_N \Phi \to \Psi$, then $\models_M \Phi \to \Psi$. For the converse, apply 4.2.1(iii) and the fact that the full subcategory of representable functors generates $\mathcal{C}$. □

The last result indicates that the case of a topos with a dominance is the most interesting one. We exploit this situation by proving that every interpretation of P in a topos is a model, and deduce the more general case from it. And to do so, we produce a kind of *canonical translation* of the calculus into the logic of the topos.

Let $\mathcal{E}$ be a topos and $\Sigma \rightarrowtail \Omega$ a dominance in $\mathcal{E}$. Suppose that an interpretation M is given of the theory P into $\mathsf{P}(\mathcal{E}, \mathcal{T})$. For $\Theta(x_1, \ldots, x_n)$ a formula in G, we define by induction the translation $\Theta^M(z_1, \ldots, z_n)$, where z_i has type $\widehat{MT_i}$, as follows

(a) $(\mathsf{E}t)^M \leftrightarrow Mt\downarrow$;

(b) $(t \equiv s)^M \leftrightarrow [Mt\downarrow \vee Ms\downarrow \rightarrow Mt = Ms]$;

(c) $(\Phi \wedge \Psi)^M \leftrightarrow \Phi^M \wedge \Psi^M$;

(d) $(\Phi \rightarrow \Psi)^M \leftrightarrow [\Phi^M \rightarrow \Psi^M]$.

4.2.3 **Theorem** *Let $\mathcal{E}$ be a topos and $\Sigma >\!\!\rightarrow \Omega$ a dominance in $\mathcal{E}$. Suppose M is an interpretation of the theory P in $\mathsf{P}(\mathcal{E},\mathcal{T})$, and let Θ be a formula in G whose free variables are $x_1,\ldots,x_n$. Then $\models_M \Theta$ if and only if $\mathcal{E} \Vdash \Theta^M$.*

Proof First, cosider the case when Θ is a conjunction of atomic formulae. For sake of simplicity, suppose $n = 1$. Let $\varepsilon\colon X \longrightarrow MT$ in C. The corresponding partial map $X \longleftarrow\!\!< \operatorname{dom}\varepsilon \xrightarrow{\varepsilon} MT$ in $\mathcal{E}$ can be extended to $\overline{\varepsilon}\colon X \longrightarrow \widehat{MT}$. In other words, $\varepsilon \in \widehat{MT}(X)$. Moreover, by definition,

$$\models_M \Phi(\varepsilon) \quad \text{if and only if} \quad X \Vdash \Phi^M(\varepsilon).$$

The general case follows from the definition of forcing in 1.3.2. □

4.2.4 **Corollary** *All the theorems of the theory P hold in the interpretation M in the p-category C.*

4.3 The syntactic p-category

As in the case of typed λ-calculus, one can generate, from the theory P, the *best* model for P: the theory P itself. The category P has the types of the theory as objects and a map $[\lambda x.t]\colon T \longrightarrow S$ is an equivalence class of closed terms of the theory P where t has type S and x has type T, under provable equivalence. The composition of two maps $[\lambda x.t]\colon T \longrightarrow S$ and $[\lambda y.s]\colon S \longrightarrow R$ is

$$[\lambda x.(\lambda y.s)(t)]\colon T \longrightarrow R,$$

where y is a variable of type S. Because of Theorem 4.3.3, we have no regret for using the same notation for the theory and the syntactic category.

We prove that P is a p-category, characterizing the total maps; finally we check that it has p-exponentials. A lemma about substitution is sufficient to prove that P is indeed a category.

4.3.1 **Lemma** *If t, s are terms of the theory P, then*

$$\mathsf{P} \vdash \mathsf{E}t \rightarrow (\lambda x.t)(s) \equiv t(x/s),$$

provided s is free for x in t.

Proof Here is a formal proof:

$$\frac{\mathsf{E}t \rightarrow (\lambda x.t)(x) \equiv t}{\mathsf{E}t \rightarrow (\lambda x.t)(s) \equiv t(x/s)}$$

It uses one axiom and an application of the rule of substitution.□

4.3.2 **Proposition** *The category P is a p-category. Moreover, a map $[\lambda x.t]\colon T \longrightarrow S$ is total if and only if $\mathsf{P} \vdash \mathsf{E}x \rightarrow \mathsf{E}t$.*

Proof Is very simple. The projections from the object $T \times S$ are $[\lambda z.pz]\colon T \times S \longrightarrow T$ and $[\lambda z.qz]\colon T \times S \longrightarrow S$. The diagonal is $[\lambda x.\langle x, x\rangle]\colon T \longrightarrow T \times T$. We shall not produce the complete proof, but show

just how to obtain the various identities required to have a p-category by proving that the projection is a natural transformation. Let $[\lambda x.t]\colon T \longrightarrow R$ be a map in P. We must show that the diagram

$$\begin{array}{ccc} T \times S & \xrightarrow{[\lambda z.pz]} & T \\ {\scriptstyle [\lambda x,y.\langle t,y\rangle]}\downarrow & & \downarrow{\scriptstyle [\lambda x.t]} \\ R \times S & \xrightarrow{[\lambda w.pw]} & R \end{array}$$

commutes. By 4.3.1, $\mathsf{P} \vdash \lambda z.pz \equiv \lambda x,y.x$. It must therefore be shown that

$$\mathsf{P} \vdash \lambda x,y.p\langle t,y\rangle \equiv \lambda x.t.$$

But this follows easily, as $\mathsf{P} \vdash \mathsf{E}y \to p\langle t,y\rangle \equiv t$. As to total maps in the p-category P, let $[\lambda x.t]\colon T \longrightarrow S$ be a map in P. Suppose it is total. By definition,

$$\mathsf{P} \vdash \lambda x.p\langle x,t\rangle \equiv \lambda x.x.$$

So, *almost* formally,

$$\dfrac{\dfrac{\dfrac{\lambda x.p\langle x,t\rangle \equiv \lambda x.x}{(\lambda x.p\langle x,t\rangle)(x) \equiv (\lambda x.x)(x)} \qquad \mathsf{E}x \to (\lambda x.p\langle x,t\rangle)(x) \equiv p\langle x,t\rangle \wedge (\lambda x.x)(x) \equiv x}{\dfrac{\mathsf{E}x \to p\langle x,t\rangle \equiv x}{\mathsf{E}x \to \mathsf{E}p\langle x,t\rangle} \qquad \mathsf{E}p\langle x,t\rangle \to \mathsf{E}\langle x,t\rangle}}{\dfrac{\mathsf{E}x \to \mathsf{E}\langle x,t\rangle}{\mathsf{E}x \to \mathsf{E}t}}$$

Conversely, suppose $\mathsf{P} \vdash \mathsf{E}x \to \mathsf{E}t$. Then

$$\dfrac{\dfrac{\mathsf{E}x \to \mathsf{E}t}{\mathsf{E}x \to p\langle x,t\rangle \equiv x}}{\lambda x.p\langle x,t\rangle \equiv \lambda x.x}$$

from which the conclusion follows. □

Because of the unrestricted substitution rule, the theory P has a clear call-by-name intention: any term can replace a variable, no matter if it is defined or not. The attestance of existence plays an essential role in this respect. The models have instead a definite call-by-value determination. Nevertheless, the closed terms of the theory are those defining the maps of the syntactic category, as a quantified variable ranges only over the *defined* elements, cf. [Sco82].

4.3.3 **Theorem** *The p-category P has all p-exponentials. Moreover, any interpretation M of the theory P into a p-category C, is a p-functor $M\colon \mathsf{P} \longrightarrow \mathsf{C}$ which preserves the terminal object and p-exponentials.*

Proof As $\vdash \mathsf{E}\lambda x.t$, there is a function

$$[\lambda x,y.t] \mapsto [\lambda y.(\lambda x.t)]\colon \mathsf{P}(S \times T, R) \longrightarrow \mathsf{P}_t(S, {}^T R).$$

The only thing to prove is that it is onto. Let $[\lambda y.s] \in \mathsf{P}_t(S, {}^T R)$. Then $\vdash \mathsf{E}y \to \mathsf{E}s$, and $\vdash \mathsf{E}y \to s \equiv \lambda x.s(x)$. Hence $\vdash \lambda y. \equiv \lambda y.(\lambda x.s(x))$. The last statement is trivial. □

Here is a complete description of total maps in the p-category P.

4.3.4 **Proposition** *The category P_t consists of those maps $[\lambda x.t]\colon T \longrightarrow S$ such that the term t is built up from variables and terms of the form $\lambda y.s$ by repeated use of p, q and $\langle\ ,\ \rangle$.*

Proof We show, by induction on the number of occurences of application, that if $[\lambda x.r(s)]\colon T \longrightarrow S$ then $\vdash \mathsf{E}x \to r(s) \equiv t$ for some term t as in the statement. In case there is none, it is trivial. Suppose $\vdash \mathsf{E}x \to \mathsf{E}r(s)$; then $\vdash \mathsf{E}x \to \mathsf{E}r \wedge \mathsf{E}s$. The only interesting case to treat is for $r = \lambda z.t$. It is then $\vdash \mathsf{E}x \to r(s) \equiv t(z/s)$, and $t(z/s)$ has a smaller number of occurences of application than $r(s)$. Hence the conclusion. □

Final remarks. It is possible to introduce other axioms of the form

$$\mathsf{E}t_1 \wedge \ldots \wedge \mathsf{E}t_k \to \psi.$$

Obviously, one expects in general that the notion of model may change. Indeed, it is possible to prove that, given any set T of formulae of the fragment G, there is a p-category T with p-exponentials, such that every model M of the theory in a p-category C lifts to a unique p-functor $M\colon \mathsf{T} \longrightarrow \mathsf{C}$ which preserves p-exponentials.

In particular, one can add the axiom of total existence

(TE) $$\mathsf{E}x$$

for all types, to obtain the usual typed λ-calculus. The equivalence with the cartesian closed categories follows from 4.2.3.

4.4 Fixed-point operators

We apply the λ-calculus theory developed in the previous two sections to study some properties of fixed point operators. Let T be a fixed type in the theory P. Add one symbol Y to theory P such that

- if t is of type ${}^{X}X$, then $\mathrm{Y}\,t$ has type X,

and add the *fixed-point axioms*

$$\mathsf{E}z \to \mathsf{E}\,\mathrm{Y}\,z \qquad \mathsf{E}\,\mathrm{Y}\,z \to \mathsf{E}z \qquad \mathsf{E}z(\mathrm{Y}\,z) \to z(\mathrm{Y}\,z) \equiv \mathrm{Y}\,z$$

Let FP denote the theory we obtain with this new axioms.

Parallelly in a p-category C, given an object X such that the p-exponential ${}^{X}X$ exists, define a total map $\mathrm{Y}\colon {}^{X}X \longrightarrow X$ to be a *fixed-point operator* if

$$\phi(\mathrm{Y}\,\phi) \downarrow \to \mathrm{Y}\,\phi = \phi(\mathrm{Y}\,\phi).$$

There is an external similar property: say that an object X satisfies the *fixed-point property* if every partial endomorphism $\phi\colon X \longrightarrow\!\!\!\!\!\rightarrow X$ has a global fixed point $e\colon 1 \longrightarrow X$ in the sense that

$$(\mathrm{dom}\,\phi)e = \phi e.$$

Remark. Given an object X, all the p-exponentials ${}^{Y}X$ have a fixed-point operator if and only if they all have the fixed-point property. The proof of sufficiency is trivial. As to necessicity, set $Z = {}^{Y}X$ and take the fixed point of the map

$$H \mapsto [\phi \mapsto \phi(H(\phi))]\colon {}^{{}^{Z}Z}Z \longrightarrow\!\!\!\!\!\rightarrow {}^{{}^{Z}Z}Z.$$

The next proposition is obvious.

4.4.1 **Proposition** *Given a p-category C, and an interpretation M of FP in C, the fixed-point axiom holds in M if and only if the map $M\,\mathrm{Y}\colon {}^{MT}MT \longrightarrow MT$ is a fixed-point operator.*

Rather more interesting is the following property that gives a necessary condition on the type T.

4.4.2 **Proposition** *In the notation above, there is a term $t(z)$ of type T whose only free variable z is of type ${}^{1}T$ such that*

$$\mathsf{FP} \vdash \mathsf{E}x \to t(\lambda u.x) = x.$$

Proof Let $t(z)$ be the term $\mathrm{Y}(\lambda x.z(\mathrm{o}))$. Note first that

$$\mathsf{FP} \vdash \mathsf{E}x \rightarrow x \equiv \mathrm{Y}(\lambda y.x).$$

Also

$$\mathsf{P} \vdash \mathsf{E}x \rightarrow \mathrm{Y}(\lambda y.(\lambda u.x)(\mathrm{o})) \equiv \mathrm{Y}(\lambda y.x),$$

hence the desired conclusion. □

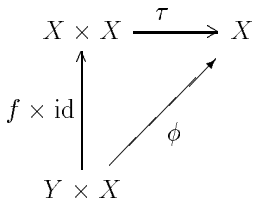
4.4.3 **Corollary** *In the syntactic p-category* FP *associated to the theory* FP*, the object* T *is a retract of* ${}^1T$.

The previous proposition has very little significance, especially if applied to the theory P, (TE) for typed λ-calculus! But it will be useful for an application in chapter 8.

4.5 Turing morphism

The work of [PH86] is directed to the study of *recursion categories*. They are dominical monoids where the (essentially unique) object X satisfies a condition which resembles those of N in the monoid of partial recursive functions, stated traditionally by the universal-map and s-m-n theorems. It is possible to prove many of the results obtained in *loc.cit.* for the case of p-categories. The essential notion was that of a Turing morphism.

Definition. A map $\tau\colon X \times X \longrightarrow X$ in the dominical category C is a *Turing morphism* if it is not a zero map and, for any map $\phi\colon Y \times X \longrightarrow X$, there is a total map $f\colon Y \longrightarrow X$ such that



—no request of uniqueness for f.

Coding. The leading example is the coding of partial recursive functions in the dominical category PR of partial recursive functions on N. We have to point out that the definition of Turing morphism is different from the one given by [Hel85], but coincides with it in the case of a dominical monoid, which is the only case treated there. It should follow from the definition that if X is endowed with a Turing morphism, then X enumerates the set of partial maps from X into itself.

4.5.1 **Theorem** *Let* C *be a p-category, and* $\mathcal{C}$ *the topos of presheaves on* $\mathsf{Dom}(C)$*. A map* $\tau\colon X \times X \longrightarrow X$ *is a Turing morphism if and only if the map* $\{\cdot\}\colon X \longrightarrow \widehat{X}^X$*, obtained by exponentially adjoining* $\overline{\tau}\colon X \times X \longrightarrow \widehat{X}$*, is an epimorphism in the topos* $\mathcal{C}$.

Proof Suppose τ is a Turing morphism: we want to show that $\{\cdot\}$ is pointwise epi. So let $\mathrm{dom}\,\psi$ be an object of $\mathsf{Dom}(C)$, say it is an endomorphism of Y in C, and let $\overline{\phi}\colon \mathrm{dom}\,\psi \times X \longrightarrow \widehat{X}$ be an element of $\widehat{X}^X$. Then $\overline{\phi}$ extends a partial map $\phi\colon \mathrm{dom}\,\phi \longrightarrow X$, with the composition

$$\mathrm{dom}\,\phi \rightarrowtail \mathrm{dom}\,\psi \times X \rightarrowtail Y \times X$$

a subdomain. Thus $\phi\colon Y \times X \longrightarrow X$ in C; hence there is a total map $f\colon Y \longrightarrow X$ such that

$$\phi = \tau(f \times \mathrm{id}\,).$$

Set

$$\vartheta = f \circ \mathrm{dom}\,\psi\colon \mathrm{dom}\,\psi \longrightarrow X.$$

in $\mathsf{Dom}(C)$ it is

$$\phi = \tau(\vartheta \times \mathrm{id}\,).$$

The extension of ϕ is then $\overline{\tau}(\vartheta \times \mathrm{id}\,)$, yielding that

$$\overline{\phi} = \overline{\tau}(\vartheta \times \mathrm{id}\,) = \{\cdot\}(\vartheta).$$

Conversely, suppose $\{\cdot\}$ is epi. As $\mathcal{C}$ is a topos of presheaves over $\mathsf{Dom}(C)$, $\{\cdot\}$ must be pointwise epi. Take a map $\phi: Y \times X \longrightarrow X$ in C. It vinduces a map $\overline{\phi}: Y \times X \longrightarrow \widehat{X}$ in $\mathcal{C}$, that is an element of $\widehat{X}^X(Y)$. As $\{\cdot\}$ is pointwise epi, $\overline{\phi} = \{\cdot\}(f)$ for some element f of $X(Y)$. By this we mean a map $f: Y \longrightarrow X$ in $\mathsf{Dom}(C)$, which is a total map $f: Y \longrightarrow X$ in C. As

$$\tau(f) = \{\cdot\}(f \times \mathrm{id}\,) = \overline{\phi},$$

they extend the same map. Hence

$$\phi = \{\cdot\}(f \times \mathrm{id}\,). \ \square$$

A Turing morphism satisfies a stronger property in the topos $\mathcal{C}$, as $\{\cdot\}$ is pointwise epi.

4.5.2 **Corollary** *Let $\tau: X \times X \longrightarrow X$ be a Turing morphism. Then "composition with $\{\cdot\}$ on the left" is epi for all representables, that is the map $\{\cdot\} \circ -: X^A \longrightarrow (\widehat{X}^X)^A$ is epi for all representable objects A in $\mathcal{C}$.*

These two results are carried over to any topos $\mathcal{E}$ of sheaves on $\mathsf{Dom}(C)$ with a subcanonical topology.

4.5.3 **Corollary** *Let $\tau: X \times X \longrightarrow X$ be a Turing morphism.*

(i) *Suppose $\mathcal{E}$ is a topos of sheaves for a subcanonical topology on $\mathsf{Dom}(C)$. Then the map $\{\cdot\} \circ -: X^A \longrightarrow (\widehat{X}^X)^A$ is epi for all representable objects A in $\mathcal{E}$.*

(ii) *Suppose C has p-exponentials. Then there are epimorphisms ${}^Y X^{Y \times X} X$.*

In 4.5.3 a price was paid: surjectivity of the coding is no longer equivalent to $\tau: X \times X \longrightarrow X$ being a Turing morphism. The problem is obviously that an epimorphism in $\mathcal{E}$ need not be pointwise surjective. In case the topology is strong enough to force global existence from local existence, it is then possible to preserve the equivalence.

4.5.4 **Corollary** *Let C be a p-category. Suppose $\mathcal{E}$ is a topos of sheaves for a subcanonical topology on $\mathsf{Dom}(C)$, coarser than the partition topology. A map $\tau: X \times X \longrightarrow X$ in C is a Turing morphism if and only if the map $\{.\}: X \longrightarrow \widehat{X}^X$ is an epimorphism in $\mathcal{E}$.*

4.6 A general recursion theorem

Again, we can use the embedding theorems of section 3.3 together with the results of the previous section to obtain properties of Turing morphisms in a general p-category. With this in mind, we will prove a series of results in the topos-theoretic setting, because from that the general case will derive. The property expressed in 4.5.3 is equivalent to the following schema of internal statements:

$$\forall \phi \in {}^{X \times X^n} X \ \exists f \in X^{X^n} \ \forall y \in X^n \ \forall z \in X.\ \phi(z, y) \equiv \tau(f(y), z)$$

where, of course, the equivalence must be read "when one side is defined, so is the other, and they are equal". The schema above can be replaced by the one sentence

$$\forall \phi \in {}^X X \ \exists x \in X \ \forall z \in X.\ \phi(z) \equiv \tau(x, z)$$

in case the axiom of choice holds for X. By 1.3.7 this happens when $\mathcal{E}$ is the topos of presheaves, or at least, the topology is smaller than the finite partition topology. As this will always be the case in the examples to follow, we will make those assumptions here. Notice that, as long as the results we proved have an equational form not involving range operators, they will hold in complete generality by the discussion following 3.3.1.

4.6.1 **Proposition** *Let $\mathcal{E}$ be a topos and Σ a dominance. Suppose $\tau\colon X \times X \longrightarrow\!\!\!\!\!\rightarrow X$ satisfies*

$$\forall \phi \in {}^{X}X\, \exists x \in X\, \forall z \in X.\ \phi(z) \equiv \tau(x,z)$$

and the axiom of choice holds for X. Suppose furthermore G is Σ-injective and enumerated by X, say $g\colon X \longrightarrow\!\!\!\!\!\triangleright\, G$. Then

$$\mathcal{E} \Vdash \forall \phi \in {}^{X}G\, \exists x \in X\ [\phi(x) \downarrow \rightarrow \phi(x) = g(x)].$$

Proof Notice that any partial map ψ in ${}^{X}G$ can be extended to a total $h\colon X \longrightarrow G$ as G is Σ-injective. Also $h\colon X \longrightarrow G$ must factor through $g\colon X \longrightarrow\!\!\!\!\!\triangleright\, G$ by the axiom of choice, as

$$\mathcal{E} \Vdash \forall x \in x\, \exists y \in X.\ hx = gy.$$

Let $f\colon X \longrightarrow G$ be a total map extending $\phi\tau\Delta\colon X \longrightarrow\!\!\!\!\!\rightarrow G$ and let $f = gk$. By hypothesis, take z an index of k, and set $x = k(z) = \tau(z,z)$. Then if $\phi(x)$ is defined,

$$\phi(x) = \phi\tau(z,z) = f(z) = gk(z) = g(x).\ \square$$

The proof of the previous proposition is essentially as in [Vis80], where the most important instances were studied, see also [Ber86]. Corollaries of 4.6.1 are all the cases of the recursion theorems considered in [PH86]: take G to be ${}^{X}X$, recalling 4.1.3 and 4.5.3.

4.6.2 **Example** Let X be representable object in the presheaf topos $\mathcal{C}$. Suppose X has a Turing morphism and $i\colon X \longrightarrow X \times X$ is an isomorphism. Let $\phi_1, \phi_2\colon X \longrightarrow\!\!\!\!\!\rightarrow X^2 \times X$ be given. There exist $t_1, t_2\colon X \longrightarrow X$ such that

$$\tau(t_1(x), y) \equiv \phi_1(x, t_1(x), t_2(x), y)$$

and

$$\tau(t_2(x), y) \equiv \phi_2(x, t_1(x), t_2(x), y),$$

for all $x, y \in X$. To find t_1, t_2, consider the map $\{\cdot\} \times \{\cdot\}\colon X \times X \longrightarrow {}^{X}X \times {}^{X}X$ which is epic. Therefore X enumerates ${}^{X}X \times {}^{X}X$ by $(\{\cdot\} \times \{\cdot\})i$, and ${}^{X}X \times {}^{X}X$ is Γ-injective by 4.1.3. Let $f_1, f_2\colon X \longrightarrow ({}^{X}X)^{X^2}$ be the maps induced by ϕ_1 and ϕ_2. Apply 4.6.1 to find

$$\forall x \in X\, \exists t \in X.\ (\{\cdot\} \times \{\cdot\})i(t) = \langle f_1(x)i(t), f_2(x)i(t)\rangle.$$

As X is representable, use the axiom of choice to find that

$$\exists t_1, t_2\colon X \longrightarrow X\, \forall x \in X.\ \langle \{t_1(x)\}, \{t_2(x)\}\rangle = \langle f_1(x)(t_1(x), t_2(x)), f_2(x)(t_1(x), t_2(x))\rangle.$$

As existence is global in $\mathcal{C}$, the conclusion follows.

Other properties. A Turing morphism also enjoys properties in tune with those of the halting set in recursion theory.

4.6.3 **Lemma** *Let $\mathcal{E}$ be a topos and Σ a dominance. Suppose $\{\cdot\}\colon X \longrightarrow \widehat{X}^X$ is onto and set $K = \{x \in X : \{x\}(x) \downarrow\}$. If $X = K \cup A$ and $K \cap A = \emptyset$, then $A \rightarrowtail X$ is not Σ.*

Proof Suppose A is a mono classified by Σ. Let α be the identity on X restricted to A. Then

$$\mathcal{E} \Vdash \exists x \in X\, \forall y \in X.\ \alpha y = \{x\}(y).$$

Hence $\alpha x = \{x\}(x)$, and $x \in A \leftrightarrow x \in K$. Contradiction. $\square$

4.6.4 **Proposition** (Rice) *Suppose $\mathcal{E}$ is a topos with a dominance Σ. Suppose $\{\cdot\}\colon X \longrightarrow \widehat{X}^X$ is onto and $\Delta = \{\bot, \mathrm{true}\} \subset \Sigma$. Let $\mathcal{U} \subset_\Delta \Sigma^X$. Then, for $V, W \subset_\Sigma X$,*

(i) $W \in \mathcal{U} \to \emptyset \in \mathcal{U}$,

(ii) $\emptyset \in \mathcal{U} \to V \in \mathcal{U}$,

(iii) $\exists W \subset_\Sigma X.\, W \in \mathcal{U} \to \mathcal{U} = \Sigma^X$.

Proof Given $A \subset_\Sigma B \subset_\Sigma X$, let $\phi_{A,B}\colon X \times X \rightharpoonup X$ be the partial map defined as

$$\overline{\phi_{A,B}}(x,y) = \{y \in X : (\{x\}(x) \downarrow \wedge y \in B) \vee y \in A\}.$$

Then $\phi_{A,B}(x,y) = f(x)(y)$ for some $f\colon X \longrightarrow \widehat{X}^X$. To obtain (i), take an f which codes $\phi_{\emptyset,W}$ so that

$$\mathcal{E} \Vdash \emptyset \notin \mathcal{U} \to K = (\mathrm{dom} \circ f)^{-1}(\mathcal{U}),$$

and the conclusion is false. Therefore $\emptyset \in \mathcal{U}$. As to (ii), take an g coding $\phi_{\emptyset,V}$ and check

$$\mathcal{E} \Vdash V \notin \mathcal{U} \to K = (\mathrm{dom} \circ g)^{-1}(\Sigma^X \setminus \mathcal{U}),$$

where the conclusion is false. So $V \in \mathcal{U}$. (iii) is trivial. □

Chapter 5

Effectiveness in a topos

If we want to study other properties of computation than those deriving from an elementary theory of partiality, we need to allow ourselves movement ability. The previous chapters 3 and 4 have shown that it is possible to approach the theory of partial maps using topoi, and not just categories, maintaining the same level of generality. As long as we use the *right* one—which seldom is the topos $\mathcal{S}$ of sets—we will conservatively extend the category, and yet have at hand all the constructs we can think of.

In this chapter we study properties of an unusual non-standard object of natural numbers in a topos. It is a model of a good portion of arithmetic, but the particular thing about it is that we are concerned not only with its first order theory, but also with the way it is related to the other objects of the topos. The idea for this comes from the many related properties of two topoi which will be introduced in chapter 6. They are related to the categorical treatment of effectiveness.

5.1 A strong model of arithmetic

From now on we shall work within a topos $\mathcal{E}$. So intuitionistic logic will be used freely, and not just as an intuition about how to develop concepts in categorical terms. We already did so in 4.6.1 and 4.6.4. The simple notion of a Turing morphism already produced interesting effects. We now want to study properties of a "non-standard" model of arithmetic.

Definition. Let $\mathbf{N} = \langle N, +, \cdot, s, 0\rangle$ be a model of HA with induction only for negated formulae, in the form

$$\forall R \subset \mathbf{N}\ [0 \notin R \wedge \forall x \notin R.s(x) \notin R] \rightarrow R = \emptyset.$$

Call $\mathbf{N}$ a *non-standard natural number object* (shortly NNO*), if equality on N is decidable:

$$\forall x, y \in N\ [x = y \vee x \neq y];$$

the *N-axiom of choice* holds in $\mathcal{E}$:

$$\forall R \subset N \times X\ [\forall x \in N\ \exists w \in X.\ \langle x, w\rangle \in R \rightarrow \exists f \in X^N\ \forall x \in N.\ \langle x, fx\rangle \in R];$$

N satisfies *Church's thesis*, in the form

$$\forall f \in N^N\ \exists e \in N\ \forall x \in N\ \exists y \in N\ [T(e, x, y) \wedge Uy = fx],$$

where T is Kleene's predicate and U is the output function; and *Markov's principle* holds:

$$\forall R \subset N\ \forall x \in N\ [Rx \vee \neg Rx] \wedge \neg\neg\exists x \in N.\ Rx \rightarrow \exists x \in N.\ Rx.$$

Adopting a rather unusual confusion, we shall often write $\mathbf{N}$ to mean either the structure or the underlying object of $\mathcal{E}$. Clearly, $\mathbf{N}$ is not just a non-standard model in a topos in the sense of model theory. It is requested to behave nicely also when interacting with the other objects of the universe $\mathcal{E}$.

The universal map $\tau : \mathbf{N} \times \mathbf{N} \longrightarrow\!\!\!\!\!\rightarrow \mathbf{N}$ is defined by

$$\overline{\tau}(x, y) = \{u \in \mathbf{N} : \exists z \in \mathbf{N}.\ T(x, y, z) \wedge Uz = u\}.$$

It is obviously a Turing morphism for the monoid of partial recursive maps on $\mathbf{N}$: we do not need to know whether there is a dominance Σ, that classifies the partial recursive maps to say that.

5.2 Elementary properties

As usual, the objects $\mathbf{N} \times \mathbf{N}$ and $\mathbf{N} + \mathbf{N}$ are both isomorphic to $\mathbf{N}$. It should also be quite obvious that $\mathbf{N}$ enjoyes many other arithmetical properties. But, before dealing with that, we concentrate our attention: let $\Sigma >\!\!\!\rightarrow \Omega$ be defined as

$$\Sigma = \{p \in \Omega : \exists f \in \mathbf{N}^{\mathbf{N}}.\ p = |\exists z \in \mathbf{N}.\ f(z) = 0|\}.$$

The first property is that Σ is a dominance: the proof we present is due to Dana Scott. We also prove that Σ classifies exactly the r.e. subsets of $\mathbf{N}$. So in the topos $\mathcal{E}$, the category of partial maps induced by Σ is a very interesting one.

5.2.1 **Proposition** *The subobject Σ is a sublattice of Ω, and a $\neg\neg$-separated image of $\mathbf{N}$. Moreover, it is a dominance.*

Proof As $s : \mathbf{N} \longrightarrow \mathbf{N}$ and $0 : \mathbf{N} \longrightarrow \mathbf{N}$, obviously $\mathrm{true}, \bot \in \Sigma$. Using the maps $\cdot : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$ and $+ : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$, it is easy to check that Σ is closed under binary intersection and union. Also, Σ is an image of $\mathbf{N}$ by Church's thesis. It is $\neg\neg$-separated, because for $s, t \in \Sigma$,

$$s = t \leftrightarrow [\mathrm{true} \in s \leftrightarrow \mathrm{true} \in t],$$

and the right-hand side is a closed formula, as all elements of Σ are $\neg\neg$-closed. Finally, we apply 3.2.7 to prove it is a dominance. Let $p \in \Sigma$, and suppose that $p \rightarrow q \in \Sigma$. Then there is $f \in \mathbf{N}^{\mathbf{N}}$ such that

$$p \leftrightarrow \exists n \in \mathbf{N}.\ f(n) = 0,$$

therefore

$$\exists n \in \mathbf{N}.\ f(n) = 0 \rightarrow \exists g \in \mathbf{N}^{\mathbf{N}}\ [q \leftrightarrow \exists m \in \mathbf{N}.\ g(m) = 0].$$

Equivalently,

$$\forall n \in \mathbf{N}\ [f(n) = 0 \rightarrow \exists g \in \mathbf{N}^{\mathbf{N}}\ [q \leftrightarrow \exists m \in \mathbf{N}.\ g(m) = 0]].$$

As $f(n) = 0$ is decidable,

$$\forall n \in \mathbf{N}\ \exists g \in \mathbf{N}^{\mathbf{N}}\ [f(n) = 0 \rightarrow [q \leftrightarrow \exists m \in \mathbf{N}.\ g(m) = 0]],$$

so that by $\mathbf{N}$-choice,

$$\exists F \in (\mathbf{N}^{\mathbf{N}})^{\mathbf{N}}\ \forall n \in \mathbf{N}\ [f(n) = 0 \rightarrow [q \leftrightarrow \exists m \in \mathbf{N}.\ F_n(m) = 0]].$$

Thus

$$p \wedge q \leftrightarrow \exists n, m \in \mathbf{N}\ [f(n) = 0 \wedge F_n(m) = 0],$$

and $p \wedge q \in \Sigma$. □

Notice that it follows immediately that induction holds for all r.e. formulae, and, by induction, one can prove that

$$\forall R \subset \mathbf{N}\ \forall x \in \mathbf{N}.\ Rx \vee \neg Rx \rightarrow \forall x \in \mathbf{N}\ [\forall y \leq x.\ Rx \vee \neg\forall y \leq x.\ Rx],$$

and

$$\forall R \subset_{\Sigma} \mathbf{N}.\ |\forall y \leq x.\ Ry| \in \Sigma.$$

5.2.2 **Proposition** *The following hold internally in $\mathcal{E}$.*

(i) $\Sigma^{\mathbf{N}}$ *is the set of r.e. subset of* $\mathbf{N}$.

(ii) $\Delta^{\mathbf{N}}$ *is the set of recursive subsets of* $\mathbf{N}$.

(iii) *The r.e. subsets of* $\mathbf{N}$ *are exactly those which can be indexed by* $\mathbf{N}$.

(iv) Σ *is closed under* $\mathbf{N}$*-indexed unions.*

Proof (i) Suppose $R \subset_\Sigma \mathbf{N}$. Then

$$\forall x \in \mathbf{N}\ \exists f \in \mathbf{N}^{\mathbf{N}}.\ Rx = |\exists z.\ f(z) = 0|.$$

Apply $\mathbf{N}$-choice to get

$$\exists g\colon \mathbf{N} \longrightarrow \mathbf{N}^{\mathbf{N}}\ \forall x \in \mathbf{N}.\ Rx = |\exists z \in \mathbf{N}.\ g(x)(z) = 0|,$$

or equivalently,

$$\exists g\colon \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}\ \forall x \in \mathbf{N}\ [x \in R \leftrightarrow \exists z \in \mathbf{N}.\ g(x,z) = 0].$$

So R is r.e., by Church's thesis. The other implication is trivial. (ii) Follows from (i). (iii) Immediate from Church's thesis. (iv) Let $R \subset_\Sigma \mathbf{N}$. Then, by (i), there is a $g\colon \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$ such that

$$x \in R \leftrightarrow \exists z \in \mathbf{N}.\ g(x,z) = 0.$$

So $\bigcup R = |\exists x, z \in \mathbf{N} \times \mathbf{N}.\ g(x,z) = 0|$ is in Σ. □

5.3 A Rice-Shapiro theorem

What we want to do is to give a description of Σ-subsets of objects in a topos with NNO*. It should be no surprise if these will resemble results in recursion theory. But the point of view is quite different. Computability is not added in from outside the universe. Rather it is an essential component; in particular, the notion of r.e. set is universal, classified, once and for all, by Σ.

$\mathbf{N}$*-finite subsets.* Yet our remote intuition remains classical. So we will concentrate our attention on the r.e. structure of objects related, in a way or another, to $\mathbf{N}$. And a main tool is that of finiteness with respect to $\mathbf{N}$.

Definition. For $x \in \mathbf{N}$, let $[x] = \{y \in \mathbf{N} : y < x\}$. Say that a subset $D \in \mathrm{P}(X)$ is $\mathbf{N}$*-finite* if

$$\exists x \in \mathbf{N}.\ [x]D.$$

Notice that $[x] \subset_\Delta \mathbf{N}$: indeed, for all $x, y \in \mathbf{N}$, it is $y < x \vee x \leq y$, so $|y < x| \in \Delta$. Therefore, if $\mathrm{Fin}^* X$ denotes the set of $\mathbf{N}$-finite subsets of X, then

$$\mathrm{Fin}^* X = \{D \subset X : \exists x \in \mathbf{N}\ \exists f \in {}^{\mathbf{N}}X.\ \mathrm{dom}\, f = [x] \wedge f\colon [x] \longrightarrow\!\!\!\!\triangleright\ D\}.$$

We will also employ the notation $D \subset_{\mathrm{fin}} X$ for $D \in \mathrm{Fin}^* X$.

5.3.1 **Lemma**

(i) $\mathbf{N} = \bigcup_{x \in \mathbf{N}} [x]$;

(ii) *if* $X \subset_\Sigma \mathbf{N}$, *then* $X = \bigcup_{x \in \mathbf{N}} X_x$, *where each* X_x *is* $\mathbf{N}$*-finite.*

(iii) *Suppose* X *is an object in* $\mathcal{E}$ *whose equality is a* Σ*-relation. Then so are inclusion and equality on* $\mathrm{Fin}^* X$.

(iv) *Suppose* Y *is an object in* $\mathcal{E}$ *whose equality is a* Δ*-relation. Then*

$$D \subset_{fin} X \leftrightarrow \exists y \in \mathbf{N}.\ [y] \simeq D.$$

(v) $\mathrm{Fin}^* \mathbf{N} \simeq \mathbf{N}$.

Proof (i) Trivial, as $x \in [x+1]$ for each $x \in \mathbf{N}$. (ii) Suppose $f\colon \mathbf{N} \longrightarrow\!\!\!\!\triangleright\ X$ is an enumeration of X. Set $X_x = f([x])$, and use (i). (iii) For $D, E \subset_{\mathrm{fin}} X$, say $D = f[x]$ and $E = g[y]$, the statement $D \subset E$ is equivalent to

$$\forall z \leq x\ \exists w \leq y.\ fz = gy$$

which is Σ. Hence the conclusion. (iv) First we prove that, for $f \in {}^{\mathrm{N}}Y$,

$$\exists y \leq x\ \exists z \in \mathbf{N}.\ f \circ \{z\}\colon [y] \simeq f[x],$$

by induction on x, as the statement is Σ. If $x = 0$, take $y = z = 0$. Suppose now the property holds for x; we need to find y' and z' for $x+1$. But $f[x+1] = f[x] \cup \{f(x)\}$. As equality on Y is Δ, either $f(x) \in f[x]$ or $f(x) \notin f[x]$. In the first case, set $y' = y$ and $z' = z$. In the other case, set $y' = y+1$ and choose z' a code for the map $\{z\} \cup \{y \mapsto x\}$. The conclusion now follows from the definition of $\mathbf{N}$-finite. (v) Obvious. □

5.3.2 **Remark** For $X \subset_\Sigma \mathbf{N}$ as in 5.3.1(ii), the sets $X_x = \{z \in \mathbf{N} : \exists y < x.\ z = fy\}$ are in Δ^{N} and form an ascending chain. So

$$y \mapsto \{x : y \in X_x \wedge \forall z < x.\ y \notin X_z\}\colon \mathbf{N} \longrightarrow \widehat{\mathbf{N}}$$

induces a partial map defined on X.

We are now in a position to prove a Rice-Shapiro-like characterization for the object $\Sigma^{\Sigma^{\mathrm{N}}}$. Recall $K = \{x \in \mathbf{N} : \exists y \in X.\ T(x,x,y)\}$, and let $\bar{K}$ denote the complement of K, which is not in Σ^{N} by 4.6.4. Let $\kappa\colon K \longrightarrow \mathbf{N}$ be the choice map defined in 5.3.2.

Definition. Given a poset S closed under joins of $\mathbf{N}$-indexed chains (simply called $\mathbf{N}$-chains), define $U \subset S$ to be *Scott-open* if

$$r \in U \wedge r \leq s \rightarrow s \in U$$
$$\bigsqcup_{n \in \mathrm{N}} r_n \in U \rightarrow \exists n \in \mathbf{N}.\ r_n \in U$$

for $(r_n : n \in \mathbf{N})$ any chain. The collection of Scott-open subsets defines a topology on S, see [GHK+81].

5.3.3 **Theorem** (Rice-Shapiro) *Let $\mathcal{U} \subset_\Sigma \Sigma^{\mathrm{N}}$. Then $\mathcal{U}$ is Scott-open.*

Proof Will descend directly from the following assertions:

(a) Let $A \in \mathcal{U}$ and $A \subset B$. Then $B \in \mathcal{U}$.

(b) Let $A \in \mathcal{U}$. Then $\exists E \subset A.\ E \in \mathrm{Fin}^*\mathbf{N} \wedge E \in \mathcal{U}$.

(i) Define a partial map of two variables as follows:

$$\overline{\phi}(x,y) = \{y : y \in A \vee (y \in B \wedge \{x\}(x)\downarrow)\}.$$

Let $h\colon x \mapsto \{y \in \mathbf{N} : \phi(x,y)\downarrow\}\colon \mathbf{N} \longrightarrow \Sigma^{\mathrm{N}}$. Then

$$h(x) = A \cup \{y \in B : \{x\}(x)\downarrow\}.$$

Notice that

$$\{x\}(x)\uparrow \rightarrow h(x) = A, \tag{5.1}$$

and

$$\{x\}(x)\downarrow \rightarrow h(x) = B. \tag{5.2}$$

From (5.1), it follows that

$$\{x\}(x)\uparrow \rightarrow h(x) \in \mathcal{U}.$$

Therefore $\bar{K} \subset h^{-1}(\mathcal{U})$. We will conclude the proof by showing that it cannot happen that $B \notin \mathcal{U}$. Suppose $B \notin \mathcal{U}$. If $h(x) \in \mathcal{U}$, then $h(x) \neq B$, hence $\{x\}(x)\uparrow$, by (5.2). This proves that

$$B \notin \mathcal{U} \rightarrow \bar{K} = h^{-1}(\mathcal{U}).$$

Therefore $\neg\neg B \in \mathcal{U}$. But the statement $B \in \mathcal{U}$ is in Σ; so $B \in \mathcal{U}$, by Markov's principle.

(ii) Define

$$\overline{\psi}(x,y) = \{y : \exists n \in \mathbf{N}\ [[x \notin K(n) \wedge y \in A_n] \vee [x \in K(n) \wedge y \in A_{\kappa(x)}]]\}.$$

Let $k\colon x \mapsto \{y : \psi(x,y)\downarrow\}$. It follows that

$$x \notin K \to k(x) = \bigcup_n A_n = A, \tag{5.3}$$

$$x \in K \to k(x) = \bigcup_{n \leq \kappa(x)} A_n. \tag{5.4}$$

So $\bar{K} \subset k^{-1}(\mathcal{U})$ by (5.3). Now, suppose

$$\neg\exists E \in \mathrm{Fin}^*\mathbf{N}\ [E \subset A \wedge E \in \mathcal{U}].$$

If $k(x) \in \mathcal{U}$, then $k(x) \neq E$ for all $\mathbf{N}$-finite $E \subset A$, by (5.4). So $x \notin K$ and $\bar{K} = k^{-1}(\mathcal{U})$. Contradiction: hence the conclusion, which is a stable statement. □

The proof is similar to the one presented by [Bee85]: no mention is made there about the "set" Σ. Moreover this kind of approach shows how the structure of r.e. subsets is employed. Indeed the same proof can be repeated to obtain a much more general statement.

5.4 Effective Scott domains

Following the ideas of [Sco80a], we investigate the behaviour of Scott domains with respect to a given NNO*. The results we obtain will be seen in later sections to have an effective content. First we set a few definitions for posets in $\mathcal{E}$.

Definition. Let S be a poset with joins of $\mathbf{N}$-chains. An element $c \in S$ is called *compact* if it satisfies

$$\forall a \in S^{\mathrm{N}}\ [\forall n,m \in \mathbf{N}\ [n \leq m \to a_n \sqsubseteq a_m]] \wedge c \sqsubseteq \bigsqcup_n a_n \to \exists n \in \mathbf{N}.\ c \sqsubseteq a_n.$$

Let C be the object of compact elements of S. We say that S is $\mathbf{N}$-*algebraic* if the relation $c \sqsubseteq s$ is a Σ-relation on $C \times S$, and,

$$\forall s \in S\ \exists c \in C^{\mathrm{N}}.\ s = \bigsqcup_n c_n.$$

Given a poset P, say that a family $X \subset P$ is *compatible* if it has an upper bound. And call S an *effective Scott domain*, or simply a Scott domain, if it is an $\mathbf{N}$-algebraic poset, where any compatible $\mathbf{N}$-finite family of compact elements has a join (remember: this is an internal statement).

We note that it follows that in any $\mathbf{N}$-algebraic poset, a $\mathbf{N}$-finite join of compact elements is compact; and, hence a Scott domain has joins of every $\mathbf{N}$-finite compatible family of elements.

The use of the word "domain" above is not directly related to the idea of domain in the sense of chapter 2, but rather it comes from [Sco80a] and the use in mathematical practice thereof.

5.4.1 **Theorem** *Let S be a $\mathbf{N}$-algebraic poset and C the subset of compact elements. Then $U \subset_\Sigma S$ if and only if U is open in the Scott topology on S and $U \cap C \subset_\Sigma C$.*

Proof Will derive from the following:

(a) Let $U \subset_\Sigma S$. Let $a \in U$ and $a \sqsubseteq b$. Then $b \in U$.

(b) Let $U \subset_\Sigma S$. Let $a \in U$. Then $\exists c \sqsubseteq a.\ c \in C \wedge c \in U$.

(c) Let $U \subset S$ be such that $U \cap C$ is Σ and U is open. Then $U \subset_\Sigma S$.

Proof (i) Define a map $h\colon \mathbf{N} \longrightarrow S$ as follows:

$$h(x) = \bigsqcup\{c_n \in C : [x \notin K(n) \wedge c_n = a] \vee [x \in K(n) \wedge c_n = b]\}.$$

As the set appearing on the right-hand side is directed and indexed by $\mathbf{N}$, the union exists. Notice that

$$\{x\}(x)\uparrow \rightarrow h(x) = a, \tag{5.5}$$

and

$$\{x\}(x)\downarrow \rightarrow h(x) = b. \tag{5.6}$$

From (5.5) it follows that

$$\{x\}(x)\uparrow \rightarrow h(x) \in U.$$

In other words, $\bar{K} \subset h^{-1}(U)$. We will conclude the proof by showing that it cannot happen that $b \notin U$. Suppose $b \notin U$. If $h(x) \in U$, then $h(x) \neq b$, hence $\{x\}(x)\uparrow$, by (5.6). This proves that

$$b \notin U \rightarrow \bar{K} = h^{-1}(U).$$

But U is Σ, and so is $h^{-1}(U)$. Therefore $\neg\neg b \in U$. But $b \in U$ is in Σ; so $b \in U$ holds.

(ii) Let $c\colon \mathbf{N} \longrightarrow C$ be a chain with sup a. Define

$$k(x) = \bigsqcup\{c_n : \exists m\ [[x \notin K(m) \wedge m = n] \vee [x \in K(m) \wedge n = \kappa(x)]]\}.$$

Again, the union exists because the set on the right-hand side is directed and indexed by $\mathbf{N}$. Therefore

$$x \notin K \rightarrow k(x) = \bigsqcup_n c_n = a, \tag{5.7}$$

$$x \in K \rightarrow k(x) = \bigsqcup_{n \leq \kappa(x)} c_n. \tag{5.8}$$

So $\bar{K} \subset k^{-1}(U)$ by (5.7). Now, suppose

$$\neg\exists c \in C\ [c \sqsubseteq a \wedge c \in U].$$

In such a case, if $k(x) \in U$, then $k(x) \neq c$ for all $c \sqsubseteq a$, because of (5.8) and part (i). So $x \notin K$ and $\bar{K} = k^{-1}(U)$. Contradiction: hence the conclusion.

(iii) An element $a \in U$ if and only if

$$\exists c \in C\ [c \sqsubseteq a \wedge c \in U].$$

The statement is in Σ, hence the conclusion. □

Consequences. The characterizations we obtained can be used to deduce some powerful continuity theorems.

5.4.2 **Corollary** *The following statements are internally valid in* $\mathcal{E}$.

(i) *Let S be a* $\mathbf{N}$*-algebraic poset. Then the Scott topology coincides with the topology σ_S induced by the family Σ^S.*

(ii) *Any function between* $\mathbf{N}$*-algebraic posets is continuous for the Scott topologies.*

(iii) *Let S and T be* $\mathbf{N}$*-algebraic posets. If the object C of compact elements of S is an image of* $\mathbf{N}$*, and T is a Scott domain, then the set ${}^S T$ can be given a structure of* $\mathbf{N}$*-algebraic poset.*

(iv) *If S and T are Scott domains, then ${}^S T$ is a Scott domain.*

Proof (i) Suppose U is Scott-open. Then $U = \bigcup\{\uparrow(c) : c \in U \cap C\}$ is a union of Σ-subsets. On the other hand, if U is a union of Σ-subsets of S, then each one is open by the previous theorem, hence their union U is such. (ii) Follows from 3.4.1. (iii) For $\phi, \psi \in {}^S T$, set

$$\phi \sqsubseteq \psi \leftrightarrow \forall a \in S\ [\phi(a)\downarrow \rightarrow \psi(a)\downarrow \wedge\ \phi(a) \sqsubseteq \psi(a)].$$

Sups of $\mathbf{N}$-chains are defined pointwise. Now, notice that, if $c \in C$ and $d \in D$ are compact elements of S and T respectively, then the partial map

$$\gamma : a \mapsto \{y \in S : c \sqsubseteq a \wedge y = d\} : S \longrightarrow\!\!\!\!\!\rightarrow T$$

is Σ-defined, as its domain is $\uparrow(c) \subset_\Sigma S$. Moreover, for $\psi \in {}^S T$,

$$\gamma \sqsubseteq \psi \leftrightarrow \psi(c)\downarrow \wedge\ d \sqsubseteq \psi(c).$$

Thus γ is a compact element. Consider now $\phi \in {}^S T$, and let $c : \mathbf{N} \longrightarrow C$ be an enumeration of C. For each $m \in \mathbf{N}$, there is an $\mathbf{N}$-chain $(d_{n,m})_{n\in\mathrm{N}}$ of compact elements in D such that

$$\phi(c_m) = \bigsqcup_n d_{n,m}.$$

Fix $k \in \mathbf{N}$ and set

$$U_k = \bigcup_{m \le k} \uparrow(c_m),$$

so that $U_k \subset_\Sigma S$. Also, for $a \in U_k$, note that the family $\{d_{n,m} : n, m \le k \wedge c_m \sqsubseteq a\}$ is bounded by $\phi(a)$, hence is compatible, and $\mathbf{N}$-finite. Let

$$\gamma_k(a) = \bigsqcup\{d_{n,m} : n, m \le k \wedge c_m \sqsubseteq a\},$$

so that γ_k is a Σ-defined partial map $\gamma_k : S \longrightarrow\!\!\!\!\!\rightarrow T$. Obviously, the sequence $(\gamma_k)_{k\in\mathrm{N}}$ is a chain, and its sup is ϕ. (iv) Trivial, by the previous statement. □

The reader should not be surprised to find many a resemblance with the results of [McC84] about information systems in the effective topos. As we shall see in section 6.3, that topos is a model of the theory of NNO*, and the results in *loc. cit.* derive from 5.4.2.

Note that any function between retracts of $\mathbf{N}$-algebraic posets is continuous for their induced topologies. We do not have a nice description of the retracts in terms of properties of the order, but, after section 6.1, the reader can refer to the work in [WD80] and [WS83] for examples.

A particular instance of 5.4.2 is Myhill-Shepherdson theorem, as $\widehat{\mathbf{N}}^{\mathrm{N}}$ and $\widehat{\mathbf{N}}$ are $\mathbf{N}$-algebraic posets. So

$$\mathcal{E} \Vdash \text{all maps from } \widehat{\mathbf{N}}^{\mathrm{N}} \text{ to } \widehat{\mathbf{N}} \text{ are continuous for the Scott topology.}$$

Moreover, all the partial types $\widehat{\mathbf{N}}^{\mathrm{N}}$, $\widehat{\mathbf{N}}^{\widehat{\mathbf{N}}^{\mathrm{N}}}$, and so on, are $\mathbf{N}$-algebraic posets.

Banach-Mazur functionals. We will pursue in greater details the study of the structure of the partial functionals in section 6.5. As far as the total functionals are concerned, a side remark about maps from $\mathbf{N}^{\mathrm{N}}$ to $\mathbf{N}$ is that they all are Banach-Mazur.

Definition. A map $f : \mathbf{N}^{\mathrm{N}} \longrightarrow \mathbf{N}$ is Banach-Mazur if for all recursive maps $g : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$ the map $x \mapsto f(g(x, -)) : \mathbf{N} \longrightarrow \mathbf{N}$ is recursive.

Then it is just by the definition of exponential adjoint that

$$\mathcal{E} \Vdash \text{all } f : \mathbf{N}^{\mathrm{N}} \longrightarrow \mathbf{N} \text{ are Banach-Mazur.}$$

Chapter 6

Effective versus recursive

The principal aim of our detailed study of categories of partial maps was to produce topoi which could model the theory of partial recursive functions. In the categorical approach to recursion theory, satisfactory results has been obtained about the higher types and the relationship with cartesian closed categories, see [Hyl82], [Sco81b], [LS83]. But the study, while saving the idea of "application", loses that of "computation". On the other hand, little has been developed, in categorical terms, of the elementary study of recursive theory. In chapter 2 we mentioned already some attempts. But, in our opinion, the central one remains Eršov's category of enumerated sets, see [Erš73], [Erš75], [Vis80]. It proved to be a very good environment for a treatment of computability, see also [Erš77b], [Hyl77b].

Two topoi were shown to embed the category of enumerated sets: the effective topos which was introduced in [HJP80] and studied in details by [Hyl82]; and the recursive topos, discovered and studied by [Mul81]. The first has had good fortune; it is in essence the realizability interpretation of [Kle73], see also [Kle45] or [Tro73]. It was successfully used in [McC84] in connection with effective domains. It seems effectiveness is built in it, so that all maps are, ideally, computable. The second has had less, but interest in it has been reawaked, see [LM84b], [Mul86], [Sce85]. Also, its construction is more resembling of a topological one; so, as a mathematical object, Mulry's topos may help in studying topological aspects of recursion.

In this chapter, we study connections between these topoi. We give new presentations for the recursive topos which enable us to characterize the theory it classifies, and produce a functor from the effective topos into the recursive topos which is useful to connect two different theories of constructive mathematics. Also, applying the theory of NNO*, we are able to give a satisfactory explanation of how these topoi may be used in problems about effective structures.

6.1 The recursive topos

As we insisted already, the first reason for the study of p-categories, is the category PR consisting of one object N and partial recursive maps. To this particular example we want to apply the results of section 3.3. The category $\mathsf{Dom}(PR)$ has r.e. subsets of N as objects, a map $\phi: R \longrightarrow S$ being a partial recursive function which is defined on all of R and takes values in S.

A site for the recursive topos. The p-category PR has ranges and coproducts, and it satisfies the weak axiom of choice. So by 3.3.6, the finite cover topology on $\mathsf{Dom}(PR)$ is subcanonical. In fact, it is the canonical topology for $\mathsf{Dom}(PR)$. As range maps are surjective, the problem is reduced to show that there cannot be an infinite covering of inclusions.

So suppose $(A_i \subset R : i \in I)$ is a covering. As constant functions are recursive, $\bigcup_{i\in I} A_i = R$. Suppose then I is infinite and there is no finite refinement of $(A_i \subset R)_{i\in I}$. As the maps of $\mathsf{Dom}(PR)$ form a countable set, I can be taken equal to N. Given *any* sequence of numbers $(a_n)_{n\in\mathsf{N}}$ with no repetitions, define a family

of functions $f_i: A_i \longrightarrow R$ as follows

$$f_0 = a_0$$
$$f_{n+1} = \begin{cases} f_n(x) & \text{if } x \in \bigcup_{k \leq n} A_k \\ \text{the first } a_k \notin f\left(\bigcup_{k \leq n} A_k\right) \end{cases}$$

—note that each f_n has a finite image. Then there is a recursive map $f: R \longrightarrow \mathsf{N}$ which lists $(a_n : n \in \mathsf{N})$. Contradiction. So there is no infinite covering in the canonical topology on $\mathsf{Dom}(PR)$. We have so proven

6.1.1 **Proposition** *The canonical topology on* $\mathsf{Dom}(PR)$ *is the finite cover topology.*

The proof we gave is a remake of Mulry's proof that, on the monoid R of total recursive functions, the canonical topology is the finite cover topology; more precisely, $(f_i: \mathsf{N} \longrightarrow \mathsf{N})_{i \in I}$ is a cover if and only if

$$I \text{ is finite and } \bigcup_{i \in I} \operatorname{im} f_i = \mathsf{N},$$

see [Mul81] for a proof. Let $\mathcal{R}$ be the *recursive topos* of sheaves for the canonical topology on $\mathsf{Dom}(PR)$.

Two more sites. It is convenient to notice that R has finite limits and binary coproducts: take your favorite couple of unpairing functions to make N into $\mathsf{N} \times \mathsf{N}$, and the bijective enumerations of odd numbers and even numbers as injections to make N into the coproduct $\mathsf{N} + \mathsf{N}$. The canonical sites on R and $\mathsf{Dom}(PR)$ are related very strongly. And there is a third one: Eršov's category En of *enumerated sets*. An object is a denumerable list $(r_n : n \in S)$ where S is an initial segment of N, possibly empty, and a map $f: (r_n : n \in S) \longrightarrow (s_n : n \in T)$ is a function $f: \{r_n : n \in S\} \longrightarrow \{s_n : n \in T\}$ such that there is a recursive function $g: \mathsf{N} \longrightarrow \mathsf{N}$ satisfying

$$f(r_n) = s_{g(n)}$$

for all $n \in S$. Notice that a set $\{r_n : n \in \mathsf{N}\}$ is either empty or the image of a function from N. Also there is no map into the empty list 0, and exactly one from it into any other object. In other words, 0 is a strict initial object in En.

Sometimes, a list in En will be confused with the listed set. In particular, the object $(n : n \in \mathsf{N})$ will be *always* confused with N. Notice that R and $\mathsf{Dom}(PR)$ are full subcategories of En. And the embedding of the second category in En was the only reason for presenting En the way we did. Indeed, there is a more familiar presentation of the category $\mathsf{En}*$ of non-empty lists which is as follows: the objects of $\mathsf{En}*$ are surjections $r: \mathsf{N} \longrightarrow\!\!\!\!\triangleright\ X$ from N, and a map from r to $s: \mathsf{N} \longrightarrow\!\!\!\!\triangleright\ Y$ is a function $f: X \longrightarrow Y$ such that

$$\begin{array}{ccc} \mathsf{N} & \xrightarrow{\ g\ } & \mathsf{N} \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

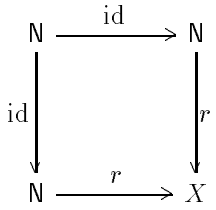
commutes for some recursive function g. It is obvious that $\mathsf{En}* \subset \mathsf{En}$ is the full subcategory of non-empty lists.

6.1.2 **Lemma**

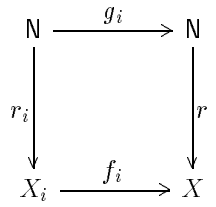
(i) *Given an object* $r: \mathsf{N} \longrightarrow\!\!\!\!\triangleright\ X$ *in* $\mathsf{En}*$, *there is an epimorphism* $r: \mathsf{N} \longrightarrow\!\!\!\!\triangleright\ r$ *in* En, *from the object* $\mathrm{id}: \mathsf{N} \longrightarrow\!\!\!\!\triangleright\ \mathsf{N}$, *which is stable effective.*

(ii) *The canonical topology on* En *is generated by the finite covers in* R.

Proof (i) It is enough to notice that r is a surjection and the diagram



commutes. (ii) Suppose $v\colon \mathsf{N} \longrightarrow\!\!\!\!\triangleright\ X$ and $r_i\colon \mathsf{N} \longrightarrow\!\!\!\!\triangleright\ X_i$ are objects of En and $(f_i\colon r_i \longrightarrow v)_{i\in I}$ is a stable effective epimorphic cover in En. Therefore there are recursive $g_i\colon \mathsf{N} \longrightarrow \mathsf{N}$ such that



commutes for each $i \in I$. By (i), it is a diagram of arrows in En. Hence $(g_i\colon \mathsf{N} \longrightarrow \mathsf{N})_{i\in I}$ is stable effective epimorphic. Therefore I must be finite and the finite covers of R generate the canonical topology on En, because r is a stable effective epimorphism. □

6.1.3 **Corollary** *The sites with the canonical topology on the categories* R, $\mathsf{Dom}(PR)$, *and* En *all define the topos* $\mathcal{R}$. *Then* R, $\mathsf{Dom}(PR)$ *and* En *are all fully embedded in* $\mathcal{R}$ *by functors which preserves existing limits, finite colimits and exponentials.*

Proof It follows from 1.5.6 and 6.1.2, as N is in R. Then take Yoneda embedding in each case. Notice that a coproduct or a coequalizer give rise to a stable effective epimorphic family. □
Properties. The previous result has been already proved, though in quite different a way, by [Mul81]. We will often make use of 6.1.3 and think of an object D of $\mathcal{R}$ as either a set $D = D(\mathsf{N})$ with an action of the monoid R satisfying a certain sheaf-condition, or as a sheaf on $\mathsf{Dom}(PR)$, or still as an extension of the concept of enumerated set with respect to a *recursion* topology. Also these topologies have nice generation properties.

6.1.4 **Lemma**

(i) *The canonical topology on* $\mathsf{Dom}(PR)$ *coincides with the finite partition topology.*

(ii) *The canonical topology on* R *is generated by* $(i\colon \mathsf{N} \longrightarrow \mathsf{N}+\mathsf{N}, j\colon \mathsf{N} \longrightarrow \mathsf{N}+\mathsf{N})$.

Proof (i) Any surjective map has a section; therefore the topology is generated by finite coverings consisting of inclusions. Suppose $(R_i \subset R)_{i\in I}$ is one such, for I a finite set. Then consider the following computation ϕ: with input $x \in R$, start at the same time, machines ϕ_i, converging exactly on R_i. If ϕ_j is the first one which converges, give output j. As $\bigcup_{i\in I} R_i = R$, ϕ converges on R and, if we let

$$S_i = \{x \in R : \phi(x) = i\},$$

then $(S_i \subset R)_{i\in I}$ is a finite partition of R, refining the given cover. (ii) By 6.1.2 (ii), the topology on $\mathsf{Dom}(PR)$ is generated by those covers which are in R. By (i), these can be reduced to finite partition. In turn, one such is a pullback of a coproduct diagram

$$i_k\colon \mathsf{N} \longrightarrow \coprod_{k\in I} \mathsf{N}, \qquad k \in I$$

for a suitable finite set I, as any finite coproduct of copies of N is N with appropriate injections. Therefore, the family $(i\colon \mathsf{N} \longrightarrow \mathsf{N}+\mathsf{N}, j\colon \mathsf{N} \longrightarrow \mathsf{N}+\mathsf{N})$ generates these covers. □

6.1.5 **Corollary** *Local existence in $\mathcal{R}$ forces global existence: if $\mathcal{R} \Vdash \exists x \in X.\ \Psi(x)$, then there is an element $a \in X(\mathsf{N})$ such that $\mathcal{R} \Vdash \Psi(a)$.*

Proof Immediate from 1.5.5. □

Generalized enumerated sets. There is yet another site for $\mathcal{R}$: the category GEn of *generalized enumerated sets*. Introduced in [LM84c], it is an extension of En, which still lies inside $\mathcal{R}$. The objects are subsets $E \subset X^{\mathsf{N}}$ of functions, whose images cover the set X and such that

$$\forall e_1, e_2 \in E\ \exists e \in E\ \exists f_1, f_2 \in \mathsf{R}.\ e_i = e f_i, \qquad i = 1, 2.$$

A map from E to $F \subset Y^{\mathsf{N}}$ is a function $f\colon X \longrightarrow Y$ such that

$$\forall e \in E\ \exists d \in F\ \exists g \in \mathsf{R}.\ fe = dg.$$

The category is an extension of En because the assignment

$$(r_n : n \in S) \mapsto \{r\} \subset \{r_n : n \in S\}^{\mathsf{N}}$$

defines trivially a full embedding of En into GEn. Actually, GEn is quite big, because it has a full picture of $\mathcal{S}$ in it: take a set X to the generalized enumerated set $X^! = X^{\mathsf{N}}$ and $f\colon X \longrightarrow Y$ to itself thought of as from $X^!$ to $Y^!$ in GEn—when so it will be denoted as $f^!$. It is trivial to see it is a full embedding.

It will follow from other results that the category GEn is equivalent to the quasitopos of $\neg\neg$-separated objects of $\mathcal{R}$. But at the moment we prepare just some of the ingredients for the proof: first, the inclusion $E \subset X^{\mathsf{N}} = X^!$ is a map in GEn, and there is a *best* representation for E. Let $E' \subset X^{\mathsf{N}}$ be defined by

$$d \in E' \text{ if and only if } d = ef \text{ for some } e \in E \text{ and } f \in \mathsf{R}, \tag{6.1}$$

so $E' \subset X^{\mathsf{N}}$. To prove it is an object in GEn, suppose $d_1, d_2 \in E'$, so that there are $e_1, e_2 \in E$ and $f_1, f_2 \in \mathsf{R}$ with $d_i = e_i f_i, \quad i = 1, 2$. As E is in GEn, there are $e \in E$ and $g_1, g_2 \in \mathsf{R}$ such that $e_1 = e g_i$; thus $d_i = e g_i f_i$. It is $E \simeq E'$, because the function $\mathrm{id}\colon X \longrightarrow X$ defines two maps between E and E', forcedly inverse of each other. The set E' satisfies good closure properties: it is closed under maps from N, in the sense that, if $f\colon \mathsf{N} \longrightarrow E'$ in GEn, then $f \in E'$; and, more generally,

$$f\colon E' \longrightarrow F' \text{ in } \mathsf{GEn} \quad \text{ if and only if } \quad f \circ E' \subset F'. \tag{6.2}$$

In diagrammatic form,

$$\begin{array}{ccc} E' & \xrightarrow{\ f\ } & F' \\ \downarrow & & \downarrow \\ X^! & \xrightarrow{\ f^!\ } & Y^! \end{array}$$

commutes in GEn. Then put an R-structure on E' by composition on the left, and denote it still as E'. Also, E' is a sheaf on R. For, if $d_1, d_2\colon \mathsf{N} \longrightarrow X$ are in E', there are $d \in E'$ and $f_1, f_2 \in \mathsf{R}$, such that

$$d_1 = d f_1 \quad \text{ and } \quad d_2 = d f_2;$$

so $d_1 + d_2$ is in E' by condition (6.2), proving E' is a sheaf by 6.1.4. Notice that property (6.1) states that E' is a subfunctor of the sheaf $X^!$, and the diagram above can be interpreted in $\mathcal{R}$. So far a functor $'\colon \mathsf{GEn} \longrightarrow \mathcal{R}$ has been produced, but, by composing it with $^!\colon \mathcal{S} \longrightarrow \mathsf{GEn}$, a functor $\Gamma^!\colon \mathcal{S} \longrightarrow \mathcal{R}$ comes to attention. It is left exact, and a property of the latter will prove the former is a full embedding.

6.2 Classifying a theory

A simple way to find a theory which $\mathcal{R}$ may classify is to check if one of the many sites that we have for $\mathcal{R}$, is a regular category. By 2.5.11 and 2.6.10, the category $\mathsf{Dom}(PR)$ satisfies these requirements. Indeed, it is the regular category of the finitary geometric fragment of the complete theory of N with a relation symbol for each recursive set; and the topology on it which defines $\mathcal{R}$ is the finite cover topology. Therefore we know from [MR77] that

6.2.1 **Theorem** *The topos $\mathcal{R}$ classifies the finitary geometric fragment of the complete theory of N with a relation symbol for each recursive set, and the functor represented by N is its generic model.*

We are not going to use the same name N for the object in $\mathcal{R}$ represented by N, because this would clash with the conventional notation for the *standard* natural number object in a topos. So we prefer to denote the generic model by $\mathbf{N}$, as we show that it is a NNO*.

The double-negation topology. As $\mathcal{R}$ classifies a fragment of the complete theory of N, the standard model on N in $\mathcal{S}$ is a model for it. Therefore, there must be a geometric morphism

$$f^* \dashv f_* : \mathcal{S} \longrightarrow \mathcal{R}$$

over $\mathcal{S}$ such that $f^*\mathbf{N} \simeq \mathsf{N}$. The functor $\Gamma : \mathcal{R} \longrightarrow \mathcal{S}$ takes $\mathbf{N}$ to N and each functor represented by a recursive set, to that recursive set. So Γ takes the generic model to the standard model on N in $\mathcal{S}$. Moreover $\Gamma\Delta \simeq \mathrm{Id}$: because, by 6.1.4, for S in $\mathcal{S}$, the sheaf $\Delta S : \mathsf{Dom}(PR)^{\mathrm{op}} \longrightarrow \mathcal{S}$ is

$$\Delta S(R) = \{f : R \longrightarrow S \mid \mathrm{im} f \text{ is finite}\}$$

acted upon by composition on the right, and $\Gamma\Delta S = \Delta S(1) \simeq S$. So $\Gamma = f^{-1}$ is the inverse image of the geometric morphism classifying the standard model if and only if Γ has a right adjoint. It was proved in [Mul81] that $\Gamma \dashv \Gamma^! : \mathcal{S} \longrightarrow \mathcal{R}$. Mulry showed also that $\Gamma \dashv \Gamma^!$ is an inclusion and the associated topology on $\mathcal{R}$ is the double-negation topology. When referring to this topology on a topos, we will refrain from writing the double-negation prefix to the words "separated" and "closed".

6.2.2 **Proposition** *The category GEn of generalized enumerated sets is equivalent to the quasitopos of separated sheaves in $\mathcal{R}$.*

Proof For an object $E \subset X^{\mathsf{N}}$ in GEn, the global sections $\Gamma E'$ coincide with X because

$$X = \bigcup_{d \in E'} \mathrm{im} d.$$

So $E' \rightarrowtail \Gamma^!\Gamma E'$, and E' is a separated sheaf. Note now that $\mathcal{R}(\mathbf{N}, \Gamma^! X) = X^{\mathsf{N}}$ for any set X. So, given a separated sheaf D, it is $D(\mathsf{N}) \subset \Gamma D^{\mathsf{N}}$ an object of GEn, and $D = D(\mathsf{N})'$. To prove fullness, recall that

$$\begin{array}{ccc} E' & \xrightarrow{f} & F' \\ \downarrow & & \downarrow \\ \Gamma^!\Gamma E' & \xrightarrow{\Gamma^!\Gamma f} & \Gamma^!\Gamma F' \end{array}$$

commutes because $\Gamma \dashv \Gamma^!$. This proves the functor $' : \mathsf{GEn} \longrightarrow \mathcal{R}$ is full and faithful onto the quasitopos of separated objects. □

A reminder about notation is mandatory: there are four full embeddings of the categories R, $\mathsf{Dom}(PR)$, En, and GEn into the topos $\mathcal{R}$. The first three were produced as Yoneda embeddings, and we agreed to confuse a category with its image in $\mathcal{R}$. An exception is made for the one object in R: it is N and is denoted $\mathbf{N}$ i $\mathcal{R}$. As to the fourth, it is essentially the identity if the representation E' for a generalized enumerated set is chosen. And one could define an equivalent category by adding the property

$$d \in E' \text{ if and only if } d = ef \text{ for some } e \in E \text{ and } f \in \mathsf{R},$$

to the definition of object of GEn. Yet we prefer to use the representation and write $'\colon \mathsf{GEn} \longrightarrow \mathcal{R}$ because we believe this helps clarity in the sequel.

We note that a subfunctor $R \rightarrowtail \mathbf{N}$ representable in $\mathsf{Dom}(PR)$ is closed, as by definition,

$$\begin{array}{ccc} R & \longrightarrow & \mathbf{N} \\ \downarrow & & \downarrow \\ \Gamma^{!}\Gamma R & \longrightarrow & \mathsf{N}^{\mathsf{N}} \end{array}$$

is a pullback in GEn, hence a pullback in $\mathcal{R}$. So $R \rightarrowtail \mathbf{N}$ is closed.

The double adjunction $\Delta \dashv \Gamma \dashv \Gamma^{!}$ will be very useful in working with the Γ-interpretation of the internal logic of the topos. We summarize a few properties which follow from the adjunction, in the line of thought drawn in §5 of [Hyl82].

6.2.3 **Corollary** *Let $\Gamma\colon \mathcal{R} \longrightarrow \mathcal{S}$ be the global section functor.*

(i) Γ *preserves all limits and colimits.*

(ii) Γ *preserves images.*

(iii) Γ *preserves interpretation of geometric sequents.*

(iv) Γ *preserves interpretation of full first order logic on closed subobjects.*

(v) Γ *preserves the evaluation map on separated objects, in the sense that, given E, F separated, F^E is also separated, so $\Gamma(F^E) \subset \Gamma F^{\Gamma E}$, and $\Gamma(\mathrm{ev})\colon \Gamma E \times \Gamma(F^E) \longrightarrow \Gamma F$ is the restriction of the evaluation map.*

Proof (i) Trivial. (ii)-(iii) Follow from 6.1.4. (iv)-(v) Particular cases of 1.2.5. □

A non-standard model. We are now in a position to prove that the object $\mathbf{N}$ is a NNO* in $\mathcal{R}$.

6.2.4 **Proposition** *The generic model $\mathbf{N}$ in $\mathcal{R}$ is a* NNO*.

Proof By 6.2.1 and the remarks following it, $\mathbf{N}$ is a model of all the axiom of HA, including induction on negated formulae by 6.2.3; for the same reason, equality is decidable. By 4.5.3, Church's thesis holds. By 1.5.5, the $\mathbf{N}$-axiom of choice holds in $\mathcal{R}$. It remains to prove that Markov's principle holds. And to prove it we restrict the site to R. So suppose $R \rightarrowtail \mathbf{N} \times \mathbf{N}$ and

$$\mathsf{N} \Vdash \ \forall x \in \mathbf{N}\ [R(x, \mathrm{id}) \vee \neg R(x, \mathrm{id})].$$

Therefore, restricting along $q\colon \mathsf{N} \times \mathsf{N} \longrightarrow \mathsf{N}$ and instantiating with $p \in \mathbf{N}(\mathsf{N} \times \mathsf{N})$

$$\mathsf{N} \times \mathsf{N} \Vdash R(p, q) \vee \neg R(p, q)$$

—we write $\mathsf{N} \times \mathsf{N}$ even though it is always N, to help intuition. So there is a pair (A, B) of disjoint r.e. sets covering $\mathsf{N} \times \mathsf{N}$, such that

$$A \Vdash \mathrm{id} \in R \quad \text{and} \quad B \Vdash \mathrm{id} \notin R.$$

Therefore $A \rightarrowtail R$ and $B \cap R = 0$ in $\mathcal{R}$. But $A \cup B = \mathbf{N} \times \mathbf{N}$ in $\mathcal{R}$. So $R = A \rightarrowtail \mathbf{N} \times \mathbf{N}$ is representable by a recursive set. So

$$\exists x \in \mathbf{N}.\ R(x, \mathrm{id})$$

is represented by an r.e. set in $\mathsf{Dom}(PR)$ by 6.1.3, and is closed. □

In the topos $\mathcal{R}$, the object Σ defined in 5.2 is the R-set $\Sigma \subset \Omega$ which consists of r.e. subsets of N acted upon by inverse image. And by 5.2.1 it is a dominance, and the category of Σ-defined partial maps in $\mathcal{R}$ fully embeds the p-category PR. Also, both Σ and $\widehat{\Sigma}$ are separated images of $\mathbf{N}$.

Functionals. We like to claim that 6.2.2 and 6.2.3 are the reason why so much of the treatment of functionals in $\mathcal{R}$ is strictly related to their external *appearance*.

As $\mathcal{S}$ is included in $\mathcal{R}$ we can try and explain this in *model-theoretic* terms. We delimit our discussion to higher type functionals: suppose Φ is a formula of higher-order arithmetic. It can therefore be interpreted on $\mathbf{N}$ in $\mathcal{R}$, say $|\Phi| \rightarrowtail \mathbf{T}$, where $\mathbf{T}$ is an appropriate product of exponentials and pre-exponentials of $\mathbf{N}$. Suppose also that we can prove that the basic compounds of Φ are closed in $\mathcal{R}$, for instance, we can manage to do that using the theory of NNO*. Then the following diagram

$$\begin{array}{ccc} |\Phi| & \longrightarrow & \Gamma|\Phi|^{\mathsf{N}} \\ \downarrow & & \downarrow \\ \mathbf{T} & \longrightarrow & \Gamma\mathbf{T}^{\mathsf{N}} \end{array}$$

is a pullback in $\mathcal{R}$. By 6.2.3, $\Gamma|\Phi|$ coincides with a *standard* interpretation of Φ because the set $\Gamma\mathbf{T}$ is a set of functionals. For a conclusive remark, draw the pullback diagram in GEn, as the four corners are separated objects. There, by just reading the pullback diagram, $|\Phi| \rightarrowtail \mathbf{T}$ is the set of *recursively parametrized* families of elements of $\Gamma|\Phi|$.

So it is so essential to know what $\Gamma\mathbf{T}$ is. Here we look at total types. We know it is internally valid that $F \in \mathbf{N}^{\mathbf{N}^{\mathbf{N}}}$ is a Banach-Mazur functional. Notice that the internal statement

$$f \in \mathbf{N}^{\mathbf{N}} \text{ is recursive}$$

is preserved, as it is equivalent to

$$\exists e \in \mathbf{N}\ \forall x, y \in \mathbf{N}\ [T(e, x, y) \leftrightarrow U_y].$$

Also, as $\mathbf{N}$ is separated, Γ transforms the map

$$g \mapsto [\langle x, y\rangle \mapsto g(x)(y)] \colon (\mathbf{N}^{\mathbf{N}})^{\mathbf{N}} \longrightarrow \mathbf{N}^{\mathbf{N}\times\mathbf{N}}$$

into the same interpretation from $\Gamma((\mathbf{N}^{\mathbf{N}})^{\mathbf{N}})$ to $\Gamma(\mathbf{N}^{\mathbf{N}\times\mathbf{N}})$. Hence Γ preserves the statement that all $F \colon \mathbf{N}^{\mathbf{N}} \longrightarrow \mathbf{N}$ are Banach-Mazur. Therefore the set

$$\mathcal{R}(\mathbf{N}^{\mathbf{N}}, \mathbf{N}) = \Gamma\left(\mathbf{N}^{\mathbf{N}^{\mathbf{N}}}\right) \subset \mathsf{N}^{\mathsf{N}^{\mathsf{N}}}$$

consists of Banach-Mazur functionals.

In fact, $R(\mathbf{N}^{\mathbf{N}}, \mathbf{N})$ *is* the set of Banach-Mazur functionals. For $\mathbf{N}$ and $\mathbf{N}^{\mathbf{N}}$ are separated, so they are objects in GEn. More precisely in GEn they look like

$$\mathbf{N}(\mathsf{N}) = \{f \colon \mathsf{N} \longrightarrow \mathsf{N} \mid f \text{ recursive}\} \subset \mathsf{N}^{\mathsf{N}},$$

and, by exponential adjunction,

$$\mathbf{N}^{\mathbf{N}}(\mathsf{N}) = \{g \colon \mathsf{N} \longrightarrow \mathsf{N}^{\mathsf{N}} \mid g(x)(y) \text{ recursive}\} \subset \mathsf{N}^{\mathsf{N}^{\mathsf{N}}}.$$

So, given a map $F \colon \mathbf{N}^{\mathbf{N}} \longrightarrow \mathbf{N}$ in $\mathcal{R}$, the diagram

$$\begin{array}{ccc} \mathbf{N}^{\mathbf{N}}(\mathsf{N}) & \xrightarrow{F} & \mathbf{N}(\mathsf{N}) \\ \downarrow & & \downarrow \\ \mathsf{N}^{\mathsf{N}^{\mathsf{N}}} & \xrightarrow{\Gamma^{!}\Gamma F} & \mathsf{N}^{\mathsf{N}} \end{array}$$

commutes, stating that F is uniquely determined by the function $\Gamma F\colon \mathsf{N}^{\mathsf{N}} \longrightarrow \mathsf{N}$ such that

$$(\Gamma F)(g) \text{ is recursive} \qquad \text{for every } g \in \mathbf{N}^{\mathrm{N}}(\mathsf{N}),$$

which says exactly that ΓF is a Banach-Mazur functional.

Hence the R-set $\mathbf{N}^{\mathrm{N}^{\mathrm{N}}}$ consists of families $F(n)$ of Banach-Mazur functionals such that, for all $g\colon \mathsf{N} \longrightarrow \mathsf{N}$ recursive, the function $F(n)(g)$ is recursive in n. The characterization is similar for the other types. The interpretation of the partial types will come later in section 6.5.

It is worth noting that in the model $\mathcal{R}$ functionals from $\mathbf{N}^{\mathrm{N}}$ to $\mathbf{N}$ are Banach-Mazur, but *can not* be effective operations. The main reason is that a parametrized version Kreisel-Lacombe-Shoenfield theorem can not hold: on the other hand, if they were, so would the global sections $\Gamma(\mathbf{N}^{\mathrm{N}^{\mathrm{N}}})$ by 6.2.3. But we know that they are all Banach-Mazur functionals, and Friedberg's counterexample in [Rog67] yields a contradiction.

Another application of 6.2.3 is to prove a representation of the category En in $\mathcal{R}$.

6.2.5 **Proposition**

(i) *The category* $\mathsf{En}*$ *of non-empty enumerated sets is equivalent to the full subcategory of* $\mathcal{R}$ *of separated images of* $\mathbf{N}$.

(ii) *The category* En *of enumerated sets is equivalent to the full subcategory of* $\mathcal{R}$ *generated by the separated objects* A *such that* $A+1$ *is an image of* $\mathbf{N}$.

Proof (i) Consider the functor $'\colon \mathsf{GEn} \longrightarrow \mathcal{R}$ restricted to $\mathsf{En}*$. After 6.2.2 and 6.1.2, the only thing to show is that, a given separated object D, which an image of $\mathbf{N}$, is (isomorphic to one of the form) $\{e\}'$ for some enumeration $e\colon \mathsf{N} \longrightarrow\!\!\!\!\triangleright\ A$. By the $\mathbf{N}$-axiom of choice if a map $d\colon \mathbf{N} \longrightarrow D$ in $\mathcal{R}$ is epi, then

$$g \in D^{\mathrm{N}} \leftrightarrow \exists f \in \mathbf{N}^{\mathrm{N}}\ [g = df \wedge f \text{ recursive}].$$

So by 6.2.3, $g \in \mathcal{R}(\mathbf{N}, D) = \Gamma(D^{\mathrm{N}}) \subset \Gamma D^{\mathsf{N}}$ if and only if there is a recursive function $f\colon \mathsf{N} \longrightarrow \mathsf{N}$ such that $g = (\Gamma d)f$. Hence $\Gamma d\colon \mathsf{N} \longrightarrow \Gamma D$ is surjective and $D \simeq \{\Gamma d\}'$. (ii) Follows from previous part, once we notice that the inverse image along $e\colon \mathbf{N} \longrightarrow\!\!\!\!\triangleright\ A+1$ of $A \rightarrowtail A+1$ is a complemented subobject of $\mathbf{N}$, hence recursive by 5.2.2. □

A way to look at the picture is to think of $\mathcal{R}$ as a large, comfortable ground where one can do any possible construction involving natural numbers, computations, and all the set-theoretic constructs. Then apply 6.2.3 and the other results in this section to bring that construct back to the *real* universe. To apply 6.2.3 (iv), a secure way to have closed subobjects is to take Σ-subsets, but also $\Pi = \neg\Sigma$ works. Therefore it seems very plausible that $\mathcal{R}$ be very useful for application in computable analysis, as in [Mul81] or [Sce85].

As an example of the method we suggested, we derive a well-known result about effectively presented algebraic cpo's. We refer to [Sco80a] for the definitions.

6.2.6 **Corollary** *Let* S *and* T *be effective* ω*-algebraic cpo's, and* C *and* D *the sets of compact elements of* S *and* T *respectively. Suppose* C *and* D *can be enumerated by* N *in such a way that the inclusions* $C \subset S$ *and* $D \subset T$ *are maps in* En*, and the subset* $\{\langle c, s\rangle : c \sqsubseteq s\} \subset C \times S$ *has an r.e. set of indices; similarly for* D *and* T*. Then all maps from* S *to* T *in* En *are continuous with respect to the Scott topologies on* S *and* T.

Proof Apply 6.2.3 to 5.4.2. □

6.3 The effective topos

A structure of NNO* may come on the standard natural number object N of a topos $\mathcal{E}$. For this to happen, the countable axiom of choice must hold in $\mathcal{E}$, and N has to satisfy Church's thesis and Markov's principle. This is the situation in the effective topos $\mathcal{F}$.

It is not difficult to introduce the effective topos, since this has already been done very clearly in [Hyl82]. We warmly refer the reader to that paper and [HJP80], for detailed expositions. For sake of completeness, we repeat the definition of the topos $\mathcal{F}$.

Set $\langle n,m\rangle = \frac{1}{2}n(n+1)+m$ forthe pairing function and n_0 and n_1 for the unique pair such that $n = \langle n_0, n_1\rangle$. First introduce the logic operations on the *truth-values*. The truth-values are the subsets of N: a set $P \subset \mathsf{N}$ ought to be thought of as the set of "proofs" of P. And in accordance to this intuition, the logical operations are defined by following the intuitionistic idea of a proof of a compound formula.

The logic of proofs. Given $P, Q \subset \mathsf{N}$, set

$n \in P \wedge Q$ if and only if $n_0 \in P$ and $n_1 \in Q$;

$n \in P \vee Q$ if and only if $n_0 = 0$ and $n_1 \in P$, or $n_0 = 1$ and $n_1 \in Q$;

$n \in P \to Q$ if and only if for all $k \in P$, the partial recursive function $\{n\}$ converges at k and $\{n\}(k) \in Q$;

$n \in \bot$ never;

$n \in \text{true}$ always;

$n \in \forall x\ P_x$ if and only if for all x, $n \in P_x$;

$n \in \exists x\ P_x$ if and only if there is x, $n \in P_x$,

where $(P_x : x \in X)$ is a family of subsets of N. Say $P \subset \mathsf{N}$ is *realized* if $P \neq \emptyset$—intuitively, $n \in P$ means "n is a proof of P".

There must be no surprise to find resemblance with realizability interpretation of [Kle73]: that is what it is meant to be.

The effective topos. The objects of the topos $\mathcal{F}$ are pairs $(X, =_X)$ of a set X and a function $=_X : X \times X \longrightarrow \mathrm{P}(\mathsf{N})$ such that

simmetry $\forall x, y\ [x =_X y \to y =_X x]$,

transitivity $\forall x, y, z\ [x =_X y \wedge y =_X z \to x =_X z]$

are realized. We use bounded quantifiers $\forall x \in X.\phi$ to mean $\forall x\ [x =_X x \to \phi]$, and similarly for the existential quantifier, and drop the subscripts of the equality signs. A map $f : (X, =) \longrightarrow (Y, =)$ is going to be the equivalence class of all the functional relations representing it. A functional relation from $(X, =)$ to $(Y, =)$ is a map $F : X \times Y \longrightarrow \mathrm{P}(\mathsf{N})$ such that it is realised that F is

defined $\forall x, x' \in X\ \forall y, y' \in Y\ [G(x,y) \wedge x = x' \wedge y = y' \to C(x', y')]$

single-valued $\forall x \in X\ \forall y, y' \in Y\ [G(x,y) \wedge G(x,y') \to y = y']$

total $\forall x \in X\ \exists y \in Y.\ G(x,y)$

and a functional relation $G : X \times Y \longrightarrow \mathrm{P}(\mathsf{N})$ is equivalent to F if

$$\forall x \in X \forall y \in Y\ [F(x,y) \leftrightarrow G(x,y)]$$

is realized. Composition of maps $f : (X, =) \longrightarrow (Y, =)$ and $g : (Y, =) \longrightarrow (Z, =)$ is defined on representatives as

$$\langle x, z\rangle \mapsto \exists y \in Y\ [F(x,y) \wedge G(y,z)] : X \times Z \longrightarrow \mathrm{P}(\mathsf{N}).$$

The proof that this definition actually gives a topos is in [HJP80], but see [Hyl82] for a clear sketch of it. Though, one can guess how to use the logic to build products, equalizers and function spaces. The object classifier Ω is the pair $(\mathrm{P}(\mathsf{N}), \leftrightarrow)$ and the internal logic of $\mathcal{F}$ is interpreted with the logical operations on $\mathrm{P}(\mathsf{N})$ given above, so that a formula is valid in $\mathcal{F}$ if and only if it is realized, see [Hyl82]. Also $\mathcal{F}$ has a natural number object $(\mathsf{N}, =)$ where

$$r \in |n = m| \text{ if and only if } r = n = m.$$

We quote a remark of [Hyl82]: a sentence of HA is recursively realised if and only if it is valid on the natural number object in $\mathcal{F}$. Also the countable axiom of choice holds in $\mathcal{F}$. So that N is a NNO*.

It is known that the effective topos is not Grothendieck because $\mathcal{F}$ does not have infinite colimits. Anyway, the global section functor $\Gamma\colon \mathcal{F} \longrightarrow \mathcal{S}$ can be defined as usual:

$$\Gamma = \mathcal{F}(1, -),$$

so that $\Gamma(X, =) = \{x \in x : |x = x| \neq \emptyset\}/\sim$, where $x \sim x'$ if $|x = x'| \neq \emptyset$. And a constant-set functor $\Delta\colon \mathcal{S} \longrightarrow \mathcal{F}$ is defined by letting

$$\Delta S = (S, =_{\Delta S})$$

where $|s =_{\Delta S} t| = \{n \in \mathsf{N} : s = t\}$, and setting $\Delta f\colon S \times T \longrightarrow \mathrm{P}(\mathsf{N})$ to be

$$\Delta f(s, t) = \{n \in \mathsf{N} : fs = t\}.$$

They are adjoint, but

$$\Gamma \dashv \Delta\colon \mathcal{S} \longrightarrow \mathcal{F},$$

and, by the clauses for disjunction and instantiation, Γ preserves finite colimits. So $\Gamma \dashv \Delta$ is a geometric morphism. Moreover, [Hyl82] proved that it is an inclusion and the topology induced on $\mathcal{F}$ is the double-negation topology. Then the geometric morphism looks more like $\Gamma \dashv \Gamma^!\colon \mathcal{S} \longrightarrow \mathcal{R}$ than $\Delta \dashv \Gamma\colon \mathcal{R} \longrightarrow \mathcal{S}$! As in 6.2, having a complete description of $\neg\neg$-sheaves helps in the study of closed properties of separated objects.

Effective objects. Thinking of $\mathcal{S}$ as a submodel of $\mathcal{F}$, one would like to stay as far away from $\mathcal{S}$ as possible, because there cannot be much effectivity there. As N is separated and closed subobjects are to be preferred, the idea is to consider *effective* objects of $\mathcal{F}$: quotients of closed subobjects of N by closed equivalence relations. As closed subobjects of N are in bijective correspondence with subsets of the *set* N, it is easy to show that an effective object A has a canonical representation of the form $(\mathsf{N}, \sim_A)$, where $\sim_A\colon \mathsf{N} \times \mathsf{N} \longrightarrow \mathrm{P}(\mathsf{N})$ satisfies the property

$$|n \sim_A m| = |n \sim_A n| \cap |m \sim_A m|.$$

We recall a few facts from [Hyl82]: for A, B effective objects, a map $f\colon A \longrightarrow B$ in $\mathcal{F}$ can be represented by the functional relation

$$F(n, m) = \bigcup \{|n \sim_A n| \wedge |m \sim_B m| : |\{e\}(n) \sim_B m| \neq \emptyset\}$$

for some appropriate code $e \in \mathsf{N}$. Therefore the functor $\Gamma\colon \mathcal{F} \longrightarrow \mathcal{S}$ induces an equivalence between the full subcategory of the effective objects and the category PER of partial equivalence relations on N: objects of PER are quotients X/R of subsets X of N maps are functions $f\colon X/R \longrightarrow Y/S$ such that, for some $e \in \mathsf{N}$,

$$f[n]_R = [\{e\}(n)]_S.$$

It is cartesian closed, the function space $(Y/S)^{(X/R)}$ being defined by the relation

$$\forall n, m \in \mathsf{N}\ [nRm \to \{e\}(n) S \{d\}(m)],$$

and it corresponds to the function space in $\mathcal{F}$ of the effective objects $(\mathsf{N}, \sim_R)$ and $(\mathsf{N}, \sim_S)$, where $|n \sim_R m| = [n]_R \cap [m]_R$.

One immediately sees that the category En is a full subcategory of PER. And PER is a subcategory of GEn, but not a full one. Indeed from the definition, $\mathsf{PER}(\mathsf{N}^\mathsf{N}, \mathsf{N})$ is the set of effective operations, while $\mathsf{GEn}(\mathsf{N}^\mathsf{N}, \mathsf{N})$ is the set of Banach-Mazur functionals. We can detect similarities among these categories by using the effective topos $\mathcal{F}$.

6.4 A functor between the two topoi

The major difference between the effective topos and the recursive topos $\mathcal{R}$ is that the object $\mathbf{N} = \mathsf{N}$ in $\mathcal{F}$ does *not* generate the topos. It is important to "measure" this difference. So consider the set

$$\mathrm{K}\, X = \mathcal{F}(\mathsf{N}, X)$$

for X an object in $\mathcal{F}$. The note-worthy fact is that $\mathrm{K}\,X$ has a canonical structure of R-set: given a recursive function $f\colon \mathsf{N} \longrightarrow \mathsf{N}$ in $\mathcal{S}$, then $f\colon \mathsf{N} \longrightarrow \mathsf{N}$ is a map in $\mathcal{F}$, and their composition is $e \circ f \in \mathcal{F}(\mathsf{N}, X)$. Moreover it is a sheaf for the canonical topology on R: consider the coproduct diagram $(i\colon \mathsf{N} \longrightarrow \mathsf{N} + \mathsf{N}, j\colon \mathsf{N} \longrightarrow \mathsf{N} + \mathsf{N})$ in R; it represents a coproduct also in $\mathcal{F}$. Therefore for each object X in $\mathcal{F}$, the presheaf $\mathrm{K}\,X$ is a sheaf for the canonical topology by 6.1.4. The assignment

$$X \mapsto \mathrm{K}\,X$$

can obviously be lifted to a functor $\mathrm{K}\colon \mathcal{F} \longrightarrow \mathcal{R}$ by letting, for $g\colon X \longrightarrow Y$ in $\mathcal{F}$,

$$\mathrm{K}\,g = g \circ -\colon \mathrm{K}\,X \longrightarrow \mathrm{K}\,Y.$$

This functor was pointed out to us at different times by Martin Hyland and Bill Lawvere.

6.4.1 **Theorem**

(i) K *preserves limits.*

(ii) K *preserves finite coproducts.*

(iii) K *preserves images.*

(iv) K *takes* N *to* $\mathbf{N}$.

(v) *For* $\Gamma \dashv \Gamma'\colon \mathcal{S} \longrightarrow \mathcal{R}$ *and* $\Gamma \dashv \Delta\colon \mathcal{S} \longrightarrow \mathcal{F}$, *it is* $\Gamma\,\mathrm{K} \simeq \Gamma$ *and* $\mathrm{K}\,\Delta \simeq \Gamma'$. *Therefore* K *preserves separated objects.*

Proof (i) Trivial. (ii) Obviuosly, K preserves the initial object. To prove it preserves all finite coproducts, we must show K preserves binary coproducts. We first prove that if $\mathsf{N} = A \cup B$ and $A \cap B = \emptyset$ in $\mathcal{F}$, then A and B are closed recursive subobjects of N. Indeed, by the clauses for implication and disjunction, there is a code number e such that $\{e\}$ is total and

$$\{e\}(n) = 0 \text{ if and only if } n \in A, \text{ and } \{e\}(n) = 1 \text{ if and only if } n \in B.$$

To fulfill the request of (ii) the map

$$\mathrm{K}\,i + \mathrm{K}\,j\colon \mathrm{K}\,X + \mathrm{K}\,Y \longrightarrow \mathrm{K}(X + Y)$$

must be an isomorphism. It is monic, as $\mathrm{K}\,i$ and $\mathrm{K}\,j$ are monics, and their pullback is 0 by the first part of the proof. Suppose then $f\colon \mathsf{N} \longrightarrow X + Y$; then $\mathsf{N} = f^{-1}X \cup f^{-1}Y$ and $f^{-1}X \cap f^{-1}Y = \emptyset$ in $\mathcal{F}$. So $f^{-1}X$ and $f^{-1}Y$ are closed recursive, and $f|_{f^{-1}X} \in \mathrm{K}\,X(f^{-1}X)$ and $f|_{f^{-1}Y} \in \mathrm{K}\,Y(f^{-1}Y)$. Hence

$$f|_{f^{-1}X} \in (\mathrm{K}\,X + \mathrm{K}\,Y)(f^{-1}X) \quad \text{and} \quad f|_{f^{-1}Y} \in (\mathrm{K}\,X + \mathrm{K}\,Y)(f^{-1}Y).$$

By the sheaf condition, there is a unique element $g \in (\mathrm{K}\,X + \mathrm{K}\,Y)(\mathsf{N})$ such that $g|_{f^{-1}X}$ and $g|_{f^{-1}Y}$; so

$$\mathrm{K}\,i + \mathrm{K}\,j(g)|_{f^{-1}X} = \mathrm{K}\,i + \mathrm{K}\,j(g|_{f^{-1}X}) = \mathrm{K}\,i(f|_{f^{-1}X}) = f|_{f^{-1}X},$$

and similarly for Y. Again by the sheaf condition,

$$\mathrm{K}\,i + \mathrm{K}\,j(g) = f.$$

(iii) Let $e\colon X \longrightarrow\!\!\!\!\triangleright\, Y$ and $f\colon \mathsf{N} \longrightarrow Y$ in $\mathcal{F}$. By countable choice, there is a map $g\colon \mathsf{N} \longrightarrow X$ such that

$$f = eg = \mathrm{K}\,e(g).$$

Thus $\mathrm{K}\,e$ is surjective. (iv) Trivial. (v) As the successor map $s\colon \mathsf{N} \longrightarrow \mathsf{N}$ is recursive, $g \in \mathcal{R}(1, \mathrm{K}\,X)$ satisfies $g = gs$ in $\mathcal{F}$. So $g \in \mathcal{F}(1, X) = \Gamma X$, proving the first equivalence. By the adjunction $\Gamma \dashv \Delta\colon \mathcal{S} \longrightarrow \mathcal{F}$, it is

$$\mathcal{F}(\mathsf{N}, \Delta S) \simeq \mathcal{S}(\Gamma\mathsf{N}, S) = S^{\mathsf{N}},$$

preserving the action. Finally, an object X in $\mathcal{F}$ is separated, if $X \rightarrowtail \Delta\Gamma X$. Applying K, it is $KX \rightarrowtail \mathrm{K}\,\Delta\Gamma X \simeq \Gamma'\Gamma(\mathrm{K}\,X)$. □

As we mentioned before, the object N is not a generator in $\mathcal{F}$. The next result expresses how close it can get to being so.

6.4.2 **Proposition** *The functor* $\mathrm{K}\colon \mathcal{F} \longrightarrow \mathcal{R}$ *is faithful on the full subcategory of countable objects of* $\mathcal{F}$*, and full and faithful on that of countable separated objects.*

Proof The first observation is trivial. The second follows from the fact that the full subcategory of countable separated objects is $\mathsf{En}*$.

A last remark about the function spaces. In [Hyl82] it was proved that the total functional on N (the objects N^{N}, $\mathsf{N}^{\mathsf{N}^{\mathsf{N}}}$, $\mathsf{N}^{\mathsf{N}^{\mathsf{N}^{\mathsf{N}}}}$, and so on) are hereditarily effective operations in $\mathcal{F}$, and the global section functor Γ preserves this property, as they are separated objects. So K cannot preserves function spaces, as $\mathbf{N}^{\mathbf{N}^{\mathbf{N}}}$ is the object of Banach-mazur functionals.

Further preservation properties hold regarding separated objects, as the topos of $\neg\neg$-sheaves is left fixed by K.

6.4.3 **Corollary**

(i) K *preserves interpretation of full first order logic on closed subobjects.*

(ii) K *preserves the evaluation map on separated objects, in the sense that, given* E, F *separated it is* $\mathrm{K}(F^E) \rightarrowtail \Gamma^!\Gamma F^{\Gamma^!\Gamma E}$*, and* $\mathrm{K}(\mathrm{ev})\colon \mathrm{K}\,E \times \mathrm{K}(F^E) \longrightarrow \mathrm{K}\,F$ *is the restriction of the evaluation map.*

Of course, the problem remains to compare $\mathrm{K}(F^E)$ and $\mathrm{K}\,F^{\mathrm{K}\,E}$. But at least we know they are both objects of functionals. The best result can be achieved when E and F are countable. In that case, by 6.4.1, the usual, simple argument works to prove that $\mathrm{K}(F^E) \simeq \mathrm{K}\,F^{\mathrm{K}\,E}$:

$$\begin{aligned}\mathcal{R}(\mathbf{N}, \mathrm{K}\,F^{\mathrm{K}\,E}) &\simeq \mathcal{R}(\mathbf{N} \times \mathrm{K}\,E, \mathrm{K}\,F)\\ &\simeq \mathcal{R}(\mathrm{K}\,\mathsf{N} \times \mathrm{K}\,E, \mathrm{K}\,F)\\ &\simeq \mathcal{R}(\mathrm{K}(\mathsf{N} \times E), \mathrm{K}\,F)\\ &\simeq \mathcal{F}(\mathsf{N} \times E, F)\\ &\simeq \mathcal{F}(\mathsf{N}, F^E)\\ &\simeq \mathcal{R}(\mathbf{N}, \mathrm{K}(F^E)).\end{aligned}$$

Total functionals on N are ruled out because N^{N} is not countable. But for partial functions something more can be done.

6.5 Variations on a partial theme

It seems as if the failure of preservation of exponentials depends upon the fact that there are recursively paramatrized families of total functions which are not effective. What we show in this section is that recursive parametrizations of partial functions are always effective. Of course, by partial functions we mean those which are Σ-defined in $\mathcal{F}$, as defined in 5.2. It is trivial to prove the following

6.5.1 **Proposition** *The functor* K *extends to a functor* $\mathsf{P}\,\mathrm{K}\colon \mathsf{P}(\mathcal{F}, \mathcal{T}) \longrightarrow \mathsf{P}(\mathcal{R}, \mathcal{T})$*, whose restriction to the full subcategory of countable separated objects is full and faithful. Therefore* K *preserves p-exponentials of countable separated objects. In particular,* K *preserves* Σ*-extensions of countable separated objects.*

Among the countable separated object, there are N, $\widehat{\mathsf{N}}$, and $\widehat{\mathsf{N}}^{\mathsf{N}}$. Actually, it makes no difference if we substitute $\mathbf{N}$ for N, and speak of separated images of $\mathbf{N}$ in $\mathcal{R}$. What about, then, the sequence of partial types? We shall find the answer to this question by using the theory of a NNO*.

6.5.2 **Lemma** *Suppose* $\mathbf{N}$ *is a* NNO* *in a topos* $\mathcal{E}$*.*

(i) *If* F *denotes the object* $\{\phi \in \widehat{\mathbf{N}}^{\mathrm{N}} : \mathrm{dom}\,\phi \subset_{fin} \mathbf{N} \times \mathbf{N}\}$*, then* $F \rightarrowtail \mathrm{Fin}^*(\mathbf{N} \times \mathbf{N})$ *has a retraction.*

(ii) *The object* $\widehat{\mathbf{N}}^{\widehat{\mathbf{N}}^{\mathrm{N}}}$ *is a retract of* $\widehat{\mathbf{N}}^{\mathrm{N}}$*.*

(iii) *Suppose* $d\colon A \rightarrowtail B$ *has a retraction* $e\colon B \twoheadrightarrow A$*. Then the maps* $(d \circ - \circ e)\colon A^A \longrightarrow B^B$ *and* $(- \circ e)\colon \widehat{\mathbf{N}}^A \longrightarrow \widehat{\mathbf{N}}^B$ *have retractions which are respectively* $(e \circ - \circ d)\colon B^B \longrightarrow A^A$ *and* $(- \circ d)\colon \widehat{\mathbf{N}}^B \longrightarrow \widehat{\mathbf{N}}^A$*.*

Proof (i) Notice first that, given $d \subset_{\mathrm{fin}} \mathbf{N}$, the formula

$$z \in d \wedge \forall x \in d.\ z \leq x$$

is Σ by 5.3.1. So the partial map

$$\overline{\mu}\colon d \mapsto \{z \in \mathbf{N} : \forall x \in d.\ z \leq x\}\colon \mathrm{Fin}^*\mathbf{N} \longrightarrow \widehat{\mathbf{N}}$$

is defined if and only if $d \neq \emptyset$. Suppose now $e \subset_{\mathrm{fin}} \mathbf{N} \times \mathbf{N}$, and consider

$$e_x = \{y \in \mathbf{N} : \langle x, y\rangle \in e\}.$$

Then

$$\mathcal{E} \Vdash \forall x \in \mathbf{N}.\ e_x \subset_{\mathrm{fin}} \mathbf{N}.$$

Set $e' = \{\langle x, y\rangle \in \mathbf{N} \times \mathbf{N} : e_x \neq \emptyset \wedge y = \mu(e_x)\}$. Then

$$\mathcal{E} \Vdash \forall e \subset_{\mathrm{fin}} \mathbf{N} \times \mathbf{N}\ [e' \subset e \wedge e' \in F].$$

Moreover, if $e \in F$, then $e' = e$. (ii) The objects $\widehat{\mathbf{N}}^{\mathbf{N}}$ and $\widehat{\mathbf{N}}$ are $\mathbf{N}$-algebraic posets with their natural partial orders, and F is the set of compact elements of $\widehat{\mathbf{N}}^{\mathbf{N}}$. Therefore by 5.4.2

$$\widehat{\mathbf{N}}^{\widehat{\mathbf{N}}^{\mathbf{N}}} \simeq \mathrm{Mono}(F, \widehat{\mathbf{N}}),$$

where $\mathrm{Mono}(F, \widehat{\mathbf{N}})$ is the object of monotone maps from F to $\widehat{\mathbf{N}}$, which is a retract of $\widehat{\mathbf{N}}^F$. So $\widehat{\mathbf{N}}^{\widehat{\mathbf{N}}^{\mathbf{N}}}$ is a retract of $\widehat{\mathbf{N}}^{\mathbf{N}}$, as the object F is a retract of $\mathrm{Fin}^*(\mathbf{N} \times \mathbf{N}) \simeq \mathbf{N} \times \mathbf{N} \simeq \mathbf{N}$. (iii) See [Sco81b]. □

Partial functionals in the topos. Properties (ii) and (iii) in the lemma above can be explained as saying that, in a treatment of partial functionals, all that continuity has to be claimed for is at step 2. After that step, it is simple set theory. It follows from the lemma that all the objects of partial functionals on $\mathbf{N}$ are retracts of $\widehat{\mathbf{N}}^{\mathbf{N}}$. In particular, by the results in [Sco81b], $\widehat{\mathbf{N}}^{\mathbf{N}}$ is a model of untyped λ-calculus. Also, as $\mathbf{N}$ enumerates it, the structure on $\widehat{\mathbf{N}}^{\mathbf{N}}$ induces a structure of partial combinatory algebra on $\mathbf{N}$.

Partial functionals in the real world. If we look at the picture in the two models, we realize there is no difference between "effective" and "recursive" partial functionals, as everything happens inside En. And we want to conclude this discussion of the functionals by calculating their global sections. To do this we prefer to use the Grothendieck topos $\mathcal{R}$. But the argument is essentially the same if carried out in $\mathcal{F}$. As we are going to consider global sections of objects of the form E^E for E an enumerated set the following two principles will be useful.

6.5.3 **Lemma** *Let E, F and G be separated objects.*

(i) *A function $f\colon \Gamma E \longrightarrow \Gamma G^{\Gamma F}$ takes values into $\Gamma(G^F) \subset \Gamma G^{\Gamma F}$ if and only if its adjoint $f'\colon \Gamma E \times \Gamma F \longrightarrow \Gamma G$ is of the form $\Gamma g'$ for some $g'\colon E \times F \longrightarrow G$ in $\mathcal{R}$.*

(ii) *Suppose $e\colon E \longrightarrow\!\!\!\!\triangleright\ F$ in $\mathcal{R}$. A function $f\colon \Gamma F \longrightarrow \Gamma G$ is of the form Γg for some $g\colon F \longrightarrow G$ in $\mathcal{R}$ if and only if $f \circ \Gamma e\colon \Gamma E \longrightarrow \Gamma G$ is so.*

Proof (i) Trivial from 6.2.3. (ii) Necessity of the condition is obvious. As for sufficiency, suppose $f \circ \Gamma e = \Gamma g$ and think of F and G as generalized enumerated sets: we must show that

$$f \circ F(\mathsf{N}) \subset G(\mathsf{N}).$$

So recall that $F(\mathsf{N}) \simeq \mathcal{R}(\mathbf{N}, F)$, and take $h\colon \mathbf{N} \longrightarrow F$. By $\mathbf{N}$-choice there is $k\colon \mathbf{N} \longrightarrow E$ such that $h = ek$. So $f = \Gamma h = f\Gamma e\Gamma k = \Gamma g \Gamma k = \Gamma(gk) \in G(\mathsf{N})$. □

We are now ready to look at global sections of functionals. First we set up the notation for our preferred core chain of partial function spaces: let $\mathbf{A}_1 = \widehat{\mathbf{N}}^{\mathbf{N}}$, and $\mathbf{A}_{n+1} = \mathbf{A}_n^{\mathbf{A}_n}$. Notice that they are p-exponentials as

$$\mathbf{A}_n^{\mathbf{A}_n} \simeq \widehat{\mathbf{N}}^{\mathbf{A}_{n-1} \times \mathbf{A}_n} \simeq {}^{(\mathbf{A}_{n-1} \times \mathbf{A}_n)}\mathbf{N}.$$

Also, by induction, $\mathbf{A}_n \times \mathbf{A}_n \simeq \mathbf{A}_n$. We shall show that $\Gamma\mathbf{A}_n$ are the *partial effective functionals* $C^{(n)}$ introduced by [LM84a], where $C^{(0)} = \mathsf{N}$, $C^{(1)}$ is the set of partial recursive functions, and by induction for $n \geq 2$, given $\tau : C^{(n)} \longrightarrow C^{(n)}$,

$$\tau \in C^{(n+1)} \text{ if and only if } \forall \phi \in C^{(n.5)}.\ \tau\phi \in C^{(n)},$$

where $\phi : C^{(n-1)} \longrightarrow C^{(n)}$ is in $C^{(n.5)}$ if the map

$$\langle x, y \rangle \mapsto \phi(x)(y) : C^{(n-1)} \times C^{(n-1)} \longrightarrow C^{(n-1)}$$

is in $C^{(n)}$. If the reader is worried about the last requirement, he can check in *loc.cit.* , but we will show in the course of the proof of next theorem that $C^{(n)} \times C^{(n)} \simeq C^{(n)}$.

6.5.4 **Theorem** *For all $n \geq 1$, $\Gamma\mathbf{A}_n \simeq C^{(n)}$.*

Proof For $n = 1$, it is trivial. Suppose it holds for all $m \leq n$. So $C^{(n-1)} \times C^{(n-1)} \simeq C^{(n-1)}$, and by 6.5.3(i),

$$C^{(n.5)} \simeq \mathcal{R}(\mathbf{A}_{n-1}, \mathbf{A}_n) \simeq \Gamma(\mathbf{A}_n^{\mathrm{A}_{n-1}}).$$

By 6.5.2, there is an epimorphism $e : \mathbf{A}_n \longrightarrow\!\!\!\!\triangleright\ \mathbf{A}_{n+1}$ in $\mathcal{R}$. By 6.5.3,

$$C^{(n+1)} \simeq \mathcal{R}(\mathbf{A}_n, \mathbf{A}_n) \simeq \Gamma\mathbf{A}_{n+1}. \ \square$$

We had a very simple proof because we could consider the set of effective functionals as *sets* of another universe. As we commented already in chapter 3, the topos may be adapted so well that the objects carry a canonical topological structure, and yet they behave like sets. For objects considered above, the topological structure mixes with an effective notion which is inheritated via a surjection from N. Even though the topological idea is clear in the many papers on enumerated sets, we believe that the correct expression of the concept can be clarified only either in the recursive topos $\mathcal{R}$ or in the effective topos $\mathcal{F}$, where one abstracts from the very bothersome enumerations. For instance, there is a definite topological resemblance of homeomorphic spaces and quotients in the *acceptable* and *relative* enumerations, respectively, defined in *loc.cit.* . And the authors use them quite appropriately, but in the haze generated by all the enumerations, they can not recognize that these are indeed expressing the notion of isomorphic objects and quotient in $\mathsf{En}*$.

Before stating the corollary we need to introduce a cartesian closed category of partial functionals: model C^k of Eršov. It is the cartesian closed category generated by the object $\widehat{\mathbf{N}}$ in $\mathcal{R}$. Note that, if we let $S = \{\{x\} \subset \mathbf{N} : x \in \mathbf{N}\} \cup \{\emptyset\}$, then

$$\widehat{\mathbf{N}}^{\widehat{\mathbf{N}}} \simeq \mathrm{Mono}(S, \widehat{\mathbf{N}}) \simeq \widehat{\mathbf{N}}^{\mathrm{N}} + \mathbf{N}.$$

So C^k is a subcategory of En by 6.5.2.

6.5.5 **Corollary**

(i) *For all $n \geq 1$, $C^{(n.5)} \simeq C^{(n)}$.*

(ii) *For all $n \geq 1$, $\mathbf{A}_n$ is a retract of an object in C^k.*

(iii) *For all $n \geq 1$, $C^{(n+1)} = \Gamma\left(\widehat{\mathbf{N}}^{\mathrm{A}_n \times \mathrm{A}_{n-1}}\right)$.*

Proof (i) Follows from $(\mathbf{A}^{\mathrm{A}})^{\mathrm{A}} \simeq \mathbf{A}^{\mathrm{A}\times\mathrm{A}}$. (ii) - (iii) Trivial. $\square$

Chapter 7

The Scott topos and the Euclidean topos

A simple way to generate categories of partial maps is to take a topological category and open subsets as domains. If one carries over that particular case and look at the topos on the category of the domains, the induced dominance should englobe the notion of *open* subsets. We do this for very small topological categories. First we consider the general case of a topological category of continuous functions on a fixed space X. The topos generated by means of the constructions in chapter 3 is a familiar one in topos theory. Then we look at two particular cases where we can give a description of some topological spaces generated by a dominance.

7.1 A topological example

Suppose X is a topological space and let X be the p-category of continous maps from cartesian products X^n to X defined on open subsets.

On $\mathsf{Dom}(X)$ take the subcanonical topology generated by open covers, i.e. $(U_i \subset V)_{i\in I}$ such that $\bigcup_{i\in I} U_i = V$. Let $\mathcal{E}$ be the topos of sheaves on $\mathsf{Dom}(X)$ for the open cover topology. Let Σ be defined as in 3.1.9, by

$$\Sigma U = \{V \subset U : V \text{ open}\}.$$

7.1.1 **Proposition** *The object Σ is a dominance,and X is fully embedded into $\mathsf{P}(\mathcal{E},\mathcal{T})$.*

Proof Follows from 3.3.3, as a union of open sets is open. □

7.1.2 **Corollary** *Let $\mathcal{Y}U$ be a representable sheaf. An element $A \in \Sigma^{\mathcal{Y}U}(V)$ is represented by an open inclusion $A' \subset V \times U$, and the function*

$$\begin{array}{rcl} \mathrm{Open}(V\times U) & \to & \Sigma^{\mathcal{Y}U}(V) \\ B & \mapsto & \mathcal{Y}B \end{array}$$

is a natural isomorphism in V. Moreover, for $a \in \mathcal{Y}U(V)$,

$$V \Vdash a \in A \quad \textit{if and only if} \quad \forall x \in V.\ \langle x, a(x)\rangle \in A'.$$

Proof The first part follows immediately from 3.3.3. Suppose, then $a\colon V \longrightarrow U$ in $\mathsf{Dom}(X)$. Recall 1.3.2 so that $\langle \mathrm{id}\,, a\rangle \in A \simeq \mathcal{Y}A'$ if and only if $\langle \mathrm{id}\,, a\rangle$ factors through A'. Hence the conclusion. □

These two properties are useful when we can prove that some internally constructed object F is representable, because we can immediately obtain a characterization of the object Σ^F.

The double-negation topology. The general situation is very similar to that of $\mathcal{R}$: the functor $\Gamma\colon \mathcal{E} \longrightarrow \mathcal{S}$ has a right adjoint $\Gamma^!$ which is defined in the same fashion:

$$\Gamma^! S(U) = S^U \quad \text{and} \quad \Gamma^! S(f) = - \circ f.$$

The proof that $\Gamma \dashv \Gamma^!$ depends on the fact that there are enough constant maps.

7.1.3 **Proposition** *The adjoint functors $\Gamma \dashv \Gamma^! : \mathcal{S} \longrightarrow \mathcal{E}$ form a geometric morphism over $\mathcal{S}$, which is an inclusion. So the induced topology on $\mathcal{E}$ is the double-negation topology.*

Proof First we note that for U in X, and S in $\mathcal{S}$, it is $\Gamma U = U$ and $\Gamma\Gamma^! S = S$. Next we prove that, for X in $\mathcal{E}$ and S in $\mathcal{S}$, a map $f : X \longrightarrow \Gamma^! S$ satisfies

$$f(h) = \Gamma(fh) \quad \text{for all } h \in X(U),$$

where we identified the sets $X(U)$ and $\mathcal{E}(\mathcal{Y}U, X)$. As $f(h)$ and $\Gamma(fh)$ are in S^U, to see they coincide we must check their values for each $x \in U$:

$$f(h)(x) = f(hx) = (fh)x = (\Gamma fh)(x).$$

Therefore any map $f : \Gamma^! X \longrightarrow S$ is completely determined by $\Gamma f : \Gamma X \longrightarrow \Gamma\Gamma^! S = S$. Moreover, given a function $g : \Gamma X \longrightarrow S$, there is a map $g' : X \longrightarrow \Gamma^! S$ in $\mathcal{E}$ defined as

$$g'(h) = g\Gamma h,$$

such that $g = \Gamma g'$. So

$$\Gamma \dashv \Gamma^!.$$

As

$$\Gamma\Delta S \simeq \Delta S(1) = S, \quad \text{and} \quad \Gamma\Gamma^! S \simeq \Gamma^! S(1) = S,$$

the assertion follows. □

There are many similarities with the work presented by [Dub79].

7.2 The Scott topos

The first instance of 7.1 is for $X = \mathrm{P}(\omega)$, the graph-model of [Sco76]. We show that in the Grothendieck topos $\mathcal{Q}$ described below, the maps from a subposet Σ^{N} of the powerobject $\mathrm{P}(\mathsf{N})$ into itself are continuous for the Scott topology.

Let D be the category of domains for Scott-continuous maps defined on Scott-open subsets of $\mathrm{P}(\omega)$ with values in $\mathrm{P}(\omega)$. Let $\mathcal{Q}$ be the topos on D generated as in 7.1.

An equivalent way to present the site is obtained by reducing the site to the basic open subsets of $\mathrm{P}(\omega)$, with the trivial topology, by 1.5.6.

As any object in D is connected, the natural number object N is the constant sheaf with value N. By 7.1.1, the functor Σ is a sheaf in $\mathcal{Q}$, and $\Sigma \rightarrowtail \Omega$ is a dominance. Moreover, Σ is a sublattice of Ω closed under countable unions. So, for each object F in $\mathcal{Q}$, the object Σ^F is a sublattice of $\mathrm{P}(F)$ closed under countable unions.

7.2.1 **Proposition** *A subobject $A \rightarrowtail X$ is classified by Σ if and only if for each representable U and each map $f : U \longrightarrow X$, the inverse image $f^{-1}A \rightarrowtail \mathcal{Y}U$ is represented by an open inclusion in D.*

Proof Follows from 3.1.1. □

7.2.2 **Proposition** *There is a bijection between Σ-subsheaves $W \rightarrowtail \mathcal{Y}U \times \mathsf{N}$ and continuous maps $w : U \longrightarrow \mathrm{P}(\omega)$, which is natural in U. Hence, the sheaf Σ^{N} is isomorphic to $\mathcal{Y}\,\mathrm{P}(\omega)$.*

Proof By 7.2.1, a subfunctor $W \rightarrowtail \mathcal{Y}U \times \mathsf{N}$ is Σ if and only if for every map $f : \mathcal{Y}V \longrightarrow \mathcal{Y}U \times \mathsf{N}$, the monic $f^{-1}W \rightarrowtail \mathcal{Y}U$ is an open inclusion. Notice that $qf : \mathcal{Y}V \longrightarrow \mathsf{N}$ must be constant because N is a constant sheaf. So the subobject W is Σ if and only if for all $n \in \mathsf{N}$, the monic

$$\langle \mathrm{id}\,, n\rangle^{-1}(W) \rightarrowtail \mathcal{Y}U$$

is represented by an open inclusion $W_n \subset U$. And W is completely described by these. In turn, the family W_n of open subsets of U is bijectively associated with the continuous function

$$w : x \mapsto \{n \in \mathsf{N} : x \in W_n\} : U \longrightarrow \mathrm{P}(\omega). \ \square$$

7.2.3 **Lemma** *Let U be an object in D. Let $R \in \Sigma^{\mathsf{N}}U$ and $\mathcal{W} \in \Sigma^{\Sigma^{\mathsf{N}}}(U)$. Let $r{:}U \longrightarrow \mathrm{P}(\omega)$ be the continuous function associated to R by 7.1.2, and $\mathcal{W}' \subset U \times \mathrm{P}(\omega)$ the open inclusion given by 7.1.2. Then, $U \Vdash R \in \mathcal{W}$ if and only if for all $x \in U$, $\langle x, r(x)\rangle \in \mathcal{W}'$. Moreover, if $U = \uparrow (e)$ is a basic open, then $U \Vdash R \in \mathcal{W}$ exactly when $\langle e, r(e)\rangle \in \mathcal{W}'$.*

Proof Immediate from 7.1.2. The second part follows, as every $x \in U$ is above e, and r is monotone. □

7.2.4 **Theorem** *The topology on Σ^{N} generated by $\Sigma^{\Sigma^{\mathsf{N}}}$ is the Scott-topology.*

Proof We shall first prove that

$$\mathcal{Q} \Vdash \ \forall\, \mathcal{W} \in \Sigma^{\Sigma^{\mathsf{N}}}\ \forall R \in \Sigma^{\mathsf{N}}\ [R \in \mathcal{W} \rightarrow\ \exists D \in \mathcal{W}\ [D \text{ finite} \wedge D \subset R]].$$

Suppose $R \in \Sigma^{\mathsf{N}}(U)$ is a Σ-subset of N at stage U and $\mathcal{W} \in \Sigma^{\Sigma^{\mathsf{N}}}(U)$. With no loss of generality, suppose $U = \uparrow (e)$, a compact open with $e \subset_{\mathrm{fin}} \mathsf{N}$. If $U \Vdash R \in \mathcal{W}$, then $\langle e, r(e)\rangle \in \mathcal{W}'$; and there is a finite subset $d \subset \omega$ with $d \subset r(e)$ and $\langle e, d\rangle \in \mathcal{W}'$. Set $D = \mathcal{Y}U \times d$. It follows that

$$U \Vdash D \subset R \qquad \text{and} \qquad U \Vdash D \in \mathcal{W}.$$

Obviously, D is finite. The proof that

$$\mathcal{Q} \Vdash \forall \mathcal{W} \in \Sigma^{\Sigma^{\mathsf{N}}}\ \forall R, S \in \Sigma^{\mathsf{N}}\ [R \in \mathcal{W} \wedge R \subset S \rightarrow S \in V]$$

is easy. Suppose $U = \uparrow (e)$ is open, and $U \Vdash R \in \mathcal{W} \wedge R \subset S$. If $s{:}U \longrightarrow \mathrm{P}(\omega)$ is the continuous function associated to S by 7.2.2, then $r \leq s$. Also $\langle e, v(e)\rangle \in \mathcal{W}'$. So $\langle e, s(e)\rangle \in \mathcal{W}'$, as $\mathcal{W}'$ is open. □

7.2.5 **Corollary** *In $\mathcal{Q}$ it is internally valid that all maps from Σ^{N} into Σ^{N} are Scott-continuous.*

We are not able to give an internal description of Σ^{N}, possibly in terms of the order of $\mathrm{P}(\mathsf{N})$. But there is hope that a better result holds in the *Scott topos* $\mathcal{D}$ of sheaves on D for the canonical topology. Even though we do not have a presentation of the canonical topology, we can prove that the result still holds.

7.2.6 **Corollary** *In the Scott topos $\mathcal{D}$, all maps from Σ^{N} into Σ^{N} are Scott-continuous.*

Proof As $\Sigma^{\mathsf{N}} \simeq \mathcal{Y}\mathrm{P}(\omega)$ is representable, it is a sheaf in $\mathcal{D}$. So, it is sufficient to show that Σ is a sheaf for the canonical topology. By definiton,

$$\begin{aligned}\Sigma(U) &\simeq \{V \subset U : V \text{ open}\}\\ &\simeq \{\chi{:}U \longrightarrow \mathrm{P}(\omega) : \chi \text{ is in } \mathsf{D} \text{ and } \mathrm{im}\,\chi \subset \{\emptyset, \mathsf{N}\}\}\,.\end{aligned}$$

As a cover $(f_i{:}U_i \longrightarrow U)_{i\in I}$ must be stable, it has to satisfy

$$\bigcup_{i\in I} f_i(U_i) = U,$$

because all constants maps into $\mathrm{P}(\omega)$ are continuous. Therefore Σ is a sheaf. □

7.3 The Euclidean topos

Another example of topological dominance which appeared in disguise already in the literature is that of the category of real-valued continuous functions in several real variables, see [Fou80]. We revisit Fourman's result that in the Euclidean topos of the real maps on many real variables are continuous for the Euclidean topology.

Consider the construction of section 7.1 for $X = \Re$, the Euclidean space of real numbers. Let E be the category of domains and let $\mathcal{U}$ be the *Euclidean topos* associatd to E by the construction in section 7.1.

The object of natural numbers is the constant sheaf $\Delta\mathsf{N}$. As the functor $\Delta{:}\mathcal{S} \longrightarrow \mathcal{U}$ is exact, the object Q of rationals is isomorphic to the constant sheaf ΔQ, where Q is the set of rationals in $\mathcal{S}$. Hence it is a separated object, by 7.1.3.

The real numbers. Consider now the object R of Dedekind real numbers in $\mathcal{U}$. It is the subobject $\mathsf{R} \rightarrowtail \mathrm{P}(\mathsf{Q}) \times \mathrm{P}(\mathsf{Q})$ consisting of those pair $\langle L, R\rangle$ such that

(a) $\forall x, y \in \mathsf{Q}\ [x \in L \wedge y \in R \rightarrow x < y]$

(b) $\forall x \in \mathsf{Q}\ [x \in L \leftrightarrow \exists z \in \mathsf{Q}\ [x < z \wedge z \in L]]$

(c) $\forall x \in \mathsf{Q}\ [x \in R \leftrightarrow \exists w \in \mathsf{Q}\ [w < x \wedge w \in R]]$

(d) $\forall n \in \mathsf{N}\ \exists x, y \in \mathsf{Q}\ [x \in L \wedge y \in R \wedge y - x < 1/n]$.

We are going to show that the object R is representable in the topos $\mathcal{U}$. First we need a lemma.

7.3.1 **Lemma** *The following hold in any Grothendieck topos $\mathcal{E}$:*

(i) $\forall x, y \in Q\ [x < y \vee x = y \vee x > y]$;

(ii) $\forall x \in Q\ [x \in L \leftrightarrow \exists y \in Q\ [x < y \wedge y \notin R]]$;

(iii) $\langle L, R\rangle = \langle L', R'\rangle \leftrightarrow R \cap L' = \emptyset = L \cap R'$.

Proof (i) Trivial, because the functor Δ is exact. (ii) Suppose $x \in L$. Then by definition

$$\exists y \in \mathsf{Q}\ [x < y \wedge y \in L].$$

So $y \notin R$, hence

$$x \in L \rightarrow \exists y \in \mathsf{Q}\ [x < y \wedge y \notin R].$$

Conversely, suppose $x < y \wedge y \notin R$. Find a natural number n, and $u \in L$ and $v \in R$, so that

$$y - x > \frac{1}{n} \wedge v - u < \frac{1}{n}.$$

If $y \geq v$, then $y \in R$. So, by (i), $y < v$, and

$$x < y - \frac{1}{n} < v - \frac{1}{n} < u.$$

As $u \in L$, it follows that $x \in L$. (iii) One implication is obvious, as $L \cap R = \emptyset$. Conversely,

$$\begin{aligned} x \in L &\leftrightarrow \exists y \in \mathsf{Q}\ [y > x \wedge y \in L] \\ &\leftrightarrow \exists y \in \mathsf{Q}\ [y > x \wedge y \notin R'] \\ &\leftrightarrow x \in L'. \end{aligned}$$

The others implications are proved similarly. □

So the object R is separated, and $\Gamma\mathsf{R} = \mathsf{R}(1) = \Re$, as the real numbers in $\mathcal{S}$ are the Dedekind cuts of the rationals. Therefore

$$\mathsf{R}(U) \subset \Gamma'(\Re)^U = \Re^U.$$

It is a folklore result that the elements of $\mathsf{R}(U)$ are the continuous functions from U to $\Re$ in E. We give a proof of this in the following

7.3.2 **Proposition** *There is a natural isomorphism $\mathsf{R} \simeq \mathcal{Y}\Re$ in the topos $\mathcal{U}$.*

Proof Let U be an object of $\mathsf{Dom}(E)$. we must prove that the image of the monomorphism

$$r \mapsto [x \mapsto r|_x] \colon \mathsf{R}(U) \longrightarrow \Re^U \tag{7.1}$$

is the set $\mathsf{E}(U, \Re)$ of real continuous functions defined on U. So let $f \colon U \longrightarrow \Re$. It is in the image of the function in (7.1), if, given $n \in \mathsf{N}$, there is a family of open subsets $U_i \subset U$ and rational numbers x_i, y_i, such that

$$\forall y \in U_i.\ x_i < fy < y_i \quad \text{and} \quad y_i - x_i < \frac{1}{n}.$$

This condition is equivalent to continuity of f. □

So, by 7.3.2 and 7.1.2,

$$\Sigma^{\mathrm{R}}(U) \simeq \{V \subset U \times \Re : V \text{ open}\}.$$

Therefore for $r \in \mathsf{R}$ and $A \in \Sigma^{\mathrm{R}}$

$$\mathcal{U} \Vdash r \in A \leftrightarrow \exists x, y \in \mathsf{R}\ [x < r < y \wedge x \in A \wedge y \in A].$$

So Σ^{R} generates the Euclidean topology on R. By 3.4.1

$$\mathcal{U} \Vdash \text{ all } f\colon \mathsf{R}^n \longrightarrow \mathsf{R} \text{ are continuous for the Euclidean topology .}$$

Chapter 8

For a synthetic theory of computation

It was shown in chapter 5 that the properties of the topological structure of countable algebraic lattices depended upon their Σ-subobjects. Indeed, one may think to make the Σ-dominance the central core of the theory: the definitely more direct approach to effectiveness is an encouragement to do so. Also, we would like to be able to restrict the model (=the topos, the p-category), in order to obtain properties which would enable us to give recursive definitions or define fixed point operators.

Note that the partial order on a countable algebraic lattice P is completely determined by its Scott topology; in turn, this depends solely on the family of Σ-subsets of P. Is it possible to specify a direct connection between the Σ-subsets and the partial order?

8.1 σ-sets

We derived the ideas exposed in this chapter from the instances in the recursive topos, and the effective topos, but the discussion can carried out in an arbitrary topos $\mathcal{E}$ with a NNO*. So let $\mathbf{N}$ be the NNO* in $\mathcal{E}$ and let $\Sigma \rightarrowtail \Omega$ be the dominance of r.e. truth values as defined in 5.2.

An adjunction. The lattice Σ is closed under $\mathbf{N}$-indexed joins. Define a partially ordered object $(H, \leq)$ to be a *σ-algebra* if it is bounded, has binary meets and joins indexed by $\mathbf{N}$ (simply called $\mathbf{N}$-joins) and meets distributes over $\mathbf{N}$-joins, in the sense that

$$s \wedge \bigvee_n t_n = \bigvee_n (s \wedge t_n).$$

Consider the category S of σ-algebras with *σ-homomorphisms*, preserving the extremes, binary meets and $\mathbf{N}$-joins.

8.1.1 **Theorem** *There is an adjunction*

$$\Sigma^{(-)} \dashv \mathit{Hom}(-, \Sigma) \colon \mathsf{S}^{\mathrm{op}} \longrightarrow \mathcal{E}.$$

Proof For all objects X, the object Σ^X has a structure of σ-algebra, defined pointwise. Moreover, let $f \colon X \times H \longrightarrow \Sigma$ be such that,

$$x \mapsto f(x, -) \colon X \longrightarrow \mathrm{Hom}(H, \Sigma). \tag{8.1}$$

That is the same as saying that

$$\forall x \in X.\ f(x, -) \text{ is a } \sigma\text{-homomorphism.}$$

As, for $w, z \in \Sigma$ and $t \colon \mathbf{N} \longrightarrow \Sigma$,

$$w \wedge z = \mathrm{true} \leftrightarrow w = \mathrm{true} \wedge z = \mathrm{true}$$

and

$$\bigvee_n t_n = \text{true} \leftrightarrow \exists n \in \mathsf{N}.\ t_n = \text{true}\,,$$

(8.1) is also equivalent to

$$h \mapsto \{x \in X : f(x,h) = \text{true}\}: H \longrightarrow \Sigma^X \text{ is a } \sigma\text{-homomorphisms.}$$

This proves the assertion. □

The σ-sets. The category-theoretic approach to computability for the definition of a suitable universe of "effective" objects dates back to work of [Erš73]. There the description was too complicated in order that category theory might be of substantial help. In [Mul81], the object Ω_{re} is considered: it is the interpretation of Σ in the recursive topos, but no recognition of its internal definition appears. The first time we saw such an internal definition was at a lecture delivered by Dana Scott in Pisa. The major stress was on the effective topos as a general ground from which to renew the attack to "effective objects". When we proved a characterization of Σ-subsets of effective algebraic posets in terms of the Scott topology, he suggested to look at the countable objects which are completely determined by their Σ-subsets. And he pointed out the notion of *σ-sets*.

Write $[X]$ for Σ^X and, again, $[H]$ for $\mathrm{Hom}(H,\Sigma)$: it will be always clear from the context which of the two function spaces we are considering. Recall the convention we adopted of writing $U \subset_\Sigma X$ in place of $U \in \Sigma^X$.

From the adjunction, it follows that there is a map of an object X into its dual $[[X]]$ which is defined as

$$\sigma_X: x \mapsto \{W \subset_\Sigma X : x \in W\}: X \longrightarrow [[X]].$$

Similarly, for a σ-algebra H, there is a map

$$s_H: h \mapsto \{\mathcal{W} \in \mathrm{Hom}(H,\Sigma) : \mathcal{W}(x) = \text{true}\}: H \longrightarrow [[H]].$$

Definition. A *σ-set* X is an object of the topos $\mathcal{E}$ such that the map $\sigma: X \longrightarrow [[X]]$ is an isomorphism.

As $[[X]] \rightarrowtail \Sigma^{[X]}$, there is an induced partial order on $[[X]]$; moreover, the join in $\Sigma^{[X]}$ of an $\mathbf{N}$-chain of σ-homomorphisms is a σ-homomorphism. Therefore, in case X is a σ-set, there is a canonical partial order structure closed under sups of $\mathbf{N}$-chains. This may seem to support the idea that in an intuitionistic universe, where a NNO* satisfies "decent" properties with respect to computations, there are special objects that carry a canonical effective partial-order structure.

Properties of σ-sets. To sustain our last assertion, in the following we give some closure and algebraic properties of σ-sets. Even though in the end, we want to concentrate only on those σ-sets which are enumerated, many of these properties can be proved in complete generality.

Notation. Given posets A, B in $\mathcal{E}$, it is a convinient notation to use $\mathrm{Mono}(A,B)$ for the object of non-decreasing maps from A into B in $\mathcal{E}$. Note that $\mathrm{Mono}(A,\Sigma)$ is isomorphic to the object of upward-closed Σ-subsets of A. Also, in case there is an onto map $\mathbf{N}E + 1$, we say that the object E is *enumerated*.

8.1.2 **Proposition** *Let S be an $\mathbf{N}$-algebraic poset. If the object C of compact elements of S is enumerated, then S is a σ-set.*

Proof By 5.4.1, the map

$$- \cap C: [S] \longrightarrow \mathrm{Mono}(C,\Sigma)$$

is an isomorphism of the object $[S]$ of Σ-subsets of S into the object $\mathrm{Mono}(C,\Sigma)$. Given a σ-homomorphism $f: \mathrm{Mono}(C,\Sigma) \longrightarrow \Sigma$, set

$$F = \{c \in C : f(\uparrow(c)) = \text{true}\} \subset C.$$

As any $D \in \mathrm{Mono}(C,\Sigma)$ is the union of the segments $\uparrow(c)$ contained in it, and they are enumerated, the σ-homomorphism f is completely determined by F, which, by definition, is Σ. As $C = \bigcup_{c\in C} \uparrow(c)$ is an $\mathbf{N}$-union, and $f(\mathrm{Mono}(C,\Sigma)) = \text{true}$, there is a $c \in F$. Hence F is enumerated, because C is so. Moreover, $F \subset C$ is upward closed. And, if $c, d \in F$, then

$$f(\uparrow(c) \cap \uparrow(d)) = f(\uparrow(c)) \cap f(\uparrow(d)) = \text{true}\,.$$

Therefore

$$\exists e \in C.\ c \sqsubseteq e \wedge c \sqsubseteq d \wedge e \in F,$$

proving that F is directed. Use $\mathbf{N}$-choice to find a chain $(c_n)_{n\in\mathrm{N}}$ filtering F. In turn, this determines an element

$$s = \bigsqcup_{n\in\mathrm{N}} c_n$$

in S, which uniquely defines f as

$$f(W \cap C) = |s \in W|,$$

for $W \in [S]$. Thus $[[S]] \simeq S$. □

Examples. The objects $\mathbf{N}$, Σ^{N}, $\widehat{\mathbf{N}}$, $\widehat{\mathbf{N}}^{\mathrm{N}}$, $\Sigma^{\Sigma^{\mathrm{N}}}$ are all σ-sets, by 5.4.2.

Before stating the next proposition, let us recall that a map $Z \to A$ in $\mathcal{E}$ is sometimes called an *internal family* of objects indexed by A. This is a straightforward generalization from the case of the topos $\mathcal{S}$ of sets, where the function $f\colon Z \longrightarrow A$ is completely determined by the fibers $(f^{-1}(a))_{a\in A}$. In tune with this particular case, we may denote an internal family $Z \to A$ just as $(Z_a)_{a\in A}$.

8.1.3 **Proposition** *Let X and Y be σ-sets and $(Z_n)_{n\in\mathrm{N}}$ an internal sequence of σ-sets. Then*

(i) *the object X is separated;*

(ii) *if $Z \rightarrowtail X$ is Σ, then Z is a σ-set;*

(iii) *if A is a retract of X, then A is a σ-set;*

(iv) *given any $f\colon X \longrightarrow Y$, the map f is monotone and preserves sups of chains;*

(v) *the coproduct $\coprod_n Z_n$ is a σ-set.*

Proof (i) Obvious, as $X \simeq [[X]] \rightarrowtail \Sigma^X$ and Σ is separated. (ii) The Σ-monomorphism $i\colon Z \rightarrowtail X$ induces two maps $i^{-1}\colon [X] \longrightarrow [Z]$ and $\exists i\colon [Z] \longrightarrow [X]$, where the first is inverse image along i, the second is extension along i, which applies $[Z]$ into $[X]$ because Σ is a dominance. Moreover, i^{-1} is a σ-homomorphism, and it is onto as

$$i^{-1} \circ \exists i = \mathrm{id}\,.$$

Let $f\colon \Sigma^Z \longrightarrow \Sigma$ be an element of $[[Z]]$. Then $f \circ i^{-1}\colon \Sigma^X \longrightarrow \Sigma$ is a σ-homomorphism; thus there is an element $x \in X$ such that for all $W \subset_\Sigma X$,

$$f(i^{-1}(W)) = |x \in W|.$$

In particular, it follows that $x \in i^{-1}(Z)$, taking $W = Z$. So there is a unique $y \in Z$ such that $x = i(y)$, and for all $W \subset_\Sigma Z$, it is

$$f(W) = f(i^{-1}(\exists i(W))) = |x \in i^{-1}(W)| = |y \in W|.$$

(iii) Similar to the previous one: this time the retraction pair $A \underset{i}{\overset{j}{\rightleftarrows}} X$ will produce a retraction pair $[A] \underset{[j]}{\overset{[i]}{\rightleftarrows}} [X]$ of σ-homomorphisms. (iv) Trivial, because the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \sim\ } & [[X]] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle [[f]]} \\ Y & \xrightarrow{\ \sim\ } & [[Y]] \end{array}$$

commutes, and $[[f]]$ preserves sups of chains. (v) Let $F:[\coprod_n Z_n] \longrightarrow \Sigma$. Each $Z_n \subset \coprod_n Z_n$ is decidable, hence Σ. So

$$\text{true} = F(\coprod_n Z_n) = \bigcup_n F(Z_n),$$

and there must be $n \in \mathbf{N}$ such that $F(Z_n) = \text{true}$. Moreover such an n is unique because summands are disjoint. Hence there is a unique $x \in \coprod_n Z_n$ such that $F = \sigma x$. □

8.2 Limits of σ-sets

A generalization of the argument used for the proof of Rice-Shapiro theorem proves that σ-sets are closed under certain limits, resembling a property in [SP82]. The main result of this section will be that, given a sequence of σ-sets and adjoint retractions, the limit of the projections is isomorphic to the colimit of the injections and is a σ-set. It would be just a particular case of the main property proven in *loc.cit.* if it was not that the sequence is *internally* indexed by $\mathbf{N}$, and the isomorphism between the two limits is produced in a category which is not an O-category.

The result can be used to give a profinite-like definition of a subclass of σ-sets, which would then be closed under products and p-exponentials, or to generate solutions for fixed-point equations.

Formally the sequence consists of a map $k: X \longrightarrow \mathbf{N}$ with a pair of maps $i, p: \{\langle n, m\rangle \in \mathbf{N} \times \mathbf{N} : n < m\} \longrightarrow \widetilde{X}^X$ such that, setting $X_n = k^{-1}(n)$,

$$\forall n \in \mathbf{N}\ \Big[[[X_n]] \simeq X_n \wedge \forall m > n\ [p_{n,m}: X_m \longrightarrow X_n \wedge i_{n,m}: X_n \longrightarrow X_m \wedge$$

$$p_{n,m} i_{n,m} = \mathrm{id} \wedge i_{n,m} p_{n,m} \sqsubseteq \mathrm{id} \wedge\ \forall l > m\ [p_{n,m} p_{m,l} = p_{n,l} \wedge i_{m,l} i_{n,m} = i_{n,l}]]\Big].$$

So much for the formalism: we know that we can be informal, so from now on we will just think that it is a sequence

$$X_0 \underset{i_0}{\overset{p_0}{\leftrightarrows}} X_1 \underset{i_1}{\overset{p_1}{\leftrightarrows}} X_2 \underset{i_2}{\overset{p_2}{\leftrightarrows}} \cdots \underset{i_{n-1}}{\overset{p_{n-1}}{\leftrightarrows}} X_n \underset{i_n}{\overset{p_n}{\leftrightarrows}} \cdots$$

of σ-sets and maps indexed by $\mathbf{N}$, satisfying $p_n i_n = \mathrm{id}$, and $p_n i_n \sqsubseteq \mathrm{id}$.

An element of the limit Z is a sequence $(x_n)_{n\in\mathrm{N}}$ such that

$$\forall n \in \mathbf{N}.\ x_n \in X_n \wedge x_n = p_n(x_{n+1}).$$

The canonical maps $\pi_n: Z \longrightarrow X_n$ from the limit are defined in the obvious way. There are left inverse maps to them $\iota_n: X_n \longrightarrow Z$ defined as

$$\iota_n(x) = \langle \ldots, p_{n-2}(p_{n-1}(x)), p_{n-1}(x), x, i_n(x), \ldots \rangle.$$

8.2.1 **Lemma** *Suppose a sequence is given as above.*

(i) *There is an induced partial order on the limit Z, and it has sups of $\mathbf{N}$-chains.*

(ii) *For each $n \in \mathbf{N}$, $\iota_n \pi_n \sqsubseteq \mathrm{id}$.*

(iii) *For $z \in Z$, $z = \bigsqcup_{n\in\mathrm{N}} \iota_n \pi_n(z)$.*

Proof (i) The order is defined pointwise, as well as sups of chains. (ii) For $m < n$,

$$\pi_m \iota_n \pi_n(z) = p_{n,m}(\pi_n(z)) = \pi_m z.$$

For $m = n$,

$$\pi_n \iota_n \pi_n(z) = \pi_n(z).$$

For $m > n$,

$$\pi_m \iota_n \pi_n(z) = i_{n,m}(\pi_n(z)) = i_{n,m} p_{n,m}(\pi_m(z)) \sqsubseteq \pi_m(z).$$

(iii) Trivial. □

8.2.2 **Lemma** *Suppose $(A, \sqsubseteq)$ is a poset in the topos $\mathcal{E}$ which has sups of $\mathbf{N}$-chains. Suppose, moreover, that $U \subset_\Sigma A$, and $c\colon \mathbf{N} \longrightarrow A$ is a chain. Then*

$$\mathcal{E} \Vdash \bigsqcup_n c_n \in U \leftrightarrow \exists n \in \mathbf{N}.\ c_n \in U.$$

Proof It is similar to 5.4.1: we shall use the same notation employed there. Suppose first the $\bigsqcup_n c_n \in U$, and define the map $h\colon \mathbf{N} \longrightarrow A$ as

$$h(x) = \bigsqcup \{c_m : \exists n \in \mathbf{N}\ [[x \notin K(n) \wedge m = n] \vee [x \in K(n) \wedge m = \kappa(x)]]\}.$$

Then

$$\{x\}(x) \uparrow \rightarrow h(x) = \bigsqcup_n c_n \in U.$$

Suppose there is no $n \in \mathbf{N}$ such that $c_n \in U$. Then

$$h(x) \notin U \rightarrow \{x\}(x) \uparrow .$$

$$\neg \exists n \in \mathbf{N}.\ c_n \in U \rightarrow \bar{K} = h^{-1}(U).$$

As U is Σ, so is $h^{-1}(U)$. Contradiction: therefore $\neg\neg\exists n \in \mathbf{N}.\ c_n \in U$. As the value $\exists n \in \mathbf{N}.\ c_n \in U$ is in Σ, it is closed; so $\exists n \in \mathbf{N}.\ c_n \in U$. Next, suppose that there is an $n \in \mathbf{N}$ with $c_n \in U$, and define

$$k(x) = \bigsqcup \{d_m : \exists k\ [[x \notin K(k) \wedge m = k \wedge d_m = c_n] \vee [x \in K(m) \wedge d_m = \bigsqcup_n c_n]]\}.$$

Again,

$$\{x\}(x) \uparrow \rightarrow k(x) = c_n \in U.$$

If $\bigsqcup_n c_n \notin U$, then

$$\{x\}(x) \downarrow \rightarrow k(x) \notin U.$$

So a contradiction follows, and $\bigsqcup_n c_n \in U$, as the statement is in Σ. □

8.2.3 **Lemma** *Suppose that*

$$X_0 \underset{i_0}{\overset{p_0}{\leftrightarrows}} X_1 \underset{i_1}{\overset{p_1}{\leftrightarrows}} X_2 \underset{i_2}{\overset{p_2}{\leftrightarrows}} \dots \underset{i_{n-1}}{\overset{p_{n-1}}{\leftrightarrows}} X_n \underset{i_n}{\overset{p_n}{\leftrightarrows}} \dots$$

is a sequence of σ-sets and maps indexed by $\mathbf{N}$, satisfying $p_n i_n = \mathrm{id}$, and $p_n i_n \sqsubseteq \mathrm{id}$. Then

(i) $[Z] \simeq \varprojlim([X_n], [i_n])$;

(ii) $[[Z]] \simeq \varprojlim([[X_n]], [[p_n]])$;

(iii) $\varprojlim([X_n], [i_n]) \simeq \varinjlim([X_n], [p_n])$ *in the category* S.

Proof (i) Let $U \in [Z]$. By 8.2.1(iii) and 8.2.2, U is completely determined by the sequence $(\iota_n {\pi_n}^{-1} U)_{n \in \mathbf{N}}$. As the maps π_n are surjective, each subset in the sequence is, in turn, completely characterize by ${\iota_n}^{-1} U$. (ii) The first part of the proof goes like for the previous assertion considering the sequence

$$[X_0] \underset{[p_0]}{\overset{[i_0]}{\leftrightarrows}} [X_1] \underset{[p_1]}{\overset{[i_1]}{\leftrightarrows}} [X_2] \underset{[p_2]}{\overset{[i_2]}{\leftrightarrows}} \dots \underset{[p_{n-1}]}{\overset{[i_{n-1}]}{\leftrightarrows}} [X_n] \underset{[p_n]}{\overset{[i_n]}{\leftrightarrows}} \dots$$

which satisfies $[i_n][p_n] = \mathrm{id}$, and $[p_n][i_n] \sqsubseteq \mathrm{id}$, and using 8.2.2 again. It remains to prove that if $\mathcal{W} \in \Sigma^{[Z]}$ is a σ-homomorphism, then so is each ${[\pi_n]}^{-1}\mathcal{W}$. But ${[\pi_n]}^{-1}\mathcal{W} = \mathcal{W} \circ [\pi_n]$. Hence the conclusion. (iii) The proof is like that of the main theorem in [SP82], as that is constructive and S is an internal O-category, see also [Ros85]. □

8.2.4 **Theorem** *It is internally true in $\mathcal{E}$ that, given a sequence indexed by* $\mathbf{N}$,

$$X_0 \underset{i_0}{\overset{p_0}{\leftrightarrows}} X_1 \underset{i_1}{\overset{p_1}{\leftrightarrows}} X_2 \underset{i_2}{\overset{p_2}{\leftrightarrows}} \dots \underset{i_{n-1}}{\overset{p_{n-1}}{\leftrightarrows}} X_n \underset{i_n}{\overset{p_n}{\leftrightarrows}} \dots$$

of σ-sets and maps satisfying $p_n i_n = \mathrm{id}$, and $p_n i_n \sqsubseteq \mathrm{id}$, the limit Z of the sequence

$$X_0 \overset{p_0}{\longleftarrow} X_1 \overset{p_1}{\longleftarrow} X_2 \overset{p_2}{\longleftarrow} \dots \overset{p_{n-1}}{\longleftarrow} X_n \overset{p_n}{\longleftarrow} \dots$$

is isomorphic to the colimit of the sequence

$$X_0 \underset{i_0}{\longrightarrow} X_1 \underset{i_1}{\longrightarrow} X_2 \underset{i_2}{\longleftarrow} \dots \underset{i_{n-1}}{\longrightarrow} X_n \underset{i_n}{\longrightarrow} \dots$$

and is a σ-set.

Proof By 8.2.3,

$$\varprojlim([[X_n]],[[p_n]]) \simeq [[Z]] \simeq [\varprojlim([X_n],[i_n])] \simeq \varinjlim([[X_n]],[[i_n]]).$$

Moreover,

$$[[Z]] \simeq \varprojlim([[X_n]],[[p_n]]) \simeq \varprojlim(X_n,p_n) \simeq Z. \ \square$$

8.3 Basic systems for σ-sets

We know that a σ-set X has a natural partial-order structure induced on it by the isomorphism $X \simeq [[X]]$. The structure may very well be trivial, as on the σ-set $\mathbf{N}$, or quite rich as for ${}^{\mathrm{N}}\mathbf{N}$. But we know next to nothing of the collection of Σ-subsets of X: there is no way to determine a non-trivial Σ-subset U of X. The fact that U has to satisfy 8.2.2 leaves us still with empty hands. So, with a biased intention, we look at those elements $b \in X$ such that

$$\uparrow b = \{x \in X : b \sqsubseteq x\} \subset_\Sigma X.$$

We call *basic* both the element b and the Σ-subset $\uparrow b$.

Note that, by 8.2.2, a basic element $b \in X$ is compact in the sense defined in 5.4:

$$\forall a \in X^{\mathrm{N}} \ [\forall n,m \in \mathbf{N} \ [n \le m \to a_n \sqsubseteq a_m]] \wedge b \sqsubseteq \bigsqcup_n a_n \to \exists n \in \mathbf{N}.\ b \sqsubseteq a_n.$$

Indeed, if we add the requirement that all the elements of X are joins of $\mathbf{N}$-chains of basic elements, then X is an $\mathbf{N}$-algebraic poset. Even better, it is another characterization of $\mathbf{N}$-algebraic posets.

8.3.1 **Proposition** *Given a σ-set X, let B be the object of basic elements of X. Every x in X is the limit of a $\mathbf{N}$-chain of basic elements if and only if X is an $\mathbf{N}$-algebraic poset with respect to the order induced by the isomorphism $X \simeq [[X]]$.*

Proof Trivial. $\square$

Basic systems. The name "basic" above has been chosen because the σ-sets that satisfy either of the equivalent conditions in 8.3.1 are completely determined by the information about their basic elements.

Consider an enumerated poset $(A,\sqsubseteq)$ in $\mathcal{E}$ such that the order relation is Σ, and let $M = \mathrm{Mono}(A,\Sigma)$ be the σ-algebra of monotone maps into Σ. Let S be the object of equivalence classes of $\mathbf{N}$-chains of elements of A with respect to mutual cofinality:

$$S = \{c \in A^{\mathrm{N}} : \forall n,m \in \mathbf{N} \ [n \le m \to a_n \sqsubseteq a_m]\}/\sim$$

where

$$c \sim c' \leftrightarrow \forall n \ \exists m \ [c_n \sqsubseteq c'_m \wedge c'_n \sqsubseteq c_m].$$

Then, by 8.2.2 and 8.1.2,

$$\Sigma^S \simeq \mathrm{Mono}(A,\Sigma) \quad \text{and} \quad S \simeq [\mathrm{Mono}(A,\Sigma)].$$

Hence S is a σ-set and A is (isomorphic to) the object of basic elements of S.

In other words, the σ-sets where every element is the join of an $\mathbf{N}$-chain of basic elements, are exactly the $\mathbf{N}$-algebraic posets with a countable set of compact elements. For X such a σ-set, call the object B of basic elements, the *basic system* of X. In the following proposition, we list a few closure properties of σ-sets with enumerated basic systems.

8.3.2 **Proposition** *Let X and Y be σ-sets, and let $(Z_n)_{n\in\mathbf{N}}$ be a sequence of σ-sets. Suppose they all have enumerated basic systems B, C and D_n respectively.*

(i) *The product $X \times Y$ is a σ-set with enumerated basic system $B \times C$.*

(ii) *If $U \subset_\Sigma X$, then U is a σ-set with enumerated basic system $U \cap B$.*

(iii) *If $A \underset{r}{\overset{p}{\rightleftarrows}} X$ is a retraction such that $\mathrm{id} \sqsubseteq rp$, then A is a σ-poset with enumerated basic system $p(B)$.*

(iv) *The coproduct $\coprod_n Z_n$ is a σ-set with enumerated basic system $\coprod_n D_n$.*

Proof (i) The poset $X \times Y$ is obviously $\mathbf{N}$-algebraic. And it is straightforward to check that $B \times C$ is the basic system. (ii) By 8.1.3, the object U is a σ-set. By 8.2.2, each of its elements is the sup of chain of basic elements contained in U. (iii) By hypothesis,

$$p(b) \sqsubseteq a \leftrightarrow b \sqsubseteq r(a).$$

Hence pb is basic in A if and only if so is b in X. Finally, as $ra = \bigsqcup b_n$ for some $\mathbf{N}$-chain in X, it is

$$a = p(r(a)) = \bigsqcup_n p(b_n).$$

(iv) By 8.1.3, the object $\coprod_n Z_n$ is a σ-set. For $\langle m, d\rangle, \langle n, z\rangle \in \coprod_n Z_n$, by definition

$$\langle m, d\rangle \sqsubseteq \langle n, z\rangle \leftrightarrow n = m \wedge d \sqsubseteq z.$$

So the element $\langle m, d\rangle$ is basic if and only if d is such in the appropriate component. □

Also closure with respect to Σ-subset spaces can be proved for σ-sets with enumerated basic systems. In fact, a stronger result can be proved about partial-function spaces.

8.3.3 **Proposition** *Let X and Y be $\mathbf{N}$-algebraic posets. If the poset C is an upper-semilattice, then ${}^X Y$ is a σ-set, the basic system consisting of the $\mathbf{N}$-finite step functions. In particular, the space Σ^X is a σ-set with enumerated basic system.*

Proof Suppose C is an upper-semilattice. Let D be the object of $\mathbf{N}$-finite strings of pairs $\langle b, c\rangle$ of basic elements in B and C respectively. For $d = (\langle b_1, c_1\rangle, \ldots, \langle b_n, c_n\rangle)$ an element of D, consider the step function

$$f_d = [x \mapsto \bigsqcup\{c_i : b_i \sqsubseteq x\}] : \bigcup_i \uparrow b_i \longrightarrow Y$$

which defined on a Σ-subset of X. Given any partial map $\phi : X \longrightarrow\!\!\!\!\!\rightarrow Y$, one has that

$$f_d \sqsubseteq \phi \leftrightarrow \forall i \leq n\ [\phi(b_i)\downarrow \wedge c_i \sqsubseteq \phi(b_i)].$$

Obviously each map in ${}^X Y$ can be approximated by an $\mathbf{N}$-chain of step functions. Therefore, the step functions f_d are the basic elements of ${}^X Y$. The last statement is a simple corollary, as the basic system of Σ is Δ. □

It is worth mentioning that what has been presented insofar in the section is new only in part. The reader of [Sco80a] and [Sco81a] must have recognized familiar constructions (and guessed the reason for using the word "system"). Also, much of what has been presented is an easy generalization of the subject in [Smy77]. It must be stressed that the point of view is different, though: there is a canonical notion of "recursively

enumerable" which does not need any painful reference to enumerations. And this is what we are exploiting: not only does the space Σ^X consist of the Scott-open subsets of X, but it must completely determine X in order that that object be interesting. A byproduct of all this is the canonical poset structure on it, and Proposition 8.1.2 assures that there are plenty. Also, one minor difference from [Sco81a] is in that, up to now, the basic systems are not supposed to be consistently complete.

One may wonder why, after an orgy of categories in previous chapters, we are now paying little attention to defining *interesting* categories of σ-sets, or of **N**-algebraic posets, or whatever.... The reason is that any one which may be proclaimed as such, must be full in $\mathcal{E}$, so that there is no reason to bother as to which maps we want to take into consideration.

8.4 Effective Scott domains revisited

The problem of higher types. A question which we have only touched in 8.3.3, is when can we assert that, for given σ-sets X and Y, the partial-function space ${}^X Y$ is a σ-set.

The goal of our investigation is to end up with a circumscribed universe of spaces on which we can straight-forwardly interpret the constructs of computation. Up to this point, everything has been smooth and easy; the next problem we want to address is about partial-function spaces.

More precisely, recall from 5.4.2 that, in case the **N**-algebraic posets X and Y are Scott domains, then the space ${}^X Y$ is a Scott domain. It need not be a σ-set, just because the object of compact elements may not be enumerated. The problem is the following: note first that, given a pair $d = \langle b, c\rangle$ of basic elements of X and Y, the map

$$f_d = [x \mapsto c]{:}{\uparrow} b \longrightarrow Y$$

is a basic element in the partial-function space ${}^X Y$, as for $\phi \in {}^X Y$,

$$f_d \sqsubseteq \phi \leftrightarrow b \sqsubseteq \phi(c).$$

Moreover, the join of two such partial maps would be again a basic element in the p-exponential, provided it is definable.

To assure definability, we need to add a consistency condition for the basic systems: let us introduce the name *σ-domain* for a σ-set X with an enumerated basic system, such that the system has joins of **N**-finite families of compatible elements. The reason for choosing the name is that a σ-domain is a σ-set and a Scott domain.

Supposing then that X and Y are σ-domains, we can consider, as basic elements of ${}^X Y$, the **N**-finite strings $d = (\langle b_1, c_1\rangle, \ldots, \langle b_n, c_n\rangle)$ of pairs of basic elements of X and Y such that

$$\forall F \subset_{\text{fin}} \{1, \ldots, n\}\ [\exists x \in X\ \forall i \in F.\ b_i \sqsubseteq x \rightarrow \exists y \in Y\ \forall i \in F.\ c_i \sqsubseteq y],$$

so that the *step function*

$$f_d = [x \mapsto \bigsqcup\{c_i : b_i \sqsubseteq x\}]{:}\bigcup_{i \le n} {\uparrow} b_i \longrightarrow Y$$

is defined. If we suppose that ${}^X Y$ is **N**-algebraic, then the step functions are exactly *all* its basic elements. The only problem remaining is that tha basic system need not be enumerated.

8.4.1 **Lemma** *Suppose X is a σ-domain.*

(i) *The compatibility relation $\exists x \in X\ [b \sqsubseteq x \wedge c \sqsubseteq x]$ between basic element of X is Σ.*

(ii) *If the compatibility relation on the basic system of X is decidable, then, given any σ-domain Y, the σ-set ${}^X Y$ is an σ-domain.*

(iii) *If ${}^X 2$ is a σ-domain, then the compatibility relation on the basic elements of X is decidable.*

Proof (i) Let B be the basic system of X. As every element of X is the sup of an $\mathbf{N}$-chain of basic elements,

$$\exists x \in X\ [b \sqsubseteq x \wedge c \sqsubseteq x] \leftrightarrow \exists a \in B\ [b \sqsubseteq a \wedge c \sqsubseteq a].$$

As B is enumerated, the compatibility relation is Σ-classified. (ii) Let A' be the object of pairs of basic elements of X and Y, and note that A' is enumerated. By hypothesis the strings $d = (\langle b_1, c_1\rangle, \ldots, \langle b_n, c_n\rangle) \in A'$ that satisfy

$$\exists x \in X\ \forall i \in F.\ b_i \sqsubseteq x \rightarrow \exists y \in Y\ \forall i \in F.\ c_i \sqsubseteq y,$$

for $F \subset_{\text{fin}} \{1, \ldots, n\}$, form a Σ-subset of A'. Hence the basic system of ${}^X Y$ is enumerated. (iii) Let $0, 1 \in 2$ be the two basic elements. For basic elements $a, b \in X$, consider the two pairs $d = \langle a, 0\rangle$ and $e = \langle b, 1\rangle$. Then

$$\exists \phi \in {}^X 2\ [f_d \sqsubseteq \phi \wedge f_e \sqsubseteq \phi] \leftrightarrow \neg\exists x \in X\ [a \sqsubseteq x \wedge b \sqsubseteq x].$$

By hypothesis the left-hand side of the equivalence is Σ; therefore the right-hand side must be Σ. Hence the conclusion. □

ω-domains. After 8.4.1(iii), we are forced to concentrate on those σ-domains for which the compatibility relation on their basic systems is decidable, as only for these the p-exponential construct can produce σ-sets with an enumerated basic system. The notation is getting heavy: let us call a σ-set with an enumerated basic system on which the restricted partial-order relation is decidable, an *ω-domain*.

Many of the closure properties are retained by ω-domains: they are closed under products, p-exponentials and $\mathbf{N}$-indexed coproducts. Moreover, a Σ-subset of ω-domain is again an ω-domain; a retract $A \underset{r}{\overset{p}{\rightleftarrows}} X$ of an ω-domain X is an ω-domain. These properties are all straightforward.

Limits of ω-domains. A question one is tempted to ask is: is there a similar result as Theorem 8.2.4 for ω-domains? The answer is positive, and part of it is essentially been proved in 8.3.2. It is sufficient to prove that, given a sequence as in 8.2.4, if all the σ-sets X_n are ω-domains, then so is the colimit σ-set $Z \simeq \lim(X_n, i_n)$. In fact, this can be proved already for σ-domains.

8.4.2 **Proposition** *Suppose a sequence*

$$X_0 \underset{i_0}{\overset{p_0}{\rightleftarrows}} X_1 \underset{i_1}{\overset{p_1}{\rightleftarrows}} X_2 \underset{i_2}{\overset{p_2}{\rightleftarrows}} \ldots \underset{i_{n-1}}{\overset{p_{n-1}}{\rightleftarrows}} X_n \underset{i_n}{\overset{p_n}{\rightleftarrows}} \ldots$$

of σ-domains and maps satisfying $p_n i_n = \mathrm{id}$, and $p_n i_n \sqsubseteq \mathrm{id}$ is given in the topos $\mathcal{E}$. Then the colimit of the sequence

$$X_0 \underset{i_0}{\longrightarrow} X_1 \underset{i_1}{\longrightarrow} X_2 \underset{i_2}{\longleftarrow} \ldots \underset{i_{n-1}}{\longrightarrow} X_n \underset{i_n}{\longrightarrow} \ldots$$

is a σ-domain.

Proof First we characterize the basic elements of Z are exactly the images of basic elements of the X_n: let $b \in Z$ be such that

$$\forall z \in Z.\ |b \sqsubseteq z| \in \Sigma.$$

Say $b = \iota_n c$. Then

$$\forall x \in X_n\ [c \sqsubseteq x \leftrightarrow \iota_n b \sqsubseteq \iota_n x].$$

Hence c is basic in X_n. Conversely, if $c \in X_m$ is basic, then $\iota_m c$ is obviously basic in Z. Moreover, given an element z in the colimit $\lim X_n$, it is of the form $\iota_n z$ for some $n \in \mathbf{N}$. And in X_n, the element $\iota_n z$ is the join of an $\mathbf{N}$-chain of basic elements; this join is preserved into Z. Therefore, the basic system is the $\mathbf{N}$-indexed union of the images in the colimit of the basic systems of the components, and hence enumerated. □

A two-step checking is required now to see that, given a sequence as in 8.4.2, if the σ-sets X_n are ω-domains,then the σ-set $Z \simeq \lim(X_n, i_n)$ is an ω-domain. Let a, b be basic elements in Z; then there is an index m and basic elements $c, d \in X_m$ such that $a = \iota_m c$ and $b = \iota_m d$. Therefore $\pi_m a = c$ and $p i_m b = d$, and

$$\begin{aligned}\exists z \in Z\ [a \sqsubseteq z \wedge b \sqsubseteq z] &\leftrightarrow \exists z \in Z\ [p_m(a) \sqsubseteq p_m(z) \wedge p_m(b) \sqsubseteq p_m(z)] \\ &\leftrightarrow \exists x \in X_m\ [p_m(a) \sqsubseteq x \wedge p_m(b) \sqsubseteq x],\end{aligned}$$

proving that the compatibility relation on the basic system of Z is decidable.

Fixed-point operators. The last tile we need to complete the intricated mosaic we have been building in this chapter is about fixed-point operators. We begin with a very simple solution which is already prepared for the p-exponentials which are σ-sets

Remark. Suppose X and Y are ω-domains. Then the object $Z = {}^X Y$ is an ω-domain, and there is a fixed-point operator $\mathrm{Y}\colon {}^Z Z \longrightarrow Z$. To define it, let $\perp$ be the totally undefined map in Z, and, for $\phi \in {}^Z Z$, set $\mathrm{Y}\,\phi$ to be the join

$$\mathrm{Y}\,\phi = \bigsqcup\{\perp, \phi(\perp), \phi(\phi(\perp)), \ldots\}$$

which exists in Z because it is a σ-set. Then

$$\phi(\mathrm{Y}\,\phi) \downarrow\!\rightarrow \mathrm{Y}\,\phi = \phi(\mathrm{Y}\,\phi).$$

Indeed, the remark is capturing the essence of the solution to the fixed point problem. We cannot expect that, say, all ω-domains have a fixed-point operator: there is no way the map $j + i\colon \mathbf{N} + \mathbf{N} \longrightarrow \mathbf{N} + \mathbf{N}$, which swaps the two components, can have a fixed point! One would argue that that is because there is no least element $\perp$ on which to start the limiting chain.

8.4.3 **Proposition** *Suppose X and Y are ω-domains such that Y has a fixed-point operator and $X \underset{j}{\overset{i}{\leftrightarrows}} Y$ is a retraction pair. Then X has a fixed-point operator. Therefore, if X is a retract of $\widehat{X}$, then X has a fixed-point operator.*

Proof The first part is trivial; the second then follows applying the remark above, as $\widehat{X} \simeq {}^1 X$. □

The previous result is essentially part of folklore: but it gains special importance if we relate it to 4.4.2. An application of 4.4.2 yields that a strong converse to 8.4.3 holds, and producesa a characterization of those ω-domains which come equipped with a fixed-point operator.

8.4.4 **Proposition** *Suppose X is a σ-set with a fixed operator $\mathrm{Y}\colon {}^X X \longrightarrow X$. Then X is Σ-injective.*

Proof Follows from 4.4.2 and 4.4.1. □

8.4.5 **Corollary** *Suppose X is an ω-domain with a fixed operator $\mathrm{Y}\colon {}^X X \longrightarrow X$. Then the least-fixed-point operator can be defined on X.*

Proof The operator Y is monotone, because it is defined between σ-sets. Moreover, it is onto as

$$\forall x \in X.\ x = \mathrm{Y}(\lambda y.x).$$

Therefore, setting $\perp$ for the totally undefined map in ${}^X X$, the element $\mathrm{Y}\,\perp$ is the least element in X: denote it with $\perp$, again. Define $\mathrm{Y}'\colon {}^X X \longrightarrow X$ as

$$\mathrm{Y}'\,\phi = \bigsqcup\{\perp, \phi(\perp), \phi(\phi(\perp)), \ldots\}.$$

Obviously, this operator is the least-fixed-point operator. □

Concluding remarks. Again we want to stress the point that σ-sets, or ω-domains, are objects of the topos $\mathcal{E}$: if we may say so, they are *sets* of the intuitionistic universe $\mathcal{E}$. Though they are just sets, they all are endowed with a canonical partial-order structure, because of the very peculiar properties of the universe itself: the notion of "recursive" computability is inherent to it, and forces the particular behaviour of the σ-sets. We do not think we have produce sufficient arguments that allow to declare that σ-sets are an interesting matter of study, but we are positively convinced that the defining property has been the guidance for much of the study about algebraic cpo's, but it is that particular property that shows that σ-sets are posets in a canonical way: because of their effective content, the partial order comes as a natural set-theoretic data. We have to recognize it, not to impose it.

We did not pause in this last chapter to compare the theory we were producing with the work done on *effectively given cpo's*, and in particular what is done in [Smy77] and [Sco81a]. It should be clear that all the result presented there are deducible from ours when the topos $\mathcal{E}$ is the effective topos $\mathcal{F}$ or the recursive topos $\mathcal{R}$. Sometimes our arguments are resembling others appearing in *loc.cit.* : it was to be so, as those were constructive. But, in addition, with our approach we have freed ourselves from the constraints of working with enumerations and presentations all the time: we believe that the examples in chapter 6 are sufficient to sustain this point of view. Besides, many properties of this last section are stated and proved in great generality, which could eventually lead, for instance, to stating necessary conditions for the existence of σ-sets of partial maps or the existence of fixed-point operators.

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