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ABSTRACTION, EQUALITY, AND UNIVALENCE

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ABSTRACT. We shall propose a conceptual-oriented discussion of the so-called Univalent Foundations Program, that is, of Martin-Löf type theory enriched with a homotopic interpretation, together with the univalence axiom proposed by Voevodsky. We shall argue that the type-theoretic notion of *propositional equality* encodes the notion of *indiscernibility*, we shall address the homotopic interpretation of Martin-Löf type theory, and we shall analyze whether Leibniz's principle of the identity of indiscernibles holds or not in Univalent Foundations. We shall finally argue that univalence can be understood as a particular implementation of a constructive notion of abstraction that resolves so to speak Fregean abstraction.

Keywords: Univalent Foundations, Homotopy Type Theory, Equality, Abstraction, Indiscernibles

1. Introduction

After set theory and category theory, *univalent foundations* (UF in what follows) can be described as a sort of third generation in the history of the intra-mathematical attempts to provide "foundations" to mathematics. Whereas set-theoretic foundations rely on the idea that every mathematical object is a *structured set* (e.g. a topological space is a set endowed with a topology), category theory does not propose a fundamental mathematical object out of which all the rest could be constructed, but rather provides a general mathematical theory of mathematical objects and their relations [14, 38, 46, 49, 50]. We could say that UF is at the crossroad of these two previous foundational frameworks, in the sense that it maintains that it is in fact possible to extract from (higher) category theory a fundamental mathematical object, namely a higher dimensional generalization of the notion of set known as ∞-*groupoid*.[1]

[1]Whereas from a category-centered perspective a groupoid is just a particular kind of category among others (namely, a category in which all morphisms are isomorphisms), Voevodsky argued that a category can be understood as a groupoid equipped with additional structure. In Voevodsky's own terms, "[t]he successes of category theory inspired the idea that categories are 'sets in the next dimension' and that the foundation of mathematics should be based on category theory or on its higher-dimensional analogues. The greatest roadblock for me was the idea that categories are 'sets in the next dimension.' I clearly recall the feeling of a breakthrough that I experienced when I understood that this idea is wrong. Categories are not 'sets in the next dimension.' They are 'partially ordered sets in the next dimension' and 'sets in the next dimension' are groupoids" [72] (see also [31, Ex.9.1.14, p.310] and [69]). Briefly, whereas a partial order set is a set X endowed with a reflexive, transitive, and antisymmetric relation, a category can be understood as a groupoid of objects (considered up to equivalence of categories) endowed with a supplementary structure that assigns to any pair of objects x and y a set $Hom(x, y)$ such that this assignment satisfies reflexivity (existence of trivial identities), transitivity (existence of compositions), and a kind of antisymmetry condition that identifies the invertible elements of $Hom(x, y)$ with the isomorphisms in the underlying groupoid.

In what follows, we shall provide a succinct philosophically-oriented discussion of UF in which we shall focus on two interrelated topics: a) the homotopy-theoretic reconceptualization and extension of the notion of mathematical *equality* (and, *a fortiori*, of mathematical identity) (see also [36, 37]) and b) the relevance of UF with respect to the understanding of the notion of mathematical *abstraction*. Very briefly, UF is based on (the intensional variant of) *Martin-Löf type theory* (MLTT).[2] This is a theory in which every term a is assigned a type X, written $a : X$. The system also has *dependent types* $B(a)$, i.e., families of types which depend on a variable $a : X$. Dependent types are used to encode predicate logic. It also has two distinct notions of equality, namely (in the notation of [31]) *definitional equality* $a \equiv b$, and *propositional equality* $a =_X b$, where $a, b : X$ [53, 54]. "Homotopy type theory" was originally introduced by Awodey as a name for intensional MLTT together with the homotopy interpretation of propositional equality introduced by Awodey and Warren [13]. Voevodsky added the *Univalence Axiom* to this system and called the result "Univalent Foundations" [74]. *Higher inductive types* (HITs) were introduced even later by Shulman and others,[3] and the resulting system consisting of MLTT + UA + HITs was called "Homotopy type theory" in the HoTT Book [31] and thereafter.[4] Fig. N°1 summarizes this sequence of successive extensions of MLTT. Mathematical introductions to UF can be found in [31, 61, 66] and more conceptual-oriented discussions in [9, 10, 18, 28, 48, 52, 67].

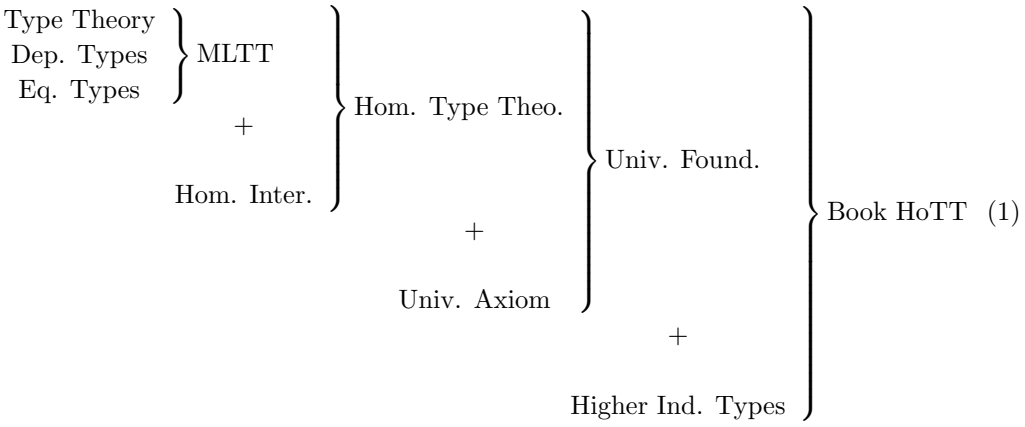
[2]The scope of the terms *intensional* and *extensional* in this type-theoretic context is convoluted. From a broad conceptual perspective, the term *intensional* means (in Frege's words [23]) that different *senses* (*Sinn*) or *modes of presentation* of the same *reference* (*Bedeutung*) are distinguished. For instance, judgmental or definitional equality is described as an intensional equality since (for instance) it distinguishes $n + m$ and $m + n$ (i.e., these expressions are not definitionally equal). By contrast, propositional equality is more extensional since (for instance) it can be shown that $m + n =_{\mathbb{N}} n + m$ [61, Prop.5.6.4, p.48]. A type theory is said to be *extensional* if it contains the so-called *equality reflection rule*:

$$\frac{a, b : X \vdash p : a =_X b}{a, b : X \vdash a \equiv b}$$

which states that two terms of a type $a, b : X$ that are propositionally equal are also definitionally equal [29]. In Fregean terms, this means that two co-referential expressions necessarily express the same sense. It can be shown that the reflection rule collapses the higher groupoid structure (see for instance [31, pp.102-3]). Indeed, the equality reflection rule implies that $p : a =_X b$ is necessarily $refl_a$ (this can be proved by induction on identity and the fact that $refl_a =_{(a =_X a)} refl_a$ is inhabited by the canonical term $refl_{refl_a}$).

[3] Higher inductive types are type-theoretic versions of important objects of homotopy theory whose terms and equality proofs are introduced inductively [31, Ch.6].

[4] It is worth noting that Voevodsky continued to prefer the system without HITs and the term "Univalent Foundations".



(1)

From a philosophical standpoint, UF was originally inspired by a particular trends of ideas in philosophy of mathematics associated to the terms *constructivism* and *intuitionism* and mainly developed by Brouwer, Heyting, Kolmogorov, Bishop, and Martin-Löf among others (see [20, 33, 68] and references therein). More precisely, the constructivist scope of UF relies on the proof-relevant conception associated to the *propositions-as-types* paradigm or the Curry-Howard correspondence (see [19, 32, 75] and [31, pp.8-11 & 41-47]).[5] In this framework, every proposition is a type, namely the type of its proofs, which means that a false proposition is just an empty type (conversely every type can be understood as a proposition, namely the proposition according to which the type is inhabited). The different proofs of a true proposition are no longer understood as a kind of mere epistemic supplement that we could discard once we know that the proposition is true. As we shall see below, this constructive understanding of propositions allows us to unpack a rich homotopical structure that is completely flattened or truncated when we understand propositions (and in particular equalities) in proof-irrelevant terms, that is (according to the standard terminology) as "mere propositions" or "truth values".

The discussion of UF that we shall propose will proceed in two steps. First we shall discuss the relationship between the intensional treatment of mathematical equalities in MLTT and Leibniz's principle of the identity of indiscernibles (PII) (see [63] and references therein). We shall then address the main thesis of UF, namely that the fundamental mathematical objects are not sets, but rather homotopy types (which, classically, are taken to be topological spaces up to homotopy equivalence). Second, we shall inscribe UF in the lineage provided by the philosophical and mathematical attempts to understand and formalize the notion of mathematical abstraction.

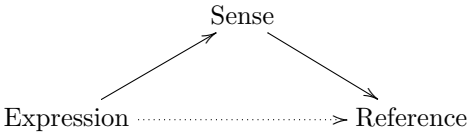
[5] It is worth stressing that this constructivist inspiration of UF does not commit the latter to assume certain features of intuitionism in the more restricted sense of the term (e.g. Heyting's "intuitionist" logic), such as the rejection of both the *axiom of choice* (AC) and the *law of the excluded middle* (LEM) (see for instance the discussion in [31, pp.8-11]). The stratification of types in h-levels opens new possibilities for both the AC and the LEM. In particular, a type-theoretic version of the axiom of choice AC_∞ is provable [31, p.32 & Th.2.15.7, p.101] and the general propositions-as-types formulation of the law of the excluded middle LEM_∞ is inconsistent with univalence [31, Th.3.2.2, p.110] (where the subscript indicates that we are considering the full logic of *propositions-as-types*). At the other extreme, the classical version of the axiom of choice AC_{-1} [31, Sect.3.8, pp.118-120] and the (-1)-truncated law of the excluded middle LEM_{-1} [31, p.113] may be freely added (where the subscript indicates that we are considering the standard (-1)-truncated logic of *mere propositions* or *propositions-as-*(-1)*-types*).

We shall argue that UF can be understood as a particular implementation of a constructive notion of abstraction that resolves so to speak Fregean abstraction (a partially related interpretation of UF as an alternative to Frege-Russell abstraction was proposed in [62]; for a constructive reconceptualization of the notion of abstraction see also [65]). Fregean abstraction relies on the formation of equivalence classes—the abstracta—within which equivalent objects are merged. In certain situations (when there are many proofs of the fact that two objects are equivalent), the formation of equivalence classes entails a loss of information (since they only retain the mere existence of an equivalence). This loss of information has as a consequence the presence of certain pathologies (like the so-called "bad quotients" associated to group actions that are not free). The constructive notion of abstraction instantiated in particular (as we shall argue) by the notion of univalence understands abstraction as a structure of equivalences added to the original domain of objects in such a way that equivalent terms are converted into indiscernible terms. By borrowing Frege's distinction between *sense* (*Sinn*) and *reference* (*Bedeutung*) [23], we shall argue that indiscernible terms can be understood as different modes of presentation of the same reference, which in turn can be identified with the corresponding abstractum. In this framework, abstraction is not understood as a *simplification* of the original domain X that proceeds by removing structure (namely the irrelevant differences between equivalent terms), but rather as an *enrichement* of X that proceeds by adding a new structure, namely the structure of equivalences.

In the following sections we shall briefly revisit the intensional treatment of mathematical equalities (Section N°2) and the homotopic interpretation of MLTT (Section N°3). We shall then discuss the notion of abstraction (Section N°4) and its constructive enhancement (Section N°5). In Section N°6 we shall argue that univalence can be understood as a particular implementation of constructive abstraction.

2. The intensional Treatment of Mathematical Equalities

In order to discuss the different notions of equality present in MLTT we shall use Frege's distinction between the notions of *expression* (*Ausdruck*), *sense* (*Sinn*), and *reference* (*Bedeutung*) [23]. Frege's main idea is that the relation between an expression and its reference is factorized through the notion of sense (which encodes what Frege calls the *mode of presentation* [*Art des Gegebenseins*]). Martin-Löf describes this factorization by means of the following diagram [51, p.501]:



In Frege's own terms, an expression *expresses* its sense and *denotes* its reference (in a sense-dependent manner). This threefold structure complexifies the interpretation of equalities as synonymy proposed by Frege in the *Begriffsschrift* [22, §8]. In this last text, an equality $a = b$ is understood as a statement asserting that a and b are different names that denote the same content. In the conceptual framework proposed by Frege in *Über Sinn und Bedeutung* [23], the semantic layer is split in two dimensions, namely the *sense* and the *reference*. Now, the introduction of two semantic dimensions implies that an equality between two expressions can encode either an equality of sense or an equality of reference. The resulting two notions of

equality match the type-theoretic distinction between *judgmental* or *definitional equality* and *propositional equality*: whereas a judgmental equality $a \equiv b$ states that a and b express the same sense, a propositional equality $a =_X b$ states that the terms $a, b : X$ denote the same reference (regarding this parallel between Frege's notions of sense and reference and type-theoretic notions of equality see [8], [31, p.102], [51], and [66, p.387]). Let's consider now these two notions of equality in more detail.

On the one hand, MLTT incorporates the so-called *judgmental* or *definitional* equality (denoted $\equiv$) that controls the syntactic overabundance of expressions over (in Frege's terms) senses. An equality of the form $a \equiv b$ means that a and b are different expressions that express the same sense. In more formal terms, definitional equality encodes the purely syntactic relation given by the fact that different expressions can be reduced to a common canonical (or fully reduced) form.[6] Besides the definitional equality, MLTT incorporates a second notion of equality, the so-called *propositional equality*. Given two terms of a type $a, b : X$, we can form the *equality type* $a =_X b$ whose terms are the proofs of the fact that a and b are equal. Several comments are here in order. First, this notion of equality is indexed by the type X of which a and b are terms (according to Martin-Löf, this idea goes back to Geach's notion of relative identity [26, 52]). This means that a proposition stating that two terms of different types are equal is not even false, it is just meaningless (like saying that a cat and the sinus function are equal). Second, the propositional equality is a *proposition*, which means that it is endowed with a truth value (by contrast, the definitional equality is a *judgement* in Martin-Löf's sense of the term, i.e., a declaration with no truth value). However—and according to the propositions-as-types paradigm—a propositional equality is not a *mere proposition*[7], but rather a type whose terms are the proofs that the proposition is true (which means that the two terms a and b are equal if the type $a =_X b$ is inhabited). However, and differently from a mere proposition, a proposition might have different inequivalent proofs.[8] Given two proofs $p, q : a =_X b$ of the fact that a and b are equal (that is, two *identifications* between a and b), we can use the fact that $a =_X b$ is itself a type to iterate the formation of equality types. This means that we can form the higher equality type $p =_{(a =_X b)} q$, which might or not be inhabited. In this way, the proposition $a =_X b$ envelops a whole hierarchy of identifications, identifications between identifications, identifications between identifications between identifications, and so on and so forth all the way up. Far from being a mere epistemic supplement that could be discarded once we know that the proposition is true, this hierarchy of higher equalities defines the intrinsic structure of the proposition-as-type. In this constructive framework, the fact that a proposition is true is not a property, but rather a structure, namely the higher structure of its proofs.

The equality types are endowed with two constructors, the *introduction constructor* and the *elimination constructor*. The introduction constructor states that $a =_X a$ is always inhabited by a canonical term denoted refl_a. In more philosophical terms, the introduction constructor

[6]The definitional equality is also called (by Voevodsky) *substitutional equality* since two terms that are definitionally equal can just be substituted to one another [71].

[7]A *mere proposition* (also called *truth value*) is a type that contains at most one term (modulo propositional equality). In other terms, a type A is a mere proposition if any two terms $x, y : A$ are propositionally equal [31, Sect.3.3]. The proposition is true if the type is inhabited and false otherwise.

[8]As it was proved by Hofmann and Streicher by constructing a model in groupoids, the property known as *uniqueness of identity proofs* (the statement that any two terms $p, q : a =_X b$ of an equality type are themselves propositionally equal [31, p.225]) is not provable in type theory [30].

guarantees that the so-called *identity principle* (i.e., the principle stating that every entity is necessarily identical to itself) holds for every a. According to Martin-Lof's factorization of Quine's precept ("No entity without identity" [59, p.23]), there is no entity a without type and there is no type X without an equality type family endowed with canonical terms $\mathit{refl}_a : a =_X a$ encoding the trivial self-identity of each term $a : X$ [52, pp.19-20]. In turn, the elimination constructor for equality types encodes the so-called *path induction principle* [31, Sect.1.12.1, 49] (see [36] for a conceptual-oriented discussion of path induction). This rule states that any proposition $C(a, b; p)$ that depends on two propositionally equal terms $a, b : X$ and a proof $p : a =_X b$ of their equality can be proved if we find a function c that assigns to every $a : X$ a proof of $C(a, a; \mathit{refl}_a)$, i.e., a proof of the specialization of the proposition $C(a, b; p)$ to a endowed with its trivial self-identity $\mathit{refl}_a : a =_X a$.[9] From a conceptual standpoint, path induction can be understood as a proof-relevant enhancement of the *principle of substitution*, which states that a two-place predicate $P(a, b)$ holds if $a = b$ and P is reflexive (that is, if $P(a, a)$ holds). In particular, if the proposition $C(a, b; p)$ is the function type $P(a) \to P(b)$, then the existence of a trivial self-identity $\mathit{refl}_{P(a)} : P(a) =_{\mathcal{U}} P(a)$ for every a (where we are using that the type $P(a)$ is itself a term of a universe type $\mathcal{U}$) guarantees that for every a and b that are propositionally equal there is a proof of $P(a) \to P(b)$. This means that any proof of $P(a)$ can be "transported" along $p : a =_X b$ to a proof of $P(b)$, i.e., that a and b are indiscernible terms. Hence, the elimination constructor for the equality types implies, among other things, a constructive version of the so-called *indiscernibility of identicals* (we should rather say *indiscernibility of (propositionally) equals*). According to this constructive version of the indiscernibility of identicals, propositional equalities are like paths along which properties can be transported.[10] All in all, propositional equality is defined as a primitive relation between terms of the same type that satisfies a) reflexivity and b) path induction (of which the indiscernibility of identicals is a particular case). This characterization of propositional equality generalizes the introduction of a primitive notion of equality in a first-order logic with equality through *reflexivity* and *substitutivity* (see for instance [55]). The main new feature of propositional equality with respect to the standard first-order logical definition of equality is that it is constructively defined as a type of identifications between the corresponding terms.

3. The Homotopic Semantics

As it was first realized by Hofmann and Streicher, the structure of identifications encoded in an identity type $a =_X b$ satisfies the laws of a groupoid, that is, of a category in which every morphism is an isomorphism (briefly, a proof $p : a =_X b$ is interpreted as an isomorphism between the corresponding objects in the category) [30]. It is worth noting that since there might be many isomorphisms between two objects in a groupoid, the model does not validate the *uniqueness of identity proofs* principle. Since a groupoid has no higher dimensional structure (i.e., no n-isomorphisms for $n > 1$), Hofmann and Streicher's groupoid model truncates the higher dimensional structure obtained by iterating the equality types. In Awodey's terms, the intensional model proposed by Hofmann and Streicher is extensional "one dimension up" [11]. In order to have models that are intensional all the way up (i.e., non-truncated models),

[9] Where the function $f(a, b; p)$ that sends $a, b : X$, and $p : a =_X b$ to a proof of $C(a, b; p)$ satisfies the normalization condition $f(a, a; \mathit{refl}_a) = c$.

[10] This explains why Voevodsky called the propositional equality *transportational equality* [71].

we need to have some notion of n-dimensional isomorphisms for all n. This can be done by using the higher dimensional enhancements of the notion of groupoid given by using some notion of weak ∞-groupoid. Indeed, it can be proved that for every type X, $(X, =_X, =_{=_X}, ...)$ forms a weak ∞-groupoid [15, 34, 45]. In turn, the Grothendieck's homotopy hypothesis states that ∞-groupoids can be geometrically understood as *homotopy types* (see [56, homotopy hypothesis] and references therein). Building on 1) the fact that weak ∞-groupoids provide models of $(X, =_X, =_{=_X}, ...)$ and 2) that an ∞-groupoid is a homotopy type, Awodey and Warren introduced a homotopic interpretation of MLTT [13] (see also [11]). In Table N°1 we summarize the main entries of the homotopic interpretation of type theory. One of the main features of this homotopic interpretation of MLTT is that the proofs of a propositional equality are given by *continuous deformations* between the corresponding terms. According to this topological interpretation of propositional equalities, terms of $a =_X b$ can be interpreted as paths between the points a and b, terms of $p =_{(a=_X b)} q$ can be interpreted as homotopies between the paths p and q (or as paths between the points p and q in the topological space associated to the type $a =_X b$), and so on and so forth all the way up.

In the previous section, we have endorsed the Fregean interpretation according to which definitional equality encodes equality of sense and propositional equality encodes equality of reference. We shall now argue that the homotopic interpretation of propositional equalities suggests that we can understand propositional equalities as a relation that encodes *indiscernibility*.[11] On the one hand, path induction guarantees the (constructive upgrading of the) indiscernibility of equals. Hence, terms that are propositionally equal are indiscernible. On the other hand, two indiscernible terms have to be in the same connected component of the corresponding topological space. Otherwise, the fact of belonging to different connected components would be a property by means of which we could discern them.[12] But if indiscernible terms necessarily belong to the same connected component, they can be connected by a path, i.e., they are propositionally equal. As we argued in [17], the predicate "x is in the same connected component as a" can be understood as the homotopic version of the so-called *haecceitas*, that is, of the predicate "$x = a$". Whereas *haecceitas* are usually discarded as a valid property to prove the validity of Leibniz's PII, its homotopic version encodes the non-trivial property that x can be path-connected to a. We can thus conclude that indiscernible terms are propositionally equal.

The fact that indiscernible terms are propositionally equal could be understood as a type-theoretic rephrase of Leibniz's PII. Let's analyze whether this is the case or not. Leibniz's PII is based on the distinction between *numerical identity* (i.e., the fact that a and b are one and the same entity) and *qualitative identity* (i.e., the fact that a and b have the same properties, that is, that they are *indiscernible*) [63]. Leibniz's PII states that if a and b are qualitatively identical, then they are also numerically identical. In other words, Leibniz's PII forbids the existence of differences *solo numero*, that is, of numerical differences that do not rely on qualitative differences. Let's consider two terms $a, b : X$ that are propositionally equal. Since path induction implies that a and b are indiscernible, their difference is *solo numero*, which means that Leibniz's PII does not hold in this formal context. Now, the situation is here a bit more convoluted since we have introduced Frege's distinction between *sense* and *reference*. Hence, Leibniz's notion of

[11] A critical assessment of the understanding of propositional equalities in terms of the notion of indiscernibility can be found in [37].

[12] This presupposes that the elements in the set of connected components can be discerned from each other. Now, this is true for every set (see footnote N°14).

TABLE 1.

Type-Theoretic Concepts	**Homotopic Semantics**
Type X	Topological Space
Term $a : X$	Point
Proof of an equality $p : a =_X b$	Path
Proof of an iterated equality $p =_{(a=_X b)} q$	Homotopy between paths
$\sum_{a,b:X} a =_X b$	Path space of A
Type Family $P : X \to \mathcal{U}$	Fibration
$\prod_{a:X} P(a)$	Space of sections of the fibration

numerical identity can mean either *identity of sense* or *identity of reference*. If we understand *numerical identity* as *identity of reference*, then Leibniz's PII does hold. Indeed, two terms $a, b : X$ that are propositionally equal (and therefore indiscernible) but definitionally different—like the terms $n+m$ and $m+n$ of the type $\mathbb{N}$ of natural numbers—express different senses of one and the same reference. However, it is worth stressing that this validity of Leibniz's PII does not preclude the possibility of having terms that are numerically different *qua* senses and indiscernible. It just imposes the condition that when this is the case, the denoted reference is one and the same. In other words, terms that are numerically different *qua* senses and indiscernible are numerically identical *qua* reference.

An important conceptual question elicited by the homotopic interpretation of MLTT is the following: why does the notion of homotopy type play such a fundamental role in mathematics?

In short, why do these geometric objects show up in the very foundations of mathematics? In order to start addressing this question it is worth noting that a set can be understood as a presentation of a particular kind of homotopy type, namely one in which the only non-trivial homotopy is given by $\pi_0(X)$ (that is, the set of connected components). In other terms, a set presents a homotopy type such that the unique homotopic non-triviality is given by the fact that it has several connected components (where each element of the set corresponds to a particular connected component). In this sense, set-theoretic foundations are 0-truncated homotopic foundations. Reciprocally, homotopic foundations stem from set-theoretic foundations when we relax the 0-truncation condition $\pi_i(X) = 0$ for all $i > 0$. Roughly speaking, removing the condition $\pi_i(X) = 0$ for all $i > 0$ amounts to blow up each "grain" of the set-theoretic "dust" into a connected space that might be topologically non-trivial at every dimension $i > 0$. In this way, the idea according to which every mathematical object has an underlying set is enriched to the thesis that every mathematical object has an underlying topological space (up to homotopy equivalence) that might have holes at every dimension. We could say in graphic terms that—according to the UF program—mathematics is not built on a set-theoretic dust, but rather on a kind of topological "foam". Whereas each grain in a set-theoretic dust is endowed with a trivial self-identity, we can understand each connected component of the foam as a grain endowed with a non-trivial self-identity given by a connected ∞-groupoid.[13]

As Anel remarks, a topological space combines two dimensions: a dimension of *differentiation* or *separation* (related to its different connected component) and a dimension of *identification* or *repetition* (related to the existence of different points in the same connected component, i.e., points that can be path-connected) [3, Sect.1.3]. Now, as Anel argues, it is only the dimension of differentiation that survives in set-theoretic foundations.[14] Since two elements of a set are either different or strictly equal, set-theoretic foundations are discrete foundations so to speak. As we have argued above, whereas points in different connected components are qualitatively different (since the property of belonging to a given connected component is a *qualia* that distinguishes them), points in the same connected component are numerically different *qua* senses and indistinguishable. By 0-truncating the fundamental mathematical objects given by ∞-groupoids to sets, we eliminate the identifications between points in the same connected component and merge all these indistinguishable points within the same equivalence class.[15] In this conceptual framework,

[13]Manin has vividly described this transition from set-theoretic to homotopy-theoretic foundations in the following terms: "Instead of sets, clouds of discrete elements, we envisage some sorts of vague spaces, which can be very severely deformed, mapped one to another, and all the while the specific space is not important, but only the space up to deformation. If we really want to return to discrete objects, we see continuous components [...]. I am pretty strongly convinced that there is an ongoing reversal in the collective consciousness of mathematicians: the right hemispherical and homotopical picture of the world becomes the basic intuition, and if you want to get a discrete set, then you pass to the set of connected components of a space defined only up to homotopy" [27, p.1274].

[14]In Anel's own terms: "[...] the notion of a set only allows one to consider a collection of objects that are different from each other. The only possible predicates within a set are $a \neq b$ or $a = b$, in other words, either a and b are two different elements or they are the same element" [3, Note.1.1].

[15]More precisely (and more generally), a n-truncated ∞-groupoid is an ∞-groupoid in which parallel j-isomorphisms for $j > n$ are isomorphic. For instance, a 0-truncated ∞-groupoid is such that parallel 1-isomorphisms are necessarily isomorphic, which means that there are no holes that could obstruct their identification. In particular, all 1-loops are contractible, i.e., the space is simply connected. Since 1-isomorphisms do not encode any non-trivial topological property, the groupoid-theoretical description of the space can be simplified by removing all the 1-isomorphisms (as well as the higher isomorphisms), i.e., by understanding the space as a set

the introduction of the notion of space into the very foundations of mathematics amounts to enrich the qualitative differences between mathematical objects of the same kind with a notion of difference *solo numero*. Hence the notion of space makes possible the existence of indiscernible terms that are numerically different *qua* senses. As we have argued above, this is compatible with Leibniz's PII since terms that are numerically different *qua* senses and indiscernible can be understood as different presentations of one and the same reference, that is, as numerically identical *qua* reference.

We could say that each connected component of a topological space is homogeneous in the sense that any point is by definition congruent (i.e., path-connectable) to any other point in the same connected component. In Leibniz's terms, (a connected) "[s]pace is something absolutely uniform, and, without the things placed in it, one point of space does not absolutely differ in any respect whatsoever from another point of space" [41, p.682]. It follows that the difference introduced by the position in a homogeneous spatial extension is purely numerical in the Leibnizian sense of the term. This is why spatial position cannot be used to discern entities located in the same connected component of a space.[16] This relation between spatial extension and the existence of differences *solo numero* has been later stressed by Weyl in the following terms:

> "Since the mere Here is nothing by itself that might differ from any other Here, space is the *principium individuationis*. It makes the existence of numerically different things possible which are equal in every respect. [...] Leibniz infers from this the ideality of space and time; for they violate the principle of the identity of indiscernibles, which [...] he postulates as necessary in the domain of substances [...]" [76, p.131].

4. On the Notion of Abstraction

We shall now make a detour in our presentation of UF by briefly addressing an apparently unrelated topic, namely the notion of *abstraction*.[17] Abstraction is a process by means of which one identifies, merges, or glues different objects by methodically "forgetting" certain differences between them. By making abstraction from such differences—i.e., we could say, by partially (co)"equalizing" the corresponding objects—it is possible to constitute new mathematical notions and objects (this is why Weyl calls definitions by abstraction "creative definitions" [76, p.8]). So, abstraction is intimately related to the problem of equality, at least in some sense of the term equality that has to be carefully defined. As Lorenzen writes, "The logic of equality is of particular importance because it is closely related to the possibility of introducing abstract objects" [44, p.44]. For instance (according to Frege's famous example), we can introduce the notion of *direction* by making abstraction from the differences in location between parallel straight lines in Euclidean space, where a given direction is defined as an equivalence class of parallel lines

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of objects with no morphisms. This amounts to uniquely considering the equivalence classes of objects. Hence, a 0-truncated ∞-groupoid is a set.

[16]This point has been clearly stated by Leibniz in the following terms: "When we perceive the existence of a thing, we perceive at the same time that it exists in space, i.e. that an infinite number of other things can exist that could in no way be distinguished from it, or what amounts to the same thing, that it can move and be in one place as well as in another [...]" [40, pp.228-31].

[17]For the history and philosophy of mathematical abstraction see [47] and references therein. For a history of the notion of equivalence relation see [7].

[24, §64, p.74]. In the field of geometry, we can define a manifold by making abstraction from the differences between certain points in the open sets U_i of an atlas, that is, by "gluing" these open sets along the intersections $U_i \cap U_j$ defined by these identifications between points in U_i and points in U_j. In general, we shall understand the process of abstraction as a transition from what we shall call *abstraction data* (typically, a domain of objects endowed with some form of equivalences between them) to the corresponding *abstracta*.

A first attempt to provide a mathematical description of the abstraction operation was introduced by the Peano school and later reconceptualized by Frege [24], Russell [64], Weyl [76] and Lorenzen [43, 44] among others. Given the abstraction data defined by a set X and an equivalence relation $\sim$ on X, the set-theoretic version of what we shall call *abstraction principle* takes the following generic form[18] :

$$\hat{a} = \hat{b} \iff a \sim b, \tag{2}$$

According to the standard understanding of the process of abstraction—which goes back at least to Russell [64, p.116-7]—the transition from the elements in the original domain to the corresponding abstracta is given by the quotient function

$$f_{\sim} : X \to X/\sim \tag{3}$$

that sends each element $a \in X$ to the equivalence class $[a]$. By doing so, the set-theoretic abstraction principle (2) becomes

$$[a] = [b] \iff a \sim b, \tag{4}$$

where the equality on the LHS is a strict equality. This means that $[a]$ and $[b]$ are just different expressions that denote the same abstractum, namely the corresponding equivalence class. In type-theoretic terms, we could say that the process of abstraction transforms an equivalence into a strict equality. Reciprocally, any function $f : X \to Y$ defines an equivalence relation $\sim_f$ on X such that $a \sim_f b$ (for $a, b \in X$) if they are in the same fiber of f. This means that the function $f : X \to Y$ classifies elements in X according to the equivalence relation "having the same value of f". By doing so, we are using the strict equality on Y to define an equivalence relation on X.

At least in the category of sets (as well as in any topos), equivalence relations are *effective*, which means that an equivalence relation $\sim$ can be recovered from the quotient map (3) as the pullback of this map with itself (for a type-theoretic definition of effectivity of equivalence relations see [31, p.346]).[19] This means that the abstraction data can be recovered from the original domain of objects together with the abstracta, i.e., that the quotient map (3) does not entail any loss of information. We could say that in this case the abstraction process fulfills the constructive prescription of not foreclosing the process of construction of the abstracta. However, equivalence relations are the simplest manner by means of which one can encode abstraction data. An equivalence relation is characterized by the fact that there is *at most* one relation

[18]The expression "definition by abstraction" comes from the Peano school [57]. Frege only uses this expression in a letter to Russell (28th, July 1902) [25, p.141] and the latter (after criticizing Peano's definitions by abstraction) uses the expression "principle of abstraction" [64].

[19]More generally, given an equivalence R on an object X of a category, the quotient object X/R is the coequalizer of the diagram $R \rightrightarrows X$ defined by the two projections. We can now consider the so-called *kernel pair* of the morphism $X \to X/R$, i.e., the pullback $X \times_{X/R} X$. The universality of the pullback implies that there is a canonical map $R \to X \times_{X/R} X$. The equivalence relation is *effective* if this map is an isomorphism [16, §2.5, p.101].

between any two elements in the set X. In type-theoretic terms, an equivalence relation on X associates to each pair $(a, b) \in X \times X$ a mere proposition that is true if a and b are equivalent and false otherwise [31, p.201]. Now, there are situations in which two elements in the original domain are related by many equivalences. For instance, a group action $G \times X \to X$ defines an equivalence relation given by $x \sim_G y$ if $y = gx$ for some $g \in G$ (i.e., if x and y are in the same orbit of the group action). If the action is not free, there might be different group elements connecting two elements in the same orbit (i.e., $y = gx = g'x$). We could say that in this case there are different witnesses or proofs of the fact that x and y are equivalent under the group action. However, the equivalence relation $\sim_G$ defined by the group action cannot encode this multiplicity of equivalences between two terms. Therefore, we cannot recover the group action from the quotient set $X/\sim_G$.

Let's consider another example of a situation in which the abstraction data contains many equivalences between two elements in the original domain. In category theory, colimits allows us to glue objects by identifying certain of their components (for the notion of colimit, see for instance [12, Ch.5.6]). For instance, the colimit of the diagram $X \xleftarrow{g} \{*\} \xrightarrow{f} Y$ is the disjoint union $X \sqcup Y$ modulo the identification of $g(*) \in X$ with $f(*) \in Y$. This colimit "glues" X and Y along a single point given by the images of $\{*\}$ in X and Y, i.e., it "glues" X and Y by making abstraction from the difference between $g(*) \in X$ and $f(*) \in Y$. Let's consider now the colimit of the diagram $\{x\} \xleftarrow{g} \{a, b\} \xrightarrow{f} \{y\}$ (we borrow this example from [4]). The colimit of this diagram is a single point, which is given by the identification of $\{x\}$ and $\{y\}$. The identification between these points is a consequence of the fact that $g(a) = x$ and $f(a) = y$. Now, there is a second reason to identify x and y, namely that $g(b) = x$ and $f(b) = y$. Hence, there are two equivalences between $\{x\}$ and $\{y\}$ that justify their identification in the quotient. However, the quotient—given by a single point—keeps no trace of this multiplicity of equivalences between $\{x\}$ and $\{y\}$.

One could argue that the process of construction of the corresponding abstracta is a kind of redundant surplus structure that we can discard without any harm. Following this argumentative line, one might think that the fact that there might be several equivalences between certain elements is irrelevant once we know that at least one equivalence exists. In other terms, we could safely (-1)-truncate sets of equivalences between two elements in the original domain to the mere proposition asserting that an equivalence exists. The problem is that this is not true. What we could call the *pathologies of (non-constructive) abstraction* (like for instance the so-called "bad quotients" arising from group actions that are not free) stem from abstraction protocols that do not take into account the different manners according to which two objects might be equivalent. In order to regularize these pathologies, we need to proceed constructively, that is, we have to take into account the whole non-truncated structure of the abstraction data.

5. Towards a Constructive Abstraction Principle

What we shall call *constructive abstraction* stems from ideas that could be traced back to Leibniz, Peano, and Lorenzen. Closer to us, we find similar ideas in Rodin [62] and Sambin and Valentini [65]. The main idea is to understand the process of abstraction as a restriction of the properties that can be attributed to the corresponding entities. More precisely, the process of abstraction defined by an equivalence relation $\sim$ is understood as a construction of a new domain of entities uniquely carrying $(\sim)$-invariant properties. According to Lorenzen, "The

essence of abstraction is precisely this restriction to invariant propositional forms" [43].[20] Since by definition equivalent terms $a \sim b$ have the same ($\sim$)-invariant properties, the new terms $\hat{a}$ and $\hat{b}$ defined by the abstraction process are indiscernible. Briefly, this alternative understanding of mathematical abstraction suggests that we can understand abstraction as a process that converts equivalent terms into *indiscernible* (but not necessarily strictly equal) terms.[21] We could say that the abstraction operation strips the terms of the original domain of all the properties that, not being ($\sim$)-invariant, could be used to distinguish them. In this way, the abstraction process converts a *relative indiscernibility* among terms in the original domain (encoded by the equivalence relation $\sim$) into an *absolute indiscernibility* among the terms of the new domain defined by the abstraction.

Now, both Peano and Lorenzen continued to use a strict equality between terms defined by equivalent elements—as in (2)—probably because they did not have at their disposal a weaker notion of equality. From a conceptual standpoint, using a strict equality between the new terms $\hat{a}$ and $\hat{b}$ amounts to impose Leibniz's PII—that is, to claim that indiscernible terms are numerically identical—under the condition of understanding Leibniz's notion of *numerical identity* as *identity of sense*. However, we are not forced to impose Leibniz's PII in this form, that is, there is no compelling reason to collapse indiscernibility into identity of sense. By making abstraction of certain differences among the original terms, what we obtain *prima facie* is a multiplicity of indiscernible terms. Interestingly enough, Leibniz himself noted that the process of abstraction by means of which we can constitute mathematical entities leads to a break of the PII:

> "[...] perfect similarity is found only in incomplete and abstract notions, where things are considered [in rations veniunt] only in a certain respect, but not in every way, as, for example, when we consider shapes alone, and neglect the matter that has shape. And so it is justifiable to consider two similar triangles in geometry, even though two perfectly similar material triangles are nowhere found." [39, p.32]

According to Leibniz, the PII only holds for metaphysical substances defined by "complete notions". Since mathematical objects are defined by means of abstraction operations, they are necessarily associated to incomplete notions (incomplete notions, write Leibniz, "cannot be tolerated unless we limit them to being abstractions of the mind" [39, p.297]). All in all, whereas Peano and Lorenzen maintained that a process of abstraction transforms discernible entities that are equivalent in some respect into indiscernible ones, we can add with Leibniz that there

[20]To my knowledge, Peano was the first one to establish a link between abstraction and a constraint on the properties carried by the corresponding abstracta. In Peano's terms, "Soit un objet u; par abstraction on déduit un nouveau objet φu [...] L'objet indiqué par φu est donc ce qu'on obtient en considérant dans u toutes et seules les proprietes qu'il a communes avec les autres objets v tels que $\varphi u = \varphi v$" [58, §38, pp.45-46]. In the same vein (but adding an explicit contrast with respect to set-theoretic abstraction), Lorenzen writes "Usually the act of abstracting (i.e., the transition from certain given objects, between which there exists an equivalence relation, to abstract objects such that the abstract objects represent the given objects as regards the equivalence) is reduced to the act of forming a set, namely, the formation of equivalence classes. [...] Non-set-theoretic abstraction occurs as a *façon de parler*; we restrict ourselves to invariant propositions concerning the given objects. The validity of the propositions must be invariant with respect to replacement of a given object by an equivalent object. We then formulate invariant propositions as if they were propositions concerning new (in fact, abstract) objects" [43].

[21]This point has been stressed by Angelleli in the following terms "The decision to abstract from, that is to ignore non-invariant predicates (with respect to the chosen relation) generates indistinguishability in the initial universe of objects" [6, p.465].

is no reason to flatten this indiscernibility onto an equality of sense. As we have argued above, Leibniz's PII still holds if we understand *numerical identity* as *identity of reference*.

When we introduce an equivalence relation $\sim$ on a set X, what we obtain before projecting to the quotient set $X/\sim$ is a set X equipped with an additional relational structure $\sim$. Rather than simplifying the original domain X by forgetting differences (which is what is done by passing to the quotient $X/\sim$), the pair $(X,\sim)$ enriches X by adding equivalences. Now—as we said above—equivalence relations cannot encode multiple equivalences between two terms. In order to encode multiple equivalence between two terms, we have to enhance the notion of equivalence relation to the notion of *groupoid*. Conceptually, the notion of groupoid resolves each mere proposition $a \sim b$ stating that a and b are equivalent into a set $Hom(a,b)$ of equivalences that witness for (or prove) the fact that a and b are equivalent. The resolution of sets into sets enriched with equivalence relations and the resolution of the latter into groupoids are summarized by the following diagram [5, p.69]:

$$\{Sets\} \underset{\hookrightarrow}{\leftarrow} \{Sets\ with\ Equivalence\ Relations\} \underset{\hookrightarrow}{\leftarrow} \{Groupoids\}$$

The right-oriented maps are inclusions: every set X can be naturally understood as a pair $(X,\sim)$ (where $\sim$ is the trivial equivalence relation) and every pair $(X,\sim)$ is a groupoid with at most one arrow between any two objects (i.e., a groupoid in which each $Hom(x,y)$ is a mere proposition). The left-oriented maps are truncations. Every groupoid has an underlying equivalence relation, where each set $Hom(x,y)$ of isomorphisms in the groupoid is substituted by a mere proposition that is true if $Hom(x,y)$ is non-empty and false otherwise. This means that the set of multiple equivalences between x and y is flattened to the mere proposition stating that they are equivalent. In turn, every pair $(X,\sim)$ defines a set, namely the quotient set $X/\sim$ of equivalence classes. In this way, Fregean abstraction can be constructively resolved by 1) unpacking each equivalence class in $X/\sim$ into a set of equivalent elements (the corresponding fiber of the quotient map) and 2) unfolding each mere proposition stating that two elements are equivalent into the set of equivalences between them. Now, this process of resolutions of sets into groupoids can be prolonged all the way up. For instance, groupoids are 1-truncations of ∞-groupoids. One level higher than groupoids, each hom-set $Hom(x,y)$ of a groupoid can be enhanced into a groupoid whose objects are the isomorphisms in $Hom(x,y)$ (that we can now call 1-isomorphisms) and whose morphisms are 2-isomorphisms between these 1-isomorphisms. In turn, the hom-set $Hom(f,g)$ of 2-isomorphisms between 1-isomorphisms $f,g : x \to y$ can be enhanced into a groupoid. If we proceed enhancing all the hom-sets into groupoids all the way up, we arrive to the full-fledged resolution of the notion of set, namely the notion of ∞-*groupoid*.

In this way, Fregean abstraction and constructive abstraction move in opposite directions so to speak. Whereas Fregean abstraction understands abstraction as a sort of simplification of the original domain that merges equivalent terms within equivalence classes, constructive abstraction understands abstraction as an additional structure of equivalences defined on the original domain X. Whereas Fregean abstraction transforms equivalent terms into abstracta that are strictly equal, constructive abstraction transforms equivalent terms into indiscernible presentations of the same abstractum. Whereas Fregean abstraction truncates sets enriched with equivalence relations $(X,\sim)$ to sets $X/\sim$, constructive abstraction enhances the former to ∞-groupoids:

$$\text{Quotient set } X/\sim \xleftarrow{\text{Fregean abs.}} (X,\sim) \xrightarrow{\text{Constructive abs.}} \infty\text{-Groupoid}$$

This constructive reconceptualization of mathematical abstraction cannot be formalized by means of an abstraction principle of the form (2). First, the abstraction data is not in general encoded by an equivalence relation, but rather by its higher enhancement, namely an ∞-groupoid. Second, equivalent objects should define indiscernible (rather than strictly equal) presentations of the corresponding abstractum. This means that we have to substitute the strict equality on the left hand side of (2) by a notion of equality *qua* indiscernibility. Finally, the set of presentations of an abstractum will be endowed with a higher structure of equalities encoding indiscernibilities that will faithfully mirror the higher structure of equivalences between the objects in the original domain. The difference between the original domain of objects (endowed with equivalences) and the domain of presentations of the abstracta (endowed with a notion of equality *qua* indiscernibility) is that whereas equivalent objects are discernible (i.e., they are similar *only in some respect* defined by the equivalences), the presentations of the same abstractum are strictly speaking indiscernible.

6. Univalence as a Form of Constructive Abstraction

After this long detour about Fregean and constructive abstraction, let's come back to our presentation of UF. We shall now argue that the type-theoretic notion of *univalence* introduced by Voevodsky can be understood as a particular instance of the constructive form of abstraction that we have sketched in the last section.

The constructive abstraction principle that we have sketched above can be understood as a correspondence between an original domain endowed with a higher structure of equivalences (the abstraction data) and a new domain—the domain of the abstracta and its presentations—endowed with a structure of (non-strict) equalities that faithfully reflect the equivalences in the original domain as indiscernibilities between the presentations of the abstracta. Translated into type-theoretic terms, the domain of abstracta and its presentations will be given by a type X. As every type, X is endowed with a structure of higher propositional equalities $(=_X, =_{=_X}, ...)$ that forms an ∞-groupoid. Since equalities are paths along which properties can be transported, terms that are propositionally equal are indiscernible. The constructive principle of abstraction will thus take the form of a correspondence between the original domain endowed with a higher structure of equivalences (the abstraction data) and a type of presentations of the abstracta. We could then informally describe constructive abstraction by means of an abstraction principle of the form:

$$\hat{a} =_X \hat{b} \iff a \sim_\infty b, \tag{5}$$

where the symbol $\sim_\infty$ encodes the fact that the abstraction data is defined by a structure of higher equivalences. By substituting a propositional equality for the strict equality on the LHS of the set-theoretic abstraction principle (2), the type-theoretic abstraction principle (5) transforms equivalent terms a and b into indiscernible presentations $\hat{a}$ and $\hat{b}$ of the same abstractum. All in all, whereas Fregean abstraction is an *equivalence-to-strict-equality principle* (where the strict equality directly relates the abstracta), constructive abstraction is an *equivalence-to-propositional-equality principle* (where the propositional equality relates different presentations of the same abstractum). The fundamental difference—as we have argued above—is that a collection of indiscernible terms related by propositional equalities might carry a non-trivial homotopic structure whose truncation might lead to pathological constructions.

A first example of this constructive abstraction principle is given by the type-theoretic definition of a 1-category $\mathcal{C}$ (see [2] and [31, §9.1]). Here the abstraction data is given by the objects of the category endowed with the equivalences given by the isomorphisms (and no higher morphisms since we are in the realm of 1-categories). The corresponding domain of presentations of the abstracta is given by a type of objects Obj, where the corresponding propositional equality $=_{Obj}$ should faithfully reflect the equivalences in the abstraction data. It can be proved that there is a map sending terms of the equality type $a =_{Obj} b$ to isomorphisms in the corresponding hom-set $Hom_{\mathcal{C}}(a, b)$.[22] If two terms are propositionally equal (i.e., indiscernible), then they are isomorphic. Conceptually, this means that there are no indiscernibilities that are not associated to an isomorphism in the abstraction data. The constructive abstraction principle (5) takes here the form of a constraint called *univalence* stating that the map (6) from propositional equalities to isomorphisms is an equivalence, i.e., that there is an inverse map sending isomorphisms in $Hom_{\mathcal{C}}(a, b)$ to terms of the equality type $a =_{Obj} b$.[23] Since properties can be transported along the paths provided by propositional equalities, isomorphic objects $a \simeq b$ are converted into indiscernible (albeit not strictly identical) presentations of abstracta in the type Obj. All in all, the constructive abstraction principle (5) coincides with the univalent condition stating that isomorphic objects define terms in the type Obj that are propositionally equal, and therefore indiscernible. In turn, different indiscernible terms can be understood as presentations of the same abstractum or, in Frege's terms, as senses of the same reference.

The second (canonical) example of the constructive abstraction principle is the UA introduced by Voevodsky in the framework of the homotopy-theoretic semantics of MLTT [35]. Here the abstraction data is given by types equipped with a notion of equivalence of types. In purely type-theoretic terms, the notion of equivalence can be defined in several manners, namely, through half-adjoint equivalences, bi-inverses (Joyal), or morphisms between types with contractible fibers (Voevodsky) [31, Ch.4]. In the homotopic interpretation of MLTT, equivalences are given by *homotopy equivalences* between topological spaces (i.e., by continuous maps that can be inverted up to homotopies, i.e., such that the two composites are homotopic to the corresponding identity maps). We now want to use the notion of equivalence of types to define abstracta such that equivalent types define indiscernible presentations of the same abstractum. According to the type-theoretic abstraction principle (5), we have to convert the elements in the abstraction data (the types) into terms of another (bigger) type. The required notion of type of small types is provided by the notion of *universe* [31, §1.3, p.24]. So, we shall consider that the presentations of the abstracta defined out of the small types in the abstraction data are terms of a universe

[22] In type-theoretic terms, this means that the following type is inhabited:

$$\text{id-to-iso} : \prod_{a,b:Obj} ((a =_{Obj} b) \to (a \simeq b)). \tag{6}$$

This can be proved by using path induction, which means that it is enough to prove that (6) is inhabited for the case $(a, a, refl_a)$. In this particular case, the *id-to-iso* map is defined by

$$(refl_a : a =_{Obj} a) \mapsto id_a : (a \simeq a).$$

[2, Lemma 3.4, p.7].

[23] According to the standard terminology, univalence is understood as an axiom that converts a *precategory* (i.e., a type of objects Obj endowed with hom-sets equipped with identity arrows $1_a : Hom(a, a)$ and satisfying an associative composition) [2, Def.3.1, p.6] into a category [2, Def.3.6, p.7]. Univalence unifies the two notions of sameness present in a precategory, namely the isomorphisms $a \simeq b$ in the hom-sets $Hom(a, b)$ and the terms of the equality types $a =_{Obj} b$ associated to the type of objects.

$\mathcal{U}$. As in the previous example, path induction can be used to show that types $A, B : \mathcal{U}$ that are propositionally equal are necessarily equivalent [31, Lemma 2.10.1, p.89]. The abstraction principle formalized by the UA encodes the reciprocal statement, namely that equivalent types define terms of the universe $\mathcal{U}$ that are propositionally equal. Since equalities are paths along which properties can be transported, equivalent types define indiscernible presentations of the same abstractum (in the terms of Ahrens et al.: "The Univalence Principle is the statement that equivalent mathematical structures are indistinguishable" [1]).

In order to conclude this brief introduction to the notion of univalence, let's note that the UA has the following important consequences:

- Propositional extensionality: propositions that are logically equivalent in the sense that they imply each other (i.e., such that the function types $P \to Q$ and $Q \to P$ are inhabited) are equivalent as types and therefore (via univalence) propositionally equal.
- Functional extensionality [31, §2.9, p.86 & §4.9, p.144]: two sections of a dependent function type $f, g : \prod_{a:X} B(a)$ are propositionally equal as terms of $\prod_{a:X} B(a)$ if and only if their values are fiberwise homotopic, i.e., if $f(a) =_{B(a)} g(a)$. Briefly, two sections are (propositionally) equal if the have the same (i.e., propositionally equal) values.
- Not every type is a set [31, Ex.3.1.9, 109]: given a type $X : \mathcal{U}$ with non-trivial automorphisms, univalence implies that the universe $\mathcal{U}$ has non-trivial equality types (since univalence $ua : (X \sim X) \to X =_{\mathcal{U}} X$ converts the self-equivalences of X into non-trivial loops in $\mathcal{U}$ based at the name of X). Since $X =_{\mathcal{U}} X$ is not a mere proposition, the type $\mathcal{U}$ is not a set.
- Univalent universes are (uniquely but not universally[24]) classifying: there is an equivalence between families of types $P : X \to \mathcal{U}$ and fibrations $\sum_{a:X} P(a) \to X$ [31, Th. 4.8.3, p. 143], where each fibration can be obtained as a pullback of a tautological fibration over $\mathcal{U}$ along the map $X \to \mathcal{U}$ [31, Th. 4.8.4, p. 143].

7. Conclusion

We have proposed a brief conceptual-oriented introduction to UF in which we have argued (in the wake of a previous work [17]) that the type-theoretic notion of propositional equality can be understood as a relation of indiscernibility. We have then maintained that Leibniz's PII holds in UF under the condition of understanding the Leibnizian notion of *numerical identity* as *identity of reference*. We have used this interpretational framework to argue that Voevodsky's notion of *univalence* can be understood as a particular instance of a constructive notion of mathematical abstraction that resolves, so to speak, Fregean abstraction (see also [62, 65]).[25] This constructive

[24] A universe $\mathcal{U}$ classifies fibrations in a unique manner but—since universes form a cumulative hierarchy [31, p.24]—this classifications is not universal. This is why the UA is called *univalent* (rather than *universal*) axiom. In Voevodsky's own terms: "In the more complex reasoning system with universes this fibration should still satisfy the 'uniqueness' part of the definition of 'universal' but not the 'for all' part.[...] I was looking for a word to use for fibrations which satisfy the uniqueness part but not the 'for all' part and decided to call them 'univalent'. And this is why the foundations are called Univalent Foundations." [73, p.7].

[25] Commenting on Awodey's structuralist reading of UF [10], Shulman has already noted that univalence can be understood as a substitute for Fregean abstraction: "[...] with univalence we no longer need any Fregean abstraction [...] Fregean abstraction is for forgetting irrelevant facts not preserved by isomorphism [...] but in HoTT/UF there are no such facts, since isomorphic types are actually already the same." [67]. In our own terms, with univalence we do not need to merge equivalent terms within a unique equivalence class (Fregean abstraction) since equivalent types are converted into absolutely indiscernible presentations of the the same abstractum.

resolution of Fregean abstraction becomes necessary when the abstraction data contains objects that might be equivalent in different manners. The resulting *constructive principle of abstraction* establishes a faithful correspondence between the abstraction data and a type of presentations of the corresponding abstracta. The equivalences encoded in the abstraction data are transformed into propositional equalities among the different presentations of the corresponding abstractum. Since propositional equalities are paths along which properties can be transformed, presentations of the same abstractum are indiscernible. Hence, the constructive principle of abstraction converts equivalent objects (where the equivalences witness for the fact that the objects are *partially* indiscernible, i.e., indiscernible only in a certain respect) into terms of a type that are *absolutely* indiscernible. In this conceptual and formal framework, abstraction is not understood as a removal of differences, but rather as an enrichment of the original domain that proceeds by adding equivalences that will then be converted into propositional equalities. We could say that constructive abstraction "forgets" inessential information in a controlled manner, that is, in such a way that the abstraction data can be fully recovered from (or faithfully encoded in) the abstracta.[26]

From a philosophical perspective, some authors—notably Awodey [10] and Tsementzis [70, 69]—have analyzed UF in the light provided by a family of interrelated trends in philosophy of mathematics enveloped by the term *mathematical structuralism* (and mainly developed, in its different eliminativist and non-eliminativist variants, by Bourbaki, Benacerraf, Putnam, Resnik, Shapiro, Hellman, and Parsons among others; see [60] and references therein). The idiosyncratic presentation of UF that we have proposed here is intended to consider UF from the standpoint provided by an alternative (and maybe complementary[27]) conceptual framework that stresses above all the constructivist elan of UF. Whereas mathematical structuralism tends to stress the importance of the structural invariants to the detriment of non-structural variants[28], constructivism tends to stress that the homotopic shape defined by the variants do matter. Whereas the structuralist reading of UF stresses that properties are invariant across propositional equalities between different presentations of the same "structure," the point that we want to underline here

[26]This point has been clearly stated by Sambin and Valentini in the following terms: "To build up an abstract concept from a raw flow of data, one must disregard inessential details; in other words, to simplify the complexity of concrete reality one must idealize over it, and this is obtained by 'forgetting' some information. To forget information is the same as to destroy something, in particular if there is no possibility of restoring that information [...] So to abstract involves a certain amount of destruction; our principle is that an abstraction is constructive, that is a reliable tool in getting knowledge which is faithful to reality, not when information is kept as much as possible, but when it is 'forgotten' in such a way that it can be restored at will at any moment. This after all is the test to show that an abstraction does not lead astray from reality; that is, that it preserves truth. It is clear that the first step, and often the only one, to be able to restore what has been 'forgotten' is to know, to be aware of, what has been forgotten, and to keep control of it. So the second principle is that constructivism does not consist of an a priori self-limitation to full information, which would tie constructivism with reluctance to abstraction (as was the case around the 1920s when finitism and intuitionism were somehow identified), but rather in the awareness of the destruction which has been operated to build up a certain abstract concept." [65, p. 224.]

[27]For an analysis of the relation between (non-eliminative) structuralism and (Fregean) abstraction see [42]. It is worth noting that Awodey's structuralist reading of UF relies on Fregean abstraction [10].

[28]For instance, Tsementzis writes: "In one way or another, structuralism in mathematics is about getting rid of inessential properties of mathematical objects. [...] to be a structuralist about mathematical objects is to believe that the only properties of those objects that matter are those we may call *structural* properties. Everything else we can say of mathematical objects is *nonsense* (or, more charitably, *non-mathematical*)" [70].

is that the collection of such presentations carries a non-trivial topological structure that cannot be always safely dismissed. Far from being a redundant surplus structure that we could just discard, modes of presentation—like for instance the terms of an equality type—encode a homotopic structure whose truncation might lead to pathological constructions (like "bad quotients"). Whereas the structuralist reading exploits the fact that isomorphic objects are propositionally equal,[29] we have tried to underscore that we cannot use this "Principle of Structuralism" (as Awodey calls it) to give in to the temptation to simply foreclose the multiplicities of presentations of the same "structure." It is thanks to the intensional treatment of mathematical equalities—i.e., to the constructivist understanding of propositional equalities as types of proofs of its truth—that the set-theoretic dust can be blown up into a homotopic-theoretic topological foam.

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